**Algebra Final**

Math 122 – Harvard University – Fall 2002

Hand in by 3pm Friday, 10 January 2003, in 325 Science Center.

Name ______________________________________________

This exam has two parts: questions below and true/false on the back. All work should be your own. Refer only to your class notes and the course textbook.

Write your answers to the following questions neatly on separate paper, stapled together and attached to this page.

1. (10 points) Prove that $D_4$ is isomorphic to the semidirect product of the Klein 4 group and $\mathbb{Z}/2$.
2. (10 points)
   (a) Give an example of a matrix $A$ in $SO_2(\mathbb{R})$ such that $A^2 = -I$.
   (b) Give an example of $A \neq 0$ in $M_2(\mathbb{R})$ such that $e^A = I$.
3. (10 points) Let $A \in M_n(\mathbb{R})$ be a symmetric, positive-definite matrix. Prove that the largest entries in $A$ occur along the diagonal; in other words, that $\max_{i \neq j} A_{ij} < \max_k A_{kk}$.
4. (10 points) Let $R \subset \mathbb{Q}$ be the set of all rationals of the form $p/q$ where $q \neq 0 \bmod 5$.
   (a) Prove that $R$ is a subring of $\mathbb{Q}$.
   (b) Find all of the ideals in $R$. Which ones are maximal? Which ones are prime?
5. (10 points) Let $G$ be a simple group of order 60.
   (a) Find the number of 3- and 5-Sylow subgroups of $G$.
   (b) Show $A_5$ has a subgroup of order 12.
   (c) Show if $G$ has a subgroup of order 12, then $G \cong A_5$.
   (d) Show $G$ is, in fact, isomorphic to $A_5$.

(50 points) Mark each assertion True (T) or False (F).

1. Given any pair of polynomials $f, g \in \mathbb{C}[x, y]$, there exists a pair of complex numbers $(z, w)$ such that $f(z, w) = g(z, w) = 0$.

2. The only unitary matrices $A \in M_2(\mathbb{C})$ with all their eigenvalues real are $A = \pm I$.

3. The center of the group $\mathrm{O}_3(\mathbb{R})$ is trivial.

4. There is a point $p$ on the cube whose orbit under the symmetries of the cube satisfies $|G \cdot p| = 4$.

5. The group $\mathrm{SL}_2(\mathbb{F}_7)$ has order 168.

6. The group $A_8$ contains an element of order 15.

7. The vector space of all continuous functions $f : [0, 1] \to \mathbb{R}$ has a countable basis.

8. The rings $(\mathbb{Z}/10)[x]$ and $\mathbb{Z}[x, y]/(10y)$ are isomorphic. (Here $(10y)$ denotes the principal ideal in $\mathbb{Z}[x, y]$ generated by $10y$.)

9. Every prime ideal in $\mathbb{Z}$ is also a maximal ideal.

10. Let $n > 0$ be a positive integer. If there exists a finite field $F$ and a ring homomorphism $\phi : \mathbb{Z}/n \to F$, then $n$ must be prime.