# 122 Solution Set 12

## 1 7.4.5

Let $A = [a_{ij}], B = [b_{ij}]$ be two hermitian matrices, and $c, d \in \mathbb{R}$. Clearly the zero matrix, which will function as the additive identity, is hermitian. We must show that $cA + dB$ is hermitian. This is equivalent to showing $ca_{ij} + db_{ij} = c\overline{a}_{ji} + d\overline{b}_{ji}$, which is trivial because $a_{ij} = \overline{a}_{ji}, b_{ij} = \overline{b}_{ji}$ and $c, d \in \mathbb{R}$.

Let $E_{ij}$ denote the $n \times n$ matrix with a one in the $ij$th entry and zeros elsewhere. Then an obvious basis for this space is $\{E_{ii}\} \cup \{E_{ij} + E_{ji}\} \cup \{i(E_{ij} - E_{ji})\}$.

## 2 7.4.16

Let $f, g, h \in P$ and $a, b \in \mathbb{C}$.

Linearity in the second variable:

$$< f, ag + bh >=$$

$$\int_0^{2\pi} \overline{f(e^{i\theta})} \left(ag(e^{i\theta}) + bh(e^{i\theta})\right) d\theta =$$

$$\int_0^{2\pi} \overline{f(e^{i\theta})} ag(e^{i\theta}) d\theta + \int_0^{2\pi} \overline{f(e^{i\theta})} bh(e^{i\theta}) d\theta =$$

$$a \int_0^{2\pi} \overline{f(e^{i\theta})} g(e^{i\theta}) d\theta + b \int_0^{2\pi} \overline{f(e^{i\theta})} h(e^{i\theta}) d\theta =$$

$$a < f, g > +b < f, g > .$$

Conjugate linearity in the first variable:

$$< af + bh >=$$

$$\int_0^{2\pi} \left(\overline{af(e^{i\theta}) + bh(e^{i\theta})}\right) g(e^{i\theta}) d\theta =$$

$$\int_0^{2\pi} \overline{a}\overline{f(e^{i\theta})} g(e^{i\theta}) d\theta + \int_0^{2\pi} \overline{b}\overline{h(e^{i\theta})} g(e^{i\theta}) d\theta =$$

$$\overline{a} < f, g > +\overline{b} < h, g >$$

Positive definite:

$$< f, f >= \int_0^{2\pi} \overline{f(e^{i\theta})} f(e^{i\theta}) d\theta =$$

$$\int_0^{2\pi} |f(e^{i\theta})| d\theta \geq 0$$

where the last inequality holds because $|f(e^{i\theta})| \geq 0$. Now, suppose $< f, f >= 0$. Then we have $\int_0^{2\pi} |f(e^{i\theta})| d\theta = 0$, which implies by elementary analysis (because $f$ is a polynomial and thus is continuous) that $|f(e^{i\theta})| = 0 \Rightarrow f(e^{i\theta}) = 0$ for $0 \leq \theta \leq 2\pi$. Thus $f$ has infinitely many zeros, and because it is a polynomial, this implies in turn that $f$ is identically zero.

(b) One can easily check that the set $\{\frac{x^k}{\sqrt{2\pi}}\}_{k=0,..,n}$ is an orthonormal basis for this space.

# 3 7.5.4

One can easily check that

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

has the property that $PAP^*$ is diagonal and $PP^* = I$. In general for this sort of problem, find a basis of eigenvectors for $A$ and normalize them so that they give a unitary change of basis matrix.

# 4 7.5.9

By the Spectral theorem, we can find some orthogonal matrix $P$ such that $PAP^T = [d_{ii}]$ is diagonal, with $d_{ii} \in \mathbb{R}$. $P$ is orthogonal, so $P^T = P^{-1}$. Thus $Pe^A P^T = Pe^A P^{-1} = e^{PAP^{-1}} = [e^{d_{ii}}](*)$. This final matrix is diagonal, and clearly positive definite because the $d_{ii}$ are real and thus the $e^{d_i i}$ are all greater than zero. Conjugating by the orthogonal matrix $P$ doesn't change the signature of $e^A$, so we're done.

(*) This calculation also offers a proof $e^A = P^T[e^{d_{ii}}]P$ is orthogonal; it's the product of three orthogonal matrices and orthogonal matrices form a group.

# 5 7.5.10

We take $\perp$ to be with respect to the standard hermitian product $<,>$. Suppose $x \in \ker A$. Then $Ax = 0 \Rightarrow < Ax, y >= 0$ for all $y$ which implies $< x, A^*y >= 0$ for all $y$, which implies $x \in (ImA^*)^\perp$.

Conversely, suppose $x \in (ImA^*)^\perp$. Then, for all $y$, $< x, A^*y >= 0 \Rightarrow < Ax, y >= 0$, which implies, since $<,>$ is nondegenerate, that $Ax = 0$.

# 6 7.6.1

We may write this quadric as $X^T AX + BX + c$, where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

and $B = (3, 0, 1)$. By the Spectral theorem, there exists an orthogonal matrix $P$ such that $PAP^T$ is diagonal. Take

$$P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 5 & 1 \end{pmatrix}$$

We will use this matrix to transform the quadric.

$$P^T AP = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

Note $(3, 0, 1)P = (3, -6, 1)$.

Our original quadric was $X^T AX + BX - 6 = 0$. With the change of basis $X = PX'$ we have the new, equivalent quadric

$$X'^T (P^T AP)X + (BP)X + c = 0 = 5x'^2 + 20y'^2 - 4z'^2 + 3x' + 2y' - 2z' - 6 = 0$$

We complete the square:

$$5(x' + \frac{3}{20})^2 - \frac{9}{20} + 20(y' + \frac{1}{20}^2 - \frac{1}{20} - 4(z' + \frac{1}{4})^2 + \frac{1}{4} - 6 =$$

$$5(x' + \frac{3}{10})^2 + 20(y' + \frac{1}{20})^2 - 4(z' + \frac{1}{4})^2 - \frac{25}{4} = 0$$

Thus we have a one-sheeted hyperboloid. Notice that in completing the squares, we affected the constant coefficient. Therefore, it is not sufficient to compute the eigenvalues of the original form to distinguish between the one-sheeted and two-sheeted hyperboloids.

## 7 7.8.4

We prove it. If $A$ is a real skew symmetric matrix, then $iA$ is hermitian, and thus there is some unitary matrix $P$ such that $PiAP^* = PiAP^{-1}$ is diagonal with all real diagonal entries. That is, the eigenvalues of $iA$ are all real. If $\lambda \in \mathbb{R}$ is an eigenvalue of $A$, then $Ax = \lambda x \iff iAx = i\lambda x$. But $i\lambda$, being an eigenvalue of $iA$, is real, so $\lambda$ must be purely imaginary.

## 8 10.1.2

Let $R$ be the smallest subring of $\mathbb{C}$ that contains the real cube root of 2. $1 \in R \Rightarrow \mathbb{Z} \in R$ because $\mathbb{C}$ has characteristic zero. Clearly all elements of the form $a + b2^{1/3} + c4^{1/3}, a, b, c \in \mathbb{Z}$ are in $R$. If we show that the set of such elements, call it $S$, form a ring, we're done. Showing that $S$ is an additive group is trivial, the multiplicative identity is 1, and distributivity follows from distributivity of $\mathbb{C}$ provided that we show $S$ is closed under multiplication. We have, for $a, b, c, d, e, f \in \mathbb{Z}$:

$$(a + b2^{1/3} + c4^{1/3})(d + e2^{1/3} + f4^{1/3}) =$$

$$(ad + 2bf + 2ed) + (bd + ae + 2cf)2^{1/3} + (cd + af + be)4^{1/3},$$

which demonstrates closure.

# 9 10.1.9

(a) Not a ring. Addition is clearly commutative, associative, and the identity is $\varnothing$. However, if $A \neq \varnothing$, $A^{-1}$ doesn't exist. This is because $A$ is contained in $A \cup B$ for all $B$. The multiplicative identity is $U$, and it is clearly associative.

Finally, distributivity:

$$A(B + C) = A \cap (B \cup C) = (A \cap B) \cup (A \cap C) =$$

$$(B \cap A) \cup (C \cap A) = (B \cup C) \cap A = (B + C)A$$

by DeMorgan's laws.

(b) A ring. Addition is clearly commutative. It has identity element $\varnothing$. A series of Venn diagrams will check that it is associative. The inverse of an element $A$ is $A$. Multiplication follows as before, and distributivity can be checked using Venn diagrams.

(c) Not a ring. Addition clearly forms an additive group with the zero function as the additive identity. Multiplication is associative because composition of functions is associative, with the function which sends $x$ to $x$ serving as the identity.

However, distributivity fails. Let $f(x) = x, g(x) = x, h(x) = x^2$. Then we have $h(f(x) + g(x)) = (2x)^2 = 4x^2$ whereas $h(f(x)) + h(g(x)) = 2x^2$.

# 10 10.2.6

(a) The fact that this is a ring follows from the fact that $R[t]$, the set of formal polynomials, is a ring, if we just note that in any sum or product of formal power series, the coefficient of $t^n$ is just a linear combination of coefficients of powers $t^k$, $k \leq n$. For example, if $a_0 + a_1 t + ...$ and $b_0 + b_1 t + ...$ are two formal polynomials in $R[[t]]$, then the coefficient of $t^n$ in their product can be computed by determining the coefficient of $t_n$ in

$$(a_0 + a_1 t + ... + a_n t^n)(b_0 + b_1 t + ... + b_n t^n)$$

if we consider the ring axioms in terms of coefficients, we just apply the fact that the formal polynomials form a ring.

(b) If $p(t)(b_0 + b_1 t + ...) = 1$ for some power series $b_0 + b_1 t + ...$, then $a_0 b_0 = 1$ which implies $a_0$ is invertible.

Now, suppose $a_0$ is invertible. Then we want to find $b_0, b_1, b_2, b_3, ... \in R$ such that $p(t)(b_0 + b_1 t + ...) = 1$. Set $b_0 = a_0^{-1}$. We have that the $n$th coefficient in the product is given by

$$\sum_{i=0}^{n}(a_i b_{n-i})$$

We may define $b_n$ inductively as $-\sum_{i=1}^{n} a_0^{-1}(a_i b_{n-i})$. Thus we have that the $n$th coefficient in the product is

$$\sum_{i=1}^{n}(a_i b_{n-i}) + a_0(-\sum_{i=1}^{n} a_0^{-1}(a_i b_{n-i})) = 0$$