Math 122, Solution Set No. 3

1 3.1 Problem 1

(a) This is a subspace. The zero matrix is symmetric, and the set is closed under addition and scalar multiplication because for symmetric matrices A, B and $c \in F$, $(A+B)^t = A^t + B^t = A + B$ and $(cA)^t = cA^t = cA$.

(b) The zero matrix is not invertible, hence this is not a subspace.

(c) This is a subspace. The zero matrix is upper triangular, and if we add any two matrices with zeroes below the diagonal, their sum will have zeroes below the diagonal. Likewise the set is closed under scalar multiplication.

Note: I required you to show that the subspaces were nonempty for (a) and (c). However, also note that the identity matrix for these as vector spaces is the zero matrix, not $I_n \in GL_n(\mathbb{R})$.

2 3.2 Problem 1

Let $a + b\sqrt{2}, c + d\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$ (a) $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + b) + (c + d)\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$
(b) $-(a + b\sqrt{2}) = -a - b\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$
(c) $(a + b\sqrt{2})(c + d\sqrt{2} = (ac + 2bd) + (ad + bc)\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$
(d) $(a + b\sqrt{2})^{-1} = (a - b\sqrt{2})/(a^2 - 2b^2) \in \mathbb{Q}[\sqrt{2}]$ (note there are no rational numbers a, b s.t. $(a^2 - 2b^2) = 0$ except for the pair $0, 0$.
(e) $1 \in \mathbb{Q} \subset \mathbb{Q}[\sqrt{2}]$
Therefore by Artin (2.1), $\mathbb{Q}[\sqrt{2}]$ is a subfield of $\mathbb{C}$.

3 3.2 Problem 4

If $W = V$, then W is the set of solutions to $AX = 0$ where X is the zero matrix. If not, choose an orthonormal basis $(w_1, \ldots, w_k)$ for W using the Gran-Schmidt process and extend this to a basis $w_1, \ldots, w_k, v_1, \ldots, v_{n-k}$ for V. Then let $A = [v_1, \ldots, v_{n-k}]^t$. If $x \in W$, then we have $x = \sum_{i=1}^k a_i w_i$ and so $Ax = A\sum_{i=1}^k a_i w_i = \sum_{i=1}^k a_i A w_i = \sum_{i=1}^k a_i [v_1 \cdot w_i, \ldots, v_{n-k} \cdot w_i]^t = 0$ since the v_i are orthogonal to the w_i by construction. Now let $x \in V$ s.t. $Ax = 0$. Then x is perpendicular to each vector v_i, so if we write $x = \sum_{i=1}^k a_i w_i + \sum_{i=1}^{n-k} c_i v_i = \sum_{i=1}^k a_i w_i$ since each of the coefficients c_i must be 0. So $x \in W$, i.e. W is exactly the space of solutions for the linear system of equations $AX = 0$.

4 3.2 Problem 11

The determinant of this matrix in $\mathbb{F}_p$ is 10 (mod p). Since we have that A is invertible when det $A \neq 0$, we must have $10 \neq 0(\text{mod p}) \Leftrightarrow p \nmid 10 \Leftrightarrow p \neq 2, 5$. Therefore A is invertible unless in $\mathbb{F}_p$ unless $p = 2, 5$.

5 3.2 Problem 15

(a) If $x \in \mathbb{F}_p$ is its own inverse, then $x^2 = 1(\text{mod p}) \Leftrightarrow (x-1)(x+1) = 0(\text{mod p}) \Leftrightarrow x = 1$ or $x = -1 = p - 1(\text{mod p})$. So since every other element has a distinct inverse, the product of all nonzero elements must be $1 \cdot 1 \cdots 1 \cdot (-1) = -1$. (b) $(p-1)!(\text{mod p})$ is the product we solved for in (a). Therefore it is equal to -1.

6 3.3 Problem 5

Let B_{ij} be the $n \times n$ matrix with a 1 in the $(i,j)^{\text{th}} and (j,i)^{\text{th}}$ entries and zeroes in all other entries. Then the set $\mathcal{B} = \{B_{ij} | i \geq j\}$ is a basis for the symmetric matrices. $\mathcal{B}$ is linearly independent because $\sum_i c_i B_{ij} = 0$ says that the matrix with entries c_i is identically zero, i.e. $c_i = 0 \forall i$. $\mathcal{B}$ spans the set of symmetric matrices because if $A = (a_{ij})$ then if A is symmetric $(a_{ij}) = a_{ji}$ and we have $A = \sum_{n \geq j \geq i \geq 0} a_{ij} B_{ij}$. Therefore $\mathcal{B}$ is a basis.

7 3.3 Problem 7

Suppose there are scalars such that $ax^3 + b\cos x + c\sin x = 0$ Then this equation must be true for all x, so if we set $x = 0$ we obtain $b = 0$. Then if we set $x = \pi/4$ we obtain $x^3 = (c\sqrt{2})/2a$ unless $a = 0$. But in the first case, we have x^3 is a constant function, which is clearly false, so we must have $a = c = 0$. Therefore these three functions are linearly independent.

8 3.3 Problem 16

(a) Let W be a subspace of V. Choose a basis $w_1, \dots, w_k$ for W and extend to a basis $w_1, \dots, w_k, v_1, \dots, v_{n-k}$ for V. Then let $U = \text{span}(v_1, \dots, v_{n-k})$. Then clearly $W + U = V$ since any vector in V can be written as the sum of the basis vectors for U and W. Also, if $w \in W \cap U$ then we have $w = \sum_{i=1}^{k} a_i w_i = \sum_{i=1}^{n-k} b_i v_i \Rightarrow \sum_{i=1}^{k} a_i w_i - \sum_{i=1}^{n-k} b_i v_i = 0 \Rightarrow w = 0$ because these vectors are linearly independent.

(b) Let W, V be as in part (a) with the same basis for V. $W \cap U = 0 \Rightarrow U \subset \text{Span}(v_1, \dots, v_{n-k} \Rightarrow$ dim $U \leq n - k$. Therefore it is impossible to have $W \cap U = 0$ and dim $W+$ dim $U \geq$ dim V.

9 3.4 Problem 10

Let $\mathbb{F}$ be a field of 81 elements. Then it is clear that if V has dimension 3 over $\mathbb{F}$, $|V| = 81^3$. From class, we know any one-dimensional subspace over a finite field has $|\mathbb{F}|$ elements. If we have two distinct one-dimensional subspaces, since their intersection is a subspace it must be the zero subspace and so we can partition V into sets of 80 nonzero elements in each subspace. So there are $(81^3 - 1)/80$ distinct one-dimensional subspaces.

10 3.5 Problem 2

Let A be the space of doubly infinite sequences. Define $\varphi : A \to \mathbb{R}^\infty$ by $\varphi(\dots, a_{-1}, a_0, a_1, \dots) = (a_1, a_{-1}, a_1, \dots)$. Since φ is defined componentwise, it is clearly bijective and linear. Therefore φ is an isomorphism.

11 4.3 Problem 4

Assume T is not multiplication by a scalar. Then T does not have a two-dimensional eigenspace, and so we can choose some v which is not an eigenvector of T, i.e. $T(v) \neq cv$ for any $c \in \mathbb{F}$. So $\{v, T(v)\}$ is linearly independent $\Rightarrow$ this set is a basis, since V has dimension 2. The columns of T w.r.t. this basis are just the coordinates of $T(v), T^2(v)$ in this basis, so $[T] = \begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}$ where $T^2(v) = (a,b)^t$.

Note: The fact that T is not multiplication by a scalar does not mean T has no eigenvectors.