122 Solution Set 8

We take the convention that s_p is the number of Sylow p- subgroups of a particular group G.

1 6.2.4

Suppose A_5 had a subgroup of order 30, say H. Then $[A_5 : H] = 2$ which implies H is normal. But A_5 is simple, so this is a contradiction.

2 6.2.5

I claim A_5 is the only proper normal subgroup of S_5. Suppose for a contradiction that S_5 had another normal proper subgroup H. Then $H \cap A_5$ is normal in A_5, which is simple. Thus $H \cap A_5 = A_5$ or $\{e\}$. If $H \cap A_5 = A_5$, then $|H| \geq 60$ which implies, by Lagrange's theorem, that $|H| = 60$, so we have $H = A_5$, a contradiction. Thus $H \cap A_5 = \{e\}$, which implies, since $HA_5 \leq S_5$, that $|HA_5| = |H||A_5| = 60|H| \leq 120$, the order of S_5. Because H is nontrivial we have $|H| = 2$. Thus H is the cyclic subgroup generated by a single element of S_5. The only such subgroups that are not contained in A_5 are those generated by a single transposition (ij). Because all transpositions are in the same conjugacy class, as proven in class, H is not normal, which provides us with our needed contradiction.

3 6.2.7

We prove the assertion. Suppose the abelian group G has prime order p. Then, because the order of a subgroup H of G must divide the order of G, we have $|H| = 1$ or p. Thus $H = \{e\}$ or $H = G$.

Now we prove the contrapositive of the other direction. Suppose an abelian group G has composite order greater than 1 (the trivial group is automatically non-simple). Then there is a prime p such that p divides $|G|$, which implies by Corollary 4.3 that there is a cyclic subgroup H of order p in G. Because G does not have prime order, H is a proper subgroup of G. In fact, H is normal in G, because all subgroups of an abelian group are normal.

4 6.2.8

(a) From a previous problem set, $|T| = 12$. There are three subgroups of order 2; the stabilizers of edges. These all have trivial intersection, so these subgroups give us 3 distinct order 2 elements. Action by conjugation is transitive on this set of elements, hence we have one conjugacy class of order 3.

Similarly, there are 4 subgroups of order 3, the stabilizers of vertices. Since the intersection of these subgroups is the identity, we have 8 distinct order 3 elements. The action of conjugation by T breaks this into two separate orbits of order 4 (you just have to think about this geometrically; draw a few pictures and convince yourself). Throwing in the trivial conjugacy class, we have

$$|T| = 1 + 3 + 4 + 4.$$

(b) The center $Z(T)$ is a subgroup of T whose order is the sum of all the 1's in the class equation. Thus $|Z(T)| = 1$, which implies $Z(T) = \{e\}$.

(c) $|T| = 2^2 3$. By the Sylow Theorems, there is a subgroup H of T of order 4. Any element of H must have order $1, 2$ or 4, and there are exactly 4 such elements in T, as discussed above. Hence H is the sole Sylow 2-subgroup.

(d) Suppose there was a subgroup $H \leq T$ such that $|T| = 6$. Then H would be normal, and hence would be some union of conjugacy classes of T. But no combination of $1, 3, 4, 4$, the orders of the 4 distinct conjugacy classes of T, sum to 6, so such a subgroup cannot exist.

5 6.4.1

We have $20 = 2^2 5$, so $s_5|4$ and is congruent to 1 (mod 5), so we have s_5=1. This implies that there is exactly one subgroup of order 5 in a group G of order 20. This subgroup must be isomorphic to C_5, and hence has exactly 4 elements of order 5. If there were any other elements of order 5 in G, they would generate their own cyclic subgroup of order 5, which can't happen; there's only one such subgroup. Thus there are exactly 4 elements of order 5 in G.

6 6.4.3

(Thanks to Zhe Lu) It suffices to show s_p or s_q is equal to one, so assume for the sake of contradiction that both are not one. The possibilities, by the third Sylow theorem, are then $s_q = p, p^2$ and $s_p = q$. Since $q = 1 \pmod p$, $p|q-1$, which implies $q \geq p+1$. If $s_q = p$, $p = 1 \pmod q$ which implies $p^2 = 1 \pmod q$, so the third Sylow theorem implies that $p^2 = 1 \pmod q$ whether $s_q = p$ or $s_q = p^2$. Thus $q|p^2-1 = (p-1)(p+1)$. But,from above, $q \geq p+1$, which implies q does not divide $p-1$ and $q = p+1$. The only way this equation can hold for p, q prime is if $p = 2, q = 3$, which corresponds to the case $2^2 3 = 12$. Artin classifies the groups of order 12 in theorem 5.1, and in particular he proves that all such groups have a proper normal subgroup; that is, they are not simple.

7 6.4.6

From class, $|GL_2(F_p)| = (p^2-1)(p^2-p) = p(p-1)^2(p+1)$. The largest power of p dividing this order is p, so we wish to find a subgroup of order p. Consider the cyclic subgroup generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Because, as you can check,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

for all $k \in F_p$, this group is exactly

$$\{\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : k \in F_p\}$$

which has order p.

8 6.4.14

It's obvious that a group G of prime order is simple; by Lagrange's theorem, it can only have the two trivial subgroups, so it can't possibly have a nontrivial normal subgroup. To prove that the rest of the groups G aren't simple, we start with a few lemmas (note p, q denote primes in these lemmas):

Lemma 1: If $|G| = p^k$, $k > 1$, G is not simple.

Proof: By Prop. 6.1.11, $Z(G) \neq \{e\}$. If $Z(G) = G$, then G is abelian, and hence, by problem 6.2.7, is not simple. If $Z(G) \neq G$, then $Z(G)$ provides a proper normal subgroup.

Lemma 2: If $|G| = p^2q$, then G is not simple.

Proof: Given above.

Lemma 3: If $|G| = pq$, then G is not simple.

Proof: If $p = q$, we're done by Lemma 1, so assume without loss of generality that $p > q$. Then $s_p = 1 \pmod p$ and $s_p|q$, which implies $s_p = 1$. Thus the sole Sylow p-subgroup of G provides a nontrivial normal subgroup.

With these lemmas in place, we've dealt with every order less than 60 other than $24, 30, 36, 40, 42, 48, 56$. We deal with these using case-by-case analysis.

24: We proved in class that no group of order 24 is simple.

30: By the Sylow theorems, $s_3 = 1 \pmod 3$,$s_3|10$. Thus $s_3 = 1$ or 10. Suppose $s_3 = 10$. The Sylow 3-subgroups are all of order 3, and have trivial intersection. Thus we have accounted for $(3-1)(10)+1 = 21$ elements of the group. Similarly, $s_5 = 1 \pmod 5$, $s_5|6$. Suppose $s_5 = 6$. Then, again, we have 6 subgroups of order 5, all with trivial intersection, and, by Lagrange, all of these subgroups have trivial intersection with the Sylow 3-subgroups. We've thus found $21 + (5-1)(6) = 45$ distinct elements of the group, which is a contradiction.

36: We have $s_3 = 1$ or 4. Suppose $s_3 = 4$. Then the action of G by conjugation on the set S of Sylow 3-subgroups induces a homomorphism

$$\varphi : G \to S_4$$

Assuming G is simple, $\ker\varphi = G$ or $\{e\}$. If it's G, then conjugation by G fixes all Sylow 3-subgroups. This is a contradiction, for the Sylow 3-subgroups are all conjugate, and we've assumed that there's 4 of them. If $\ker\varphi = \{e\}$, then φ is injective, which is a contradiction, for $|G| = 36 > 24 = |S_4|$.

40: We have $s_5|8, s_5 = 1 \pmod 5$, which implies $s_5 = 1$.

42: We have $s_7|6, s_7 = 1 \pmod 6$, which implies $s_7 = 1$.

48: We have $s_3 = 1$ or 4. Suppose $s_3 = 4$. Then the action of G by conjugation on the set S of all Sylow 3-subgroups induces a homomorphism

$$\varphi : G \to S_4$$

Assuming G is simple, $\ker\varphi = G$ or $\{e\}$. If it's G, then conjugation by G fixes all Sylow 3-subgroups. This is a contradiction, for the Sylow 3-subgroups are all conjugate, and we've assumed that there's 4 of them. If $\ker\varphi = \{e\}$, then φ is injective, which is a contradiction, for $|G| = 348 > 24 = |S_4|$.

54: We have $s_3|2, s_3 = 1 \pmod 3$, which implies $s_3 = 1$.

56: We have $s_7 = 1$ or $s_7 = 8$, and $s_2|7$. Suppose $s_7 = 8$ and $s_2 = 7$. In this case a Sylow 7-subgroup has order 7, so any two Sylow 7-subgroups have trivial intersection. Thus, the set of all Sylow 7-subgroups account for $(7-1)(8)+1 = 49$ elements of G. Consider any two Sylow 2-subgroups, say K, H. These subgroups are of order 8, and they can't share all of their elements. Thus $|K \cup H| > 8$. Further, K, H must have trivial intersection with any Sylow 7-subgroup by Lagrange's theorem. Thus K, H give us an additional $k > 7$ distinct elements of G, which means we've $49 + k > 56$ elements of G, a contradiction.

9 6.4.15

Let G be a group such that $|G| = 33$. By the Sylow theorems, $s_3 = 1 \pmod 3$ and $s_3|11$, which implies $s_3 = 1$. Similarly, $s_11 = 1 \pmod 1)1$ and $s_11|3$, which implies that $s_1 = 1$. Thus we have two subgroups $H, K \leq G$ such that $|H| = 3, |K| = 11$, and H, K are both normal in G. From Lagrange's theorem, $H \cap K = \{e\}$. We also have $HK \leq G$. Noting that $|HG| = |H||G| = 33$, this implies $HK = G$. Putting these facts together and applying Prop. 2.8.6, we have that G is isomorphic to $H \times K$. Since H is isomorphic to C_3 and K is isomorphic to C_{11}, we have G is isomorphic to $C_3 \times C_{11}$. Thus there is only one group (up to isomorphism) of order 33.