Homework 7
Official Solutions
Math 154

III.10.8 We can find $P\{S_1 \geq 0, \ldots, S_{2n-1} \geq 0\}$ by considering all paths that never touch the line $x = -1$. The possible end points for these paths ranges from 0 to $2n - 1$. To count the number of paths that do not touch $x = -1$, we will subtract out paths that touch $x = -1$. The number of paths that end at value k and touch the line $x = -1$ can be found with the reflection principle. By reflecting the origin over $x = -1$, we find this value to be $N_{2n-1,k+2}$. The total number of paths that do not touch $x = -1$ is given by:

$$\sum_{k=0}^{2n-1} N_{2n-1,k} - N_{2n-1,k+2} \tag{1}$$

Note that only terms with odd k are nonzero in this sum. In the sum, the negative part of the kth term is cancelled by the positive part of the $(k+2)$th term, such that the only term remaining is $N_{2n-1,1}$ (the term $N_{2n-1,2n+1} = 0$ since there aren't enough steps to make it to the value $2n + 1$). Dividing $N_{2n-1,1}$ by the total number of paths gives the probability of this event: $N_{2n-1,1}/2^{2n-1} = \binom{2n-1}{n}2^{-(2n-1)} = \binom{2n}{n}2^{-2n} = u_{2n}$ as we wanted to show.

IV.6.6 We want the probability that in r throws of two dice we see all the throws $(1,1),(2,2),\ldots,(6,6)$ at least once. Instead we will calculate the probability that we do not see at least one of the 6 throws. We will call the event that we do not see (i,i) by A_i. In this case then $p_r = 1 - A_1 \cup A_2 \cup \ldots \cup A_6$. The probability of those unions is given by:

$$P_1 = S_1 - S_2 + S_3 - S_4 + S_5 - S_6 \tag{2}$$

Above, $S_1 = \sum p_i, S_2 = \sum p_{ij}$, etc. In this case all of the probabilities $p_i = (\frac{35}{36})^r$, since there are 35 rolls besides (i,i) at any given throw. Likewise $p_{ij} = (\frac{34}{36})^r$, $p_{ijk} = (\frac{33}{36})^r$, etc., for the same reason. In general, $S_i = \binom{6}{i}(\frac{36-i}{36})^r$. So,

$$\boxed{p_r = 1 - \left(\binom{6}{1}(\frac{35}{36})^r - \binom{6}{2}(\frac{34}{36})^r + \ldots - \binom{6}{6}(\frac{30}{36})^r\right)} \tag{3}$$

V.8.8 We have $P\{A|B\} = P\{AB\}/P\{B\}$. In this case, $P\{B\}$ is the probability that two cells are empty and $P\{AB\}$ is the probability that two cells are empty and there is a triple occupancy. The probability that exactly two cells are empty was found in general in chapter 4 of the book and is:

$$\begin{aligned}p_2(7,7) &= \binom{7}{2}\sum_{v=0}^{7-2}(-1)^v\binom{7-2}{v}(1-\tfrac{2+v}{7})^7 \\ p_2(7,7) &= \binom{7}{2}\sum_{v=0}^{5}(-1)^v\binom{5}{v}(1-\tfrac{2+v}{7})^7\end{aligned} \tag{4}$$

Now the probability that two cells are empty and there is a triple occupancy is relatively easy to find. Of the 5 cells that are occupied, 4 are required to have 1 ball and the last must have 3 to have two cells be empty with a triple occupance. In this way, there are $\binom{7}{2}$ ways of arranging the two empty cells, and 5 different cells that can hold the triple.

There are $\binom{7}{3}$ ways of picking the three balls in the triple and 4! ways of arranging the remaining 4 balls. The probability of such an event is $5 \cdot 4! \cdot \binom{7}{3}\binom{7}{2}/7^7$. Our conditional probability is then:

$$P\{A|B\} = \frac{5 \cdot 4! \cdot \binom{7}{3}\binom{7}{2}}{7^7} / \binom{7}{2}\sum_{v=0}^{5}(-1)^v \binom{5}{v}(1 - \frac{2+v}{7})^7$$

$$= 5 \cdot 4! \cdot \binom{7}{3} / \sum_{v=0}^{5}(-1)^v \binom{5}{v}(5-v)^7$$

$$= 5 \cdot 4! \cdot \binom{7}{3}/16800 = 4200/16800 = \boxed{\frac{1}{4}} \qquad (5)$$

In the table, there are only two sets of occupancy numbers with two cells empty: $\{2,2,1,1,1,0,0\}$, and $\{3,1,1,1,1,0,0\}$. The probability of the second occupancy numbers is $0.107098/(0.107098+ .321295) \approx 0.25$.

VI.10.27 The probability that an insect lays r eggs is $p(r; \lambda)$ and the probability that an individual egg develops is p. We want the probability of k total survivors. Let's assume that we lay some n eggs initially. Then, the probability of achieving k survivors is given by the binomial distribution: $\binom{n}{k}p^k q^{n-k}$. If n is less than k this probability will be 0. To get the total probability of having k survivors we just need to cycle through all the possible values of n:

$$p_k = \sum_{n=k}^{\infty} p(n;\lambda)\binom{n}{k}p^k q^{n-k}$$

$$p_k = \sum_{n=k}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\frac{n!}{k!(n-k)!}p^k q^{n-k}$$

$$p_k = \sum_{n=k}^{\infty} e^{-\lambda}\frac{(\lambda p)^k}{k!}\frac{(\lambda q)^{n-k}}{(n-k)!}$$

$$p_k = e^{-\lambda}\frac{(\lambda p)^k}{k!}e^{\lambda q} = \boxed{e^{-\lambda p}\frac{(\lambda p)^k}{k!} = p(k, \lambda p)} \qquad (6)$$

VII.7.1 We will show that for all odd k:

$$n(x)(x^{-1} - x^{-3} + 1 \cdot 3 \cdot x^{-5} - \ldots + (-1)^k(1 \cdot 3 \ldots (2k-1))x^{-(2k+1)}) <$$
$$< 1 - N(x) <$$
$$< n(x)(x^{-1} - x^{-3} + 1 \cdot 3 \cdot x^{-5} - \ldots + (-1)^{k-1}(1 \cdot 3 \ldots (2k-3))x^{-(2k-1)}) \qquad (7)$$

If we take a derivative of the left side of inequality we will get:

$$(-x^{-2} + 3x^{-4} - 1 \cdot 3 \cdot 5x^{-6} + \ldots + (-1)^{k+1}(1 \cdot 3 \ldots (2k+1))x^{-2k})n(x)+$$
$$+(-x)(x^{-1} - x^{-3} + 1 \cdot 3 \cdot x^{-5} - \ldots + (-1)^k(1 \cdot 3 \ldots (2k-1))x^{-(2k+1)})n(x)$$
$$= (-1 + (-1)^{k+1}(1 \cdot 3 \ldots (2k+1))x^{-(2k+2)})n(x) \qquad (8)$$

Similarly if we take a derivative of the right side of the inequality we will get:

$$(-1 + (-1)^k (1 \cdot 3 \ldots (2k-1)) x^{-(2k)}) n(x) \qquad (9)$$

For odd k it is clear that:

$$(1 - (1 \cdot 3 \ldots (2k+1)) x^{-(2k+2)}) n(x) < n(x) < (1 + (1 \cdot 3 \ldots (2k-1)) x^{-(2k)}) n(x) \quad (10)$$

As we calculated before, the left and right sides of this inequality are the negatives of the derivatives of the inequality we are trying to show, so the correctness of our original inequality follows by integrating this between x and ∞. From the first inequality, we see that $1 - N(x)$ tends to the correct quantity. In the problem statement, an even k corresponds to the right side of the inequality (it overestimates $1 - N(x)$) and an odd k corresponds to the left side of the inequality (it underestimates $1 - N(x)$).

VIII.7.4 We need to examine the sum of the $P\{A_n\}$. We are looking for the probability of a run of n consecutive successes between the 2^nth and the 2^{n+1}st trials. So we have a total of $2^{n+1} - 2^n = 2^n$ trials in which we want to fit a run of at least n successes. We can estimate this by considering the sequence S^n, a run of n successes. There are $2^n - n + 1$ different places to put such a sequence, so the probability of having this sequence is roughly $(2^n - n + 1)p^n$. This will be an overestimate since we are double counting situations that have multiple runs of length n. Using this $P\{A_n\}$ we find that the sum:

$$\sum P\{A_n\} \le \sum_{n=1}^{\infty} (2^n - n + 1) p^n \le \sum_{n=1}^{\infty} (2p)^n < 1$$

if $p < 1/2$. If that is the case, only finitely many A_n occur with probability 1. Otherwise, if $p \ge 1/2$, we have $P(A_n) > 1 - (1-p^n)^{2^n/n} > 1 - e^{-(2p)^n/2n}$ and when we sum this up it diverges. Hnce infinitely many A_n's occur with probability 1. The last set of inequalities is obtained b/c the probability that A_n doesn't happen is at most the probability that the $2^n/n$ consecutive blocks of length n satisfy the condition. The probability that a block satisfies the condition is $(1 - p^n)$ hence overall we get that the probability of the complement of A is at most $(1-p^n)^{2^n/n}$ so $P(A_n) > 1 - (1-p^n)^{2^n/n}$.

VIII.7.5 The probability A_n that the nth trial is followed by a run of more than $a\log_{1/p} n$ is about $p^{a\log_{1/p} n} = n^{-a}$. for $a > 1$, $\sum n^{-a}$ converges. By BC, finitely many A_n occur, so $\sup N_n / \log n < 1$.
For $a < 1$, consider the subsequence of trials $a_i = \lfloor nLogn \rfloor$, $i \ge 1$. then $a_{n+1} - a_n \ge Logn$ for large enough n. Hence A_{a_n} for $a < 1$ are independent. A_{a_n} has probability approximately $p^{aLog(nLogn)} = (nLogn)^{-a} \ge (nLogn)^{-1}$. This is proportional to $(n\log_2 n)^{-1}$. By Cauchy Condensation test, the series $\sum (n\log_2 n)^{-1}$ converges iff $\sum 2^n (n2^n)^{-1} = \sum 1/n$, which doesn't. Hence $\sum P(A_n) \ge \sum P(A_{a_n})$ diverges. By BC this means that A_n occurs infinitely many times. So $\sup N_n / Logn \ge 1$. Combined with the above, this implies the equality.

VIII.7.6 Let's assume $p = 1/2$. Let A_n be the event that S_{n*} is possitive and let B_n be the event that $A_n, A_{n+1}, \ldots A_{17n}$ happen. Then if we prove that $\sum P(B_n)$ diverges, we are done

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by BC. Now, note that S_{n^*} is positive iff $S_n - np > 0$. Let $w_n = S_n - np$. Then with probability p, w_{n+1} is $S_n + 1 - p(n+1)$ and with probability $(1-p)$, $w_{n+1} = S_n - p(n+1)$. In the case $p = 2$ we have $w_{n+1} = w_n + 1/2$ or $w_{n+1} = w_n - 1/2$. Hence w_n follows a random walk and $P(B_n)$ is just the probability that a path of length $16n + 1$ is positive. But we have proved before, by ballot theorem that $P/N = x/n$ where x is the point hight of the endpoint of the random walk. Now the probability that a random walk of length $16n + 1$ is positive is $\frac{\sum P_x}{\sum N_x} = \frac{1}{16n+1} \frac{\sum N_x * x}{\sum N_x} > \frac{1}{(16n+1)}$. Hence when we sum up over n we get that $\sum P(B_n)$ is rougly $1/16$ times the harmonic series which diverges. Hence the required event occurs infinitely many times.

Note that for the general case when p is not $1/2$ we observe a generalized random walk that at each step would go up by $(1-p)$ with probability p and down by p with probability $(1-p)$. It is intuitively true that the proportion of positive random walks should remain the same, hence greater than $1/n$ b/c on expectation at each step w goes up by $p(1-p)$ and down by $p(1-p)$.

Note that 17 didn't have any role; the problem remains true for any constant.