

Homework 2

Real Analysis

Math 212a – Harvard University – Fall 1998

Due Friday, 2 October 1998

Do the following problems from Royden:

Chapter 2: 54. Chapter 3: 2, 5, 11, 12, 17.

1. Let \mathbb{Z}_{10} be the ring of 10-adic integers (all formal sums $x = \sum_0^\infty a_n 10^n$ with $a_n \in \{0, 1, 2, \dots, 9\}$, with addition and multiplication defined as for decimal numbers). Show \mathbb{Z}_{10} has zero divisors; that is, construct $x, y \in \mathbb{Z}_{10}$ such that $xy = 0$ even though $x \neq 0$ and $y \neq 0$.
2. Let $A \subset [0, 1]$ be a (measurable) set of positive measure.
 - (a) Prove A contains a Cantor set (a compact subset homeomorphic to $\{0, 1\}^{\mathbb{N}}$).
 - (b) Prove that for any $\epsilon > 0$, there is an interval I of positive length such that $m(A \cap I)/m(I) > 1 - \epsilon$.
 - (c) Prove $A - A = \{a - b : a, b \in A\}$ contains an open interval.
 - (d) Prove $A + \mathbb{Q} = \mathbb{R}$ up to a set of measure zero (i.e. $m(\mathbb{R} - (A + \mathbb{Q})) = 0$.)
 - (e) Prove A contains a non-measurable set.
3. Let M denote the set of measurable subsets of $[0, 1]$, modulo the equivalence relation $A \sim B$ if $m(A \Delta B) = 0$. Let $d(A, B) = m(A \Delta B)$. Show (M, d) is a complete metric space.
(The *symmetric difference* of two sets is defined by $A \Delta B = (A - B) \cup (B - A)$.)