Homework 3
Real Analysis
Math 212a – Harvard University – Fall 1998
Due Friday, 9 October 1998

1. Show that if $m^*(E) = 0$ then E is measurable.

2. Given an example of a closed set $F \subset [0, 1]$ with empty interior but with positive measure.

3. Give an example of a set E of measure zero such that

$$E - E = \{x - y \,:\, x, y \in E\}$$

contains an interval.

4. Let $\mathcal{U} = \{U \subset \mathbb{R} \,:\, U \text{ is open}\}$ be the collection of all open subets of the real numbers. What is the cardinality of $\mathcal{U}$? For example, is there a bijection between $\mathcal{U}$ and $\mathbb{R}$?

5. Let C be the Cantor middle-thirds set.
   (a) Show C has measure zero.
   (b) Construct a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f(C)$ has positive measure.
   (c) Using the preceding f, show there is a measurable set A such that $f(A)$ is not measurable. (Hint: any $A \subset C$ is measurable.)
   (d) Show that $f(B)$ is measurable whenever B is a Borel set. Thus A is an example of a Lebesgue measurable set that is not Borel.
   (e) Construct measurable functions f and g such that $f(g(x))$ is not measurable.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function vanishing outside $[0, 1]$.
   (a) Show there is a sequence of continuous functions such that $f(x) = \lim f_n(x)$ for almost every x.
   (b) Prove that $f_n(x) = f(x + 1/n)$ converges to f in measure; that is, for every $\epsilon > 0$, $m\{x \,:\, |f_n(x) - f(x)| > \epsilon\} \to 0$.
   (c) Construct a measurable f which cannot be expressed as a limit $f(x) = \lim f_n(x)$ of an increasing sequence of continuous functions, $f_1 \leq f_2 \leq \cdots$.