**Homework 6**
Real Analysis
Math 212a – Harvard University – Fall 1998
Due Friday, 30 October 1998

Royden:
Chapter 5: 23, 27, 28.

1. Let $\mathbb{Q} = \{q_1, q_2, \ldots\}$ be an enumeration of the rational numbers, and define $f(x) = \sum_{q_i < x} 2^{-i}$. Show that $f'(x) = 0$ a.e.

2. Let $h : [0,1] \to [0,1]$ be a homeomorphism.

   (a) Show that for any open set $U$,
   $$\int_U |h'(x)|\, dx \leq m(h(U)).$$

   (b) Suppose there are $E_n \subset [0,1]$ with $m(E_n) \to 1$ but $m(h(E_n)) \to 0$. Prove there is a measurable set $A \subset [0,1]$ with $m(A) = 1$ but $m(h(A)) = 0$.

   (c) We say $h$ is *singular* if $h'(x) = 0$ a.e.
   Prove that $h$ is singular iff there is a set $A \subset [0,1]$ with $m(A) = 1$ but $m(h(A)) = 0$.

3. Define $a_i : [0,1] \to \{0,1\}$ by setting $a_i(x)$ equal to the $i$th digit in the binary expansion of $x$. All integrals below are over $[0,1]$.

   (a) Prove that $\int a_i = \int a_i^2 = 1/2$ and $\int a_i a_j = 1/4$ for $i \neq j$.

   (b) Letting $s_n(x) = \sum_1^n a_i(x)$, compute $\int (s_n/n - 1/2)^2$.

   (c) Show that for any $g : [0,1] \to \mathbb{R}$, we have
   $$m\{x \,:\, |g| > \epsilon\} \leq \frac{1}{\epsilon^2} \int g^2.$$

   (d) Show that $s_n(x)/n \to 1/2$ in measure. Explain the sense in which this means 'most $x$ in $[0,1]$ have 50% zero's and 50% one's in their binary expansion.'

4. Fix $0 < p < 1$ and $q = 1 - p$; one can think of $p$ and $q$ are the probabilities of heads and tails for a (maybe poorly balanced) coin.

   (a) Show there is a (unique) increasing homeomorphism $h : [0,1] \to [0,1]$ such that for any binary interval $I = [a, a + 2^{-n}]$ with $a = 0.a_1 a_2 \ldots a_n$ in base 2, we have $m(h(I)) = p^h q^t$ where $h + t = n$ and $h = \sum_1^n a_i$. (Here $h$ and $t$ can be thought of as the number of heads and tails in a trial of $n$ flips.)

   (b) Show that $s_n(h^{-1}(x))/n \to p$ in measure, where $s_n(x)$ is the number of one's in the first $n$ binary digits of $x$, as in the previous exercise.

   (c) Show that if $p \neq 1/2$, then $h$ is a singular homeomorphism.