Riemann Surfaces Homework 2

Math 213b — Harvard University — Spring 2001

Due 4 April 2001

1. Let $\omega \in \Omega(X)$ be a holomorphic 1-form on a Riemann surface X. Show that locally $\omega = \partial u$, where u is a real-valued harmonic function.

2. Show that for any $p \in X$ and $\omega \in \Omega(X)$, there is a local coordinate z at p with $z(p) = 0$, in which we have $\omega = z^n \, dz$ for some $n \geq 0$.

3. Show that for any nonconstant holomorphic map $f : X \to Y$ and any meromorphic 1-form ω on Y, we have $\operatorname{Res}(p, f^*(\omega)) = \operatorname{mult}(f,p) \cdot \operatorname{Res}(f(p), \omega))$.

4. Let $\mathcal{Q}$ denote the sheaf of meromorphic 1-forms on a Riemann surface X with zero residue at every pole. Show that $0 \to \mathbb{C} \to \mathcal{M} \xrightarrow{d} \mathcal{Q} \to 0$ is an exact sequence of sheaves. Interpret the connecting homomorphism $\mathcal{Q}(X) \to H^1(X, \mathbb{C})$ in terms of periods.

5. Prove *de Rham's theorem*: for any n-dimensional manifold M, we have
$$H^p_{DR}(M) \cong H^p(M, \mathbb{C})$$
for all $p \geq 0$. (Hint: letting $\mathcal{Z}^q$ denote the sheaf of closed q-forms, observe that the sequence of sheaves $0 \to \mathcal{Z}^q \to \mathcal{E}^q \xrightarrow{d} \mathcal{Z}^{q+1} \to 0$ is exact.)

6. Let $H^2(\Delta)$ be the space of holomorphic functions on the unit disk such that $\|f\|_2^2 = \int_\Delta |f|^2$ is finite. Prove that $H^2(\Delta)$ is a Hilbert space, and that for any $r < 1$, the map $T : H^2(\Delta) \to H^2(\Delta)$ given by $(Tf)(z) = f(rz)$ is a compact operator.

7. Suppose $\Delta f = u$ where u is a smooth function on $\mathbb{C}$ with compact support, and $f(z) \to 0$ as $|z| \to \infty$.
   (a) Prove that f is proportional to $u * \log|z|$.
   (b) Now suppose u is just a continuous function with compact support, and $f = u * \log|z|$. Prove that $f \in C^1(\mathbb{C})$, i.e. f has a continuous first derivative.

8. Why is there no exponential sequence for meromorphic functions, $0 \to \mathbb{Z} \to \mathcal{M} \to \mathcal{M}^* \to 0$?

9. Define the sheaf of (locally finite) divisors Div on a Riemann surface X, and show $H^1(X, \mathrm{Div}) = 0$. (Hint: use a discontinuous, integer-valued partition of unity.)

10. Show that on any Riemann surface X, the sequence $0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathrm{Div} \to 0$ is exact. Conclude there is a natural map $\alpha : \mathrm{Div}(X) \to H^1(X, \mathcal{O}^*)$ on any Riemann surface X.

    (In fact $H^1(X, \mathcal{M}^*) = 0$ and α is surjective; cf. Forster, exercise 30.1.)

11. Let X be a compact Riemann surface. Show that $H^2(X, \mathbb{Z})$ is naturally isomorphic to $\mathbb{Z}$ for any compact Riemann surface, and construct a natural map $\delta : H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$.

12. Composing the maps from the preceding problems, we obtain a map $\delta \circ \alpha : \mathrm{Div}(X) \to H^2(X, \mathbb{Z}) = \mathbb{Z}$. Show this map simply sends a divisor to its degree; that is, $\delta \circ \alpha(D) = \deg D$.

13. Let X be a compact Riemann surface of genus 2.

    (a) A point $P \in X$ is a *Weierstrass point* if there exists a nonzero $\omega \in \Omega(X)$ with a double zero at X. Show that X has exactly 6 Weierstrass points. (Hint: a basis $\langle \omega_1, \omega_2 \rangle$ for $\Omega(X)$ determines a degree two meromorphic function $f = \omega_1/\omega_2 : X \to \mathbb{P}^1$, and the Weierstrass points coincide with the critical points of f.)

    (b) Compute $\dim H^0(X, \mathcal{O}_{nP})$ for every $n \in \mathbb{Z}$ and $P \in X$.