Riemann Surfaces Homework 3

Math 213b — Harvard University — Spring 2001

Due 18 April 2001

Throughout X is a compact Riemann surface.

1. Let $V \subset \mathcal{M}(X)$ be a finite-dimensional subspace of positive dimension. Show there is a single divisor $D \geq 0$ such that $(f)_\infty = D$ for most $f \in V$; in particular, most $f \in V$ have the same degree. Describe the set of $f \in V$ such that $(f)_\infty \neq D$.

   (Here $(f)_\infty = \sum n_P \cdot P \geq 0$ is the *polar divisor* of f; P ranges over the poles of f, and $n_P = -\operatorname{ord}_P(f)$.)

2. Let D be a divisor on X such that $h^0(D) > 0$ but $|D|$ is *not* base-point free. Describe the hyperplane sections of the natural map $\phi_D : X \to \mathbb{P}H^0(X, \mathcal{O}_D)^*$. (Hint: consider the largest divisor B such that $B \leq E$ for all $E \in |D|$.)

3. Prove that for any pair of effective divisors D, E on X, we have

$$\dim |D| + \dim |E| \leq \dim |D + E|.$$

4. Let $D \geq 0$ be an effective divisor. We say D is *special* if there exists a holomorphic 1-form $\omega \neq 0$ with $(\omega) \geq D$; equivalently, if there exists a canonical divisor K with $D \leq K$.

   Prove that we have $\dim |D| \geq \deg D - g$, and equality holds unless D is special.

5. Prove that any effective divisor with $\deg D < g$ is special, and any special divisor satisfies $\deg D \leq 2g - 2$.

6. Prove that any special divisor $D \geq 0$ satisfies

$$\deg D - g < \dim |D| \leq \frac{1}{2} \deg D.$$

   (Hint: for the upper bound, apply (3) with $E = K - D$, where $K = (\omega)$.)

7. State and prove a theorem saying that a 'typical' effective divisor of degree g is not special. (Hint: $D = P_1 + \ldots + P_g$ is special if and only if the points $f(P_i)$ lie on a hyperplane under the canonical map $f : X \to \mathbb{P}^{g-1}$.)

8. Show that the space $X^{(n)} = X^n/S_n =$ the space of unordered n-tuples of points on X, carries the natural structure of a complex n-manifold. (Hint: there is an isomorphism between $\mathbb{C}^{(n)}$ and $\mathbb{C}^n$ sending $(r_1, \ldots, r_n)$ to the coefficients of the polynomial $(z - r_1)\cdots(z - r_n)$.)

9. Fix a divisor D of degree $n > 0$ on a X. Define $\phi : X^{(n)} \to \mathrm{Jac}(X)$ by $\phi(P_1, \ldots, P_n) = [-D + \sum P_i]$. Show for each $x \in \mathrm{Jac}(X)$, fiber $\phi^{-1}(x)$ is a projective space $\mathbb{P}^k$ (or empty). Express k in terms of (P_i) when $x = \phi(P_i)$ (using Riemann-Roch).