Advanced Complex Analysis
Homework 4
Due Tuesday, 26 February 2013

You may assume the Riemann existence theorem ($\mathcal{M}(X)$ separates points).

1. Let $p(x,y) = \sum_{0\leq i+j\leq d} a_{ij}x^i y^j \in \mathbb{C}[x,y]$ be a polynomial defining a smooth Riemann surface $X^* \subset \mathbb{C}^2$ of degree d. Let $\pi : X^* \to \mathbb{C}$ be projection to the x-coordinate.

   (i) Show that for 'typical' p (i.e. for generic coefficients a_{ij}), the map π is proper of degree d.

   (ii) Determine the 'typical' number of critical points $|C(\pi)|$.

   (iii) Let $\overline{\pi} : X \to \widehat{\mathbb{C}}$ denote the compact branched covering obtained by completing $\pi : X^* \to \mathbb{C}$. Show that $\pi^{-1}(\infty)$ 'typically' consists of d points.

   (iv) Derive a formula for the 'typical' genus of X in terms of d.

2. Let $P = \{2, 3, 5, \ldots\}$ be the prime numbers, endowed with the cofinite topology (a set is closed iff it is finite or the whole space). Given an open set $U = P - \{p_1, \ldots, p_n\}$, let $\mathcal{F}(U)$ be the ring $\mathbb{Z}[1/p_1, \ldots, 1/p_n] \subset \mathbb{R}$, and let $\mathcal{F}(\emptyset) = (0)$.

   Show that with the natural restriction maps, $\mathcal{F}$ is a sheaf of rings over P. What is the stalk $\mathcal{F}_p$?

3. Prove that for any sheaf of abelian groups $\mathcal{F}$ on a space X and any open set $U \subset X$, the group $\mathcal{F}(U)$ is naturally isomorphic to the group of continuous sections of the espace étalé $|\mathcal{F}|$ over U.

4. Let $U \subset \mathbb{C}$ be an open set. Prove that every complex-linear ring homomorphism $\chi : \mathcal{O}(U) \to \mathbb{C}$ is given by $\chi(f) = f(p)$ for some $p \in U$.

5. Show that every compact Riemann surface X arises by analytic continuation. That is, given X show there is a power series $f(z) = \sum a_n(z-p)^n$ whose maximal analytic continuation lives on a Riemann surface X^* which embeds as the complement of a finite set in X.

6. Let $z : X \to \widehat{\mathbb{C}}$ be a cyclic branched covering with Galois group $\mathbb{Z}/n$, generated by $g : X \to X$. Suppose $\mathcal{M}(X) = \mathbb{C}(z, f)$. Using f and g and perhaps a particular point on X, construct a rational function $F \in \mathbb{C}(z)$ such that $\mathcal{M}(X) \cong \mathbb{C}(z, \sqrt[n]{F})$. What points are included among the zeros and poles of F?