Advanced Complex Analysis
Homework 8
Due Tuesday, 2 April 2013

1. Let $\mathcal{Q}$ denote the sheaf of meromorphic 1-forms on a Riemann surface X with zero residue at every pole. Show that $0 \to \mathbb{C} \to \mathcal{M} \xrightarrow{d} \mathcal{Q} \to 0$ is an exact sequence of sheaves.

2. Let X be a compact Riemann surface. Prove that for every homomorphism $\rho : \pi_1(X) \to \mathbb{C}$, there exists an $\omega \in \mathcal{Q}(X)$ such that $\rho(C) = \int_C \omega$.

3. Let $X = \mathbb{C}/\Lambda$ be a Riemann surface of genus one. Find an explicit pair of forms $\omega_1, \omega_2 \in \mathcal{Q}(X)$ whose periods form a basis for $H^1(X, \mathbb{C}) \cong \mathbb{C}^2$.

4. Suppose $\Delta f = u$ where u is a smooth function on $\mathbb{C}$ with compact support, and $f(z) \to 0$ as $|z| \to \infty$.

   (i) Prove that f is proportional to $u * \log|z|$.
   (Here $f * g(z) = \int_{\mathbb{C}} f(w) g(z - w) \, |dw|^2$.)

   (ii) Now suppose u is just a continuous function with compact support, and $f = u * \log|z|$. Prove that $f \in C^1(\mathbb{C})$, i.e. f has a continuous first derivative.

5. Using the $\overline{\partial}$-operator, define the Dolbeault cohomology group $H^{p,q}(\mathbb{C}^n)$ and prove $H^{0,n}(\mathbb{C}^n) = 0$. Given an example of a smooth $(0,1)$-form ω on $\mathbb{C}^2$ such that the equation $\overline{\partial} f = \omega$ has no solution.

6. Let $q(x)$ be a monic polynomial of degree $2g + 2$ with simple zeros, $g \geq 2$, and let X be the hyperelliptic curve $y^2 = q(x)$. Let $P_0, P_\infty \in X$ be points with $y(P_0) = 0$ and $y(P_\infty) = \infty$.

   (i) Compute $h^0(nP_0)$ and $h^0(nP_\infty)$ for all $n \in \mathbb{Z}$.

   (ii) Find a divisor $D = P - Q$ on X which is not principal, such that $2D$ is principal.

   (iii) Find a divisor D on X such that $2D$ is a canonical divisor. Compute $h^0(D)$ for your example.