Advanced Complex Analysis
Homework 1
Due Thursday, 6 February 2014

Prepare your answers neatly, stapled together, with the problems clearly separated, and with your name on the first page. Your are encouraged to work with others, but you must cite your collaborators and references.

This first homework is meant to jog your memory about some background for this course, and offer a topological challenge (last problem).

1. Let $P(x)$ be a polynomial of degree $d > 1$ with $P(x) > 0$ for all $x \geq 0$. For what values of $\alpha \in \mathbb{R}$ does the integral
$$I(\alpha) = \int_0^\infty \frac{x^\alpha}{P(x)}\,dx$$
converge? Given a formula for $I(\alpha)$ in terms of residues. Compute $I(\alpha)$ for $P(x) = 1 + x^4$.

2. Let $X = \mathbb{C} - \{0, 1\}$. Give a pair of closed differential forms on X that furnish a basis for the deRham cohomology group $H^1(X, \mathbb{C}) \cong \mathbb{C}^2$.

3. Let $f(z) = z(e^z - 1)$. Prove there exists an analytic function $h(z)$ defined near $z = 0$ such that $f(z) = h(z)^2$. Find the first 3 terms in the power series expansion $h(z) = \sum a_n z^n$. Does $h(z)$ extend to an entire function on $\mathbb{C}$?

4. Let $f_t(z)$ be family of entire functions depending analytically on $t \in \Delta$. Suppose $f_t(z)$ is nonvanishing on S^1 for all t. Prove that for each $k \geq 0$,
$$N_k(t) = \sum_{|z|<1, f_t(z)=0} z^k$$
is an analytic function of t. (The zeros of $f_t(z)$ are taken with multiplicity.)

5. Let $B \subset \mathbb{C}$ be the union of the circles of radius 1 centered at ± 1. (i) Describe the universal cover of B as a topological space. (ii) Draw a (connected) covering space B' of B with deck group S_3. (iii) Draw a picture of $B'' = B'/A_3$. What is the deck group of B''/B?

6. Let S be a closed smooth surface of genus two. Let us say a surface T is *large* if it arises as a (connected) regular covering space of S with an infinite deck group. Describe as many different large surfaces as you can (preferrably five), and prove no two surfaces on your list are diffeomorphic to one another.