Advanced Complex Analysis
Homework 9
Due Tuesday, 8 April 2014

1. Prove that the sheaf of divisors on a compact Riemann surface satisfies $H^k(X, \mathrm{Div}) = 0$ for $k \geq 1$.

2. Show that the natural map $H^1(X, \mathcal{O}^*) \to H^1(X, \mathcal{M}^*)$ is surjective.

3. Let $q(x)$ be a monic polynomial of degree $2g + 2$ with simple zeros, $g \geq 2$, and let X be the hyperelliptic curve $y^2 = q(x)$. Let $P_0, P_\infty \in X$ be points with $y(P_0) = 0$ and $y(P_\infty) = \infty$.

   (i) Compute $h^0(nP_0)$ and $h^0(nP_\infty)$ for all $n \in \mathbb{Z}$.

   (ii) Find a divisor $D = P - Q$ on X which is not principal, such that $2D$ is principal.

   (iii) Find a divisor D on X such that $2D$ is a canonical divisor. Compute $h^0(D)$ for your example.

4. Let X be the compact Riemann surface defined by $y^2 = 1 - x^6$, and let $P = (0, 1) \in X$ in the coordinates (x, y).

   (i) Find the least $n > 0$ such that $h^0(nP) > 1$.

   (ii) For this value of n, find an explicit rational function $f(x, y)$ such that $f \in H^0(X, \mathcal{O}_{nP})$.

   (iii) Compute the principal part of this function,

   $$f(x) = \frac{a_n}{x^n} + \cdots + \frac{a_1}{x} + O(1),$$

   using the local coordinate x at P.

   (iv) Verify that $\mathrm{Res}_P(f\omega) = 0$ for all $\omega \in \Omega(X)$.

5. Let D be a divisor on a compact Riemann surface X. Show that every element of $H^1(X, \mathcal{O}_D)$ can be represented by Mittag-Leffler data, i.e. by a cocycle of the form $g_{ij} = f_i - f_j \in \mathcal{O}_D(U_{ij})$ where $f_i \in \mathcal{M}(U_i)$. (You may use the isomorphism $H^1(X, \mathcal{O}_D)^* \cong \Omega_{-D}(X)$.)

6. A *canonical divisor* $K \subset \mathbb{P}^n$ is the formal sum of hypersurfaces determined by the zeros and poles of a meromorphic canonical form $\omega = \omega(z)\, dz_1 \wedge \cdots \wedge dz_n$. Give an example of a canonical divisor on $\mathbb{P}^2$. What is the degree of a canonical divisor K on $\mathbb{P}^n$? (I.e. for what value of d is dH a canonical divisor, where H is a hyperplane?)