

Conformal Dynamics Homework 4
Math 275 — Harvard University — Fall 2001

1. Let $X = \mathbb{H}/\Gamma$ where Γ is generated by $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. Prove that X is isomorphic to the hexagonal torus $\mathbb{C}/\mathbb{Z} \oplus \omega\mathbb{Z}$ with a point removed (here $\omega^3 = 1$).
2. Let S be the punctured torus S with $\pi_1(S) = \langle a, b \rangle$. Is the word $ab^2ab^3a^2b$ represented by a simple closed curve on S ?
3. Prove that the punctured square torus can be represented as $\mathbb{H}/\langle A, B \rangle$, where $\text{tr } A = \text{tr } B = 2\sqrt{2}$. What is $\text{tr } AB$? Show that A and B can be chosen to lie inside the image of $\text{GL}_2(\mathbb{Q})$ in $\text{PGL}_2(\mathbb{R})$, but not inside the image of $\text{GL}_2(\mathbb{Z})$.
4. Show that the group of all outer automorphisms of $G = \mathbb{Z} * \mathbb{Z}$ is generated by three ‘reflections’ (i.e. elements of order 2). Describe the natural action of these reflections on (a) the space of simple closed curves on the punctured torus, (b) on \mathbb{H} , and (c) on the representation variety $\mathcal{V}(G)$.

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1. Let $\phi : A(\mathbb{R}) \rightarrow A(\mathbb{R})$ be a quasiconformal automorphism of an annulus, commuting with the action of an irrational rotation. Prove that ϕ is Lipschitz.
 2. Let $f(z)$ be a polynomial of degree d , and define $h : \mathbb{C} \rightarrow \mathbb{R}$ by

$$h(z) = \lim_{n \rightarrow \infty} |f^n(z)|^{1/d^n}.$$

Show that the limit above exists, that $h(f(z)) = h(z)^d$, that $U = \{z : h(z) > 0\}$ coincides with the attracting basin of infinity, and that $\partial U = J(f)$.

Show that for $z, w \in U$, we have $h(z) = h(w)$ iff the closures of the small orbits of z and w agree. Thus h gives the foliated small orbit quotient on the basin of infinity.

3. Consider the family of cubic polynomials

$$f_t(z) = -z^3 + 12z + t$$

over $X = \{t : |t| < 1\}$.

Show that f_t is J -stable but not topologically stable. Sketch a picture of the topologically stable regime. What is happening dynamically at the topologically unstable points?