Homework 2
Geometric Topology
Math 99r – Harvard University
Due Monday, 6 October 2003

1. Let $A : \mathbb{Z}^a \to \mathbb{Z}^b$ be a group homomorphism, specified by a $b \times a$ matrix A_{ij}, and let $G_A = \mathbb{Z}^b / A(\mathbb{Z}^a)$. Describe a sequence of elementary moves on matrices, such that G_A is isomorphic to G_B iff A and B are related by a sequence of your moves.

2. Let $G = \langle g_1, \ldots, g_n \ : \ r_1, \ldots, r_m \rangle$. Let A_{ij} be the total exponent with respect to which g_i occurs in the relation r_j. Using Tietze moves, show that $n - \text{rank}(A_{ij})$ is an invariant of G. Can you describe this invariant without using a presentation for G?

3. Which graphs are covered by the complete graphs K_5 and $K_{3,3}$ (the utility graph and the pentagram)?

4. Let $H \subset G$ be groups with $[G : H] < \infty$. Prove there is a normal subgroup $N \subset H$ with $[G : N] < \infty$. (Hint: consider the action of G on G/H.) Conclude that if $Y \to X$ is a finite covering, then there is a finite cover $Z \to Y$ such that $Z \to X$ is Galois.

5. Let $A_4 \subset S_4$ be the alternating group. Pick generators (a, b) for A_4 with $a^2 = b^3 = \text{id}$, and draw the Cayley graph of A_4 with respect to these generators. Finally give a presentation for A_4 in terms of your generators.

6. Up to conjugacy, the group $G = A_4$ has unique subgroups H_n of orders $n = 1, 2, 3, 4$ and 12. Let $\phi : \langle a, b \rangle \to A_4$ be the map from the free group to A_4 given by your generating set, and let $H'_n = \phi^{-1}(H_n) \subset \langle a, b \rangle$. Draw the covering space X_n of the bouquet of 2 circles corresponding to each subgroup $H'_n \subset \langle a, b \rangle$. (Hint: X_1 is the Cayley graph of A_4, and X_i covers X_j when i divides j.)

7. Let $H \subset \langle a, b \rangle$ be the subgroup generated by nontrivial elements x, y of the free group on 2 generators. Invent an algorithm to determine if H is isomorphic to $\mathbb{Z}$ or not.

8. Show there is a (geographical) map on the Möbius strip that requires more than 4 colors to make sure bordering countries have different hues.

9. The *real projective plane*, $\mathbb{RP}^2$, is obtained by identifying the gluing together a 2-disk and a Möbius band along their edges.

   Prove that $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$, and that the universal cover of $\mathbb{RP}^2$ is isomorphic to S^2. Show how $\mathbb{RP}^2$ can be identified with the space of 1-dimensional subspaces of $\mathbb{R}^3$.