Homework 7

Geometric Topology

Math 99r – Harvard University

Due Monday, 10 November 2003

In exercises 1 through 5, K is a knot with a fixed knot projection, and $G = \pi_1(S^3 - K) = \langle g_1, \ldots, g_n : r_1, \ldots, r_n \rangle$ is the corresponding Wirtinger presentation.

1. Prove that $H_1(S^3 - K, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

2. Prove that all the generators of G are conjugates of one another.

3. Prove that if you change the knot projection by a Reidemeister move, then the Wirtinger presentation changes by one or more Tietze moves. Conclude that G is an invariant of the knot (without using the fact that $G = \pi_1(S^3 - K)$).

4. Let K be the figure eight knot. Compute the Wirtinger presentation for G. How many surjective homomorphisms $G \to S_3$ are there?

5. Prove that the fundamental group of the figure eight knot complement is not abelian. (Hint: find a conjugacy class in A_4 with four elements.)

6. Let L_1 be the Whitehead link, picture in Figure 1.30 of Adams. Find another link L_2, different from L_1, such that $S^3 - L_1$ and $S^3 - L_2$ are homeomorphic. (You do not have to prove that L_1 and L_2 are really different, but you do have to prove that their complements are homeomorphic.)

7. Let $X = \mathbb{R}^2/\mathbb{Z}^2$ be a torus, let $p/q \in \mathbb{Q}$ be a rational number in lowest terms, and let $T(p/q) \subset X$ be the simple closed curve obtained by projecting a line of slope p/q from $\mathbb{R}^2$ to X.

   Give a formula, in terms of p/q and r/s, for the number of points of in the intersection $T(p/q) \cap T(r/s)$ of two such simple curves.

8. Given an explicit formula for a homeomorphism between S^3 and a pair of solid tori, $S^1 \times D^2$, glued along their boundaries. (Hint: think of S^3 as the unit sphere in $\mathbb{C}^2$, and begin by showing it is homeomorphic to $\partial(\Delta \times \Delta)$, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.)