Algebra and Dynamics

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1 Dynamics on $\mathbb{P}^1$ and the theory of equations

1.1 Tools

Lebesgue density. Let $E \subset \mathbb{R}^n$ be a measurable set. Then for almost every $x \in \mathbb{R}^n$ we have

$$\lim_{r \to 0} \frac{m(B(x,r) \cap E)}{m(B(x,r))} = \chi_E(x).$$

Schwarz lemma. A Riemann surface X is *hyperbolic* if its universal cover is isomorphic to $\mathbb{H}$. In this case the conformal metric $|dz|/y$ of constant curvature -1 descends to give the complete *hyperbolic metric* on X.

Every holomorphic map $f : X \to Y$ between hyperbolic Riemann surface is *non-expanding* for the respective hyperbolic metrics: we have $d(f(x), f(x')) \leq d(x, x')$, and $\|Df\| \leq 1$. Moreover the following are equivalent:

- f is a covering map;
- f is a local isometry; and
- $\|Df_x\| = 1$ for some $x \in X$.

In particular, any proper inclusion $X \hookrightarrow Y$ is a contraction.

Koebe distortion. The space of *univalent* or *schlicht* functions is given by:

$$S = \left\{ f : \Delta \to \mathbb{C} \ : \ f(z) = z + \sum_2^\infty a_n z^n \right\}.$$

It is given the topology of uniform convergence on compact sets. Many results flow from the basic fact that S *is compact.* For example if $|z| \leq r < 1$ then we have:

$$K_r^{-1} \leq |f'(z)| \leq K_r$$

for all $f \in S$, where $K_r \to 1$ as $r \to 0$. This implies that f distorts the density of a set $E \subset \Delta_r$ by only a bounded amount.

One of the strongest forms of this compactness is the *Bieberbach conjecture*, proved by de Branges: we have $|a_n| \leq n$.

1.2 Dynamics of rational maps

In one complex dimension, the behavior of an iterated rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d > 1$ is controlled by its $2d-2$ critical points. A basic feature of this control is the following result.

An *attracting cycle* is a collection of points $(a_1, \cdots, a_n)$ cyclically permuted by f, such that $|(f^n)'(a_1)| < 1$. Any such cycle evidently attracts an open set of $\widehat{\mathbb{C}}$.

Theorem 1.1 *Any attracting cycle of f attracts a critical point of f.*

Proof. Replacing f with f^n, we can assume the attracting cycle consists of a single point $a \in \widehat{\mathbb{C}}$. Let $U \subset \widehat{\mathbb{C}}$ be the open set of all points attracted to a, and U_0 the component of U containing a. Then $f : U_0 \to U_0$ is a proper map. If U_0 contains no critical points, then $f|U_0$ is also a covering map, and hence an isometry for the hyperbolic metric. This contradicts the fact that $|f'(z)| < 1$.

(Note: we have tacitly assumed that $\widehat{\mathbb{C}} - U_0$ contains 3 or more points so the hyperbolic metric is defined. In the other cases, U_0 is isomorphic to $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{C}^*$, and self-coverings of these spaces are easily analyzed.) ∎

Corollary 1.2 *A rational map of degree d on $\mathbb{P}^1$ has at most $2d-2$ attracting cycles.*

The attractor of f. Let $A(f)$ denote the union of the attracting cycles for f. By the results above, $A(f)$ is a finite set. We wish to determine conditions under which most points on $\widehat{\mathbb{C}}$ converge to $A(f)$.

The Fatou and Julia sets. The *Fatou set* $\Omega(f)$ is the largest open set on which the set of iterates $\mathcal{F} = \{f, f^2, f^3, \ldots\}$ form a *normal family.* (This means that for any compact set $K \subset \Omega(f)$, any sequence in $\mathcal{F}|K$ has a uniformly convergent subsequence.)

The complement of the Fatou set is the *Julia set* $J(f)$. By definition, $J(f)$ is a closed, f-invariant set (the forward and backward orbit of any $z \in J(f)$ is also contained in $J(f)$). It is known that $J(f)$ is perfect, that repelling periodic points are dense in $J(f)$, and that $J(f)$ is either the whole sphere or its interior is empty.

Expanding dynamics. The next result shows that the behavior of the critical points of f controls the behavior of most points on the sphere. We say a rational map is *hyperbolic*, or *expanding*, if all its critical points converge to attracting cycles.

Theorem 1.3 *If f is hyperbolic, then:*

- *The Julia set $J(f)$ has measure zero,*
- *All $z \notin J(f)$ converge to attracting cycles, and*
- *There is a smooth metric defined near $J(f)$ with respect to which $\|f'(z)\| > 1$ for all $z \in J(f)$.*

Proof. Let $B \supset A(f)$ be a union of closed balls around the attracting cycles such that $f(B) \subset \mathrm{int}(B)$. Replacing B by $f^{-n}(B)$ for n large enough, we can assume B contains all the critical points of f.

Now let $V = \widehat{\mathbb{C}} - B$ and $U = f^{-1}(V)$. Then we have $\overline{U} \subset V$ and $f : U \to V$ is a covering map. By a Schwarz Lemma argument, for all $z \in U$ we have $\|f'(z)\| > 1$ in the hyperbolic metric on U.

Let $R = \bigcap f^{-n}(U)$ be the set of points that never escape from U. Clearly R consists exactly of the points in $\widehat{\mathbb{C}}$ that do not converge to $A(f)$. But $\|(f^n)'z\| \to \infty$ for all $z \in R$, and therefore $R = J(f)$.

To see $J(f)$ has measure zero, use the expansion of f and Koebe distortion to blow up any point of Lebesgue density to definite size, contradicting the fact that $J(f)$ is nowhere dense. ■

A useful generalization of the result above, that can be proved using orbifolds, is:

Theorem 1.4 *Suppose all critical points of f are either pre-periodic or converge to the set of attracting cycles $A(f)$. Then either:*

- *$J(f) = \widehat{\mathbb{C}}$ and $A(f)$ is empty; or*
- *$J(f)$ has measure zero, and $f^n(z) \to A(f)$ for all $z \notin J(f)$.*

1.3 Rigidity

Dynamics from complex tori. Let $E = \mathbb{C}/\Lambda$ be a Riemann surface of genus 1, and let $F : E \to E$ be the endomorphism given by $F(z) = nz$, $n > 1$. The Weierstrass $\wp$-function provides a natural degree two map $\wp : E \to \widehat{\mathbb{C}}$, whose fibers have the form $\{z, -z\}$. Since $F(-z) = -F(z)$, there is a rational map f making the diagram

$$\begin{array}{ccc} E & \xrightarrow{F} & E \\ {\scriptstyle I_\wp}\downarrow & & {\scriptstyle I_\wp}\downarrow \\ \widehat{\mathbb{C}} & \xrightarrow{f} & \widehat{\mathbb{C}} \end{array}$$

commute. It is easy to see $J(f) = \widehat{\mathbb{C}}$. These are among the simplest examples of rational maps with Julia set the whole sphere.

When f arises via the construction above, we say f is *covered by integral multiplication on a torus.*

Let V be an irreducible algebraic variety. An *algebraic family* of rational maps over V is given by a map $f : V \to \mathrm{Rat}_k$, sending a generic $\lambda \in V$ to the rational map $f_\lambda(z)$. The family *has bifurcations* if for any n, there is an $f_\lambda(z)$ with an attracting cycle of period greater than n. Otherwise the family is *stable.*

Theorem 1.5 *Let $f_\lambda(z)$ be an algebraic family of rational maps. Then either:*

- *There is a fixed rational map $g(z)$ to which every $f_\lambda(z)$ is conformally conjugate; or*
- *Every member is covered by integral multiplication on a complex torus (whose modulus varies in the family); or*
- *The family has bifurcations.*

We refer to this result as *rigidity of stable algebraic families*; the rigidity meaning the family is essentially constant once it is stable.

A proof can be found in [Mc1]. As a plausibility argument, suppose $f_\lambda(z)$ has no bifurcations. Then there is an N independent of λ such that all points z_λ with period $n \geq N$ are repelling. That is, the multiplier $m_\lambda = (f_\lambda^n)'(z_\lambda)$ satisfies $|m_\lambda| > 1$. Using the fact that bounded holomorphic functions (such as $1/m_\lambda$) on V are constant, one can conclude that the multipliers at all points of high period are constant in the family. A little extra argument shows all multipliers are constant, independent of the period.

It is plausible that the multipliers serve as local moduli for rational maps up to conformal conjugacy. This turns out to be true, *except* in the case of the torus construction: there the multipliers stay constant as the modulus of the torus varies.

1.4 Newton's method

One of the classical interactions between algebra and dynamics is the problem of locating the roots of a polynomial $p(z) \in \mathbb{C}[z]$.

Newton's method. Newton's method provides an iteration scheme $z \mapsto f(z)$ for improving a guess for a root of $p(z)$. The value $f(z_0)$ is simply the zero of the linear approximation $p(z) = p(z_0) + p'(z_0)(z - z_0)$; that is, Newton's method is given by

$$f(z) \quad = \quad z - \frac{p(z)}{p'(z)}.$$

This method is *generally convergent for* p if

$$f^n(z) \to [\text{a root of } p]$$

for all z in an open, dense, full measure subset of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

How can we determine the global behavior of a rational map such as Newton's method?

Theorem 1.6 *Newton's method is generally convergent for a polynomial $p(z)$ provided the points of inflection of $p(z)$ are pre-periodic or converge to roots of p.*

Proof. Since $f'(z) = p(z)p''(z)/p'(z)^2$, the zeros of $p''(z)$, together with the zeros of $p(z)$, comprise the critical points of Newton's method. So this result follows from Theorem 1.4. ■

Computing nth roots. For the special case $p(z) = z^2 - a$, Newton's method gives the classical averaging method for computing square roots: $f(z) = (z + a/z)/2$. For $p(z) = z^d - a$ we get a weighted average:

$$f(z) = \left(1 - \frac{1}{d}\right) z + \frac{1}{d}\left(\frac{a}{z^{d-1}}\right).$$

Corollary 1.7 *Newton's method is a reliable technique for extracting radicals.*

Proof. For degree $d = 2$ there are no points of inflection, while for $d \geq 3$ the only point of inflection of $p(z) = z^d - a$ is $z = 0$, which lands on the fixed point $z = \infty$ after one iteration. ■

Success and failure of Newton's method. Let $\mathrm{Poly}_d \cong \mathbb{C}^d$ denote the space of monic polynomials of degree d, and $\mathrm{Rat}_k \cong \mathbb{P}^{2k+1}$ the space of rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree k.

A *purely iterative algorithm* is a rational map $T : \mathrm{Poly}_d \to \mathrm{Rat}_k$, assigning to a polynomial p a rational $T_p(z)$ computed from the coefficients of p. The algorithm is *generally convergent* if

$$T_p^n(z) \to [\text{a root of } p]$$

for all (p, z) in an open, dense, full-measure subset of $\mathrm{Poly}_d \times \widehat{\mathbb{C}}$.

Theorem 1.8 *Newton's method is generally convergent for degree $d = 2$ but not for degree $d \geq 3$.*

Proof. A degree two polynomial has no points of inflection, so Newton's method is generally convergent.

For degree three, consider the polynomial $p(z) = 4z^3 - 2z + 1$; it has a point of inflection at $z = 0$. Newton's method for p is given by $f(z) = (8z^3 - 1)/(16z^2 - 2)$. We have $f(0) = 1/2$ and $f(1/2) = 0$, so f has a superattracting cycle of order 2. The basin of this cycle consists of initial guesses that do not converge to roots of p, and thus f fails to be generally convergent for this particular p. By transversality, the attracting cycle of period two persists under a small perturbation of p, and thus Newton's method in degree 3 fails on an open set in $\mathrm{Poly}_3 \times \widehat{\mathbb{C}}$. Therefore it is not generally convergent.

It is immediate the Newton's method fails to be generally convergent in higher degrees as well, by considering polynomials of the form $p(z) = 4z^3 - 2z + 1 + \epsilon z^d$. ∎

1.5 Inventing algorithms

Quadratics. Here is a pure-thought approach to constructing a generally convergent algorithm to solve quadratic polynomials.

Let $p(z) = (z - a)(z - b)$. We would like to construct a rational map $T_p(z)$ whose attractor coincides with the roots $\{a, b\}$ of p.

Now a simple example of a rational map whose attractor consists of two fixed points is $f(z) = z^2$. For this map, we have

$$f^n(z) \to \begin{cases} 0 & \text{if } |z| < 1, \text{ and} \\ \infty & \text{if } |z| > 1; \end{cases}$$

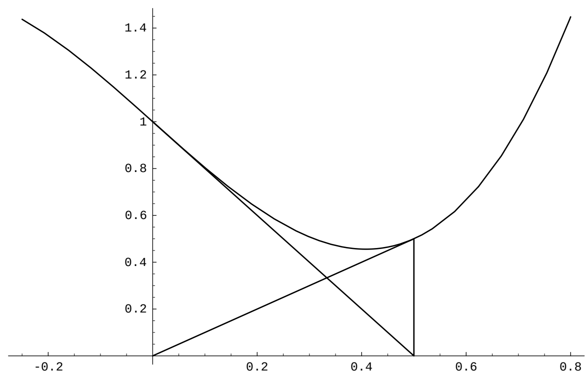

Figure 1. Newton's method for $4z^3 - 2z + 1$.

the circle $S^1 = \{z : |z| = 1\}$ is invariant and coincides with the Julia set of f.

Suppose we construct $T_p(z)$ by conjugating f so its attractor $\{0, \infty\}$ is sent to the roots $\{a, b\}$ of p. More precisely, let $T_p = MfM^{-1}$, where $M(z)$ is the unique Möbius transformation sending $\{0, \infty, 1\}$ to $\{a, b, \infty\}$. Then clearly $A(T_p) = \{a, b\}$, and $T_p^n(z)$ tends to a root of p for all z outside $M(S^1)$. Thus T_p is a reliable algorithm for finding the roots of p.

The only potential problem with this algorithm is that to compute T_p by the prescription above, we already need to know the roots of p. But notice that $f(z)$ is symmetric with respect to the symmetry $z \mapsto 1/z$ interchanging its attracting fixed points. Because of this symmetry, we obtain the same result for T_p if we interchange the *order* of the roots a and b for p. Thus T_p only depends on *symmetric functions of the roots*, and hence it is a rational function of the coefficients of p.

Thus T_p gives a purely iterative algorithm, and by construction it is generally convergent.

In fact this algorithm coincides with Newton's method for quadratics.

Cubics. Now let us undertake a similar program for cubics. To do this, we need to find a rational map f whose attractor $A(f)$ consists of three points — say the cube roots of unity, $\{1, \omega, \omega^2\}$, where $\omega = \exp(2\pi i/3)$. Moreover, we would like f to be symmetric with respect to permutations S_3 of $A(f)$. That is, we would like to have $f(1/z) = 1/f(z)$ and $f(\omega z) = \omega f(z)$.

By a direct calculation, it is easy to see that maps of the form

$$f(z) = \frac{z^4 + az}{az^3 + 1}$$

have the desired symmetries. This map *fixes* the cube roots of unity. We have $f'(1) = (4-2a)/(1+a)$; the derivative at the other cube roots of unity is the same, by symmetry.

Now set $a = 2$; then $f'(1) = 0$, so each cube root of unity is a superattracting fixed point for f. Moreover $f''(1) = 0$ as well, so the six critical points of f *coincide* with the cube roots of unity — each has multiplicity two. It follows that $J(f)$ has measure zero, and all points outside $J(f)$ converge under iteration to a root of $z^3 - 1$.

As in the quadratic case, we can now define $T_p = MfM^{-1}$, where M is a Möbius transformation sending the roots of $z^3 - 1$ to the roots of $p(z)$. Although there are 6 choices for M, they all give the same map T_p, because of the symmetries of f. Thus T_p is a rational function of the coefficients of p, and we have shown:

Theorem 1.9 *There exists a generally convergent algorithm for solving cubic equations.*

Quartics. The symmetry construction breaks down for polynomials of degree $d = 4$. Since the cross-ratio varies, there is generally no Möbius transformation sending the roots of p to the roots of q for $p, q \in \mathrm{Poly}_4$. On the other hand, the rigidity of stable algebraic families can be used to show that any successful purely iterative algorithm must be based on the symmetry construction. Thus we have:

Theorem 1.10 *There is no generally convergent purely iterative algorithm for solving polynomials of degree $d \geq 4$.*

1.6 Quintic polynomials: classical theory

To proceed to polynomials of still higher degree, we first recall the classical solution to the quintic equation via the icosahedron and modular functions.

Reduction to the icosahedron. Let $s : \mathbb{C}^5 \to \mathbb{C}^5$ be the map sending $(r_1, \ldots, r_5)$ to the coefficients $(a_1, \ldots, a_5)$ of the polynomial

$$p(z) = \prod_1^5 (z - r_i) = z^5 + a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 z + a_5.$$

This maps gives the quotient of $\mathbb{C}^5$ under the action of S_5 permuting coordinates.

Passing to projective space, we obtain an action of S_5 on $\mathbb{P}^4$. By a change of variable of the form $z \mapsto z + a$, we can arrange that $a_1 = -\sum r_i = 0$. The set of roots normalized so $\sum r_i = 0$ forms an S_5-invariant hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$.

After a further change of variable of the form $z \mapsto z^2 + az + b$ – known as a Tschirnhaus transformation – we can normalize our polynomial so that $a_2 = 0$ as well. In terms of roots, we have now further restricted to the quadric $Q \subset \mathbb{P}^3$ defined by $\sum r_i^2 = 0$.

The smooth quadric surface Q admits a pair of rulings that are interchanged by the odd elements of S_5. Note that Q/S_5 is isomorphic to the weighted projective space of quintics normalized so that $a_1 = a_2 = 0$.

Now we extract the square-root of the discriminant of $p(z)$. This results in the reduction of the Galois group from S_5 to A_5, the subgroup that preserves the rulings. (Geometrically, the pullback of Q splits into two components.)

Now the original polynomial p determines 60 points on the quadric Q, and hence 60 lines ℓ_i in one of the rulings. (Each line corresponds to an even reordering of the roots (r_i).) The space of all lines in a given ruling is isomorphic to $\mathbb{P}^1$. From p we can determine the common image q of the $\ell_i \in \mathbb{P}^1$ in the quotient space $\mathbb{P}^1/A_5 \cong \mathbb{P}^1$.

Thus, if we can invert the degree 60 *icosahedral map*,

$$f : \mathbb{P}^1 \to \mathbb{P}^1/A_5 \cong \mathbb{P}^1,$$

we can solve the original polynomial p. (Once the equation for a specific line ℓ_i is determined, elimination of variables gives the roots of p.)

Modular forms Let $\Gamma(n) = \{A \in \mathrm{SL}_2(\mathbb{Z}) : A \sim I \bmod n\}$, and let $X(n) = \mathbb{H}/\Gamma(n)$. Then $\Gamma(1)/\Gamma(5) = \mathrm{PSL}_2(\mathbb{Z}/5)$ is isomorphic to A_5. Moreover, the quotient space $X(1)$ is the $(2, 3, \infty)$ orbifold, while the quotient space $Y = \mathbb{P}^1/A_5$ is the $(2, 3, 5)$ orbifold (with underlying space the sphere). With suitable normalizations, we obtain a commutative diagram:

$$\begin{array}{ccc} X(5) & \longrightarrow & \mathbb{P}^1 \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ X(1) & \longrightarrow & Y = \mathbb{P}^1/A_5. \end{array}$$

Let $j : \mathbb{H} \to X(1)$ be the modular j-function, and $h : \mathbb{H} \to X(5)$ a suitable modular function uniformizing the quotient by $\Gamma(5)$. To compute $q = f^{-1}(p)$, we first compute a point $\tau \in j^{-1}(p)$, and then set $q = h(\tau)$.

The transcendental function j intervenes in this calculation much as the exponential function intervenes in the calculation of roots, when the latter are computed by

$$z^{1/n} = \exp(\log(z)/n).$$

The Brauer group. To compute the values of a and b for the Tschirnhaus transformation that makes a_2 vanish, one needs to extract a square-root. This 'accessory irrationality' does not diminish the Galois group — we still have S_5 acting on Q. Rather, it serves to kill an element in the Brauer group.

In general, if L/K is a field extension with Galois group G, we have an exact sequence in group cohomology

$$H^1(G, \mathrm{GL}_{n+1}(L)) \to H^1(G, \mathrm{PGL}_{n+1}(L)) \to H^2(G, L^*). \qquad (1.1)$$

(Recall that group cohomology is the derived functor of the fixed subgroup functor. If A is a group equipped with an action of G, then $H^1(G, A)$ consists of the *crossed homomorphism* $\rho : G \to A$, satisfying

$$\rho(gh) = \rho(g)^h \rho(g),$$

modulo the coboundaries of the form $\rho(g) = a^g a^{-1}$.)

To interpret these groups geometrically, suppose L/K corresponds to a finite map $\pi : Y \to X$ between projective varieties. The first two of these 'Galois cohomology groups' classify rank $n+1$ vector bundles and $\mathbb{P}^n$ bundles over X that become trivial when lifted to Y.

For example, let $P \to X$ be a $\mathbb{P}^n$-bundle that pulls back to $P' \to Y$. Then G acts on P' compatibly with its action on Y. If P' is (birationally) trivial, then after choosing an isomorphism $P' \cong Y \times \mathbb{P}^n$, we can write the action of $g \in G$ on P' by:

$$g \cdot (y, \xi) = (g(y), m_g(y) \cdot \xi)$$

where $m_g : Y \to \mathrm{Aut}(\mathbb{P}^n)$. Since $m_g(y)$ depends rationally on Y, we can regard $\rho(g) = m_g$ as a twisted homomorphism $\rho : G \to \mathrm{PGL}_{n+1}(L)$. Changing the trivialization of P' changes ρ by a coboundary, and P is trivial iff P' can be trivialized so the action of G becomes trivial.

By the exact sequence (1.1), the obstruction to writing a projective space bundle as $\mathbb{P}V$ for a vector bundle V is an element of $H^2(G, L^*)$. When L is the algebraic closure of K this group is known as the *Brauer group* $\mathrm{Br}(K)$. Every element of $\mathrm{Br}(K)$ arises from some $\mathbb{P}^n$ bundle [Ser1, X.5].

It is know that $H^1(G, \mathrm{GL}_{n+1}(L)) = 0$. Geometrically, this comes from the fact that every vector bundle of rank $n+1$ $V \to X$ admits $n+1$ rational sections that are generically linearly independent. These sections trivialize V over a Zariski opens set.

Thus a $\mathbb{P}^n$-bundle is trivial iff its image in the Brauer group is zero.

It is also known that a $\mathbb{P}^n$ bundle is trivial iff it admits a section [Ser1, Ex. 1, §X.6]. For example, given $p \in P \cong \mathbb{P}^1$, we can canonically write $P = \mathbb{P}V^*$ where

$$V = \mathcal{O}(p) = \{\text{rational functions } f \text{ on } \mathbb{P}^1 \text{ with at worst a simple pole at } p.\}.$$

Similarly, a section of a $\mathbb{P}^1$-bundle P over X allows one to canonically construct a vector bundle $V \to X$ with $V_x = \Gamma(P_x, \mathcal{O}(p))^*$; then $P = \mathbb{P}V$.

A_5 and $\mathbb{P}^1$-bundles. We can now ask, given an A_5 extension of varieties $Y \to X$, if there is a map $f : X \to \mathbb{P}^1$ such that Y is the pullback of the icosahedral covering $\mathbb{P}^1 \to \mathbb{P}^1$. That is, can we find a map $f : X \to \mathbb{P}^1$ with a lift to Y making the diagram:

$$\begin{tikzcd} Y \arrow[r, "\widetilde{f}"] \arrow[d, "A_5"'] & \mathbb{P}^1 \arrow[d, "A_5"'] \\ X \arrow[r, "f"] & \mathbb{P}^1 \end{tikzcd}$$

commute?

Let $L/K = K(Y)/K(X)$, and let $G = \mathrm{Gal}(L/K) \cong A_5$,. The icosahedral representation $\rho : G \to \mathrm{PSL}_2(\mathbb{C})$ determines an element of $H^1(G, \mathrm{PSL}_2(L))$, as well as a flat $\mathbb{P}^1$-bundle

$$P = (Y \times \mathbb{P}^1)/A_5 \to X.$$

Constructing a map f as above, with the required lift to Y, is the same as finding a section of P/Y. This is the same as writing $P = \mathbb{P}V$ for a rank two vector bundle $V \to X$, and hence killing $[\rho]$ in $H^1(X, \mathrm{PSL}_2(L))$.

A universal instance of this obstruction arises when we try to solve the general quintic polynomial: we take $X = Y = \mathbb{C}^5$, with A_5 acting by permutation of coordinates. The space Y is the space of quintic polynomials, and the space $X$is the space of their roots.

To reduce the solution of the quintic to inversion of degree 60 icosahedral map $\mathbb{P}^1 \to \mathbb{P}^1$, we must trivialize the bundle $E \to Y$ by a (cyclic, rational) extension of the base. The obstruction to this trivialization comes from the nontrivial $\mathbb{Z}/2$ central extension of A_5, which is classified by $H^2(A_5, \mathbb{Z}/2) \cong \mathbb{Z}/2$ and which determines an element in the Brauer group $H^2(G, L^*)$.

Twisting $\mathbb{P}^1$ over a torus. Here is an easily-visualized example of a nontrivial $\mathbb{P}^1$-bundle. Let $V = \mathbb{Z}/2 \times \mathbb{Z}/2$ be the Klein 4-group, acting on $\mathbb{P}^1$ via $z \mapsto -z$ and $z \mapsto 1/z$. Let $X = \mathbb{C}/\mathbb{Z}[i]$ be a complex torus, and let

$$\rho : \pi_1(X) \cong \mathbb{Z} \times \mathbb{Z} \to V$$

be a surjective homomorphism. Then from V we obtain a flat (and hence holomorphic) $\mathbb{P}^1$-bundle $P \to X$, with monodromy ρ.

Suppose P is isomorphic to the trivial bundle $\mathbb{P}^1 \times X$. Then the original flat connection determines a unique conformal isomorphism between any two nearby fibers. Integrating the connection from a basepoint, we obtain a map

$$c : X \to \mathrm{PSL}_2(\mathbb{C})/V.$$

Noting that $\mathrm{SL}_2(\mathbb{C})$ is the universal cover of $\mathrm{PSL}_2(\mathbb{C})$, we find that the fundamental group of the target is the quaternion group $\widetilde{V}$ given by the preimage of V in $\mathrm{SL}_2(\mathbb{C})$.

Thus $\pi_1(c)$ gives a lifting of ρ to a homomorphism $\widetilde{\rho} : \pi_1(X) \to \widetilde{V}$. But since $\pi_1(X)$ is abelian, no such lifting exists.

Degree 4. A similar obstruction arises for polynomials of degree 4. Here S_4 acts on $\mathbb{P}^1$ by the symmetries of the cube. Letting $D \subset \mathrm{Poly}_4$ denote the discriminant locus, and B_4 the braid group, we have a map

$$\rho : \pi_1(\mathbb{C}^4 - D) \cong B_4 \to S_4$$

recording the permutation of roots under monodromy.

We can ask if one can construct a family of rotating cubes over $\mathbb{C}^4 - D$ that exhibits the same monodromy. The answer is *no.* Such a family would give a lifting of the map $B_4 \to S_4$ to the binary octahedral group $\widetilde{S_4} \subset \mathrm{SL}_2(\mathbb{C})$, by the same argument as above. But there are two commuting braids that give the permutations (12)(34) and (13)(24) respectively, whose lifts to $\widetilde{S_4}$ can never commute, since they too generate a quaternion group.

Thus ρ determines a nontrivial $\mathbb{P}^1$-bundle over $\mathbb{C}^4 - D$. It can be shown that this bundle remains trivial when restrict to any Zariski open subset of $\mathbb{C}^4 - D$, and thus it cannot even be birationally trivialized.

The Brauer group obstruction for the quintic, killed by the accessory irrationality, is similar.

Bundles over curves. It is known that the Brauer group of the function field of a curve X over $\mathbb{C}$ is trivial (cf. [Ser1, p.156]). That is, every $\mathbb{P}^n$

bundle over X has a rational section and can be trivialized over a Zariski open subset of X.

1.7 Quintic polynomials: dynamics

The classical theory of equations concerns solution by radicals. That is, we enrich our arithmetic constructions by adjoining the operation of forming nth roots. As we have seen, Newton's method provides a reliable computational scheme for computing radicals.

How is it, then, that quartic polynomials can be solved by radicals, but not by purely iterative algorithms?

The reason is that the solution to the quartic has *nested* radicals. Although each individual radical can be computed by iteration, there is no single dynamical system in one variable to compute the entire expression.

This suggest that we should broaden our perspective, and enrich our arithmetic operations by including *all* algebraic functions (such as radicals) that can be reliably computed by purely iterative algorithms.

Computable extensions. To formulate a result using Galois theory we use the language of birational geometry and field extensions.

Let V be an irreducible (quasi-)projective variety over $\mathbb{C}$, and $K = K(V)$ its function field. We can visualize a finite extension L/K, $L = K[z]/p(z)$, as the function field $L = K(W)$ of the graph

$$W = \{(\lambda, z) \;:\; p_\lambda(z) = 0\} \subset V \times \widehat{\mathbb{C}}$$

of the (rational) *correspondence*

$$\lambda \mapsto [\text{the zeros of } p_\lambda(z)].$$

Now let $T : V \to \mathrm{Rat}_d$ be a generally convergent purely iterative algorithm. Then the map

$$\lambda \mapsto A(T_\lambda),$$

sending a parameter $\lambda \in V$ to the attractor of the corresponding map, determines a correspondence $V \to \widehat{\mathbb{C}}$. Let us suppose the graph $W \subset V \times \widehat{\mathbb{C}}$ of this correspondence is irreducible. Then T determines a finite extension $L = K(W)$ of V. We refer to L as the *output field* of T.

A *tower* of algorithms is described by a sequence of fields $K = K_1 \subset K_2 \subset \cdots \subset K_n$ as above, and rational maps $T_i \in K_i(z)$, $i < n$, such that K_{i+1} is the output field of T_i.

A field extension L/K is *computable* if it is isomorphic over K to a subfield of K_n for some tower of algorithms.

Theorem 1.11 *A Galois field extension L/K is computable iff its Galois group G is within A_5 of solvable.*

This means $G = \mathrm{Gal}(L/K)$ admits a subnormal series

$$G = G_n \rhd G_{n-1} \rhd \ldots \rhd G_1 = \mathrm{id}$$

such that each quotient G_{i+1}/G_i is cyclic *or* A_5. (One also says G is *nearly solvable.*)

Rational maps with symmetry. Let $\mathrm{Aut}(f)$ denote the finite subgroup of $\mathrm{Aut}(\widehat{\mathbb{C}})$ commuting with a rational map f. Any finite subgroup of $\mathrm{Aut}(\widehat{\mathbb{C}})$ is isomorphic to $\mathbb{Z}/n$, D_n, A_4, S_4 or A_5. Of these, all but A_5 are solvable. The key to the result above is to construct a rational map with A_5 symmetry.

Let G be a finite group of Möbius transformations. Here are 3 ways to construct f such that $\mathrm{Aut}(f) \supset G$.

1. *Projectively natural Newton's method.* (Also known as *Halley's method.*) If we use the osculating Möbius transformation instead of the osculating linear function, we get

$$N_p(z) = z - \frac{p(z)p'(z)}{p'(z)^2 - \frac{1}{2}p''(z)p(z)}.$$

If $p(z)$ is a G-invariant rational function, then $N_p(z)$ is a G-invariant rational map (with attracting fixed points at the roots of p.)

On the other hand, if p is G-invariant then its degree is $d \cdot |G|$ for some d. So p would have degree at least 60 for the icosahedral group, and hence N_p would be of even higher degree.

2. *Riemann maps.* Consider the sphere tiled by 12 circular pentagons in the usual dodecahedral pattern, symmetric under A_5. Let F' denote the pentagon antipodal to F. There is a unique conformal map from F to $\widehat{\mathbb{C}} - F'$, sending vertices to antipodal vertices. These maps piece together to give a degree 11 rational map f with A_5 symmetry.

A similar construction yields A_5-invariant maps of degrees 19 and 29, using tilings of the sphere by equilateral triangles and by rhombuses, respectively. But how would we every find formulas for these functions?

3. *Invariant theory.* Let us lift G to a subgroup of $\mathrm{SL}(E)$, where $E \cong \mathbb{C}^2$ (a central extension may be required for this lifting). A homogeneous polynomial (or form) $f : E \to \mathbb{C}$ is *invariant* if $g^*(f) = \chi(g) \cdot f$ for some character $\chi : G \to \mathbb{C}^*$; it is *absolutely invariant* if χ is trivial.

For example, the forms $\lambda = (x\,dy - y\,dx)/2$ and $\omega = dx\,dy$ are absolute invariants of any group G. Note that $d\lambda = \omega$.

Since E is a vector space, $T_p(E)$ is naturally isomorphic to E, and thus $T_p^*(E) \cong E^*$. Thus a homogeneous 1-form on E is the same thing as a homogeneous polynomial map $\alpha : E \to E^*$. Projectivizing, from α we obtain a rational map $f : \mathbb{P}E \to \mathbb{P}E^* \cong \mathbb{P}E$, which inherits the symmetries of α.

Concretely, if $\alpha = \alpha_1(x,y)\,dx + \alpha_2(x,y)\,dy$, and $z = x/y$, then the corresponding rational map is given by $f(z) = -\alpha_2(z,1)/\alpha_1(z,1)$. The formula comes from the fact that α annihilates the vector $(x,y) = (-\alpha_2, \alpha_1)$ with $z = -\alpha_2/\alpha_1$. (The choice of z is compatible with the usual action of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ by $(az+b)/(cz+d)$.)

Theorem 1.12 *To any finite set $S \subset \mathbb{P}^1$, there is canonically associated a rational map $f_S : \mathbb{P}^1 \to \mathbb{P}^1$ with $\deg(f_S) = |S| - 1$. Every symmetry of S gives rise to an automorphism of f_S.*

Examples. For $S = \{0\}$, $\{0, \infty\}$ and $\{1, \exp(2\pi i/3), \exp(-2\pi i/3)\}$, the corresponding maps are $f_S(z) = 0$, $f_S(z) = -z$ and $f_S(z) = 1/z^2$. In general S coincides with the fixed points of f_S.

Corollary 1.13 *Any orbit S of G on $\mathbb{P}^1$ gives rise to a G-invariant rational map f with $\deg(f) = |S| - 1$.*

Theorem 1.14 *A homogeneous 1-form α is invariant if and only if*

$$\alpha = g(v)\,\lambda + dh(v),$$

where g and h are invariant homogeneous polynomials with the same character, and $\deg(g) = \deg(h) + 2$.

Example: the icosahedral group. The group G has 3 special orbits, of cardinalities 12, 20 and 30. The corresponding invariant polynomials are:

$$\begin{aligned} f &= x^{11}y + 11x^6y^6 - xy^{11} \\ H &= -x^{20} - y^{20} + 228(x^{15}y^5 - x^5y^{15}) - 494x^{10}y^{10} \\ T &= x^{30} + y^{30} + 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + x^{10}y^{20}). \end{aligned}$$

Note that the degree 60 invariants f^5, H^3 and T^2 satisfy a linear relation, since a generic orbit of the A_5 action moves in a linear system.

Theorem 1.15 *Every homogeneous polynomial invariant under the binary icosahedral group is a polynomial in f, H and T.*

From df we obtain the invariant rational function

$$f_{11}(z) = \frac{z^{11} + 66z^6 - 11z}{-11z^{10} - 66z^5 + 1}$$

of degree 11. Similarly, formulas for the invariant functions of degrees 19 and 29 constructed geometrically above can be computed from dH and dT.

Theorem 1.16 *The critical points of the map $f_{11}(z)$ are periodic, and $A(f_{11})$ coincides with the 20 vertices of the dodecahedron, each of which has period 2.*

Corollary 1.17 *The quintic equation can be solved by a tower of purely iterative algorithms.*

Proof. Let L/K be the icosahedral extension, where $K = \mathbb{C}(x)$ and $G = \mathrm{Gal}(L/K) \cong A_5$. By the classical theory of the quintic, if this extension is computable, then the quintic can be solved.

Let

$$\rho : G \to \mathrm{PSL}_2(\mathbb{C})$$

be the icosahedral action of G on the Riemann sphere. By the vanishing of the Galois cohomology group $H^1(G, \mathrm{PSL}_2(L))$, there is a $\phi \in \mathrm{PSL}_2(L)$ such that

$$\phi^g = \rho(g) \circ \phi$$

for all $g \in G$. (Note that G acts trivially on $\mathbb{C} \subset L$, so ρ can be considered as a crossed homomorphism from G to $\mathrm{PSL}_2(L)$.)

Now let $f(z)$ be the degree 11 rational map invariant under $\rho(G)$. Let

$$T = \phi^{-1} \circ f \circ \phi \in L(z).$$

Then $T^g = T$ for all g so in fact $T \in K(z)$.

The map $T \circ T(z)$ is our desired algorithm. The output field for T is given by $K(\alpha) \subset L$, where $\alpha \in L$ parameterizes one of the 20 attractors of T. Since A_5 acts transitively on these 20 attractors, with A_3 stabilizers, the field $K(\alpha)$ is the fixed field of an $A_3 \subset A_5$. Thus $K(\alpha)$ contains the fixed field of an $A_4 \subset A_5$; that is, we have $K \subset K(r_i) \subset K(\alpha)$, where r_i is one of the roots of the 5th degree equation.

This shows a root of the original quintic can be computed from the output of the purely iterative algorithm $T \circ T$. ∎

1.8 Higher degree

Degree 6: The Valentiner group. In 1889 Valentiner discovered an action of A_6 on $\mathbb{P}^2$, generalizing the action of A_5 on $\mathbb{P}^1$. (These examples are sporadic — they do not exist for general A_n). In the 1990s Crass and Doyle proposed an algorithm for solving *sextic* equations based on the iteration of a degree 19 rational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ with Valentiner symmetry.

There are 12 copies of A_5 in A_6, each preserving a conic (coming from the representation of A_5 in SO(3)). (There are 6 copies that act like permutations stabilizing a point, and 6 copies that act like A_5 does on the 6 pairs of opposite faces of the dodecahedron.) The degree 19 map leaves each of these conics invariant and restricts to the degree 19 map mentioned above (coming from the 20 vertices of the dodecahedron). It is conjectured that these points attract an open, full measure subset of $\mathbb{P}^2$, but this statement has only been checked experimentally.

Theoretical tools for analyzing the dynamics of rational maps on $\mathbb{P}^n$, $n > 1$, will be discussed below. Because the critical points of such a map form an infinite set (they are a divisor of positive dimension), it is possible to have infinitely many attractors.

Degree 7: Hilbert's 13th problem. Is every algebraic function of 3 variables a composition of algebraic functions of 2 or fewer variables?

This question is perhaps the original motivation for Hilbert's 13th problem. As a specific function that might truly require 3 variables, Hilbert considered the map that sends (a, b, c) to the roots of the 7th degree polynomial $z^7 + az^3 + bz^2 + cz + 1$.

1.9 Exercises

1. Show that if f expands some Riemannian metric on its Julia set, then f is hyperbolic.

2. Show directly that Newton's method for a quadratic polynomial $p(z)$ converges to the closer root — and thus its Julia set is the line of points equidistant from the roots of p.

3. Sketch a topological picture of the Julia set of the S_3-symmetric map $f(z) = (z^4 + 2z)/(2z^3 + 1)$. (Hint: what are the preimages of the cube roots of unity?)

4. Give an example of a rational map $f(z)$ of degree two whose Julia set is the whole Riemann sphere. (Hint: arrange that c_1 maps to c_2, and

then c_2 lands in two iterations on a fixed point, where c_1 and c_2 are the two critical points of f.)

5. Let E be the complex torus $\mathbb{C}/\mathbb{Z}[i]$ and let $F : E \to E$ be the degree two map given by $F(z) = (1+i)z$. Show that F covers the map f above; that is, construct a degree two map $p : E \to \widehat{\mathbb{C}}$ such that $p(F(z)) = f(p(z))$. (Hint: p should be branched over the four points that form the forward orbit of the critical points of f.)

6. Find all the rational maps of degree $d \leq 3$ that commute the group of Möbius transformations S_3 generated by $z \mapsto 1/z$ and $z \mapsto \omega z$.

7. Let $p(z) = z^3 + az + b$. Find an explicit formula for the algorithm $T_p(z)$ for solving cubics given by Theorem 1.9.

8. Show that $T_p(z)$ is nothing more than Newton's method for the *rational* function
$$r(z) = \frac{z^3 + az + b}{3az^2 + 9bz - a^2}.$$
Check that the points of inflection of $r(z)$ coincide with the roots of $p(z)$.

9. Show the two zeros of the denominator of $r(z)$ coincide with the fixed points of the group of Möbius transformations cyclically permuting the three roots of the numerator.

10. Compute the ring of invariants for the group S_4 of symmetries of the cube, and use its generators to construct rational maps of degrees 5, 7 and 11 with octahedral symmetry.

11. Let p_n and f_n be the dimension of the space of homogeneous, degree n, $2 \cdot A_5$-invariant polynomials and 1-forms on $\mathbb{C}^2$, respectively. Compute the generating functions for $\sum p_n t^n$ and $\sum f_n t^n$.

12. Let $p(z) = z^5 + z^3 + 1$. (a) Compute the quintic polynomial $q(w)$ satisfied by $w = z^2 + az + b$. (b) Show that a and b can be chosen to make the coefficients of w^4 and w^3 both equal to zero.

13. Let $B = \mathbb{C}^* \times \mathbb{C}^*$, and let $E \to B$ be the $\mathbb{P}^1$ bundle whose fiber over $(a, b) \in X$ is the conic in $\mathbb{P}^2$ with affine equation
$$E_{(a,b)} = \{(x, y) \in \mathbb{C}^2 \, : \, ax^2 + by^2 = 1\}.$$

(a) Show that E admits no rational section. (b) Show that E admits a smooth section. (c) Show that E cannot be smoothly trivialized over B.

14. Let $Q_r \subset \mathbb{R}^3$ be the affine quadric defined by $q(x,y,z) = r$, where $q(x,y,z) = x^2 + y^2 - z^2$. The locus Q_{-1} can be interpreted as (two copies of) the hyperbolic plane $\mathbb{H}$, with $\mathrm{SO}(q)$ acting isometrically.

    (a) Show that Q_1 can be interpreted as the space of oriented geodesics in $\mathbb{H}$. (b) Interpret the two rulings of Q_1 as pencils of geodesics in $\mathbb{H}$. (c) Interpret the isomorphism of algebraic varieties

    $$Q_1 \cong \mathbb{RP}^1 \times \mathbb{RP}^1 - (\text{diagonal})$$

    in terms of hyperbolic geometry.

15. Given $S \subset \mathbb{P}^1$ with $|S| = d+1$, show that for all $p \in S$ the canonically associated rational map satisfies $f_S(p) = p$ and $f'_S(p) = -d$.

16. A *kite* is a quadrilateral (in Euclidean, spherical or hyperbolic geometry) which is symmetric about the line joining some pair of opposite vertices. Show that any two kites are conformally equivalent, by a map taking vertices to vertices.

17. Construct geometrically all the rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ with A_5 symmetry such that f has degree 59, and every critical point of f has nontrivial stabilizer in A_5. (Hint: use the result on kites above, and the fact that $116 = 36 + 80 = 36 + 20 + 30 = 96 + 20$.)

18. Prove Theorem 1.14. (Hint: use the fact that if $g(x,y)$ is a homogeneous polynomial of degree D, and $\lambda = x\,dy - y\,dx$, then $d(g \cdot \lambda) = (D+2)\,g(x,y)\,dx\,dy$.)

1.10 Unsolved problems

1. Is the open set of hyperbolic rational maps dense among all rational maps of degree d? Among all polynomials of degree d? (Even the case of quadratic polynomials is open).

2. Show that a suitable map $f : \mathbb{P}^2 \to \mathbb{P}^2$ with A_6 symmetry is generally convergent, thus verifying that the sextic can be solved by iteration in two variables.

1.11 Notes

A systematic treatment of the dynamics of rational maps on $\widehat{\mathbb{C}}$ can be found in [Mil], [CG] and [MNTU].

The problem of constructing generally convergent algorithms for solving polynomials was formulated by Smale in [Sm] and resolved in [Mc1]. The generalization to towers of iterations, and the solution to the quintic polynomial, are given in [DyM]. The approach to sextic polynomials using iteration on $\mathbb{P}^2$ and the Valentiner group is detailed by Crass and Doyle in [CD].

A modern perspective on the classical theory of the quintic equation (as in Klein's lectures on the icosahedron [Kl]) appears in a letter of Serre [Ser2]. See also [Ser1] for more on Galois cohomology and the Brauer group. Hilbert's 13th problem, on polynomials of degree 7, and its algebraic reformulation by Arnold and Shimura, are discussed in [Br] (see pp. 20 and 45).

2 Dynamics on $\mathbb{P}^k$

In this section we turn to the study of dynamics in several complex variables. To facilitate the discussion of forms and cohomology, we will study *holomorphic* dynamics on *compact* complex manifolds X. Because of compactness, it will be easy to take limits of functions, forms, measures and currents. We will also avoid the difficulties posed by the locus of indeterminacy of rational maps.

The simplest higher-dimensional complex manifolds are projective spaces, so we will concentrate in this chapter on dynamics on $\mathbb{P}^k$.

It must be mentioned that a great deal of work has been done on polynomial automorphisms of $\mathbb{C}^n$, especially the Hénon maps on $\mathbb{C}^2$, and that these examples fall outside of the present considerations.

2.1 Overview and examples

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism. Such a map can always be written in homogeneous coordinates in the form

$$z = [z_i] \mapsto F(z) = [F_i(z)]$$

where $F_i \in \mathbb{C}[z_0, \ldots, z_k]$ are homogeneous polynomials, all of the same degree d and with no common factor. The fact that f is not just a rational map, but is well-defined everywhere, is reflected in the fact that the only common zero of the polynomials F_i is at the origin in $\mathbb{C}^{k+1}$.

Degree. We refer to d as the *(algebraic) degree* of f. It is characterized by the property that for any hyperplane $H \subset \mathbb{P}^k$, the preimage $f^{-1}(H)$ is a hypersurface of degree d. In other words, f acts on $H^2(\mathbb{P}^k)$ by $x \mapsto d \cdot x$.

The *topological degree* of f is given by d^k. This is the number of preimages of a generic point in $\mathbb{P}^k$.

Critical points. The *critical locus* $C(f) \subset \mathbb{P}^k$ is the hypersurface along which $\det Df$ vanishes; it has degree $(k+1)(d-1)$.

Julia set. Just as for $\mathbb{P}^1$, we define the *Fatou set* $\Omega(f) \subset \mathbb{P}^k$ to be the set of points where $\mathcal{F} = \{f, f^2, f^3, \ldots\}$ forms a normal family. Its complement is the *Julia set* $J(f)$.

Examples.

1. Let $F[z_0, z_1, z_2] = [z_0^d, z_1^d, z_2^d]$. Then f leaves the lines in $\mathbb{P}^2$ defined by $z_i = 0$ invariant, and acts on each by $z \mapsto z^d$.

In an affine chart we have $f(z,w) = (z^d, w^d)$. The Julia set in these coordinates is the union of the 3 sets defined by $|z| = 1 \geq |w|$, $|w| = 1 \geq |z|$ and $|z| = |w| \geq 1$. Note that $J(f) \cap \mathbb{C}^2$ is definitely not equal to $(J(z^d) \times \mathbb{C}) \cup (\mathbb{C} \times J(w^d))$. Moreover, the closure of the repelling periodic points is strictly smaller than $J(f)$ — it is a sort of 'fine Julia set', given by $\{(z,w) : |z| = 1, |w| = 1\} \cong S^1 \times S^1$.

In this case, whether or not (z,w) lies in $J(f)$ depends only on the absolute values of the coordinates. Thus $J(f)$ can be described pictorially by its projection to $\mathbb{R}^2$ under (z,w) to $(|z|,|w|)$; see Figure 2.

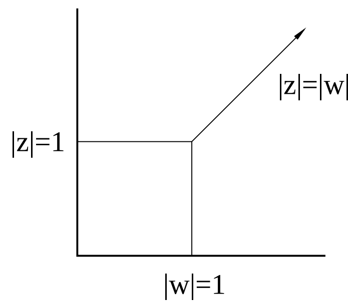

Figure 2. The Julia set of (z^d, w^d) as a circled domain.

2. Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a homogeneous polynomial map of degree $d > 1$ lifting a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d. Since F is proper, it extends to an endomorphism of $\mathbb{P}^2$, leaving the line at infinity $L \subset \mathbb{P}^2$ invariant. Moreover, $F|L$ is isomorphic to $f|\mathbb{P}^1$.

   Since $F(\lambda z) = \lambda^d F(z)$, the origin $z = 0$ is a superattracting fixed point for F. Let $B \subset \mathbb{C}^2$ denotes its basin of attraction. Let $K \subset \mathbb{P}^2$ denote the cone with base $J(f) \subset L$ and vertex $z = 0$. Then we have

$$J(F) = (K - B) \cup \partial B.$$

   In particular, when $J(f) = \mathbb{P}^1$, the Julia set $J(F)$ contains a neighborhood of L but excludes a neighborhood of the origin $z = 0$. This shows:

   For an endomorphism $F : \mathbb{P}^k \to \mathbb{P}^k$, $k \geq 2$, the Julia set can contain a nonempty open set even when $J(F) \neq \mathbb{P}^k$.

3. Given a pair of monic polynomials of the same degree, the map $f(z_1, z_2) = (p(z_1), q(z_2))$ extends from $\mathbb{C}^2$ to an endomorphism of $\mathbb{P}^2$. When z_1 and z_2 are large, we have $p(z_1)/q(z_2) \approx z_1^d/z_2^d$, and thus f acts on the line at infinity by $z \mapsto z^d$.

4. Recall that $\mathbb{P}^k$ is the same as the k-fold symmetric product $(\mathbb{P}^1)^{(k)}$. Given a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$, we obtain an endomorphism of $\mathbb{P}^k$ by letting $F(z_i) = (f(z_i))$ for any unordered k-tuple of points $z_i \in \mathbb{P}^1$. Compare [Ue].

5. Let $E = \mathbb{C}/\Lambda$ be an elliptic curve, and let $F : E^k \to E^k$ be the endomorphism $F(x) = dx$. Let $G = S_k \ltimes (\mathbb{Z}/2)^k$ acting by permutation and negation of coordinates on E^k. Then $E^k/G \cong P^k$, and since F commutes with G it descends to a degree d endomorphism of $\mathbb{P}^k$.

   This map can also be thought of as a k-fold symmetric product of a Lattés example on $\mathbb{P}^1$. Using the fact that F is expanding on E^k it is easy to see that $J(f) = \mathbb{P}^k$.

6. The *post-critical set* of f is defined by

$$P(f) = \overline{\bigcup_1^\infty f^n(C(f))}.$$

   An endomorphism of $\mathbb{P}^k$ is *critically finite* if $P(f)$ is an algebraic variety, i.e. if every component of $C(f)$ eventually cycles.

   A nice example of such a map on $\mathbb{P}^2$, due to Fornaess and Sibony, is given in affine coordinates by

$$f(z, w) = \left(\left(\frac{z - 2w}{z} \right)^2, \left(\frac{(z-2)}{z} \right)^2 \right). \tag{2.1}$$

   The map f has degree two, so $C(f)$ has degree 3; it consists of the lines

$$(C_1, C_2, C_3) \quad = \quad (z = 2w, z = 2, z = 0).$$

   (These are the lines along which the numerator or denominator in one of the squared expressions for f vanishes.)

   Under iteration, each of these lines eventually enters the cycle

$$(P_1, P_2, P_3) \quad = \quad (z = 1, w = 1, z = w),$$

   satisfying $P_1 \mapsto P_2 \mapsto P_3 \mapsto P_1$. One can show that the Julia set of f is all of $\mathbb{P}^2$ [For, p.15]

Cohomology and fixed-points.

Theorem 2.1 *A holomorphic map $f : \mathbb{P}^k \to \mathbb{P}^k$ of algebraic degree d acts on $H^{2i}(\mathbb{P}^k, \mathbb{Z})$ and on $H_{2i}(\mathbb{P}^k, \mathbb{Z})$ by multiplication by d^i.*

Proof. That $f|H^2(\mathbb{P}^k, \mathbb{Z})$ is multiplication by d is the definition of the algebraic degree. The general case follows from Poincaré duality and the fact that if x generates $H^2(\mathbb{P}^k, \mathbb{Z})$, then x^i generates $H^{2i}(\mathbb{P}^k, \mathbb{Z})$. ∎

Theorem 2.2 *The fixed points of an endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of degree $d > 1$ are isolated.*

Proof. If not, then $f(z) = z$ on some curve $C \subset \mathbb{P}^k$. Since the map $f : C \to C$ has degree one, we have $f_*([C]) = [C]$ in $H_2(\mathbb{P}^k, \mathbb{Z})$. On the other hand we have just seen that $f_*([C]) = d[C]$. Since $[C] \neq 0$, this is a contradiction. ∎

For another, more analytical proof, see [FS, Thm 3.1].

Corollary 2.3 *The map f has $1 + d + d^2 + \cdots + d^k$ fixed points, counted with multiplicity.*

Proof. The cohomology ring of $X = \mathbb{P}^k \times \mathbb{P}^k$ is given by

$$H^*(X) = \mathbb{Z}[\alpha, \beta]/(\alpha^{k+1} = \beta^{k+1} = 0),$$

where $\alpha^i \beta^j = [\mathbb{P}^{k-i} \times \mathbb{P}^{k-j}]$. The graphs of f and of the identity map satisfy:

$$\begin{aligned} \mathrm{gr}(\mathrm{id}) &= \sum_0^k \alpha^{k-i}\beta^i, \\ \mathrm{gr}(f) &= \sum_0^k d^i \alpha^{k-i}\beta^i, \end{aligned}$$

as can be verified using the intersection pairing in the middle-dimensional cohomology of X. Thus their product is given by

$$(1 + d + d^2 + \cdots + d^k)\alpha^k \beta^k.$$

∎

It can be shown that the multiplicity of a given isolated fixed point remains bounded under iteration. Therefore we have:

Theorem 2.4 *The map f has infinitely many periodic cycles.*

In stark contrast to the case of rational maps on $\mathbb{P}^1$, it is known that an endomorphism of $\mathbb{P}^2$ can have *infinitely many attracting cycles* [Ga].

Theorem 2.5 *The critical points $C(f) \subset \mathbb{P}^k$ are a divisor of degree $(k+1)(d-1)$.*

2.2 The escape rate for polynomials on $\mathbb{C}$

Subharmonic functions. Recall that a function h is *subharmonic* if the distributional Laplacian $\Delta h = \mu$ is a positive measure. A continuous function is subharmonic iff it satisfies the sub-mean-value property:

$$h(p) \leq \frac{1}{2\pi} \int_{S^1} h(p + re^{i\theta})\, d\theta.$$

The subharmonic functions on $\mathbb{R}$ are just the convex functions.

Example. Let $h(z) = \log^+|z| = \max(0, \log|z|)$; it is a continuous, subharmonic function on $\mathbb{C}$, since it is the maximum of two harmonic functions. Clearly $\mu = \Delta h$ is supported on the unit circle S^1 and rotationally invariant. Integrating $\nabla \log|z|$ over a large circle, we conclude that the total mass of μ is 2π and thus μ coincides with arclength on S^1.

Like harmonic functions, subharmonic functions are preserved under composition with holomorphic maps.

Escape rates. Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of degree $d > 1$. Its *filled Julia set*, $K(f)$, is the set of points with bounded orbits in $\mathbb{C}$. It is easy to see that $K(f)$ is compact and $\partial K(f) = J(f)$.

The *escape rate* of f is the function $\phi : \mathbb{C} \to \mathbb{R}$ given by

$$\phi(z) = \lim_{n\to\infty} d^{-n} \log^+|f^n(z)|.$$

The limit of the sequence $\phi_n(z)$ on the right exists uniformly on $\mathbb{C}$. To see this, note that $|f(z)|/|z|^d \to 1$ as $z \to \infty$, and thus $d^{-1}\log^+|f(z)| - \log^+|z|$ is uniformly bounded on $\mathbb{C}$; therefore $|\phi_{n+1}(z) - \phi_n(z)| = O(d^{-n})$, and the limit converges geometrically fast.

Using the fact that uniform limits of subharmonic functions are subharmonic, we find:

Theorem 2.6 *The escape rate $\phi(z) \geq 0$ is a subharmonic function, vanishing on $K(f)$ and harmonic on $\mathbb{C} - K(f)$. It satisfies:*

$$\phi(z) = \log|z| + O(1)$$

for z large, and $\phi(f(z)) = d\phi(z)$.

Charge. The *equilibrium measure* is given by $\mu = (1/2\pi)\Delta\phi$. It is a probability measure supported on $J(f)$. The terminology comes from the fact that a unit charge confined to $K(f)$ will distribute itself according to μ to minimize energy. From the properties of ϕ we have immediately:

Theorem 2.7 *The measure μ satisfies $f^*\mu = d \cdot \mu$ and $f_*\mu = \mu$.*

Proof. The first assertion follows from the functional equation $\phi(f(z)) = d\phi(z)$, and the second from the fact that $f_* f^*(\nu) = d\nu$ for any measure ν. ■

The measure μ records the distribution of almost any inverse orbit. Let us say a point $z \in \widehat{\mathbb{C}}$ is *exceptional* if its grand orbit is finite. The point $z = \infty$ is exceptional for polynomials, and $z = 0$ is exceptional for $f(z) = z^n$; these are essentially the only kinds of exceptional points for rational functions.

Theorem 2.8 *Let $p \in \mathbb{C}$ be any non-exceptional point, and let*

$$\mu_n = d^{-n} \sum_{f^n(z)=p} \delta_z.$$

Then $\mu_n \to \mu$ (weakly) as $n \to \infty$.

Sketch. For subharmonic functions, uniform convergence $\phi_n \to \phi$ implies weak convergence of the corresponding measures: $\Delta\phi_n \to \Delta\phi$. Let $\phi_0(z) = (1/2\pi)\log^+|z|$ and $\phi_n(z) = (\phi_0 \circ f^n(z))/d^n$. By uniform convergence of the escape function, we have

$$\nu_n = \Delta\phi_n \to \mu.$$

Here $\nu_n = d^{-n}(f^n)^*(\nu_0)$, and ν_0 is arclength measure on S^1 normalized to have total mass one.

Now the Theorem asserts that $d^{-n}(f^n)^*(\delta_p) \to \mu$. To indicate the main ideas, we give the argument in the case where the critical points satisfy $C(f) \subset K(f)$, and $z \notin K(f)$. (Later we will establish a much more general result.)

First, suppose we replace $\log^+|z|$ with another continuous, subharmonic function $h(z)$, also satisfying $h(z) = \log|z|$ for z large. Then we still have $\phi(z) = \lim d^{-n} h(f^n(z))$ uniformly. (Indeed, if z escapes to infinity, it enters the regime where $h(z) = \log|z|$ as before, while if z stays bounded, so does $h(z)$ and hence $\phi(z) = 0$.)

Now for $r > 0$ small, let $h(z) = \log^+|z-p|/r$. Then $\nu_0 = \Delta h$ is normalized arclength on the small circle

$$S^1(p,r) = \{z \,:\, |z-p| = r\}$$

By the same argument as above, we have $\nu_n = d^{-n}(f^n)^*(\nu) \to \mu$.

Choose r small enough that $B(p,2r) \cap K(f) = \emptyset$. Since $C(f) \subset K(f)$, the inverse map f^{-n} has d^n distinct univalent branches defined on $B(p,2r)$. Each branch has bounded distortion on $B(p,r)$. The set $f^{-n}(B(p,r))$ lies in an ϵ_n-neighborhood of $K(f)$, with $\epsilon_n \to 0$. Since the area of such a neighborhood, minus $K(f)$ itself, goes to zero, so does the area of $f^{-n}(B(p,r))$.

By bounded distortion, the diameter of each component of $f^{-n}(S^1(p,r))$ is also bounded by a constant $d_n \to 0$. But each such component encloses a point of $f^{-n}(p)$. Thus μ_n and ν_n have the same weak limits, and therefore $\mu_n \to \nu$. ∎

Brownian motion and Riemann maps. The measure $\mu(E)$ can also be interpreted as the probability that a Brownian particle starting at $z = \infty$ first hits $K(f)$ in the set E.

When $K(f)$ is connected (which is equivalent to $C(f) \subset K(f)$), we can construct a canonical *Riemann mapping*

$$h : \mathbb{C} - K(f) \to \mathbb{C} - \overline{\Delta},$$

normalized by $h(z) = z + O(1)$ for z large (in other words, by $h'(\infty) = 1$). Then h conjugates $f(z)$ to z^d; that is, we have $h(f(z)) = h(z)^d$; and the escape rate is given simply by $\phi(z) = \log|h(z)|$. Indeed, the Riemann mapping can be given by the formula:

$$h(z) = \lim_{n\to\infty} (f^n(z))^{1/d^n}$$

with a suitable choice of d^nth root.

Examples.

1. For $f(z) = z^d$, the measure μ is normalized arclength on $J(f) = S^1$.

2. The map $f(z) = z^2 - 2$ is semiconjugate to $s(z) = z^2$; indeed, $h(s(z)) = f(h(z))$ where $h(z) = z + z^{-1}$. Thus $J(f) = h(J(s)) = [-2, 2]$, and the measure $\mu_f = h_*(\mu_s)$ is given by
$$\mu_f = \frac{dx}{\pi\sqrt{4 - x^2}}.$$

3. For $f(z) = z^d + M$, M large, $J(f)$ is a Cantor set isomorphic to the 1-sided d-shift $\Sigma_d = (\mathbb{Z}/d)^{\mathbb{N}}$, and μ is isomorphic to the standard measure on σ_d, i.e. the product of the probability measures on $\mathbb{Z}/d$ that assign equal weight to each point.

4. For a Lattès example coming from multiplication by n on an elliptic curve $E = \mathbb{C}/\Lambda$, the escape-rate approach above does not make sense to define μ. Nevertheless, the inverse images of points are clearly distributed according to the push-forward of normalized Lebesgue measure on E. This measure is proportional to
$$\frac{|dz|^2}{|(z - a_1)(z - a_2)(z - a_3)(z - a_r)|}$$
if the 2-fold semiconjugacy $p : E \to \widehat{\mathbb{C}}$ is ramified over $\{a_1, a_2, a_3, a_4\}$.

2.3 Forms, currents, divisors and Chern classes

Currents. Let X be a complex manifold of dimension n. The space of (p, q) *currents* on X is the dual to the space of smooth, compactly supported $(n - p, n - q)$ forms.

A smooth (p, q)-form ω is also a (p, q) current, with pairing given by
$$\langle \omega, \eta \rangle = \int_X \omega \wedge \eta.$$
The space of (locally finite, Borel) measures on X is canonically a subspace of the space of (n, n)-currents. A (p, q)-current is essentially a (p, q)-form with distributional coefficients.

As usual we can extend differential operators to currents by integration by parts. A current is *closed* if $d\omega = 0$; equivalently, if $\langle \omega, d\alpha \rangle = 0$ for all compactly supported smooth forms α.

A smooth (p, p)-form α is *positive* if, for any complex submanifold $Y \subset X$ of dimension p, the restriction $\alpha|Y$ defines a non-negative volume element on Y. A current is positive if $\langle \omega, \alpha \rangle \geq 0$ for every smooth positive form.

Currents analogous to the delta function come from submanifolds. If $Z \subset X$ is a properly embedded complex submanifold of codimension p, then the *current of integration* over Z is the (p,p)-current given by:

$$\langle [Z], \eta \rangle = \int_Z \eta$$

for any smooth, compactly supported $(n-p, n-p)$-form η.

Proposition 2.9 *For any properly embedded codimension p complex submanifold $Z \subset X$, the (p,p)-current $[Z]$ is closed and positive.*

In particular, hypersurfaces in X give rise to closed, positive $(1,1)$-currents.

Variants of the Laplacian. Next we introduce a variant of the Laplacian whose values are also $(1,1)$-currents.

To keep the algebra simple, we begin with complex dimension one. On $\mathbb{C}$ we have the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and we can write $d = \partial + \overline{\partial}$, where on functions we have

$$\partial f = \frac{\partial f}{\partial z} dz \quad \text{and} \quad \overline{\partial} f = \frac{\partial f}{\partial \overline{z}} d\overline{z},$$

and similarly for forms. Note that $\partial\overline{\partial} f = (1/4)\Delta f$ and $dz \wedge d\overline{z} = -2i\, dx \wedge dy$. Thus we have

$$\partial\overline{\partial} f = \frac{1}{2i} \Delta f \, dx \wedge dy.$$

To recover reality, we introduce the operator

$$d^c = \frac{i}{2\pi}(\overline{\partial} - \partial).$$

It is real in the sense that $\overline{d^c f} = d^c \overline{f}$. Moreover, we have

$$dd^c f = \frac{i}{\pi} \partial\overline{\partial} f = \frac{1}{2\pi} \Delta f \, dx \wedge dy.$$

In particular, f is subharmonic iff $dd^c f$ is a positive $(1,1)$-current.

Now recall that $\Delta \log |z| = 2\pi \delta_0$ as a distribution. The normalization of dd^c is particularly convenient here: we have $dd^c \log |z| = \delta_0$ as well, where δ_0 is now interpreted as a $(1,1)$-current, or measure. More generally we have:

Theorem 2.10 *Let f be a meromorphic function on a Riemann surface, with divisor $(f) = \sum n_P \cdot P$. Then we have*

$$dd^c \log|f| = \sum n_P \delta_P. \tag{2.2}$$

Several complex variables. The differential operators above generalize in the expected way to several complex variables. For example, formula (2.2) works just as well in higher dimensions, giving the *Poincaré-Lelong formula* (see [GH, p.388]):

Theorem 2.11 *For any meromorphic function f on a complex manifold with divisor $(f) = \sum n_Z \cdot Z$, we have*

$$dd^c \log|f| = \sum n_Z [Z],$$

where $[Z]$ denotes the current of integration over the hypersurface Z.

(Note: our definition of d^c differs by a factor of 1/2 from [GH, p.109].)

Pluriharmonic functions. Let $f : X \to \mathbb{R}$ be a locally integrable function. We say f is *pluriharmonic* if $dd^c(f) = 0$; equivalently, if the restriction of f to every complex disk $\Delta \subset X$ is harmonic. (In this case f is actually smooth.)

Similarly, f is *plurisubharmonic* if $dd^c(f) \geq 0$ as a current; equivalently, if $f|\Delta$ is subharmonic for every holomorphic disk $\Delta \subset X$.

Theorem 2.12 *A function $f : X \to \mathbb{R}$ is pluriharmonic iff f is locally the real part of a holomorphic function.*

Proof. If $f = (g + \overline{g})/2$ is the real part of a holomorphic function, then $\overline{\partial} g = \partial \overline{g} = 0$; hence $\partial\overline{\partial}(f) = 0$ and $dd^c(f) = 0$.

Conversely, if $dd^c(f) = 0$, then $\alpha = \partial f$ satisfies $\overline{\partial}\alpha = 0$. Thus α is a holomorphic $(1,0)$-form; integrating, we can locally write $\alpha = dg$ for some holomorphic function g. Then we have $\overline{\partial} g = 0$ and $\partial g = \alpha = \partial f$. Thus:

$$d(g + \overline{g}) = (\partial g) + \overline{\partial}\overline{g} = (\partial f) + \overline{\partial f} = (\partial f) + \overline{\partial} f = df.$$

It follows that $f(z) = \mathrm{Re}(2g) + c$. ∎

Subharmonic functions and positive measures are (locally) interchangeable. The same is true for plurisubharmonic functions and closed, positive $(1,1)$-currents by:

Theorem 2.13 (The $\partial\overline{\partial}$-Poincaré Lemma) *Every closed, positive $(1,1)$-current is locally of the form $\omega = dd^c f$, where f is a plurisubharmonic function.*

See [GH, Chap. 3.2]

Remark. On a complex n-manifold, the equation $dd^c(f) = 0$ imposes $n(n+1)/2$ conditions on f, since a $(1,1)$-form has $n(n+1)/2$ components. The fact that we have more than one condition when $n > 1$ reflects the fact that the Cauchy-Riemann equations are overdetermined on complex manifolds of dimension two or more.

Positivity and metrics. If ω is a smooth positive $(1,1)$-form on X, it can be used to define a *Hermitian metric* on X. For real tangent vectors, the metric is given by:

$$g(v)^2 = \omega(v, Jv)$$

where J gives the complex structure on the real tangent space to X. For (the more usual) vectors in the holomorphic tangent bundle $\mathrm{T}X$, the metric is given by:

$$g(v)^2 = -2i\,\omega(v, \overline{v}).$$

If we write the form in local coordinates as

$$\omega = \frac{i}{2}\sum a_{ij}\, dz_i \wedge d\overline{z}_j,$$

then the metric is given by:

$$g(v)^2 = \sum a_{ij} v_i \overline{v}_j.$$

Thus ω is positive iff (a_{ij}) is a positive definite Hermitian matrix. (The condition that ω is *real*, i.e. that $\overline{\omega} = \omega$, already implies $a_{ji} = \overline{a}_{ij}$.)

Kähler manifolds. A *Kähler manifold* is a pair (X, ω) consisting of a complex manifold X equipped with a smooth, *closed*, positive $(1,1)$-form ω. For example, the Euclidean metric on $\mathbb{C}^n$ (satisfying $g(\partial/\partial z_i) = 1$) is attached to the closed $(1,1)$-form:

$$\omega = \frac{i}{2}\sum_1^n dz_i \wedge d\overline{z}_i = \sum dx_i \wedge dy_i.$$

Thus $\mathbb{C}^n$ is Kähler. The restriction of ω to a complex coordinate line gives the standard area form $dx_i \wedge dy_i$. On the other hand, of the 2^n cross terms arising in the formula for ω^n, only those $n!$ resulting from a permutation of the coordinates are nonzero, and thus

$$\omega^n = n! \cdot dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

That is, $\omega^n/n!$ agrees with the volume form coming from the metric determined by ω.

The same computation yields *Wirtinger's theorem*: for any complex submanifold Y^k of a Kähler manifold (X, ω) we have

$$\operatorname{vol}(Y^k) = \frac{1}{k!} \int_Y \omega^k.$$

It is a remarkable fact that positivity makes sense for forms on a complex manifold. The submanifolds of a real manifold are not canonically oriented, so positivity does not make sense in the real setting, nor is their any real analogue of Wirtinger's theorem.

GAGA for analysts. Let X be a compact complex manifold. To compute the cohomology of X, one can replace the de Rham complex of smooth forms with the complex of currents. The result ends up being the same; the two groups are naturally isomorphic. In particular, every closed current is cohomologous to a smooth form.

Compactness of currents. The set of all probability measures on a compact Hausdorff space is itself compact.

In local coordinates, a positive (p,p)-current is a matrix of measures. Thus with a suitable boundedness statement, one expects compactness. Here is a precise statement.

Theorem 2.14 *Let (X, ω) be a compact complex n-manifold, equipped with a smooth, positive $(1,1)$-form such that ω^n is everywhere nonzero. Then the space of all positive (p,p)-currents α on X with total mass satisfying*

$$M(\alpha) = \int_X \alpha \wedge \omega^{n-p} \leq 1$$

is compact.

Proof. For any smooth real form β of type complementary to α, we can find a constant $C > 0$ such that $C\omega^{n-p} \pm \beta$ is positive. Then we have

$$\left| \int \alpha \wedge \beta \right| \leq CM(\alpha),$$

and hence α ranges in a compact subset of the dual to the space of smooth forms. ∎

In the case of a Kähler manifold, we can take ω to be a *closed*, positive $(1,1)$-form. Then the mass $M(\alpha)$ depends only on the cohomology class of α. This shows:

Corollary 2.15 *Let X be a Kähler manifold, and let $K \subset H^{p,p}(X)$ be a compact set. Then set of all closed, positive $(1,1)$-forms representing cohomology classes in K is compact.*

Line bundles. Now let $L \to X$ be a holomorphic line bundle over a compact Riemann surface X. A (Hermitian) metric on L is given locally by

$$\|\xi\| = g(z) \cdot |\xi|,$$

where (z, ξ) are coordinates on a holomorphic trivialization $L|U \cong U \times \mathbb{C}$.

Theorem 2.16 *The first Chern class of L is represented by the form given locally by $\omega = -dd^c \log(g)$.*

Note that under a change of coordinates on L, $g(z)$ is replaced by $g(z)h(z)$ with $h \in \mathcal{O}^*(U)$; since $\partial\overline{\partial} \log |h(z)| = 0$, the value of ω does not change, and hence it is globally well-defined.

Sketch. Here is an intuitive proof of the Theorem above. First, if g_1 and g_2 are two metrics, then g_1/g_2 is a globally well-defined, nowhere vanishing smooth function, and the corresponding $(1,1)$ forms satisfy

$$\omega_1 - \omega_2 = -dd^c \log(g_1/g_2).$$

Thus the de Rham cohomology class of ω is independent of the choice of metric.

Secondly, if $f : X \to L$ is any meromorphic section, then its divisor represents $c_1(L)$ and is given by the current $dd^c \log |f|$, where we have locally expressed the section as a map $f : U \to \mathbb{C}$ with respect to a trivialization $L|U \cong U \times \mathbb{C}$.

Finally, a section determines a singular metric on L by declaring $\|f(z)\| = 1$, or in other words by setting $\|\xi\| = |\xi|/|f(z)|$ locally. For this 'metric' we have $g(z) = 1/|f(z)|$, and thus the first Chern class is represented by $(f) = dd^c \log |f| = -dd^c \log g$. ∎

Absence of divisors. We should remark that if we work outside the context of projective varieties, not every line bundle admits a meromorphic section, so the sketch doesn't quite apply. E.g. if $X = \mathbb{C}^2/\Lambda$ is a complex torus with no divisors, we still have nontrivial (flat) line bundles on X coming from representations $\rho : \pi_1(X) \to S^1$; of these, only the trivial bundle admits a meromorphic section.

Example. Let X be a Riemann surface of genus $g \geq 2$, and let $\mathrm{T}X$ be the tangent bundle to X. The hyperbolic metric $|dz|/y$ on $\mathbb{H}$ of constant curvature -1 gives a Hermitian metric on $\mathrm{T}X$. The volume form for this metric is given by

$$\alpha = \frac{|dz|^2}{y^2} = \frac{dx \wedge dy}{y^2}$$

and satisfies $\int_X \alpha = 4\pi(g-1)$ by Gauss-Bonnet. Since $g(z) = 1/y$ on $\mathbb{H}$, the first Chern class of $\mathrm{T}X$ is represented by the form

$$\begin{aligned} \omega &= dd^c \log(y) = \frac{i}{\pi}\partial\overline{\partial}\log(z-\overline{z}) = \frac{i}{\pi}\frac{dz \wedge d\overline{z}}{(z-\overline{z})^2} \\ &= \frac{i}{\pi}\frac{(-2i\, dx \wedge dy)}{(-4y^2)} = -\frac{1}{2\pi}\frac{dx \wedge dy}{y^2} = -\frac{1}{2\pi}\alpha. \end{aligned}$$

Thus $\int_X \omega = (-1/2\pi)\int_X \alpha = 2 - 2g = \chi(X)$, as expected.

Upstairs on the line bundle. Here is a more global formulation of the result on $c_1(L)$.

Consider the function $G : L \to \mathbb{R}$ defined by $G(z,\xi) = \|\xi\|_g$. Then $\omega_1 = -dd^c \log G$ is a closed $(1,1)$-form on the complement of the zero section in L. Since locally $G(z,\xi) = g(z)|\xi|$, and $dd^c \log|z| = 0$, the form G is pulled back from the base manifold X (it only depends on z). Writing $\omega_1 = \pi^*(\omega)$, we have $c_1(L) = [\omega]$.

Fubini-Study metric. The most basic example of the global formula for the first Chern class comes from the tautological bundle $L \to \mathbb{P}^k$. Recall that the complement of the zero section of L can be identified with $\mathbb{C}^{k+1} - 0$. The standard metric on L is given simply by

$$G(z) = G(z_0, \ldots, z_k) = |z| = \sqrt{\sum |z_i|^2}.$$

Since $-dd^c \log G$ gives $c_1(L)$, we have:

Theorem 2.17 *The form $\omega_1 = dd^c \log|z|$ on $\mathbb{C}^{k+1}$ is pulled back from a form ω on $\mathbb{P}^k$ with*

$$[\omega] = c_1(L^*) \in H^2(\mathbb{P}^k, \mathbb{Z}).$$

In particular for any hyperplane H we have

$$\int_H \omega = 1.$$

The form ω is the symplectic form associated to the *Fubini-Study metric* on $\mathbb{P}^k$. This metric is normalized so that the volume of $\mathbb{P}^k$ is equal to 1.

Example. By a direct calculation, we obtain

$$\omega_1 = \frac{i}{2\pi}\partial\overline{\partial}\log|z|^2 = \frac{i}{2\pi}\cdot\frac{\langle z,z\rangle\langle dz,dz\rangle - \langle dz,z\rangle\langle z,dz\rangle}{|z|^4},$$

where $\langle a,b\rangle = \sum a_i\overline{b}_i$. In the case of $\mathbb{P}^1$ with affine coordinate z, we can pull back ω_1 to $\mathbb{P}^1$ via $z \mapsto (z,1) = (z_0,z_1)$ to obtain:

$$\omega = \frac{i}{2\pi}\cdot\frac{(1+|z|^2-|z|^2)\,dz\wedge d\overline{z}}{(1+|z|^2)^2} = \frac{1}{\pi}\cdot\frac{|dz|^2}{(1+|z|^2)^2}$$

(since $dz\wedge d\overline{z} = -2i|dz|^2$). The usual metric of constant curvature 1 on $\mathbb{P}^1$ has area form $\alpha = 4|dz|^2/(1+|z|^2)^2$, satisfying $\int\alpha = 4\pi$; thus $\int\omega = 1$ as expected.

Primitives. It is intuitively reasonable that if we have a charge distribution μ on a closed Riemann surface X with $\int_X \mu = 0$, then there should be a harmonic function h such that $\Delta h = 0$. This function h is simply the potential for this charge distribution. For example, if μ has finite support, then ∇h gives the minimum energy area-preserving flow with the specified sources and sinks. The sinks and sources must exactly cancel so that area can be conserved.

In higher dimensions we have an analogous result.

Theorem 2.18 *Let M be a compact Kähler manifold. Then a smooth, closed $(1,1)$-form ω represents $0 \in H^2(M,\mathbb{C})$ if and only if $\omega = dd^cf$ for some smooth function f.*

Proof. We will show equivalently that $\omega = \partial\overline{\partial}f$.

We first treat the case $M = \mathbb{P}^k$, which is the only case needed for our applications to dynamics. Then the Dolbeault cohomology group $H^{0,1}_{\overline{\partial}}(\mathbb{P}^k)$ vanishes, since it appears as a factor in the Hodge decomposition of $H^1(\mathbb{P}^k) = 0$.

We can assume $\omega = \overline{\omega}$. Since ω is exact and real, we can write

$$\omega = d(\alpha+\overline{\alpha}) = (\partial+\overline{\partial})(\alpha+\overline{\alpha}),$$

where α is a $(1,0)$-form. Then $\overline{\partial}\overline{\alpha} = 0$, since this form represents the $(0,2)$ part of $d(\alpha + \overline{\alpha})$. By the vanishing of $H^{0,1}_{\overline{\partial}}(\mathbb{P}^k)$, we have $\overline{\alpha} = \overline{\partial} f$ for some smooth function f. Thus we have

$$\omega = (\partial + \overline{\partial})(\partial \overline{f} + \overline{\partial} f) = \partial\overline{\partial}(f - \overline{f}),$$

completing the proof.

In the general case, when $H^{0,1}_{\overline{\partial}}(M) \neq 0$, we again appeal to Hodge theory to assert that the de Rham and Dolbeault decompositions of the cohomology of M agree. In dimension one this gives the isomorphisms:

$$H^1(M) \cong H^{0,1}_{\overline{\partial}}(M) \oplus H^{1,0}_{\overline{\partial}}(M) \cong \Omega(M) \oplus \overline{\Omega}(M),$$

where $\Omega(M)$ is the space of holomorphic 1-forms on M. (Every such form is closed.)

Thus the class $[\overline{\partial}\overline{\alpha}] \in H^{0,1}_{\overline{\partial}}(M)$ is represented by the complex conjugate $\overline{\beta}$ of a closed holomorphic 1-form. This means we can write $\overline{\alpha} = \overline{\beta} + \overline{\partial} f$. Since $d(\beta) = d(\overline{\beta}) = 0$, we obtain $\omega = \partial\overline{\partial}(f - \overline{f})$ just as before. ∎

Compare [GH, p. 149].

Corollary 2.19 *If there is a line bundle $L \to M$ such that $[\omega] = c_1(L)$, then there is a Hermitian metric g on L whose curvature form is $\omega = -dd^c \log g$.*

Proof. Choose an arbitrary Hermitian metric h on L and solve the equation $dd^c f = \omega + dd^c \log h$. Then ω gives the curvature of the metric $g = e^{-f} h$. ∎

Products of currents. It is a notorious fact that a product of distributions, or a product of currents, does not make sense in general. However it is clear geometrically that the intersection of analytic submanifolds often *does* make sense. This suggest that certain currents *do* admit products.

Indeed, let α and β be a pair of *closed, positive* currents. Suppose in addition that α is locally given by a *continuous* potential; that is, we can locally write $\alpha = dd^c(\gamma)$ for some continuous form γ.

Then $\alpha \wedge \beta$ is a well-defined current. Its pairing with a smooth form ξ of complementary dimension is defined by

$$\int \xi \wedge \alpha \wedge \beta = \int \xi \wedge (dd^c\gamma) \wedge \beta = \int (dd^c\xi) \wedge \gamma \wedge \beta.$$

Here we have formally used integration by parts and the fact that β is closed. Also we have assume (without loss of generality) that the support of ξ is small enough that we can express α in terms of the continuous potential γ over the support of ξ.

Finally we note that the last expression above is well-defined because the positive form β, being like a measure, pairs with the continuous form $(dd^c\xi) \wedge \gamma$.

Stein manifolds. A complex manifold X is *Stein* if X admits a proper holomorphic embedding in $\mathbb{C}^n$ for some n. Thus Stein manifolds are analogous to, and include, affine algebraic varieties.

Theorem 2.20 *Let T be a closed, positive $(1,1)$-form on a Stein manifold X. Then the complement of the support of T is also a Stein manifold.*

This result can be compared to the fact that the complement of a hypersurface in an affine variety is again an affine variety. That is, one can imagine that the support of T is like a possibly diffuse divisor on X. See [Ue] for a proof.

2.4 The escape rate on $\mathbb{C}^{k+1}$

Experience with complex manifolds suggests that in passing from dimension one to higher dimensions, it is often advantageous to replace *points* by *divisors*. Dynamically, this means that understanding the orbit of a divisor might be as useful as understanding the orbit of a point.

In this section we discuss a potential-theoretic approach to the equilibrium measure already discussed for polynomials on $\mathbb{C}$. This approach, introduced by Hubbard and Papadopol, has several advantages:

1. It applies immediately to rational maps on $\mathbb{P}^1$ and to endomorphisms of $\mathbb{P}^k$.

2. It suggests that the natural generalization of the equilibrium measure μ_f to $\mathbb{P}^k$ is a positive $(1,1)$-current T_f, rather than a measure.

3. It makes it easy to show that T_f gives the asymptotic distribution of the backwards orbit of a *diffuse* divisor.

Lifting. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map of algebraic degree $d > 1$. We begin by lifting f to a homogeneous polynomial map

$$F : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}.$$

The map F is uniquely determined by f up to composition with $z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$.

Since $F(\lambda z) = \lambda^d F(z)$, the map F has a superattracting fixed point at $z = 0$. Moreover, letting

$$B(F) = \{z \,:\, f^n(z) \to 0\}$$

denote the basin of attraction of the origin, we have $f^n(z) \to \infty$ for all z outside $\overline{B(F)}$.

Thus the lifted map F has plenty of attracting and wandering behavior, even in the cases where $J(f) = \mathbb{P}^k$. The advantages of passing to F are reminiscent of the advantages of considering the action of a Kleinian group Γ on $\mathbb{H}^3$, where its action is properly discontinuous even when the limit set $\Lambda(\Gamma) = \widehat{\mathbb{C}}$.

Escape rate on $\mathbb{C}^{k+1}$. We now define, as in the case of polynomials in one variable, the *escape rate* of F by

$$\phi(z) = \lim \phi_n(z) = \lim d^{-n} \log \|F^n(z)\|.$$

(By convention, $\phi(0) = -\infty$.)

Theorem 2.21 *The escape rate converges uniformly on $\mathbb{C}^{k+1}$, and defines a plurisubharmonic function satisfying $\phi(\lambda z) = \log|\lambda| + \phi(z)$ and*

$$\phi(F(z)) = d\phi(z).$$

Proof. Note that $b(z) = \log|F(z)| - \log|z|^d$ is homogeneous of degree zero and continuous, hence bounded (by its max on $|z| = 1$.) Because of this, we have

$$\phi_n(z) - \phi_{n-1}(z) = d^{-n}\left(\log|F(F^{n-1}(z))| - \log|F^{n-1}(z)|^d\right) = O(d^{-n}).$$

Thus $\phi_n(z)$ converges uniformly and exponentially fast.

Since $\log|z|$ is plurisubharmonic, so is $\log|F^n(z)|$, and hence so is the limit $\phi(z)$. The functional equations satisfied by ϕ are immediate. ∎

Metrics and currents. Note that the function $g(z) = \exp(\phi(z))$ can be interpreted as a *continuous metric* on the tautological bundle over $\mathbb{P}^k$. Thus there is a closed, positive $(1,1)$-current T_f on $\mathbb{P}^k$, defined by

$$dd^c \phi(z) = \pi^*(T_f),$$

representing the cohomology class of a hyperplane section as well as the induced metric on the dual of the tautological bundle. We refer to T_f as the *canonical current* for f.

Theorem 2.22 *The current T_f on $\mathbb{P}^k$ satisfies $f^*(T_f) = dT_f$, $f_*(T_f) = T_f$ and*

$$d^{-n}(f^n)^*(\omega) \to T_f$$

for any smooth, closed $(1,1)$-form ω on $\mathbb{P}^k$ normalized such that $\int_{\mathbb{P}^k} \omega^k = 1$.

(The last condition means simply that $[\omega]$ represents the positive generator of $H^2(\mathbb{P}^k, \mathbb{Z})$.)

Proof. The functional equations satisfied by T_f follow from those satisfied by ϕ. As for the convergence, we have already verified this for the special case of the symplectic form of the Fubini-Study metric, $\omega_0 = dd^c \log |z|$.

But for any other normalized form, $\omega - \omega_0$ is cohomologous to zero, which implies $\omega - \omega_0 = dd^c(h)$ for some smooth function h on $\mathbb{P}^k$. Since we have

$$d^{-n}(f^n)^*(dd^c(h)) = dd^c(d^{-n} h \circ f^n) \to 0,$$

convergence also holds for ω. ∎

Other norms and metrics. Here is an alternative perspective on the result above, that makes it clear that we can have

$$d^{-n}(f^n)^*(\omega) \to T_f$$

even when ω is not smooth.

Let $h : (\mathbb{C}^{k+1} - 0) \to \mathbb{R}$ be any positive, continuous function satisfying $h(\lambda z) = |\lambda| h(z)$. Define the associated curvature form by $\pi^*(\omega_h) = dd^c \log h(z)$. By compactness of the unit sphere, the ratio $h(z)/\|z\|$ is uniformly bounded on $\mathbb{C}^{k+1} - 0$, and hence we have:

Theorem 2.23 *For any continuous metric h on the tautological bundle to $\mathbb{P}^k$, we have:*

$$\phi(z) = \lim d^{-n} \log h(F^n(z))$$

uniformly, and hence

$$d^{-n}(f^n)^*(\omega_h) \to T_f.$$

We have seen that *any* smooth $(1,1)$-form ω in the correct cohomology class arises as the curvature of a (smooth) metric h, so we have an alternative proof that such forms converge to the canonical current under iterated pullback.

Consistency with polynomials. For example, take the metric given by the L^∞ norm:

$$h(z) = \|z\|_\infty = \sup |z_i|.$$

In the case of $\mathbb{C}^2$, the current on $\mathbb{P}^1$ given by $dd^c \log \|z\|_\infty$ is simply normalized arclength measure along the unit circle $|z| = 1$. Moreover, we have

$$\log \|(z,1)\|_\infty = \log^+|z|.$$

From this it follows that T_f coincides the measure μ_f defined previously for polynomials on $\mathbb{C}$.

Moreover, if we lift a polynomial $p(z) = \mathbb{C} \to \mathbb{C}$ via $F(z,w) = (P(z,w), w^d)$ — where $P(z,w)$ is the homogenization of $p(z)$ — then the line $L = \{(z,1) : z \in \mathbb{C}\}$ is invariant under F. It follows that

$$\phi(z,1) = \lim d^{-n} \log^+|p^n(z)| = \phi_f(z)$$

agrees on this affine slice with the escape rate previously defined for $p : \mathbb{C} \to \mathbb{C}$.

Julia sets via currents. We can now relate T_f to $J(f)$.

Theorem 2.24 *The Julia set of f coincides with the support of the closed, positive $(1,1)$-current T_f.*

Proof. Let $p \in \mathbb{P}^k$ belong to the Fatou set of f. Then there is a neighborhood U of p such that $f^n|U$ converges uniformly to a holomorphic function $g : U \to \mathbb{P}^k$, at least along a subsequence $n_k \to \infty$.

Shrinking U if necessary, we can assume that $g(U)$ is contained in a small ball $B \subset \mathbb{P}^k$. Let ω be a normalized smooth closed $(1,1)$-form on $\mathbb{P}^k$ such that $\int_U \omega = 0$. Then it is clearly that $\omega_n = d^{-n}(f^n)^*(\omega) = 0$ on U, at least along a subsequence $n_k \to \infty$. But we have $\omega_n \to T_f$, so $T_f|U = 0$.

Conversely, suppose $T_f = 0$ on an open ball $V \subset \mathbb{P}^k$. Let

$$\pi : \mathbb{C}^{k+1} - \{0\} \to \mathbb{P}^k$$

be the natural projection, let $F : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ be a lift of f, and let $\phi(z)$ be its escape rate. Recall that $B(F) = \{z : \phi(z) < 0\}$ is the basin of attraction of $z = 0$ for F.

We claim there is a holomorphic section $s : V \to \mathbb{C}^{k+1}$ of π such that $s(V) \subset \partial B(F)$.

Indeed, since $T_f|V = 0$, the escape rate ϕ is pluriharmonic on the open cone $U = \pi^{-1}(V) \subset \mathbb{C}^{k+1} - \{0\}$. Thus we can write $\phi = \log |g|$ for a holomorphic function g on U. (First write $\phi = \operatorname{Re} h$, then $g = e^h$; and use the fact that U is simply-connected.) Since $\phi(\lambda z) = \phi(z) + \log |\lambda|$, we have

$$g(\lambda z) = \lambda g(z).$$

In particular, $dg \neq 0$.

Let $V_0 = \{z \in U \ : \ g(z) = 1\}$. Then $\phi | V_0 = 0$, so we have $V_0 \subset \partial B(F)$. By the homogeneity condition above, V_0 meets every line $\mathbb{C}^* \cdot z \subset U$ in a unique point, and V_0 is a complex hypersurface since $dg \neq 0$. Thus there is a unique holomorphic section $s : V \to V_0$; it is characterized by

$$\{s([z])\} = (\mathbb{C}^* \cdot z) \cap (V_0).$$

Finally, we can write $f^n | V = \pi \circ F^n \circ s$. Since we have $s(V) = V_0 \subset \partial B(F)$, and $B(F)$ is bounded and F-invariant, the iterates $F^n \circ s$ form a normal family. Thus $f^n | V$ also forms a normal family, and hence V is contained in the Fatou set of f. ∎

We can state the last part of the argument more intrinsically.

Theorem 2.25 *The part of $\partial B(F)$ lying over the Fatou set of f on $\mathbb{P}^k$ admits a foliation $\mathcal{F}$ by complex k-manifolds, with*

$$\mathrm{T}\mathcal{F} = \mathrm{Ker}(\partial \phi).$$

Any simply-connected region $U \subset \mathbb{P}^k - J(f)$ admits a holomorphic lifting to a leaf of $\mathcal{F}$ inside $\partial B(F)$.

Note that $\alpha = \partial \phi$ is a holomorphic 1-form on the cone over $\Omega(f)$ in $\mathbb{C}^{k+1}$.

The invariant measure. Given $f : \mathbb{P}^k \to \mathbb{P}^k$, we obtain a natural invariant probability measure μ_f by setting

$$\mu_f = T_f^k.$$

This measure satisfies $f_*(\mu_f) = \mu_f$ and $f^*(\mu_f) = d^k \mu_f$. The kth power of the current T_f is well-defined because T_f is closed and positive, and because it admits a continuous potential coming from the escape rate.

Theorem 2.26 *For any normalized smooth symplectic form ω on $\mathbb{P}^k$, we have*

$$\mu_f = \lim d^{-nk} (f^n)^* (\omega^k).$$

Proof. We have already seen that $d^{-n}(f^n)^* \omega \to T_f$, so this result follows from suitable continuity results regarding wedge products. ∎

Stratified Julia sets. In higher dimensions it seems useful to *stratify* the Julia set, by defining $J_1(f) = J(f) = \mathrm{supp}(T_f)$ and

$$J_i(f) = \mathrm{supp}(T_f^i).$$

For example, we have seen that on $\mathbb{P}^2$ we can have examples where $\emptyset \neq \mathrm{int}(J(f)) \neq \mathbb{P}^2$. But we will later see (Corollary 2.54) that $J_k(f) = \mathbb{P}^k$ whenever $\mathrm{int}\, J_k(f) \neq \emptyset$.

Metrics on $\mathbb{P}^1$. The escape rate for a rational map on $\mathbb{P}^1$ gives a Hermitian metric on $\mathcal{O}(-1)$, which in turn gives a Kähler metric on $\mathbb{P}^1$ using the isomorphism $\mathcal{O}(2) \cong \mathrm{T}\mathbb{P}^1$. This metric g is determined up to scale, and is characterized by the property that its *curvature* $K(g)$, thought of as a measure with total integral 4π (by Gauss-Bonnet), satisfies $K(g) = 4\pi\mu_f$.

The metric g is flat outside of the Julia set. Because of this, the unit tangent bundle for g is foliated by holomorphic curves. The holonomy of this flat metric determines a homomorphism

$$\rho : \pi_1(\Omega(f)) \to S^1$$

which just gives the total mass of μ_f enclosed by a loop (on either side). Under parallel transport along a closed loop $\gamma \subset \Omega(f)$, a unit vector is rotated by $\rho(\gamma)$.

For an easily-visualized example, consider the case $f(z) = z^2 + c$ with $c < -2$. Then $J(f)$ is a Cantor set on the real axis. With respect to the flat metric g, the upper half-plane is isometric to a bounded 'polygon' $P \subset \mathbb{C}$ with a Cantor set's worth of 'vertices'. The double of P gives $(\mathbb{P}^1, g)$.

2.5 Kobayashi hyperbolicity

General theory. Let X be a complex manifold, and let Δ be the upper halfplane equipped with the hyperbolic metric

$$\|\xi\|_\Delta = \frac{|dz(\xi)|}{1 - |z|^2}$$

of constant curvature -4. The *Kobayashi Finsler metric* on X is the length function $\|\xi\|_X$, $\xi \in \mathrm{T}X$, defined by

$$\|\xi\|_X = \inf\{\|\xi'\|_\Delta \;:\; \exists \text{ a holomorphic } f : \Delta \to X \text{ with } Df(\xi') = \xi.\}$$

Define a distance function on X by:

$$d_X(x, y) = \inf_\gamma \int_0^1 \|\gamma'(t)\|_X \, dt,$$

with the infimum over all paths $\gamma : [0,1] \to X$ joining x to y.

We say X is *Kobayashi hyperbolic* if $d_X(x,y)$ is actually a metric. (The only danger is that we might have $d_X(x,y) = 0$ even though $x \neq y$.)

Note: because of its normalization, the Kobayashi metric on $\mathbb{H}$ is given by $|dz|/(2y)$. An advantage of this normalization is that the Kobayashi and Teichmüller metrics agree on $\mathcal{M}_g$.

Schwarz lemma. The following result is immediate from the definitions.

Theorem 2.27 *Let X be a Kobayashi hyperbolic manifold. Then any holomorphic map $f : \Delta \to X$ is distance decreasing; that is, we have*

$$\|Df(\xi)\|_X \leq \|\xi\|_\Delta.$$

More generally, any holomorphic map between Kobayashi hyperbolic manifolds is non-expanding.

In other words, the Schwarz lemma holds by fiat for the Kobayashi metric.

Local behavior. It is known that $d_X(x,y)$ is continuous on $X \times X$, and that when it is a metric, it induces the usual topology on X. (Proof: the open balls in (X,d) are path connected, and the intersection of $B_X(p,r)$ for $r > 0$ is just $\{p\}$; thus the balls must eventually be disjoint from any small sphere around p. See [Bar].)

From these statements we obtain:

Theorem 2.28 *If X is Kobayashi hyperbolic, then there are Riemannian metrics g and h such that*

$$\|\xi\|_g \leq \|\xi\|_X \leq \|\xi\|_h$$

for all $\xi \in \mathrm{T}X$.

Proof. It suffices to prove the result locally. The result is immediate for a polydisk, using the fact that any two Finsler norms invariant under $\mathrm{Aut}(\Delta^k)$ are comparable. The upper bound then follows, using the fact that inclusions are contracting.

For the lower bound, let p be a point of X and U a neighborhood of p such that (U,p) is biholomorphic to $(\Delta^k, 0)$. As we have just observed, $\|\xi\|_U$ is bounded below by a Riemannian metric. Thus it suffices to show that $\|\xi\|_X$ is locally bounded below by a multiple of $\|\xi\|_U$.

Since the Kobayashi metric d induces the usual topology, there is an $r > 0$ such that $B(p,r) \subset U$. Consider $\xi \in \mathrm{T}_q X$, $q \in B(p, r/2)$, and let $f : (\Delta, 0) \to (X, q)$ be a holomorphic map with $Df(\xi') = \xi$. Then there is an $s > 0$, depending only on r, such that $f(\Delta(s)) \subset U$. Indeed, we need only choose s such that the hyperbolic radius of $\Delta(s)$ about $z = 0$ is equal to $r/2$. By considering $f(sz)$ as a map from the unit disk into U, we find:

$$\|\xi'\| \geq s\|\xi\|_U.$$

Taking the infimum, we conclude that $\|\xi\|_X \geq s\|\xi\|_U$. ∎

If X is a domain in a compact complex manifold Y, we say X is *hyperbolically embedded* if there is a Riemannian metric g on Y such that $\|\xi\|_X \geq \|\xi\|_g$.

Theorem 2.29 (Brody) *A compact manifold X is Kobayashi hyperbolic if and only if any holomorphic map $f : \mathbb{C} \to X$ is constant.*

Theorem 2.30 *Let $H \subset X$ be a hypersurface in a compact complex manifold such that both H and $X - H$ are hyperbolic. Then $X - H$ is hyperbolically embedded in X.*

Examples.

1. The upper halfplane $\mathbb{H}$ is Kobayashi hyperbolic, and $\|\xi\|_{\mathbb{H}} = |dz(\xi)|/(2\operatorname{Im}(z))$ is a multiple of the usual hyperbolic metric.
2. The spaces $\mathbb{C}$ and $\widehat{\mathbb{C}}$ are not Kobayashi hyperbolic.
3. Kobayashi hyperbolicity is preserved under passing to covering spaces. Thus a Riemann surface is Kobayashi hyperbolic iff its universal cover is $\mathbb{H}$.
4. The Kobayashi metric on a product is given by

   $$\|\xi\|_{X\times Y} = \max(\|(\pi_X)_*\xi\|_X, \|(\pi_Y)_*\xi\|_Y).$$

   In particular the Kobayashi metric on the polydisk Δ^n gives an ℓ^∞ norm on each tangent space.
5. Any subdomain of a Kobayashi hyperbolic manifold is also Kobayashi hyperbolic.
6. Any bounded domain in $\mathbb{C}^n$ is Kobayashi hyperbolic.
7. Any manifold that can be immersed into a Kobayashi hyperbolic manifold is itself Kobayashi hyperbolic.

8. In contrast to the case of $\mathbb{C}$ — where the complement of 2 points is already Kobayashi hyperbolic — the complement of a compact set in $\mathbb{C}^2$ is never hyperbolic, since it still contains many complex lines.

9. Similarly, the Kobayashi metrics on $X = \mathbb{H} \times \mathbb{H}$ and on $X - \{p\}$, $p \in X$, are the same; thus the Kobayashi metric in higher dimensions need not be complete.

10. The Kobayashi and Teichmüller metric on Teichmüller space $\mathcal{T}_g$ coincide.

11. The complement of 4 lines in $\mathbb{P}^2$ is never hyperbolic. Indeed, we can always find a 5th line that meets the original 4 in just two points. This gives a copy of $\mathbb{C}^*$ in the complement.

12. Let $V = \bigcup_1^{2k+1} H_i$ be a union of hyperplane in general position in $\mathbb{P}^k$. Then the complement $X = \mathbb{P}^k - V$ is Kobayashi hyperbolic and hyperbolically embedded.

    For a heuristic proof, note that any line L meets V in $2k+1$ points, and L moves in a $2k-2$-dimensional family. By imposing $2k-2$ conditions on L, we can arrange that $2k-1$ points collapse to a single point. There still remain 2 other points, so $|L \cap V| \geq 3$ for all L. Thus $L - V$ is 'uniformly hyperbolic'.

    (The detailed proof is a little delicate and turns on Borel's Lemma: if $h_i(z)$ are *units*, that is entire functions without zeros, and $h_1 + \cdots + h_n = 0$, then there is a subsum that is also identically zero and whose terms satisfy $h_i = c_i f$ for some fixed unit f and some $c_i \in \mathbb{C}^*$.)

13. If X is a submanifold of a complex torus $\mathbb{C}^g/\Lambda$, and X does not contain the translate of a subtorus, then X is Kobayashi hyperbolic (Green).

Blowup at divisors. The hyperbolic metric on the punctured disk Δ^* is given by $\rho = |dz|/(|z| \log(1/|z|))$. Similarly, if X is a hyperbolic Riemann surface, the hyperbolic metric on $X - \{p\}$ is asymptotic to $|dz|/(|z| \log(1/|z|))$ in an appropriate chart z near p with $z(p) = 0$.

It is useful to have a similar picture in higher dimensions. Such a picture is provided by:

Theorem 2.31 *Let $D \subset X$ be a divisor in a Kobayashi hyperbolic manifold. Let $dV_{X-D} = f(x)\, dV_X$ relate the Kobayashi volume forms on $X - D$ and X. Then $f(x) \to \infty$ as $x \to D$.*

Here the *Kobayashi volume element* dV_X on $\mathrm{T}_p(X)$ is defined so that the measure of the unit norm ball,

$$\{\xi \in \mathrm{T}_p(X) \,:\, \|\xi\|_X < 1\},$$

agrees with the Euclidean volume of the unit ball in $\mathbb{C}^k$, $k = \dim(X)$.

Lemma 2.32 *Let $D \subset X$ be any closed subset of a Kobayashi hyperbolic manifold. Let $B \subset X$ be an open set. Then the Kobayashi metrics $\|\xi\|_{X-D}$ and $\|\xi\|_{B-D}$ are comparable on any compact subset of B.*

Proof. Let $K \subset B$ be a compact set. Choose $s > 0$ such that the diameter of $\Delta(s)$ in the hyperbolic metric on Δ is less than $d_X(K, \partial B)$. Given a vector $\xi \in \mathrm{T}_p(X)$, $p \in K - D$, let $f : (\Delta, 0) \to (X - D, p)$ be a holomorphic map with $Df(\xi') = \xi$. Then we have $f(\Delta(s)) \subset B - D$, so $f(sz)$ is among the maps competing to determine $\|\xi\|_{B-D}$. It follows that

$$\|\xi\|_{X-D} \le \|\xi\|_{B-D} \le (1/s)\|\xi\|_{X-D}$$

on K. ■

Proof of Theorem 2.31. Given a smooth point $p \in D$, choose a local chart $B \cong \Delta^k$ in which $p = 0$ and $B - D \cong (\Delta^*) \times \Delta^{k-1}$. The Kobayashi volume form on $B - D$ blows up at D since the hyperbolic metric on Δ^* blows up at zero. By the preceding Lemma, the metrics d_{X-D} and d_{B-D} are comparable near p, so d_{X-D} blows up at D as well.

The case of a divisor with normal crossings can be handled similarly. For the general case one can appeal to embedded resolution of singularities. ■

Theorem 2.33 *Let $H \subset \mathbb{P}^k$ be an algebraic hypersurface such that $\mathbb{P}^k - H$ is Kobayashi hyperbolic. Then the volume of $\mathbb{P}^k - H$ in the Kobayashi metric is finite.*

Proof. Let $z = 0$ be the origin in Δ^k. Then the volume of a small neighborhood of z in $U = (\Delta^*)^j \times \Delta^{k-j}$ is finite in the Kobayashi metric on U, since the volume of a neighborhood of 0 in Δ^* is finite. But a neighborhood of z in U models a neighborhood of $p \in H$ in $\mathbb{P}^k - H$ (again using resolution of singularities if necessary). Since inclusions are contracting, $\mathbb{P}^k - H$ has finite volume. ■

2.6 Hyperbolicity and Fatou components

For rational maps on $\mathbb{P}^1$, it is clear that every component of the Fatou set $\Omega(f) = \widehat{\mathbb{C}} - J(f)$ is hyperbolic. Indeed, $J(f)$ is uncountable — so it contains at least 3 points! — and the triply-punctured sphere is already hyperbolic.

To prove the analogous result in higher dimensions, we use the escape rate.

Theorem 2.34 *Every component of $\mathbb{P}^k - J(f)$ is Kobayashi hyperbolic and hyperbolically embedded.*

Proof. Let $B(F) \subset \mathbb{C}^{k+1}$ be the attracting basin of $z = 0$ for a lift F of f, and let $\mathcal{F}$ be the holomorphic foliation of the part of $\partial B(F)$ that lies over a given component U of $\Omega(f)$. The natural projection $\pi : \mathcal{F} \to U$ is proper, so every leaf L of $\mathcal{F}$ is a covering space of U. Since L is immersed in the bounded set $\overline{B(F)}$, it is Kobayashi hyperbolic, so U is as well.

The Kobayashi metric on L is bounded below by a multiple of the Kobayashi metric on a large ball (since inclusions are contracting), which in turn is bounded below by a multiple of the pullback of the Fubini-Study g metric on $\mathbb{P}^k$. Thus the Kobayashi metric on U is also bounded below by a multiple of g, and therefore U is hyperbolically embedded. ■

Corollary 2.35 *The immediate basin B of any attracting cycle $A \subset \mathbb{P}^k$ contains a critical point of f.*

Proof. Otherwise $f : B \to B$ is a covering map, and hence an isometry for the Kobayashi metric. ■

Number of attracting cycles. Of course in dimension $k > 1$ the critical divisor $C(f)$ has infinitely many points; this is why f can have infinitely many attracting cycles when $k > 1$.

Critically finite maps. Recall that f is *critically finite* if the postcritical set $P(f)$ is an algebraic variety (i.e. if every component of $C(f)$ eventually cycles.) In this case, setting $V = \mathbb{P}^k - P(f)$, and $U = f^{-1}(V)$, we obtain a *covering map*

$$f : U \to V$$

as well as an inclusion

$$\iota : U \hookrightarrow V.$$

Now assume V (and hence U) is Kobayashi hyperbolic. Since f is a Kobayashi isometry, and ι is a contraction, we see $\|f'(x)\|_U \geq 1$ when it is defined (i.e. when x and $f(x)$ both lie in U). This expansion greatly aids the analysis of the dynamics of f.

Theorem 2.36 *Let f be critically finite, with $\mathbb{P}^k - P(f)$ Kobayashi hyperbolic. Then $f^n(x) \to P(f)$ for all x in the Fatou set of f.*

Lemma 2.37 *If f is critically finite, and the complement of $P(f)$ is Kobayashi hyperbolic, then $f^{-1}(P(f))$ is strictly larger than $P(f)$.*

Proof. The map $f : U \to V$ is a covering map of degree d^k. Since V is the complement of a hypersurface, its Kobayashi volume is finite, and thus the volume of U is d^k-times larger. Thus $U \neq V$. ∎

Proof of Theorem 2.37. Suppose to the contrary that there exists an $x \in \Omega(f)$ such that along a subsequence, we have $f^n(x) \to y \notin P(f)$. Then we can find a ball B_x containing x and a limit h of a subsequence of $f^n|B_x$ such that $h(x) = y$.

Now f expands the metric ρ_U and the associated volume element. On the other hand, h expands volume only a finite amount, despite the fact that h is a limit of compositions involving high iterates of f near y. It follows that there is a neighborhood B_y on which all iterates of f are *isometries.* Moreover, y is *recurrent*, since we can find $f^n(x) \in B$ close to y such that $f^{n+m}(x)$ is also close to y, and hence $f^m(y)$ is close to y.

It follows that there is a subsequence along which $f^n|B_y$ converges to the identity. By considering inverses lifted to the universal covers, we obtain at the same time a subsequence such that

$$f^{-n} : \widetilde{V} \to \widetilde{V}$$

converges to the identity globally on $\widetilde{V}$. (Note that these maps are contractions, hence equicontinuous.) But f^{-n} sends $\widetilde{V}$ into $\widetilde{U}$, a proper subset of $\widetilde{V}$, so this is a contradiction. ∎

Corollary 2.38 *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be critically finite. Then every point in $\Omega(f)$ tends under iteration to a superattracting cycle. In particular, if f has no such cycles, then $J(f) = \mathbb{P}^1$.*

Corollary 2.39 *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a critically finite map, such that $\mathbb{P}^2 - P(f)$ is Kobayashi hyperbolic.*

Suppose each periodic component X of $P(f)$ is a rational curve, and $f^i : X \to X$ is itself critically finite with Julia set equal to X. Then $J(f) = \mathbb{P}^2$.

This Corollary can be applied to the example of a critically finite map given in equation (2.1) above.

Fatou–Bieberbach domains. It is worth noting that domains which are *not* Kobayashi hyperbolic also arise in complex dynamics. Extreme examples are provided by *Fatou-Bieberbach domains.* These are proper subdomains $U \subset \mathbb{C}^n$ that are biholomorphic to $\mathbb{C}^n$ itself.

Such domains can only exist when $n > 1$. Their existence is guaranteed by the following linearization result. Let us say n numbers $\lambda_i \in \mathbb{C}^*$ have a *resonance* if for some λ_j, we can write

$$\lambda_j = \prod_1^n \lambda_i^{n_i}$$

with $n_i \geq 0$ and $\sum n_i \geq 2$.

Theorem 2.40 *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic automorphism with $f(p) = p$. Suppose the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $Df(p)$ have no resonance and satisfy $|\lambda_i| < 1$. Then:*

1. *the map f is linearizable at p; and*
2. *the immediate basin of attraction of p is isomorphic to $\mathbb{C}^n$.*

See [St].

Now let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be an automorphism of the form

$$f(z, w) = (p(z) + \alpha w, \beta z),$$

where $0 < |\alpha|, |\beta| < 1$ and where $p(z)$ is a polynomial. Suppose p has attracting fixed-points at z_1, z_2. Then f has attracting fixed points near $(z_i, 0)$, so long as α and β are sufficiently small. Generically the eigenvalues at these fixed points have no resonances, so we obtain a pair of disjoint attracting basins $B_i \cong \mathbb{C}^2$. Since they are disjoint, each is a Fatou-Bieberbach domain.

2.7 Classification of Fatou components

In this section we summarize what is known about the possible dynamics of f on the Fatou set.

Classification on $\mathbb{P}^1$. The classification is rather complete in dimension one.

Theorem 2.41 (Sullivan) *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map. Then every component of $\Omega(f)$ eventually cycles, and there are only finitely many periodic components.*

Theorem 2.42 (Classification of stable regions) *A component Ω_0 of period p in the Fatou set of a rational map f is of exactly one of the following five types:*

1. *An* attractive basin*: there is a point x_0 in Ω_0, fixed by f^p, with $0 < |(f^p)'(x_0)| < 1$, attracting all points of Ω_0 under iteration of f^p.*
2. *A* superattractive basin*: as above, but x_0 is a critical point of f^p, so $(f^p)'(x_0) = 0$.*
3. *A* parabolic basin*: there is a point x_0 in $\partial\Omega_0$ with $(f^p)'(x_0) = 1$, attracting all points of Ω_0.*
4. *A* Siegel disk*: Ω_0 is conformally isomorphic to the unit disk, and f^p acts by an irrational rotation.*
5. *A* Herman ring*: Ω_0 is isomorphic to an annulus, and f^p acts again by an irrational rotation.*

Classification on $\mathbb{P}^k$. It is not known if a holomorphic map $f : \mathbb{P}^k \to \mathbb{P}^k$ can have a *wandering domain*; that is, a component Ω_0 of $\Omega(f)$ whose forward iterates never cycle. However, $\Omega(f)$ *can* have infinitely many periodic components, since f can have infinitely many attracting cycles when $k > 1$.

Now suppose Ω_0 is a periodic component of $\Omega(f)$. Replacing f with a suitable iterate, we can assume Ω_0 is an *invariant* component. Then $f : \Omega_0 \to \Omega_0$ is non-expanding for the Kobayashi metric.

We say Ω_0 is *recurrent* if for any $z_0 \in \Omega_0$, there is a compact set $K \subset \Omega_0$ such that $f^n(z_0) \in K$ for infinitely many n.

Rotation domains. We say Ω_0 is a *rotation domain* if there the maps $f^n|\Omega_0$ converge to the identity along a subsequence. For example, it is known that a holomorphic map in one variable of the form

$$f(z) = e^{2\pi i\theta} z + O(z^2)$$

is linearizable near $z = 0$, provided θ is Diophantine (Siegel). The Diophantine condition means there is an N such that $|\theta - p/q| > 1/q^N$ for all rationals p/q. A generalization of this result holds in higher dimensions (Sternberg).

Thus f has a rotation domain centered at every fixed point whose derivative is a sufficiently Diophantine isometry. Explicit examples in higher dimensions can also be obtained by taking products of 1-dimensional examples.

Theorem 2.43 *Let $\Omega_0 \subset \Omega(f) \subset \mathbb{P}^k$ be an invariant, recurrent Fatou component. If $\Omega_0 \cap C(f) = \emptyset$, then Ω_0 is a rotation domain and $\langle f \rangle \subset \mathrm{Aut}(\Omega_0)$ is contained in a 1-parameter subgroup.*

Proof. In this case f is a covering map and an isometry for the Kobayashi metric. By recurrence we can find a point z_0 such that $f^n(z_0) \to z_0$ along a subsequence. Using compactness of the set of isometries, we can a further subsequence such that $f^n|\Omega_0$ converges to the identity. It follows that f is an automorphism of Ω_0. By compactness again, the subgroup $\langle f \rangle \subset \mathrm{Aut}(\Omega_0)$ contains elements accumulating to the identity; thus Ω_0 is a rotation domain. A 1-parameter group of automorphisms arises by considering the closure of $\langle f \rangle$. ■

This result accounts for the Siegel disk and Herman rings cases in dimension $k = 1$. A geometric classification of rotation domains in higher dimensions is still lacking.

Recurrence with critical points. In dimension one, the dynamics on a recurrent invariant domain Ω_0 that *meets* $C(f)$ is strictly contracting: we have $d(f(x), f(y)) < d(x, y)$ for $x \neq y$, by the Schwarz lemma. From this one easily deduces that Ω_0 contains an attracting or superattracting fixed point.

In higher dimensions, mixed behavior is possible. For example, if θ is Diophantine then the map

$$f(z, w) = (e^{2\pi z} z + z^2, w^2)$$

has a Fatou component centered at $(0, 0)$ and isomorphic to Δ^2, on which f is conjugate to $(z, w) \mapsto (e^{2\pi\theta} z, w^2)$. The dynamics is recurrent, but there is no attracting fixed-point: instead, orbits converge to the disk $\Delta \times \{0\}$ on which f acts by a rotation. This is a general phenomenon.

Theorem 2.44 (Fornaess-Sibony) *Let $\Omega_0 \subset \Omega(f) \subset \mathbb{P}^2$ be an invariant, recurrent component that meets $C(f)$. Then either:*

1. *f has an attracting or superattracting fixed-point in Ω_0; or*
2. *there is a properly embedded disk, punctured disk or annulus $X \subset \Omega_0$, such that $f^n(z) \to X$ for all $z \in \Omega_0$, and $f|X$ is an irrational rotation.*

It seems likely that a similar result holds in higher dimensions, by considering $X = g(\Omega_0)$ where g is the limit of a subsequence of $f^n|\Omega_0$.

Non-recurrent components. A fixed-point z_0 for $f : \mathbb{P}^1 \to \mathbb{P}^1$ is *parabolic* if $f'(z_0)$ is a root of unity. In this case an iterative of f in an appropriate chart with $z_0 = 0$ has the form

$$f^n(z) = z + z^p \ldots$$

and f has p *attracting petals* at $z = 0$. These petals lies in $\Omega(f)$, while $z = 0$ itself lies in the Julia set.

If $\Omega_0 \subset \mathbb{P}^1$ is a non-recurrent component, then one can show f has a fixed point on $\partial\Omega_0$ by considering the accumulation points of $f^n(z_0)$. The argument uses the fact that the hyperbolic metric on Ω_0 blows up relative to the spherical metric, to deduce that

$$d_{\mathbb{P}^1}(f^{n+1}(z_0), f^n(z_0)) \to 0.$$

A higher-dimensional generalization of this result is as yet unknown. It is not even known if a non-recurrent component has a fixed point on its boundary.

The one-dimensional proof breaks down because the Kobayashi metric on a domain $U \subset \mathbb{P}^k$, $k > 1$, need not blow as one tends to ∂U. For example, when $U = \Delta \times \Delta$, the Kobayashi length of the vector $(1, 0)$ at $p_n = (0, x_n)$ is bounded below as $x_n \to 1$.

2.8 Lelong's theorem

Let $X \subset B(0, r) \subset \mathbb{R}^2$ be a properly embedded arc passing through the origin. It is then clear that the length of $X \cap B(0, r)$ is at least $2r$, and this value is achieved when X is a diameter.

In the complex setting one has a similar statement, valid in arbitrary dimensions, intuitively related to the fact that a complex submanifold of $\mathbb{C}^n$ is volume-minimizing.

Theorem 2.45 (Lelong) *Let $X \subset \mathbb{C}^n$ be a complex analytic variety of dimension k passing through the origin $z = 0$, such that $X \cap B(0, R)$ is closed. Then the Euclidean volume of X^k satisfies*

$$\mathrm{vol}(X \cap B(0, R)) \geq \mathrm{vol}(\mathbb{C}^k \cap B(0, R)).$$

Proof. For simplicity assume X is smooth. By the maximum principle, X is transverse to $\partial B(0,r)$ for $0 < r \leq R$. The Euclidean symplectic form on $\mathbb{C}^k$ is given by:

$$\omega = \frac{\pi}{2} dd^c |z|^2 = \frac{i}{2} \sum dz_i \wedge d\overline{z}_i = \sum_1^k dx_i \wedge dy_i.$$

Define

$$F(r) = \frac{1}{r^{2k}} \int_{X \cap B(0,r)} (dd^c|z|^2)^k$$

Clearly $F(r) = C_k$ is constant if $X = \mathbb{C}^k$, and in any case $F(r) \to C_k$ as $r \to 0$ since X is approximated by its tangent plane at the origin. Thus to complete the proof, we need only show that $F(r)$ is increasing.

To further simplify, assume $k = 1$. By Stokes' theorem we have

$$F(r) = \frac{1}{r^2} \int_{B(0,r) \cap X} dd^c |z|^2 = \int_{S(0,r) \cap X} \frac{d^c |z|^2}{|z|^2} = \int_{S(0,r) \cap X} d^c \log |z|^2,$$

and thus by Stokes' theorem again, for $r_2 > r_1$,

$$F(r_2) - F(r_1) = \int_{X \cap (B(0,r_2) - B(0,r_1))} dd^c \log |z|^2.$$

Taking the limit as $r_1 \to 0$ we obtain:

$$F(r) = F(0) + \int_{B(0,r) \cap X} dd^c \log |z|^2.$$

We recognize the integrand on the left as (twice) the pullback of the Fubini-Study form from $\mathbb{P}^{n-1}$. Thus, letting $\pi : \mathbb{C}^n - \{0\} \to \mathbb{P}^{n-1}$ denote the natural projection, and letting $B(0,r)^*$ denote the ball with the origin removed, we find:

$$F(r) = F(0) + 2\,\mathrm{area}(\pi(X \cap B(0,r)^*))).$$

Here the area is measured with multiplicity. Since $\pi : X \to \mathbb{P}^{n-1}$ has positive degree everywhere, we see $F(r)$ is increasing, completing the proof.

The case $k > 1$ is similar. For singular varieties, one uses the fact that the smooth part of X behaves like a manifold for the purposes of integration. ∎

Multiplicity. If $z = 0$ is a singular point of X, the *multiplicity* of X at 0 can be defined as the degree of the $(k-1)$-dimensional Zariski tangent cone to X at the origin. The proof of Lelong's theorem yields an alternative geometric definition:

$$\mathrm{mult}_p(X) = \lim_{r \to 0} \frac{\mathrm{vol}(X \cap B(p,r))}{\mathrm{vol}(\mathbb{C}^k \cap B(0,r))}.$$

This formula makes sense with X replace by an arbitrary closed, positive (k,k)-current T; the resulting multiplicity (which may be real-valued) is known as the *Lelong number* of T at p. Siu has shown that the set of points where T has Lelong number $\geq c > 0$ is a subvariety of pure dimension p [Siu].

Excess formula. The proof of Lelong's theorem gives an exact formula relating the volume of $X \cap B(0,r)$ to its image in projective space. In the one-dimensional case we obtain:

$$\mathrm{vol}(B(0,r) \cap X) = \pi r^2 \left(\mathrm{mult}_0(X) + \mathrm{area}(\pi(X \cap B(0,r)^*))\right), \qquad (2.3)$$

with the normalization $\mathrm{area}(\mathbb{P}^1) = 1$ for a line in $\mathbb{P}^{n-1}$.

2.9 Entropy

Definition. Let $f : X \to X$ be a continuous map of a compact metric space (X, d_0) into itself. Define a sequence of metrics by

$$d_n(x,y) = \max{}_{i=0}^{n} \, d(f^i(x), f^i(y)).$$

A subset E of a metric space is *ϵ-separated* if $d(x,y) > \epsilon$ for all $x \neq y$ in E. A subset E of X is *(n,ϵ)-separated* if it is ϵ-separated for the metric d_n. The *topological entropy* of (X,f) is defined by

$$h(f) = \sup_{\epsilon > 0} \limsup_{n} \frac{\log \max(|E| \ : \ E \subset X \text{ is } (n,\epsilon)\text{-separated})}{n}.$$

It is easy to show that $h(f)$ depends only on the topology of X, not on the metric d.

Example. Let $\Sigma_k = (\mathbb{Z}/k)^{\mathbb{N}}$, and let $f(x_i) = (x_{i+1})$ be the shift operator. It is convenient give Σ_k the metric:

$$d(x,y) = k^{-\min(j \,:\, x_j \neq y_j)}.$$

Then $E \subset \Sigma_k$ is (n, ϵ)-separated if for any $x \neq y$ in E we have $x_i \neq y_i$ for some $i \leq n + \log(1/\epsilon)/\log(k)$. It follows easily that $h(f) = \log k$.

More generally, if $X \subset \Sigma_k$ is a closed, f-invariant set, then $h(f|X) = \lim(\log W_n)/n$, where W_n (words of length n) is the number of different sequences $(x_1, \ldots, x_n)$ that occur as the first n coordinates of $x \in X$.

Theorem 2.46 (Yomdin) *The entropy satisfies $h(f) \geq \log \sigma(f)$, where $\sigma(f)$ is the spectral radius of f on $H^*(M, \mathbb{R})$.*

Theorem 2.47 (Gromov) *The topological entropy of a degree d map $f : \mathbb{P}^k \to \mathbb{P}^k$ is $h(f) = \log d^k$.*

Proof. We make $X = \mathbb{P}^k$ into a metric space by taking $d(x, y)$ to be the distance in the Fubini-Study metric with symplectic form ω.

Consider the embedding

$$f_n : \mathbb{P}^k \to X_n = (\mathbb{P}^k)^n$$

given by $f_n(x) = (x, f(x), \ldots, f^{n-1}(x))$. Let ω_i be the pullback of ω from the ith factor of X_n, and define a Kähler metric on X_n by taking the symplectic form

$$\eta = \omega_1 + \cdots + \omega_n.$$

Note that Lelong's theorem holds for η, since on small balls it is comparable to a Euclidean metric.

We have $f_n^*(\omega_i) = d^i \omega_i$, and thus the volume of the image of f_n is given by:

$$V_n = \int_{\mathbb{P}^k} f_n^*(\eta^k) = \int_{\mathbb{P}^k} ((1 + d + \cdots + d^{n-1})\omega)^k = \left(\frac{d^n - 1}{d - 1} \right)^k \cdot$$

Thus we have $V_n \asymp d^{nk}$.

We claim that for fixed ϵ, any ϵ-separated set E in $(\mathbb{P}^k, d_n)$ satisfies $|E| = O(V_n)$. To see this, note that for any pair of distinct points $x, y \in f_n(E)$, we have

$$d_\eta(x, y) = \left(\sum d(x_i, y_i)^2 \right)^{1/2} \geq \max d(x_i, y_i) > \epsilon.$$

Thus in the metric d_η, the balls

$$\{B(p, \epsilon) \ : \ p \in E\}$$

are disjoint. But by Lelong's theorem, the η-volume of $f(\mathbb{P}^k) \cap B(p, \epsilon)$ is bounded below by a constant depending only on ϵ. Thus we have $|E| = O(V_n) = O(d^{nk})$, and therefore $h(f) \leq \log(d^k)$. ■

Remark. The same argument show the $h(f) = \log(\sigma(f|H^*(X)))$ for any endomorphism $f : X \to X$ of a compact Kähler manifold. The proof ultimately relies on the fact that the graphs of *all iterates* of $f^n : X \to X$ are minimal surfaces, and hence the 'simplest' representatives of their isotopy classes.

The proof also brings to light the importance of geometric bounds, such as those provided by Lelong's theorem, which apply with uniform constants across spaces of arbitrarily large dimension. A crucial point is that the comparison between ℓ^2 and ℓ^∞ metrics on a large products works, in this case, in a favorable direction.

2.10 Equidistribution on $\mathbb{P}^1$

In this section we specialize again to the setting of rational maps on the projective line. In this case of $f : \mathbb{P}^1 \to \mathbb{P}^1$, the canonical current and the invariant measure for f coincide: we have $\mu_f = T_f$. Using the Koebe distortion theorem, we will investigate this measure in more detail.

Theorem 2.48 (Lyubich) *Let $p \in \mathbb{P}^1$ be a point whose backward orbit consists of 3 or more points. Then we have*

$$\mu_n = d^{-n} \sum_{f^n(z)=p} \delta_z \to \mu_f$$

as $n \to \infty$.

In this sum points are weighted according to their multiplicities.

Once the backward orbit of p consists of 3 or more points, it is infinite. The backward orbit contains no periodic points unless p itself is periodic. Even in this case one may easily verify:

Lemma 2.49 *Suppose p has an infinite backward orbit. Then we have*

$$\frac{|\{z \in f^{-n}(p) \ : \ z \text{ is periodic}\}|}{d^n} \to 0$$

as $n \to \infty$.

Proof of Theorem 2.48. The measure μ_{n+m} is a weighted average of the measures μ_m obtained from $z \in f^{-n}(p)$. The preceding Lemma then reduces the analysis to the case where p is not periodic. Once p is aperiodic, the sets $f^{-n}(p)$ are disjoint as n varies. Thus we can further reduce to the case where the backward orbit of p contains no critical points; equivalently, we can assume that $p \notin P(f)$. Then $f^{-n}(p)$ consists of d^n distinct points, each mapping to p with multiplicity one.

Choose a small ball $B_0 = B(p, r)$ and a smooth $(1,1)$-form ω_0 supported in $B(p, r/2)$ of total mass one. We have seen that $\omega_n \to \mu_f$, where $\omega_n = d^{-n} f^*(\omega_0)$. We will show that when B_0 is very small, ω_n and μ_n are close for $n \gg 0$.

Let $B_n = f^{-n}(B_0)$, and let U_n denote the number of components of B_n mapping univalently to B_0 by f^n. At most $2d-2$ components of B_n meet $C(f)$, so we have

$$U_{n+1} \geq dU_n - (2d-2).$$

Thus the fraction of univalent components, $u_n = U_n/d^n$, satisfies

$$u_{n+1} \geq u_n - \frac{2d-2}{d^{n+1}}.$$

Choose $N \gg 0$. By making B_0 small enough, we can insure that B_N consists of d^N disjoint disks, each mapping univalently to B_0. That is, we can assume $u_n = 1$. Then by induction we find:

$$U_{N+k} \geq 1 - \sum_1^\infty \frac{2d-2}{d^{N+i}} \geq 1 - O(d^{-N}).$$

In other words, for any n, for all but a very small percentage of the points z in the support of μ_n, we have a ball B around z mapping univalently by f^n to B_0. Thus ignoring the part of μ_n where no such B exists makes a very small change in the measure μ_n.

On the other hand, the components of $f^{-n}(B_0)$ are disjoint, so almost all of them have small area — say area $O(d^{-n})$. Ignore those points in the support of μ_n lying in components of large area again makes a only a very small change in the measure μ_n.

Finally, recall that ω_0 is supported in the ball $B(p, r/2) \subset B(p, r) = B_0$. By the Koebe distortion theorem, if $f^n : B \to B_0$ is univalent, then the diameter of

$$B' = B \cap f^{-n}(B(p, r/2))$$

is controlled by the *area* of B.

Summing up, to almost every point z in the support of μ_n, we can associate a set B' of small diameter such that $\int_{B'} \omega_n = \mu_n(B')$ and $z \in B'$. Moving the mass of ω_n from B' to z makes only a small change in the measure ω_n (as detected by integration against a uniformly continuous function on the sphere).

Thus ω_n is close to μ_n for all $n \gg 0$. But $\omega_n \to \mu_f$, so the same is true of μ_n. ∎

Corollary 2.50 *The support of μ_f coincides with $J(f)$.*

Proof. Construct μ_f by the pullback procedure, starting with $p \in J(f)$. ∎

Mixing. Recall that an f-invariant probability measure μ is *ergodic* if $\mu(E) = 0$ or 1 for any Borel set E satisfying $f^{-1}(E) = E$. A stronger condition than ergodicity is *mixing*, which means that:

$$\int \phi(f^n(z))\psi(z)\, d\mu(z) \to \int \phi \int \psi \tag{2.4}$$

for any pair of functions $\phi, \psi \in L^2(\mathbb{P}^k, \mu)$. (To see that mixing implies ergodicity, set $\phi = \psi = \chi_E$.)

Example. The map $f(z) = z^d$, $d > 1$, is mixing on S^1 with respect to linear measure, as can be checked using Fourier series.

Corollary 2.51 *The measure μ_f is mixing.*

Proof. It is enough to establish equation (2.4) when ϕ and ψ are continuous functions on $\mathbb{P}^1$. Let $\mu_{n,p} = d^{-n} f^*(\delta_p)$; these measures tend to μ_f for all $p \in J(f)$. Then we have:

$$\int \phi(f^n(z))\psi(z)\, d\mu(z) = \int \phi(p) \left(\int \psi(z)\, d\mu_{n,p}(z) \right) d\mu_f(p).$$

The inner integral, which is a function of z, converges pointwise to the constant function $\int \psi d\mu$. By the dominated convergence theorem, the limit of the outer integral is then $(\int \phi\, d\mu_f)(\int \psi\, d\mu_f)$. ∎

Equidistribution. The Koebe distortion theorem allows one to pass from equidistribution of the pullbacks of a smooth form to equidistribution of the backwards orbit of a point. A similar argument, using the 'wavefront lemma', allows one to pass from mixing of the geodesic flow on a hyperbolic manifold to equidistribution theorems for spheres and horocycles [EsM].

2.11 Equidistribution on $\mathbb{P}^k$

As another application of Lelong's theorem, we sketch some of the ideas behind the proof in [BD2] of:

Theorem 2.52 (Briend-Duval) *There exists a proper algebraic subset $E \subset \mathbb{P}^k$ such that $\mu_f(E) = 0$ and for all $p \notin E$, we have:*

$$\mu_f = \lim\ d^{-nk} \sum_{f^n(z)=p} \delta_z.$$

Corollary 2.53 *The fine Julia set $J_k(f) = \operatorname{supp} \mu_f$ is the minimal, closed, f-invariant set not contained in a proper subvariety.*

Corollary 2.54 *If the interior of $J_k(f) = \operatorname{supp} \mu_f$ is nonempty, then $J_k(f) = \mathbb{P}^k$.*

Proof. Suppose $J_k(f) \neq \mathbb{P}^k$. Then its complement is a nonempty, open, f-invariant set U. Choose a point $p \in U$ whose weighted preimages converge to μ_f. Since the preimages of p never enter the interior of the fine Julia set, they deposit no charge there, so the interior must be empty. ■

Area and diameter. Recall that Lyubich's proof for $\mathbb{P}^1$ uses the Koebe distortion theorem. For a parallel argument in higher dimensions, we need to show area controls diameter.

Lemma 2.55 *Let $h : \Delta \to \mathbb{P}^k$ be a holomorphic map, and let $r < 1$. Then with respect to the Fubini-Study metric, we have a bound*

$$\operatorname{diam}(h(\Delta_r))^2 < C_r \cdot \operatorname{area}(h(\Delta)).$$

Proof. Once the area is small, we have $\int |h'|^2$ small. A familiar length-area argument, based on the Cauchy-Schwarz inequality

$$\left(\int |h'|\right)^2 \leq \int 1 \int |h'|^2,$$

and Fubini's theorem, shows we there is an s with $r < s < 1$ such that

$$\operatorname{length}(h(S^1_s))^2 \leq C_r \cdot \operatorname{area}(h(\Delta)).$$

Thus we are done, unless we have $D = \operatorname{diam}(h(\Delta_s)) \gg \operatorname{diam}(h(\partial\Delta_s))$. But in this case we can find a $p \in \Delta_s$ such that $h(\partial\Delta_s)$ lies outside the ball

$B(h(p), D/2)$ At the same time we can replace the Fubini-Study metric with a flat metric, since we are working in a small region of $\mathbb{P}^k$. Then by Lelong's theorem, we have

$$\pi(D/2)^2 \leq \text{area}(h(\Delta_s) \cap B(h(p), D/2)),$$

so D, and hence $\text{diam}(h(\Delta_r))$, are controlled by the area of $h(\Delta)$. ∎

We also need to control the area of preimages.

Lemma 2.56 *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a rational map of degree d. Let $h : \Delta \to \mathbb{P}^k$ be a holomorphic disk, whose image $U = h(\Delta)$ is disjoint from the postcritical set $P(f)$ and contained in a line $L \cong \mathbb{P}^1 \subset \mathbb{P}^k$. Then the average area of a component of $h^{-n}(U)$ is $O(d^{-n})$.*

Proof. Recall that f acts by multiplication by d^i on $H^{2i}(\mathbb{P}^k)$. Since L represents a class in $H^{2k-2}(\mathbb{P}^k)$, we have

$$\text{area}(f^{-n}(U)) \leq \text{area}(f^{-n}(L)) = d^{n(k-1)}$$

in the normalized Fubini-Study metric on $\mathbb{P}^k$. But the degree of f is d^k, so $f^{-n}(U)$ has d^{nk} components. Thus their average area is d^{-n}. ∎

Given $x \in \mathbb{P}^k$, let

$$\mu_{n,x} = d^{-nk} \sum_{f^n(y)=x} \delta_y$$

as usual. Then the preceding Lemmas easily imply:

Corollary 2.57 *If $U \subset \mathbb{P}^k - P(f)$ is a linear disk, and $x, y \in U$, then $\mu_{n,x} - \mu_{n,y} \to 0$ as $n \to \infty$.*

Sketch of the proof of Theorem 2.52. The preceding Corollary can be enhanced to show that $\mu_{n,x} - \mu_{n,y} \to 0$ so long as neither measure 'charges the critical set': that is, so long as $\mu_{n,x}(C(f)) \to 0$, and similarly for y. Then one can show that there is a proper closed algebraic subset $E \subset \mathbb{P}^k$ such that all x that charge $C(f)$ lie in E.

Now fix $x \notin E$ and let ν be a weak limit of $\mu_{n,x}$. We will show that $\nu = \mu_f$, and hence $\mu_{n,x} \to \mu_f$ as desired.

Passing to a subsequence, we can assume $\mu_{n,x} \to \nu$, and hence $\mu_{n,y} \to \nu$ for any $y \notin E$.

Let η be a unit mass smooth, positive (k, k)-form supported in a ball disjoint from E. Assume that $\eta = \omega_1 \wedge \omega_2 \cdots \omega_k$ where ω_i are normalized closed $(1,1)$-forms. Since $d^{-n}(f^n)*(\omega_i) \to T_f$, we have $\eta_n = d^{-nk}(f^n)^*(\eta) \to \mu_f$.

But as a measure, we have

$$\eta_n = \int \mu_{n,y}\, d\eta(y).$$

Thus $\eta_n \to \nu$ along a subsequence, and thus $\nu = \mu_f$. ∎

Repelling cycles. Briend and Duval have shown that the *smallest* Lyapunov exponent of f with respect to μ_f is at least $(1/2)\log\deg(f)$; in particular, the support of μ_f is a "repeller" [BD1]. Using the fact that f is also mixing, they show that repelling periodic points (those with all eigenvalues of modulus > 1) are uniformly distributed with respect to μ_f. It would be interesting to find a more direct proof that an endomorphism of $\mathbb{P}^k$, $k > 1$, always has at least one repelling periodic cycle.

2.12 Exercises

1. Prove that a the critical locus $C(f)$ for an endomorphism of degree d on $\mathbb{P}^k$ has degree $(k+1)(d-1)$.

2. Verify that the map f on $\mathbb{P}^2$ given by equation (2.1) is critically finite, and compute the first return map $f^3 : L \to L$ on a periodic line $L \cong \mathbb{P}^1$ in the post-critical set $P(f)$. Prove that $J(f|L) = L$.

3. Draw a scatter plot of $f^{-n}(z_0)$ for $f(z) = ((z-i)/(z+i))^2$. Give a formula for the measure with respect to which inverse images are distributed.

4. Using computer graphics, draw a scatter plot of $f^{-n}(z_0)$ to approximate the equilibrium measure μ for the following quadratic polynomials: (a) $f(z) = z^2 + i$; (b) $f(z) = z^2 - 1$; (c) $f(z) = z + z^2$.

5. Draw pictures of the escape rate functions for the same quadratic polynomials; that is, color or shade the complex plane according to the value of $\lim 2^{-n}\log^+|f^n(z)|$.

6. The level sets of the escape rate function for a polynomial $f(z)$ foliate the attracting basin of infinity, U. Show that the same foliation can be defined in terms of the *small orbit equivalence relation*, $z \sim w$ if $f^n(z) = f^n(w)$ for some $n \geq 0$.

7. Describe the foliation of the basin of infinity for $f(z) = z^2 - 100$.

8. Let $P(z)$ and $Q(z)$ be polynomials of the same degree $d > 1$. What is the Julia set of the map $f : \mathbb{P}^2 \to \mathbb{P}^2$ given in an affine chart by $(z, w) \mapsto (P(z), Q(w))$?

9. What happens if the degrees of P and Q are different?

10. Compute the branch locus (critical values) of the map $E^2 \to \mathbb{P}^2$ sending an elliptic curve $E = \mathbb{C}/\Lambda$ to the quotient of E^2 by $S_2 \ltimes (\mathbb{Z}/2)^2$.

11. Let $V \subset \mathbb{P}^k$ be a hypersurface defined by the homogeneous polynomial equation $P(z) = 0$ on $\mathbb{C}^{k+1}$. Show that $T = dd^c \log |P(z)|$ is the pullback of the $(1,1)$-current represented by V.

12. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be the polynomial map $f(z) = z^2 + c$ with c outside the Mandelbrot set. Then $J(f)$ is a Cantor set, and there is unique conformal metric g on $\mathbb{P}^1$ (up to scale) with curvature $4\pi\mu_f$. Parallel transport gives a foliation $\mathcal{F}$ of $T_1\mathbb{P}^1$ by Riemann surfaces.

    Show that every leaf of $\mathcal{F}$ has infinite genus. (Cf. [HP].)

13. Show that every domain $\Omega \subset \mathbb{P}^1$ with $|\widehat{\mathbb{C}} - \Omega| \geq 3$ is Kobayashi hyperbolic and hyperbolically embedded.

14. Let $X = \mathbb{P}^2 - H$ where H is the union of the line at infinity and the lines defined by $z = 0$, $w = 0$, $w = 1$ and $z = w$ in $\mathbb{C}^2 \subset \mathbb{P}^2$.

    (a) Show that X is hyperbolic. (Hint: show X is isomorphic to $(\mathbb{C} - \{0, 1\})^2$.)

    (b) Show that X is not hyperbolically embedded in $\mathbb{P}^2$. (Hint: consider the Kobayashi metric on the submanifolds $L_r = X \cap (z = r)$ as $r \to 0$.)

15. Let $X = \{(z, w) \ : \ |z| < 1, |zw| < 1 \text{ and } |w| < 1 \text{ if } z = 0\}$.

    (a) Show that X is not Kobayashi hyperbolic, but any holomorphic map $f : \mathbb{C} \to X$ is constant.

    (b) Let H be the hypersurface in X defined by $z = 0$. Show that $X - H$ is Kobayashi hyperbolic.

    (c) Show that $X - H$ is not hyperbolically embedded in X, even though H is hyperbolic.

    (Cf. [Ko, Example (3.6.6)].)

16. Show that the resonant map $f(x,y) = (\lambda x, \lambda^2 y + x^2)$, $0 < |\lambda| < 1$, is not linearizable at the origin in $\mathbb{C}^2$.

17. Show that a holomorphic 1-form θ on a compact complex surface X is always closed. (Hint: consider $\int d\theta \wedge d\overline{\theta}$.)

18. Show that a holomorphic form $(p,0)$-form θ on a compact Kähler manifold X of dimension n is always closed. (Hint: consider $\int d\theta \wedge d\overline{\theta} \wedge \omega^{n-p-1}$.)

19. Prove Lelong's theorem on $\mathrm{vol}(X \cap B(0,r))$ when $\dim X > 1$.

20. Consider the curve $X \subset \mathbb{C}^2$ defined by $y = x^2$. Show that as $r \to \infty$ we have
$$\mathrm{area}(X \cap B(0,r)) \sim 2\pi r^2$$
(a) by pulling back the area form on $\mathbb{C}^2$ under the map $t \mapsto (t, t^2)$; (b) by equation (2.3) coming from the proof of Lelong's theorem.

21. Let $X \subset \mathbb{C}^2$ be an affine curve of degree d. Show that we have
$$\mathrm{area}(X \cap B(0,r)) \sim d\pi r^2$$
as $r \to \infty$.

22. Let $X = G/\Gamma$ be the *Iwasawa manifold*, where
$$G = \left\{ \left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix} \right) \; : \; a,b,c \in \mathbb{C} \right\} \subset \mathrm{GL}_3(\mathbb{C})$$
and where $\Gamma = G \cap \mathrm{GL}_3(\mathbb{Z}[i])$.

    Show that X is a compact complex manifold, and that X carries a holomorphic 1-form that is not closed.

23. Let $\Sigma_2 = (\mathbb{Z}/2)^{\mathbb{N}}$ equipped with the shift map $\sigma(x_i) = (x_{i+1})$. Let
$$X = \{(x_i) \in \Sigma_2 \; : \; \forall i \; ((x_i, x_{i+1}) \neq (0,0)\}$$
be the set of all sequences without consecutive zeros.

    What is the topological entropy of $\sigma | X$?

24. Let $f : M \to M$ be a degree d self-covering map of a compact manifold M. Prove its entropy satisfies $h(f) \geq \log d$.

25. Invent a interesting holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$ over degree $d > 1$, defined over $\mathbb{R}$, and draw a contour plot of itshe natural $(1,1)$-current T_f? (This is true for $k = 1$ by [Ly], and for $k = 2$ by [FJ].)

2.13 Unsolved problems

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map of algebraic degree $d > 1$ in dimension $k > 1$.

1. Complete the classification of periodic Fatou components for f.
2. Can f have a wandering Fatou component? (It cannot when $k = 1$.)
3. Let $H \subset \mathbb{P}^k$ be a generic hyperplane. Does $(f^n)^*(H)/d^n$ converge to the natural $(1,1)$-current T_f? (This is true for $k = 1$ by [Ly], and for $k = 2$ by [FJ].)

2.14 Notes

Basic references for dynamics on $\mathbb{P}^k$ include [For], [FS], [Si], [HP], [Ue] and [BD2].

The theory of currents, including Lelong's theorem, is presented in [GH, Ch. 3]. See [KH] for more on the notion of topological entropy.

The basic reference for the Kobayashi metric is [Ko]. See for example [Ko, 3.5.41, p.99] for a version of Theorem 2.28 above.

3 Dynamics of surface automorphisms

3.1 Automorphisms of curves

What is the simplest interesting dynamical system?

Let us interpret this question more precisely. By a *dynamical system* we will mean a diffeomorphism of a compact manifold, $f : X \to X$. The reason to focus on *invertible* maps is that these arise as cross-section of flows, and hence contribute to the analysis of differential equations. Indeed, Poincaré's original works on dynamics were motivated in part by questions of celestial mechanics such as the 3-body problem.

A map on a real manifold can sometimes be complexified, especially when the map is simple enough that it is given by an algebraic equation. The complex setting is more rigid and admits more definitive tools, so we will further require from the outset that f is a *holomorphic* diffeomorphism of a compact *complex* manifold.

The word 'interesting' is subjective, but should include the requirement that the dynamics is essentially nonlinear.

Poncelet's iteration. Algebraic geometry provides a source of simple dynamical systems. Here is a classical example, due to Poncelet.

Let C and D be a pair of smooth conics in $\mathbb{RP}^2$. For concreteness, assume C and D are disjoint ellipses, with D encircling C.

Given a point p on the outer ellipse D, we can construct a new point $f(p)$ on D by drawing a tangent line to C through p and recording its new point of intersection with D. Iterating this process, we obtain a 'polygon' with vertices $p, f(p), f^2(p), \ldots$, inscribed in D and circumscribed in C.

Theorem 3.1 (Poncelet) *If this polygon closed for some point $p \in D$, then it closes for all $p \in D$.*

Theorem 3.2 *Let X be a compact Riemann surface of genus g. Then either:*

- *$g = 0$, we have $X \cong \mathbb{P}^1$ and $\mathrm{Aut}(X) = \mathrm{PGL}_2(\mathbb{C})$ acts via linear automorphisms; or*
- *$g = 1$, $X \cong \mathbb{C}/\Lambda$ and $\mathrm{Aut}(X)$ is a compact group of Euclidean isometries; or*
- *$g \geq 2$ and $\mathrm{Aut}(X)$ is finite.*

Proof. For $g = 0$ we have $X \cong \mathbb{P}^1$ by Riemann-Roch, and hence its automorphisms come from Möbius transformations. Note that for $g = 0$, $\mathrm{Aut}(X)$ is *not* compact.

For $g = \dim \Omega(X) > 0$, one can immediately see that $\mathrm{Aut}(X)$ *is* compact, because it preserves the *Bergman metric* on X.

To define this metric, note that $\Omega(X)$ carries a natural L^2-norm:

$$\|\omega\|^2 = \frac{i}{2} \int_X \omega \wedge \overline{\omega}.$$

(This norm measures area of the image of X under the locally defined function $f : X \to \mathbb{C}$ with $df = \omega$.) From this L^2 norm we obtain a metric on X by:

$$\|v\| = \sup\{|\omega(v)| \ : \ \|\omega\| = 1\}.$$

(Alternatively, the L^2-norm on 1-forms gives a metric on the the Jacobian $\mathrm{Jac}(X) \cong \Omega(X)^*/H_1(X, \mathbb{Z})$, which pulls back to the above metric on X under the natural map $X \to \mathrm{Jac}(X)$.)

In the case of genus one it is clear that the Bergman metric is the Euclidean metric coming from the universal cover. For higher genus the Bergman metric generally has variable curvature. On the other hand, the automorphism group of X is discrete because $\chi(X) < 0$, and thus by compactness it is finite. ■

Remark. The finiteness of $\mathrm{Aut}(X)$ for genus $g \geq 2$ can also be deduced using the hyperbolic metric. The proof above, however, by virtue of working with forms and the Bergman metric, also works in higher dimensions; for example a similar argument can be applied to show $\mathrm{Aut}(X)$ is finite for surfaces of general type.

3.2 Automorphisms of surfaces

Let X be a compact, connected, complex surface. We can attempt to use sections of powers of the canonical bundle K_X and its powers to find a natural model for X as a projective variety. Let

$$X \to X_m \subset \mathbb{P}^N$$

be the map to projective space given by the linear system $|mK_X|$. (Here X_m denotes the Zariski closure of the image.) Since K_X is canonical, $\mathrm{Aut}(X)$

respects the map $X \to X_m$ and acts on X_m via linear automorphisms of the ambient projective space $\mathbb{P}^N$.

The *Kodaira dimension* of X is defined by

$$\operatorname{kod}(X) = \max_{m>0} \dim(X_m).$$

If $h^0(X, K_X^m) = 0$ for all $m > 0$, we set $\operatorname{kod}(X) = -\infty$.

A surface is *minimal* if has no rational curves E with $E^2 = -1$. Such a curve can always be blown down to obtain a simpler surface birationally equivalent to X.

A *fibration* is a regular surjective map $\pi : X \to C$ where C is a smooth curve. If the generic fiber is an elliptic curve, then X is an *elliptic surface*. If all fibers are isomorphic to $\mathbb{P}^1$, then X is a *ruled surface*. Every ruled surface has the form $X = \mathbb{P}E$ where $E \to C$ is a rank-two vector bundle.

Note that for ruled surface such as $X = \mathbb{P}^1 \times C$, $g(C) > 1$, K_X^m has no sections at all, regardless of the sign of m.

A minimal surface is of *general type* if $\operatorname{kod}(X) = 2$. In this case it is known that the pluricanonical map is regular and birational for $m \gg 0$, although the image might have some double points.

We can now briefly sketch the Enriques–Kodaira classification of *minimal* complex surfaces. Here are the possibilities, organized by the Kodaira dimension (cf. [BPV, Ch. VI]).

1. $\operatorname{kod}(X) = -\infty$. (a) $X \cong \mathbb{P}^2$. (b) X is ruled. (c) X has $b_1(X) = 1$ (and hence is not projective).

   A ruled surface $X \to C$ is rational iff $C \cong \mathbb{P}^1$. In this case X is a *Hirzebruch surface* of the form $\Sigma_d = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(d))$. (Since $\mathcal{O}(d)$ is birationally trivial, Σ_d is birationally equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$ and hence to $\mathbb{P}^2$.)

   The Hopf surfaces,

   $$X = (\mathbb{C}^2 - (0,0)/z \sim \lambda z,$$

   $|\lambda| > 1$, are examples of (c).

2. $\operatorname{kod}(X) = 0$. (a) complex tori $X = \mathbb{C}^2/\Lambda$. (b) K3 surfaces (K_X is trivial, $b_1(X) = (1)$). (c) surfaces finitely covered by (a) and (b).

   In case (c), the only surfaces properly covered by K3 surfaces are *Enriques surfaces*; for these the covering is of degree two.

3. $\mathrm{kod}(X) = 1$. These are all 'properly elliptic' surfaces, i.e. each admits a canonical 'fibration' $X \to X_m \cong C$ with generic fiber an elliptic curve, but X is not covered by a complex torus.

   A typical example is $E \times C$ where E is an elliptic curve and $g(C) > 1$. For an example with a rational base, let C be a hyperelliptic curve and let X be the quotient of $E \times C$ by $(e, c) \mapsto (e + p, \eta(c))$, where $2p = 0$ in E and where η is the hyperelliptic involution.

4. $\mathrm{kod}(X) = 2$. These are surfaces of general type. They include, for example, all smooth hypersurfaces in $\mathbb{P}^3$ of degree $d \geq 5$.

   For m large, the pluricanonical map $\phi_m : X \to X_m$ for a surface of general type is regular and birational, but it may not be an isomorphism. For example, consider a quintic hypersurface $Y \subset \mathbb{P}^3$ with a double point, but otherwise smooth, and blow up to obtain a smooth surface X (with a rational -2-curve). Then X has general type, but its canonical images X_m, $m \geq 1$ are all isomorphic to Y rather than X. (Note that $K_Y = \mathcal{O}_Y(1)$ by the adjunction formula.)

Canonical models and automorphisms. The autmorphism group $\mathrm{Aut}(X)$ respects any canonical map, and acts via automorphisms of $\mathbb{P}^N$ on its image. We thus have:

Theorem 3.3 *If X is an algebraic surface with $\mathrm{kod}(X) > 0$, then $\mathrm{Aut}(X)$ acts either 'linearly' on X or as a skew-product over automorphisms of a curve.*

In the case of general type we can say more:

Theorem 3.4 *The automorphism group of a surface of general type is finite.*

Proof. There is a natural L^2-norm on the space of sections of the canonical bundle:

$$\|\omega\|^2 = \int_X |\omega|^2.$$

Similarly, $\int |\sigma|^{2/m}$ provides a natural measurement of the size of a section σ of K_X^m for $m > 0$. Because this size is preserved by $\mathrm{Aut}(X)$, we see that for a surface of general type, $\mathrm{Aut}(X)$ is compact. (That is, it acts on X_m via automorphisms of $\mathbb{P}^N$ which preserve the size function on the space of sections, and hence lie in a compact group.) Any compact complex subgroup of $\mathrm{Aut}(\mathbb{P}^N)$ is finite. ∎

It is less popular, but also useful, to try to find a model for X using sections of negative powers of the canonical bundle. For example, if $X = \mathbb{P}^2$, then we have $K_X^{-1} = \mathcal{O}(3)$, and thus $|-K_X|$ gives the Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^9$ by the linear system of cubic curves.

In this case $\mathrm{Aut}(X)$ is of course not compact, but its action is linearized by the 'anticanonical embedding'. In other words, we have:

Theorem 3.5 *If K_X is ample, then* $\mathrm{Aut}(X)$ *is linearizable.*

These result indicates that $\mathrm{Aut}(X)$ is rather uninteresting dynamically, both when X is fairly irrational ($\mathrm{kod}(X) > 0$) and when X is fairly rational (X is an algebraic surface with $\mathrm{kod}(X) = -\infty$.) On the other hand, when K_X is *trivial* there is no hope of using the canonical bundle to forge a projective model of X.

Theorem 3.6 (Cantat) *Let $f : X \to X$ be an automorphism of a minimal,compact complex surface. If f has positive entropy, then the X is either a K3 surface, an Enriques surface or a complex torus.*

Note: the minimal model of an irrational surface is canonical, so automorphisms of such surfaces pass to their minimal model and are covered by the theorem above. On the other hand, certain non-minimal rational surfaces do admit automorphisms of positive entropy, and these surfaces have not yet been completely classified.

3.3 Real dynamics on K3 surfaces

***K*3 surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.** Let $X \subset (\mathbb{P}^1)^3$ be a smooth surface of degree $(2, 2, 2)$. Then X is simply-connected by the Lefschetz hyperplane theorem, and K_X is trivial by the adjunction formula. Thus X is a K3 surface.

As a concrete family of examples, we consider the surfaces X_A defined by the equation

$$(1 + x^2)(1 + y^2)(1 + z^2) + Axyz = 2$$

for $A \in \mathbb{R}$. Each such surface carries 3 natural involutions, coming from sheet interchange under the 2-fold projection to a coordinate plane. The product of these involutions,

$$f_A = \iota_x \circ \iota_y \circ \iota_z$$

gives a positive-entropy automorphism of X_A.

Experimentally the dynamics of $f_A : X_A(\mathbb{R}) \to X_A(\mathbb{R})$ exhibits all the features of a typical area-preserving map. In [Mak] we read:

> A few hours playing at a computer terminal is sufficient to convince one that almost all area preserving maps have essentially the same features...
>
> Firstly, one often sees stable periodic orbits. Their stability comes from being surrounded by closed invariant curves.
>
> Secondly, in between some of the larger invariant circles are "island chains", strings of alternating stable and unstable periodic points.
>
> Thirdly, when they are large enough, one can see that they are surrounded by a "sea of stochastic orbits".

KAM theory. Let (X, ω) be a compact smooth surface of real dimension two, equipped with a smooth area form ω.

Let $f : X \to X$ be a C^4 area-preserving automorphism of X. If $f(p) = p$, and $Df|T_p$ is conjugate to a rotation, then p is an *elliptic fixed point* of f.

In suitable polar coordinates with $p = 0$, one can express f in *Birkhoff normal form*:

$$f(r, \theta) = (r, \theta + \alpha + \beta r) + F(r, \theta),$$

where the derivatives of $F(r, \theta)$ of order ≤ 3 all vanish. We say p is a *nondegenerate* elliptic point if $\beta \neq 0$, i.e. if there is a nontrivial 'twist' in the dynamics near p.

Theorem 3.7 (Kolmogorov–Arnold–Moser) *Let p be a nondegenerate elliptic fixed-point with $\alpha \neq 0, \pm\pi, \pm\pi/2, \pm 2\pi/3$. Then there exists a positive measure set $A \subset X$ with p as a point of density, such that A is foliated by invariant circles on each of which f acts by an irrational rotation.*

These invariant circles bound 'elliptic islands' which are trapped near p.

Let $\mathrm{Diff}^k_\omega(X)$ denote the space of C^k diffeomorphisms of X preserving the area form ω.

Corollary 3.8 *For $k \geq 4$, having a dense orbit is not a generic condition in $\mathrm{Diff}^k(X)$. On the contrary, there is a nonempty open set $U \subset \mathrm{Diff}^k(X)$ such that every $f \in U$ has elliptic islands.*

Remark. For KAM theory see, for example, [Me, Theorem 5.1]. Oxtoby and Ulam proved that a generic C^0 area-preserving map (in the sense of

Baire category) has a dense orbit (see [Ox]). It is not known what happens in the C^k case for $1 \leq k \leq 3$.

Returning to our specific examples (3.3) of area-preserving maps on surfaces, we can now formulate the following:

Conjectures.

1. There is an open dense set of $A \in \mathbb{R}$ such that $f_A : X_A \to X_A$ has a dense set of elliptic periodic points.

2. There is a positive measure set of $A \in \mathbb{R}$ such that $f_A : X_A \to X_A$ is ergodic and mixing.

3. So long as f_A has infinite order, it has an ergodic component of positive measure (a stochastic sea).

4. For any value of A, periodic points of f_A are dense.

3.4 Topology and Hodge theory of complex tori

A *complex torus* is a compact complex n-manifold of the form $X = \mathbb{C}^n/L$ where $L \cong \mathbb{Z}^{2n}$ is a lattice in $\mathbb{C}^n$.

In dimension one, complex tori are just Riemann surfaces of genus one. In higher dimensions a typical complex torus is not projective — its Picard group is trivial. The projective complex tori are *Abelian varieties.*

Topology. A complex torus X of dimension n is homeomorphic to the standard real torus $(S^1)^{2n}$. The cohomology of X is thus easily described. Indeed, if we let $L = H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2n}$, then we have canonical isomorphisms:

$$H^k(X, \mathbb{Z}) \cong \wedge^k(L^*)$$

and hence $b_k(X) = \binom{2n}{k}$.

An integral basis of $H^k(X)$ is given by the real subtori of $(S^1)^{2n}$ of codimension k obtained by fixing k coordinates.

Intersection form. The complex structure on X determines an *orientation* of L, that is a choice of generator for $\wedge^{2n}L \cong \mathbb{Z}$.

Theorem 3.9 *For n even, the intersection pairing makes $H^n(X, \mathbb{Z})$ into an even unimodular lattice of signature (k, k), where $k = \binom{2n}{n}$.*

In fact it is easy to decompose $H^n(X, \mathbb{Z})$ as a direct sum of hyperbolic planes. To do this, let $(e_i)_1^n$ be a basis for L^*. Then as I ranges over ordered subsets of $\{1, 2, \ldots, n\}$, the elements

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_n}$$

form a basis for $H^n(X,\mathbb{Z})$. We have

$$|e_I \cdot e_J| = \begin{cases} 1 & \text{if } I \text{ and } J \text{ are disjoint, and} \\ 0 & \text{otherwise.} \end{cases}$$

Adjunction formula. For an example of the pairity of the intersection number, consider the adjunction formula for a curve C on an Abelian surface X: we have

$$\deg(K_C|C) = 2g(C) - 2 = C \cdot K_C = C \cdot (K_X + C) = C^2.$$

This shows:

Theorem 3.10 *If $C \subset X$ is a smooth curve of genus g, then $C^2 = 2g - 2$. In particular, rational curves satisfy $C^2 = -2$ and elliptic curves satisfy $C^2 = 0$.*

This result applies equally well to K3 surfaces; it only depends on triviality of the canonical bundle.

Stiefel-Whitney classes. On any orientable 4-manifold X, we have

$$C^2 = w_2(X) \cdot C \bmod 2$$

on $H^2(X)$, where $w_2(X)$ is the second Stiefel-Whitney class of TX. For a complex manifold, we have $w_2(X) = c_1(X) \bmod 2$. Thus the intersection form is also even on any complex surface with even first Chern class.

Hodge structure. A complex torus is uniquely determined by its 1-dimensional *Hodge structure*.

Let $L \cong \mathbb{Z}^{2n}$ be a free abelian group, and let $L^* = \mathrm{Hom}(L,\mathbb{Z})$ be its dual. A *Hodge structure* on L means a complex splitting

$$L^* \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$$

such that $H^{1,0} = \overline{H^{0,1}}$. A Hodge structure determines a unique complex torus

$$X = (H^{1,0})^*/L \cong \mathbb{C}^n/L,$$

with a canonical isomorphism $L \cong H_1(X,\mathbb{Z})$, such that the Hodge decomposition

$$H^1(X,\mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X)$$

agrees with the given Hodge structure on L, under the isomorphism $H^1(X,\mathbb{C}) \cong L^* \otimes \mathbb{C}$.

Orientation. It is conventional to fix an orientation of L to start out with, and require that the Hodge structure gives a compatible orientation. (This convention makes the space of Hodge structures connected; otherwise, the space of Hodge structures for elliptic curves would be parameterized by $\mathbb{H} \cup -\mathbb{H}$.)

Canonical bundle. Since a complex torus X is a Lie group, its canonical bundle K_X is trivial. The form $\eta = dz_1 \cdots dz_n$ on $\mathbb{C}^n$ descends to a nowhere-vanishing section of K_X, and spans $H^{n,0}(X) \cong \mathbb{C}$. The (n,n)-form $\eta \wedge \overline{\eta}$ gives a canonical volume element on X.

Middle-dimensional Hodge structure. We wish to consider surfaces that arise as complex tori, so now assume $n = 2$.

Given $L \cong H_1(X, \mathbb{Z})$, we have a canonical isomorphism

$$\wedge^2 L^* \cong H^2(X, \mathbb{Z}).$$

The (symmetric) intersection form is given by

$$\langle \alpha, \beta \rangle = \alpha \wedge \beta \in \wedge^4 L^* \cong \mathbb{Z},$$

using the orientation of L. As we have seen above, the intersection form is the even unimodular form with signature $(3,3)$.

The Hodge structure on L determines the Hodge structure on $H^2(X)$:

$$\left(\wedge^2 L^*\right) \otimes \mathbb{C} = H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}.$$

Here the summands are orthogonal for the intersection form, and satisfy $H^{0,2} = \overline{H^{2,0}}$ and $H^{1,1} = \overline{H^{1,1}}$.

Since the holomorphic $(2,0)$-form on X satisfies $\int \eta \wedge \overline{\eta} > 0$, the intersection pairing has signature $(2,0)$ on $H^{2,0} \oplus H^{0,2}$, and hence signature $(1,3)$ on $H^{1,1}$.

Kähler cone. Since $\mathbb{C}^n$ admits (many) translation-invariant Kähler metrics, every complex torus is a Kähler manifold.

Let $H^{1,1}(X, \mathbb{R})$ denote the space of cohomology classes represented by closed $(1,1)$-forms satisfying $\overline{\omega} = \omega$. The intersection form makes $H^{1,1}(X, \mathbb{R})$ into a Minkowski space of signature $(1,3)$, and hence a model for hyperbolic space $\mathbb{H}^3$.

The *Kähler cone* $C_X \subset H^{1,1}(X, \mathbb{R})$ consists of the cohomology classes of sympletic forms of Kähler metrics on X.

Theorem 3.11 *The Kähler cone*

$$C_X \subset H^{1,1}(X, \mathbb{R}),$$

coincides with one component of the space of timelike vectors, i.e. of those classes satisfying $[\omega^2] > 0$.

In fact, each cohomology class in $H^{1,1}(X, \mathbb{R})$ is represented by the symplectic form of a unique, translation invariant Hermitian form on TX. The signature of this form is $(2, 0)$ or $(0, 2)$ for timelike vectors, and $(1, 1)$ for spacelike vectors; it is $(1, 0)$ or $(0, 1)$ on the light cone. (The sign of $[\omega^2]$ is the determinant of the form.)

3.5 Dynamics on complex tori

Now let $f : X \to X$ be an automorphism of a complex torus of dimension two. Two natural invariants of f are its *determinant*

$$\delta(f) = \mathrm{Tr}(f|H^{2,0})$$

and its *leading eigenvalue* $\lambda(f)$, given by the spectral radius

$$\lambda(f) = \sigma(f|H^2) = \sigma(f|H^{1,1}).$$

(Note that f acts unitarily on $H^{2,0} \oplus H^{0,2}$.) We have $|\delta(f)| = 1$ and $\lambda(f) \geq 1$.

Theorem 3.12 *At any fixed-point p of f we have* $\det Df_p = \delta(f)$.

Theorem 3.13 *The entropy of f is given by $h(f) = \log \lambda(f)$.*

Corollary 3.14 *The map f has positive entropy iff f determines a hyperbolic (as opposed to parabolic or elliptic) isometry of the space $\mathbb{H}^3$ attached to the Kähler cone of X.*

In fact, once f is normalized to fix the origin in $X = \mathbb{C}^2/L$, it becomes an automorphism of X as a *group*. We denote subgroup of such normalized automorphism by $\mathrm{Aut}_0(X)$. Upon lifting to the universal cover, we obtain a complex linear automorphism $F : \mathbb{C}^2 \to \mathbb{C}^2$ satisfying $F(L) = L$.

Let α, β denote the complex eigenvalues of F, with $|\alpha| = |\beta|^{-1} \geq 1$. We then have:

$$\delta(f) = \alpha\beta, \quad \text{and} \quad \lambda(f) = |\alpha|^2.$$

The eigenvalues of f on $H^{1,1}$ are $|\alpha|^{\pm 2}$, $\alpha\overline{\beta}$ and $\beta\overline{\alpha}$.

Salem numbers. An algebraic integer $\lambda > 1$ is a *Salem number* if λ is a unit, and its conjugates other than $\lambda^{\pm 1}$ lie on the unit circle.

The irreducible polynomial $S(t)$ for λ always has even degree $2d$, and its roots are invariant under $t \mapsto 1/t$. Thus there is a corresponding *Salem trace* *polynomial* $R(t)$ of degree d such that $S(t) = t^d R(t + 1/t)$. The condition that $S(t)$ has only two roots outside S^1 translates into the condition that $R(t)$ one root $\tau = \lambda + \lambda^{-1} > 2$ and its remaining roots lie in $[-2, 2]$.

Theorem 3.15 *If f has positive entropy then $\lambda(f)$ is a Salem number.*

Proof. Since $f|H^2(X)$ preserves the integral cohomology, $\lambda(f)$ is an algebraic integer; and it is a unit because f is invertible. Since f acts unitarily on $H^{2,0} \oplus H^{0,2}$, and $H^{1,1}$ has signature $(1, 4)$, the eigenvalues of $f|H^2(X)$ other than $\lambda(f)^{\pm 1}$ must lie on the unit circle. ∎

Examples from $\mathbf{SL_2}(\mathbb{Z})$. Let $X = E \times E$ be a product of two elliptic curves. Then any element $A \in \mathrm{SL}_2(\mathbb{Z})$ determines an automorphism $f_A : X \to X$. Its invariants are $\delta(f) = 1$ and $\lambda(f) = \sigma(A)^2$.

Examples from Riemann surfaces with automorphisms. Let C be the Riemann surface of genus two defined by $y^2 = x^5 - 1$. This C has a symmetry $g : C \to C$ order 5, satisfying $g(x, y) = (\zeta x, y)$, where $\zeta^5 = 1$. Taking dx/y and $x\,dx/y$ as a basis for $\Omega(X)$, we see g action on $H^{1,0}(X) \cong \Omega(X)$ with eigenvalues (ζ, ζ^2).

Now let $X = \mathrm{Jac}(C)$. Then we obtain a subring

$$\iota : \mathbb{Z}[\zeta] \subset \mathrm{End}(\mathrm{Jac}(X))$$

by defining $\iota(\zeta) = g$. Given $\alpha \in \mathbb{Z}[\zeta]$, we let α' denote its image under the Galois automorphism sending ζ to ζ^2; then $\iota(\alpha)$ acts on $H^{1,0}(\mathrm{Jac}(X)) \cong \Omega(X)$ with eigenvalues α and α'.

Any *unit* $\alpha \in \mathbb{Z}[\zeta]$ gives an *automorphism* $f = \iota(\alpha)$ of X. For example, the golden mean $\gamma = (1 + \sqrt{5})/2$ gives an automorphism $f = \iota(\gamma)$ satisfying $\lambda(f) = \gamma^2$ and $\delta(f) = \gamma\gamma' = -1$.

Examples from rings of algebraic integers. More generally, let

$$I \subset \mathcal{O}_K \subset \mathbb{C}$$

be an ideal in the ring of integers in a complex extension of a real quadratic field k. (We can also take $I = \mathcal{O}_K$.) Let $\alpha \mapsto \alpha'$ be a Galois automorphism of $K/\mathbb{Q}$ which is nontrivial on K. Then the map $\iota(\alpha) = (\alpha, \alpha')$ sends I to a lattice $L = \iota(I) \subset \mathbb{C}^2$. Since $\mathcal{O}_K \cdot I = I$, we have $\mathcal{O}_K \subset \mathrm{End}(X)$ and $\mathcal{O}_K^* \subset \mathrm{Aut}(X)$. The units of infinite order act on X by automorphisms of positive entropy.

Theorem 3.16 *If X is projective, then $\delta(f)$ is a root of unity.*

Proof. Let $S = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$. If X is projective, then a hyperplane section C provides an integral class $C \in S$ with $C^2 > 0$. Thus the signature of S has the form $(1,q)$ for some $q \geq 0$. Clearly $f|H^2(X,\mathbb{Z})$ preserves S, so it also preserves $S^\perp$. The lattice $S^\perp$ has signature $(2, 3-q)$. Now f also preserves the positive-definite subspace $H^{2,0} \oplus H^{0,2}$ of $S^\perp \otimes \mathbb{C}$, with signature $(2,0)$, so $f|S^\perp$ lies in a compact group of the form $\mathrm{SO}(2) \times SO(3-q)$. But $\langle f|S^\perp \rangle$ is a discrete group, since $S^\perp \cong \mathbb{Z}^{5-q}$ is a lattice. Thus f has finite order on $S^\perp \otimes \mathbb{C} \supset H^{2,0}(X)$. ∎

Synthesis of dynamics. Here is a general method for constructing examples of torus automorphisms with prescribed eigenvalues.

Theorem 3.17 *Let $p(t) = t^4 + a_1 t^3 + a_2 t^2 + a_3 t + 1 \in \mathbb{Z}[t]$ be an irreducible polynomial whose roots $\alpha, \overline{\alpha}, \beta, \overline{\beta}$ occur in conjugate pairs. Then there exists a complex torus X and an $f \in \mathrm{Aut}_0(X)$ with $p(t) = \det(tI - f|H^1(X))$.*

Proof. Realize p as the characteristic polynomial of an element $F \in \mathrm{SL}_4(\mathbb{Z})$; use the condition on the eigenvalues of F to construct an F-invariant splitting

$$\mathbb{Z}^4 \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1} = \overline{H^{0,1}}$$

such that F acts on $H^{1,0}$ by $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$; project the lattice $\mathbb{Z}^4$ to the first factor, and let f be the induced automorphism of $X = H^{1,0}/\mathbb{Z}^4$. ∎

Examples: degree 6 Salem numbers. For any integer $a \geq 0$, it is easy to check that

$$P(t) = t^4 + at^2 + t + 1$$

has only complex roots, say $\{\alpha, \beta, \overline{\alpha}, \overline{\beta}\}$ with $|\alpha| > |\beta|$. By the preceding result, there is a complex 2-torus X and an automorphism $f \in \mathrm{Aut}(X)$ such that

$$\delta(f) = \alpha\beta, \quad \lambda(f) = |\alpha|^2,$$

and $P(t)$ is the characteristic polynomial of $f^*|H^1(X)$.

The products of pairs of distinct roots of $P(t)$ give the roots of the characteristic polynomial

$$S(t) = t^6 - at^5 - t^4 + (2a-1)t^3 - t^2 - at + 1$$

of $f^*|H^2(X) = \wedge^2 H^1(X)$. Thus $\delta(f)$ and $\lambda(f)$ are roots of $S(t)$. Similarly, $\tau = \lambda(f) + \lambda(f)^{-1} > 2$ is a root of the cubic Salem trace polynomial

$$R(t) = (t-a)(t^2-4) - 1.$$

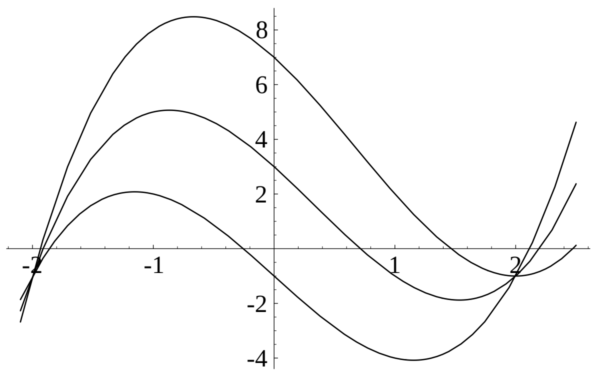

Figure 3. Salem trace polynomials $R(t) = (t - a)(t^2 - 4) - 1$ for $a = 0, 1, 2$.

See Figure 3. (The formulas for $S(t)$ and $R(t)$ come from a straightforward calculation with determinants and the companion matrix of $P(t)$.)

Since $R(-2) = R(2) = -1$ while $R(-1) > 0$, the roots of $R(t)$ other than τ lie in the interval $[-2, 2]$, and since $R(n) \neq 0$ for $n \in \mathbb{Z}$, $R(t)$ is irreducible. Thus $R(t)$ is a Salem trace polynomial. Therefore $\lambda(f)$ is a sextic Salem number, and $S(t)$ is a Salem polynomial. This shows:

Theorem 3.18 *For every $a = 0, 1, 2 \ldots$, there exists a positive entropy automorphism of a complex torus $f : X \to X$ such that $\lambda(f)$ and $\delta(f)$ are roots of the Salem polynomial $S(t) = t^6 - at^5 - t^4 + (2a - 1)t^3 - t^2 - at + 1$.*

Remark. It is clear that $\delta(f)$ in the examples above is not a root of unity, since it has a conjugate $\lambda(f) > 1$. Thus the complex torus X is *not* projective.

Invariant currents. From now on we assume $h(f) > 0$. Thus F has a pair of distinct eigenvalues α, β on the universal cover of X. Choose coordinates on the universal cover such that $F(x, y) = (\alpha x, \beta y)$. The forms dx and dy descend to X, and determine a pair of closed positive $(1, 1)$-forms:

$$T_+ = \frac{i}{2}\, dx \wedge d\overline{x} \quad \text{and} \quad T_- = \frac{i}{2}\, dy \wedge d\overline{y}.$$

We will refer to the forms $T_\pm$ as *currents*, since in more general settings such as K3 surfaces they need not be smooth.

These forms satisfy:

$$f^*(T_\pm) = \lambda(f)^{\pm 1} T_\pm.$$

Thus the measure $\mu_f = T_+ \wedge T_-$ is f-invariant. In the case at hand (a complex torus), the measure μ_f is a constant multiple of the canonical volume element $\eta \wedge \overline{\eta}$.

Note that $[T_\pm^2] = 0$ in cohomology — these forms lie on the *boundary* of the Kähler cone, and hence on the light cone for the Minkowski form.

Dynamics on currents. When a complex torus X is actually projective — that is, when X is an Abelian surface — it then carries lots of algebraic curves, giving closed, positive $(1,1)$-currents. But even when X is not projective, the Kähler cone C_X gives lots of closed, positive $(1,1)$-forms.

Theorem 3.19 *The current T_+ is the unique positive representative of its cohomology class. Moreover T_+ determines an extreme ray in the convex cone of positive currents.*

Proof. Let ω be a Kähler form on X. Recall that the *mass* of a closed, positive $(1,1)$-current can be defined by:

$$M(T) = \int T \wedge \omega.$$

Since ω is closed, the mass depends only on the cohomology class of T. The space of closed, positive currents with $M(T) \leq M_0$ is compact.

Let K denote the convex set of all closed, positive $(1,1)$-currents cohomologous to T_+. All elements of K have the same mass, so K is a compact set as well. Moreover, any $T \in K$ can be uniquely expressed in the form

$$T = T_+ + dd^c(\phi)$$

where ϕ belongs to $L^1(X)$ and $\int \phi = 0$. (Here we integrate with respect to the invariant volume element $\eta \wedge \overline{\eta}$.)

Define $L : K \to \mathbb{R}$ by $L(T) = \int |\phi|$. Then L is a bounded, continuous function on a compact space, and hence it assumes its maximum.

Now define $R : K \to K$ by $R(T) = \lambda^{-1} \cdot f^*(T)$. Clearly R is an automorphism of K, with T_+ as a fixed-point. On the other hand, if $T = T_+ + dd^c(\phi)$, then we have

$$R(T) = T_+ + \lambda^{-1} \cdot dd^c(\phi \circ f).$$

Thus $L(R(T)) = \lambda^{-1} L(T)$. Since the functions $L(T)$ and $L(R(T))$ have the same maximum on K, we conclude that the maximum is zero and therefore K reduces to the single point T_+.

The extremality of T_+ now follows from the extremality of its cohomology class. ■

Corollary 3.20 *Let T be any closed, positive $(1,1)$-current on X. Then we have*

$$T_n = \lambda(f)^{-n}(f^n)^*(T) \to \alpha T_+$$

for some $\alpha \geq 0$.

Proof. Since the dynamics on cohomology is hyperbolic, with expanding eigenvector $[T_+]$, we can find an $\alpha \geq 0$ such that $[\alpha T_+] = \lim [T_n]$. Since X is Kähler, the mass of a positive $(1,1)$-form is controlled by its cohomology class, so T_n accumulates on a set of positive representatives for the cohomology class $[\alpha T_+]$. But αT_+ is the only positive representative of its cohomology class, so it is the limit of T_n. ∎

Note: we have $\alpha > 0$ unless T is actually proportional to T_-.

Question. Which cohomology classes on the boundary of the Kähler cone have a unique positive representative? Note: if E is the smooth fiber of an elliptic fibration of X, then $E^2 = 0$ and the many equivalent fibers provide many different representatives for $[E] \in \partial C_X$.

Measurable dynamics. The measure

$$\mu_f = T_- \wedge T_+$$

is f-invariant, because T_- and T_+ scale by reciprocal factors. In fact μ_f is proportional to the natural volume element $\eta \wedge \overline{\eta}$ on X, or equivalently to the obvious Euclidean measure coming from the presentation of X as a quotient $\mathbb{C}^2/L$.

Theorem 3.21 *Let f be an automorphism of a complex torus $X = \mathbb{C}^2/L$. If $f : X \to X$ has positive entropy, then f is ergodic and mixing with respect to the natural invariant measure μ_f.*

Proof. Fourier analysis gives an isomorphism between $L^2(\mathbb{C}^2/L)$ and $\ell^2(L^*)$, sending the unitary action of f to the linear action of

$$F = f^*|H^1(X,\mathbb{Z}) \cong L^*.$$

Since F has no eigenvalues on the unit circle (recall its eigenvalues are $\alpha, \overline{\alpha}, \beta, \overline{\beta}$ with $|\alpha| > 1 > |\beta|$), we have $F^n(x) \to \infty$ for every $x \neq 0$ in L^*. Thus if $\phi, \psi \in \ell^2(L^*)$ are functions with finite support, and $\phi(0) = \psi(0) = 0$, we have

$$\langle \phi \circ F^n, \psi \rangle = 0$$

for all n sufficiently large. Since such functions are dense in $\ell^2(L^*)$, their transforms are dense in $L^2(\mathbb{C}^2/L)$.

We conclude that for any $\phi, \psi \in L^2(\mathbb{C}^2/L)$ with $\int \phi = \int \psi = 0$, we have

$$\langle \phi \circ f^n, \psi \rangle \to 0,$$

and thus f is mixing. ∎

3.6 Topology and Hodge theory of K3 surfaces

Theorem 3.22 *Let X be a K3 surface. Then we have $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{22}$, and the intersection form is even and unimodular with signature $(3, 19)$.*

Proof. Noether's formula for a complex surface states:

$$1 - q(X) + p_g(X) = \frac{c_1(X)^2 + c_2(X)}{12}.$$

In the case of a projective surface, this formula gives $\chi(\mathcal{O}_X)$ in terms of topological invariants of X. Here:

- $q(X) = \dim \Omega^1(X)$, the *irregularity* of X, is the dimension of the space of holomorphic 1-forms;
- $p_g(X) = \dim \Omega^2(X)$, the *geometric genus* of X, is the dimension of the space holomorphic sections of the canonical bundle of X;
- $c_1(X)^2 = K_X^2$ is the self-intersection number of a canonical divisor; and
- $c_2(X) = \chi(X) = 2 + b_2(X) - 2b_1(X)$ agrees with the topological Euler characteristic of X.

For a K3 surface, we have $q(X) = 0$ (since $b_1(X) = 0$), $p_g(X) = 1$ and $c_1(X)^2 = 0$ (since the canonical bundle is trivial). Thus $c_2(X) = 24$, which implies $b_2(X) = 22$.

According to the Thom-Hirzebruch index theorem, the index of the intersection form on $H^2(X)$ is given by

$$\text{index}(X) = \frac{c_1(X)^2 - 2c_2(X)}{3} = -16,$$

so $H^2(X)$ has signature $(3, 19)$.

Next we check $H^2(X,\mathbb{Z})$ has no torsion. If it did, then $H_1(X,\mathbb{Z})$ would have torsion too, so we could form a finite-sheeted cover $Y \to X$ of degree $d > 1$. The canonical bundle for Y would still be trivial, so Noether's formula would give:

$$2 - q(Y) = c_2(Y)/12 = d\,c_2(X)/12 = 2d > 2,$$

a contradiction.

To see the intersection form is even, note that X is orientable, and its second Stiefel-Whitney class vanishes: we have $w_2(X) = c_1(X) \bmod 2 = 0$. But for any oriented surface $C \subset X$, we have a 'mod 2 adjunction formula':

$$C^2 = w_2(X) \cdot C \bmod 2,$$

and thus the intersection form is even. ∎

Corollary 3.23 *The intersection form on $H^{1,1}(X)$ has signature* $(1, 19)$.

Corollary 3.24 *The lattice $H^2(X,\mathbb{Z})$ is isomorphic to* $\mathrm{II}_{3,19}$.

Kummer surfaces. The simplest *instrinsically* constructed K3 surface is the Kummer surface X attached to a complex torus $Y = \mathbb{C}^2/L$.

To construct X, form the 2-fold quotient $X_0 = Y/\iota$ using the involution $\iota(p) = -p$ in the group law on Y. The 16 points $Y[2]$ of order 2 on Y comprise the fixed-points of ι, and give rise to double points of X_0. Blowing up these double points, we obtain the Kummer surface X. Alternatively, we can first blowup $Y[2]$ to obtain a surface $\widehat{Y}$ with 16 exceptional curves fixed pointwise by ι. Then $\widehat{Y}$ admits a regular, 2-to-1 map to X. In summary, we have a commutative diagram:

$$\begin{tikzcd}
\widehat{Y} \arrow[r] \arrow[d, "\text{blowup } Y[2]"'] & X \arrow[d, "\text{blowup double points}"] \\
Y = \mathbb{C}/L \arrow[r, "\text{quotient by } \iota"] & X_0
\end{tikzcd}$$

We have $b_1(X) = 0$ because ι acts by -1 on $H_1(Y,\mathbb{Q})$; there is no ι-invariant first cohomology, and blowing up adds none.

Geometric representatives for two 2-dimensional cohomology classes of a K3 surface are particularly transparent in the Kummer case. Each double point gives rise to a -2 curve on X. Also $H^2(Y)$ is invariant under ι, so it descends rationally to give:

$$H^2(X, \mathbb{Q}) \cong H^2(Y, \mathbb{Q}) \oplus \mathbb{Q}^{16}.$$

The intersection form on the first factor has signature $(3, 3)$, as we have seen; the second has signature $(0, 16)$, so we obtain $(3, 19)$ altogether.

The above rational decomposition does not correspond to an integral splitting of $H^2(X, \mathbb{Z})$; instead, when intersected with $H^2(X, \mathbb{Z})$ we obtain the lattices

$$L_1 \oplus L_2 = \mathrm{II}_{3,3}(2) \oplus \mathbb{Z}^{16}(-2)$$

of determinant 2^6 and 2^{16} respectively. (Here $L(n)$ denotes the lattice L with its inner form multiplied by n.) To obtain the full integral cohomology, the lattices L_1 and L_2 must be 'glued': vectors from $(L_1 \oplus L_2)^*/(L_1 \oplus L_2)$ must be adjoined to make the lattice unimodular. Compare [BPV, Ch. VIII.5].

Canonical form. The canonical form $\eta_Y = dz_1 \wedge dz_2$ on Y descends to give a nowhere vanishing canonical form η_X on X.

To see this, note first that $\iota^*(\eta_Y) = \eta_Y$. Thus η_Y descends to a form which is holomorphic and nonvanishing away from the -2-curves on X that result from $Y[2]$. To check its behavior along one of these curves, note that we can choose local coordinates (x, y) near the origin in $Y = \mathbb{C}^2/L$ such that $\eta_Y = dx\,dy$, $\iota(x, y) = (-x, -y)$ and such that the rational map to X is given locally by $(u, v) = \phi(x, y) = (x^2, y/x)$. (Note that this map blows up the origin. Also we do not use $(y^2, y/x)$ since the level sets $y^2 = 0$ and $y/x = 0$ are not transverse.) We then have:

$$\phi^*(du\,dv) = d(x^2)\,d(y/x) = (2x\,dx)(dy/x) = 2\,dx\,dy = 2\eta_Y.$$

This shows η_X is locally proportional to $du\,dv$ and hence holomorphic and nowhere zero.

Alternatively, one can first pull η_Y back to $\widehat{Y}$, and observe that it vanishes to order 1 along the 16 exceptional curves there. These zeros disappear upon passing to the 2-fold quotient X.

Kähler cone. Every K3 surface is Kähler. The *Kähler cone*

$$C_X \subset H^{1,1}(X, \mathbb{R})$$

is the set of all classes represented by the symplectic forms of Kähler metrics on X. (Here $V_{\mathbb{R}} = \{v \in V \,:\, v = \overline{v}\}$.)

The shape of the Kähler cone can be made more explicit as follows. Let

$$\begin{aligned}\Delta(X) &= \{D \in \operatorname{Pic}(X) \,:\, D^2 = -2\}, \quad \text{and} \\ W(X) &= \{\omega \in H^{1,1}(X)_{\mathbb{R}} \,:\, \omega^2 > 0 \text{ and } \omega \cdot D \neq 0 \text{ for all } D \in \Delta(X)\}.\end{aligned}$$

Since the intersection form on $H^{1,1}(X)_{\mathbb{R}}$ has signature $(1, 19)$, $W(X)$ is the cone over two copies of the hyperbolic space $\mathbb{H}^{19}$, with a configuration of hyperplanes corresponding to $\Delta(X)$ deleted.

It is known that the Kähler cone C_X coincides with a component or *chamber* of $W(X)$. The automorphisms of $H^2(X, \mathbb{Z})$ preserving the intersection form and Hodge structure act transitively on the set of chambers. (Observe that such automorphisms include the reflections $C \mapsto C + (C \cdot D)D$ through the hyperplanes defined by $D \in \Delta(X)$.)

Notes. The Kähler cone of a K3 surface is very unstable: slight deformations of X completely change $\operatorname{Pic}(X)$ and hence $\Delta(X)$. A complex torus carries no rational curves, and hence no smooth -2-curves; this explains why its Kähler cone has a simpler structure.

Torelli theorem. The Torelli theorem asserts that a K3 surface is determined up to isomorphism by its Hodge structure. More precisely we have:

Theorem 3.25 *Let X and Y be K3 surfaces, and let*

$$F : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$

be an isomorphism preserving the intersection pairing. Extend F to $H^2(X, \mathbb{C})$ by tensoring with $\mathbb{C}$; then:

1. *If F sends $H^{2,0}(X)$ to $H^{2,0}(Y)$, then X and Y are isomorphic.*
2. *If F also sends C_X to C_Y, then $F = f^*$ for a unique isomorphism $f : Y \to X$.*

Marked K3 surfaces. Next we discuss the space of all possible Hodge structures on a K3 surface.

Let L be a fixed even, unimodular lattice of signature $(3, 19)$. A *Hodge structure* on L is a splitting

$$L \otimes \mathbb{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

such that $\dim H^{2,0} = 1$, $H^{2,0} = \overline{H^{0,2}}$, $H^{2,0} \oplus H^{0,2}$ has signature $(2, 0)$ and $H^{1,1} = (H^{2,0} \oplus H^{0,2})^{\perp}$. The space of Hodge structures on L is parameterized by the *period domain*

$$\Omega(L) = \{[\eta] \in \mathbb{P}(L \otimes \mathbb{C}) \,:\, \eta \cdot \eta = 0 \text{ and } \eta \cdot \overline{\eta} > 0\},$$

via the correspondence

$$H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \quad = \quad \mathbb{C} \cdot \eta \oplus \{\eta, \overline{\eta}\}^{\perp} \oplus \mathbb{C} \cdot \overline{\eta}.$$

The period domain is an open subset of a smooth, 20-dimensional quadric hypersurface in $\mathbb{P}^{21}$.

Now let X be a K3 surface. A *marking* for X is an isomorphism

$$\iota : H^2(X, \mathbb{Z}) \cong L$$

preserving the intersection pairing. Every K3 surface admits a marking. Two marked surfaces (X_1, ι_1), (X_2, ι_2) are *equivalent* if there is an isomorphism $f : X_1 \to X_2$ such that $\iota_2 = \iota_1 \circ f^*$.

Let $\mathcal{M}(L)$ be the *moduli space* of equivalence classes of K3 surfaces marked by L. The *period mapping*

$$\pi : \mathcal{M}(L) \to \Omega(L) \subset \mathbb{P}(L \otimes \mathbb{C})$$

is defined by $\pi(X, \iota) = [\iota(\eta)]$, where $\eta \neq 0$ is a holomorphic (2,0)-form on X. The image of π lies in the period domain because $\eta \wedge \eta = 0$ and $\eta \cdot \overline{\eta} > 0$.

The next result complements the Torelli theorem by showing all possible Hodge structures on K3 surfaces actually arise:

Theorem 3.26 *The period mapping $\pi : \mathcal{M}(L) \to \Omega(L)$ is surjective.*

Remarks. The period mapping is not injective, because the marked Hodge structure on $H^2(X)$ does not uniquely determine the Kähler cone C_X. Note also that the discrete group $\mathrm{Aut}(L) = \mathrm{O}(\mathrm{II}_{3,19})$ acts on $\Omega(L)$ with *dense orbits.* In particular, Kummer surfaces are dense, and any pair of K3 surfaces are 'nearly isomorphic'.

For similar reasons, the moduli space of *unmarked* K3 surfaces, $\Omega(L)/\,\mathrm{O}(\mathrm{II}_{3,19})$, does not exist as a reasonable space. This should be contrasted to the case of Riemann surfaces, whose moduli space $\mathcal{M}_g$ is a variety. In the Riemann surface case the sympletic form on $H_1(X, \mathbb{Z})$ provides a polarization, making $\mathrm{Aut}(X)$ finite and rendering its moduli space separated.

3.7 Dynamics on K3 surfaces

Let $f : X \to X$ be an automorphism of a K3 surface X. Then f preserves the Hodge structure on $H^2(X)$. As for tori we define

$$\delta(f) = \mathrm{Tr}(f|H^{2,0})$$

and

$$\lambda(f) = \sigma(f|H^2).$$

Many features of dynamics on complex 2-tori continue to hold on K3 surfaces. For example:

- The determinants at fixed-points are synchronized: we have $\delta(f) = \det Df_p$ whenever $f(p) = p$.

  (On the other hand, on a K3 surface, $\mathrm{Tr}(Df_p)$ is generally different for different fixed-points.)

- If X is projective, then $\delta(f)$ is a root of unity.

- The entropy of f is given by $h(f) = \log \lambda(f)$.

- The intersection form makes $H^{1,1}(\mathbb{R})$ into a Minkowski space of signature $(1, n)$, and f has positive entropy iff it determines a hyperbolic (as opposed to elliptic or parabolic) isometry of $\mathbb{H}^n$.

- If f has positive entropy, then $\lambda(f)$ is a Salem number (of degree at most 6 for 2-tori, at most 22 for K3 surfaces.)

- In the positive entropy case, there are eigenvectors $\xi_\pm$ in the boundary of the Kähler cone $C_X \subset H^{1,1}(X)$, satisfying

  $$f^*(\xi_\pm) = \lambda(f)^{\pm 1}\xi_\pm.$$

- Each eigenvector is uniquely represented by a positive $(1,1)$-current, $[T_\pm] = \xi_\pm$.

- For any other closed positive current, we have

  $$\lambda(f)^{-n}(f^n)^*(T) \to \alpha T_+$$

  for some $\alpha \geq 0$.

- The measure $\mu_f = T_+ \wedge T_-$ is f-invariant, mixing, and it is the measure of maximal entropy for f.

Invariant currents. With one exception, the proofs of the assertions above follow along exactly the same lines as in the case of tori; they rely mostly on triviality of the canonical bundle.

The exception is the existence of the invariant currents $T_\pm$. For complex tori we could write these currensts down directly, using the linear form of f. For K3 surfaces the existence is also easy, but comes from an averaging argument.

Theorem 3.27 (Cantat) *If f has positive entropy, then there is a closed, positive $(1,1)$ eigencurrent $T_+ \neq 0$ on X such that $f^*(T_+) = \lambda(f)T_+$.*

Proof. To construct T_+, let ω be a Kähler form on X and let $\omega_n = \lambda^{-n}(f^n)^*(\omega)$. After suitably scaling the eigenclass, we have $[\omega_n] \to \xi_+$ in $H^2(X)$.

Recall that the mass of a closed, positive current can be defined by $M(\alpha) = \int \alpha \wedge \omega$; it depends only on the cohomology class $[\alpha]$. Since $[\omega_n]$ is bounded in $H^2(X)$, the mass of ω_n is uniformly bounded.

Let T_+ be any accumulation point of the currents $T_N = (1/N)\sum_1^N \omega_n$. Since $[T_+] = \xi_+$ on the level of cohomology, we have $T_+ \neq 0$. By construction we have

$$f^*(T_N) = \lambda(f)T_N + \frac{\lambda(f)(\omega_{N+1} - \omega_1)}{N}.$$

The fraction above tends to zero since $M(\omega_n)$ is bounded; thus $f^*(T_+) = \lambda(f)T_+$. ∎

Examples from Kummer surfaces. Let $Y = \mathbb{C}^2/L$ and let X be the Kummer surface associated to Y. Then any automorphism $F \in \mathrm{Aut}_0(Y)$ (fixing $z = 0$) commutes with $z \mapsto -z$, and hence induces an automorphism $f : X \to X$.

The isomorphism:

$$H^2(X, \mathbb{Q}) \cong H^2(Y, \mathbb{Q}) \cong \mathbb{Q}^{Y[2]}$$

relates the action of f to the action of F. Thus we have $\delta(f) = \delta(F)$ and $\lambda(f) = \lambda(F)$. The action of f on the final 16-dimensional factor is simply a permutation representation. Since (X, f) is a measurable quotient of (Y, F), we have:

Theorem 3.28 *Let $f : X \to X$ be a positive entropy automorphism of a Kummer surface, arising from an automorphism of a complex torus. Then f is ergodic and mixing for Lebesgue measure on X.*

Examples from surfaces of degree $(\mathbf{2,2,2})$. Let X be a $(2,2,2)$-hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as before. Projections to coordinate axess present X as an elliptic fibration over $\mathbb{P}^1$ in 3 different ways. Thus the Picard group $\mathrm{Pic}(X)$ contains at least the subgroup

$$S = \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3 \subset H^{1,1}(X, \mathbb{R})$$

generated by 3 different elliptic fibers. For $i \neq j$, we have $E_i \cdot E_j = 2$ since $E_i \cap E_j$ coincides with the intersection of X with a coordinate line. Thus the intersection pairing on S has matrix

$$(E_i \cdot E_j) \quad = \quad \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

with signature $(1, 2)$.

Thus the projectivization of $S \otimes \mathbb{R}$ contains a hyperbolic plane $\mathbb{H}$. The elliptic curves E_i give 3 points in $\partial\mathbb{H}$ and determine an ideal triangle T. The 3 involutions on X described before are nothing more than reflections through the sides of T.

For example, the reflection that fixes E_1 and E_2 sends E_3 to $2(E_1+E_2)-E_3$, as can be verified using the fact that the intersection form is preserved. That is, its action on S is given by the matrix:

$$\iota_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

Taking the product of 3 such matrices, we find that

$$f = \iota_1 \circ \iota_2 \circ \iota_3 = \begin{pmatrix} -1 & -2 & -6 \\ 2 & 3 & 10 \\ 2 & 6 & 15 \end{pmatrix}$$

has eigenvalues $(-1, \lambda, \lambda^{-1})$ with $\lambda = 9 + 4\sqrt{5}$. Thus its entropy is given by $h(f) = \log(9 + 4\sqrt{5})$. (Note: $\lambda = \gamma^6$ where $\gamma = (1 + \sqrt{5})/2$ is the golden ratio.)

Siegel disk examples. Let us say a linear map $F(z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2)$ is an *irrational rotation* if $|\lambda_1| = |\lambda_2| = 1$ and F has dense orbits on $S^1 \times S^1$. A domain $U \subset X$ is a *Siegel disk* for f if $f(U) = U$ and $f|U$ is analytically conjugate to $F|\Delta^2$ for some irrational rotation F. (Here $\Delta = \{z : |z| < 1\}$.)

Like an elliptic island on $X(\mathbb{R})$, a Siegel disk on X is an obstruction to ergodicity and to the existence of dense orbits. In the next section we will show:

Theorem 3.29 *There exist K3 surface automorphisms with Siegel disks. Every such automorphism has positive entropy and resides on a non-projective K3 surface.*

Differences between K3 surfaces and complex tori.

1. *Form of the currents.* For complex tori we can write explicit formulas for the currents $T_\pm$; in general this is impossible on a K3 surface.

2. *Invariant curves.* On a complex torus, an automorphism of positive entropy has no invariant curves. On the other hand, such an automorphism of a K3 surface can have many invariant curves. For example, in the Kummer surface construction we obtain 16 smooth rational -2-curves satisfying $f(C) = C$. (Any invariant curve must have $C^2 < 0$).

3. *Invariant measure.* For a complex torus, the dynamical measure μ_f agrees with the invariant Lebesgue measure $\eta \wedge \overline{\eta}$ up to scale. This property is shared by the Kummer surface examples, but fails in the presence of Siegel disks, because μ_f is supported on $J(f)$.

4. *Julia set.* We have $J(f) = \mathbb{C}^2/L$ for any positive entropy map on a complex torus. On the other hand in the Siegel disk examples we have $J(f) \neq X$.

Conjecture 3.30 *Let $f : X \to X$ be a positive-entropy automorphism of a general K3 surface. Then $J(f) = X$ iff X has no Siegel disks.*

In particular, $J(f) = X$ if X is projective.

Conjecture 3.31 *Let $f : X \to X$ be a positive entropy automorphism of a K3 surface. Then μ_f is singular with respect to Lebesgue measure, unless X is a Kummer surface and f comes from an automorphism of a complex torus.*

Cantat has shown that the stable manifold $W_p \cong \mathbb{C} \subset X$ of a hyperbolic fixed-point $p \in X$ is uniformly distributed with respect to the current T_+. The speculation that μ_f is singular with respect to Lebesgue measure is supported in part by the picture of the real points of the stable manifold, $W_p(\mathbb{R})$, in Figure 4. (The point p is in the center of the picture.)

This image suggests that $W_p(\mathbb{R})$ is very unevenly distributed on $X(\mathbb{R})$, and hence the current T_+ is transversally very singular. The measure $\mu_f = T_- \wedge T_+$ is therefore also likely to be singular.

We remark that for rational maps f on $\mathbb{P}^1$, Zdunik has shown that μ_f is absolutely continuous with respect to Lebesgue measure iff f is a Lattès example [Zd]. See also [Be] for related results on $\mathbb{P}^k$.

Lyapunov exponent. It would be interesting to numerically calculate the Lyapunov expansion factor of $f : X \to X$ with respect to normalized

Lebesgue measure ν. If f happens to be ergodic with respect to ν, then this expansion factor is given by:

$$\Lambda(f, \nu) = \lim_{n\to\infty} \|(Df^n)_p\|^{1/n}$$

for almost every $p \in (X, \nu)$. Here the norm of the derivative can be measured with any smooth metric on X.

In the Kummer surface examples, we have $\Lambda(f, \nu)^2 = \lambda(f)$. We also have $\Lambda(f, \mu_f)^2 = \lambda(f)$ quite generally (Cantat). Thus a difference between these exponents would also indicate that μ_f is singular with respect to ν.

Figure 4. The stable manifold of a hyperbolic fixed-point for a K3 surface automorphism.

3.8 Siegel disks on K3 surfaces

In this section we conclude our discussion of K3 surfaces by sketching a construction from [Mc2]. This construction shows there exist K3 surface automorphisms with positive entropy that are not ergodic.

Siegel disks. Let us say a linear map $F(z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2)$ is an *irrational rotation* if $|\lambda_1| = |\lambda_2| = 1$ and F has dense orbits on $S^1 \times S^1$. A domain $U \subset X$ is a *Siegel disk* for f if $f(U) = U$ and $f|U$ is analytically conjugate to $F|\Delta^2$ for some irrational rotation F. (Here $\Delta = \{z : |z| < 1\}$.)

Like an elliptic island on $X(\mathbb{R})$, a Siegel disk on X is an obstruction to ergodicity and to the existence of dense orbits. The main result of this seciton is:

Theorem 3.32 *There exist K3 surface automorphisms with Siegel disks.*

Every such automorphism has positive topological entropy.

Unfortunately, these Siegel disks are invisible to us: they live on non-projective K3 surfaces, and we can only detect them implicitly, through Hodge theory and dynamics on the cohomology. Indeed, we have:

Theorem 3.33 *There are at most countably many K3 surface automorphisms with Siegel disks, up to isomorphism; and there are no Siegel disks on projective K3 surfaces.*

Synthesis of automorphisms. The next theorem provides the key to building examples of K3 surface automorphisms. It reduces the construction of automorphisms to a problem in integral quadratic forms. It also represents the first step towards determining which Salem numbers can arise as $\lambda(f)$. For more progress in this direction, see [GM].

Theorem 3.34 (Synthesis) *Let $F : L \to L$ be an automorphism of an even, unimodular lattice of signature $(3, 19)$. Suppose $S(t) = \det(tI - F)$ is a Salem polynomial. Then there is:*

- *A K3 surface automorphism $f : X \to X$, and*
- *A marking $\iota : H^2(X, \mathbb{Z}) \to L$, such that $F = \iota \circ f^* \circ \iota^{-1}$.*

Proof. Since F has only two eigenvalues off the unit circle, while the signature of L is $(3, 19)$, there exists an eigenvector $\eta \in L \otimes \mathbb{C}$ such that $T(\eta) = \delta\eta$, $|\delta| = 1$ and $\eta \cdot \overline{\eta} > 0$. By surjectivity of the period mapping (Theorem 3.26), there exists a K3 surface X and a marking $\iota : H^2(X, \mathbb{Z}) \to L$ such that $\iota(H^{2,0}(X)) = \mathbb{C} \cdot \eta$.

Let $T : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ be the automorphism given by $T = \iota^{-1} \circ F \circ \iota$. Then T respects the intersection paring and the Hodge structure on $H^2(X)$, and its characteristic polynomial is also $S(t)$.

We claim that $\mathrm{Pic}(X) = 0$. Indeed, since $S(t)$ is irreducible, T has no proper rational invariant subspace, and thus $H^{1,1}(X) \cap H^2(X, \mathbb{Z}) = (0)$. In particular, $\Delta(X) = \emptyset$, so the Kähler cone $C_X \subset H^{1,1}(X)_{\mathbb{R}}$ is simply one of the two components of the space $W(X) = \{\omega : \omega^2 > 0\}$.

Since the leading eigenvalue of T is a Salem number $\lambda > 1$, T does not interchange the components of $W(X)$, and therefore $T(C_X) = C_X$. By the Torelli theorem (Theorem 3.25), there is a unique automorphism $f : X \to X$ such that $f^*|H^2(X, \mathbb{Z}) = T$. ■

Remarks. The marked K3 surface X constructed above is unique up to complex conjugation. (We can replace η with $\overline{\eta}$, thereby swapping $H^{2,0}$ with $H^{0,2}$.) Note that X is non-projective, since $\mathrm{Pic}(X) = (0)$.

Construction of automorphisms. Here is a sketch of the construction of K3 surface automorphisms with Siegel disks.

1. Let $\lambda > 1$ be a degree 22 Salem number with minimal polynomial $S(x) \in \mathbb{Z}[x]$. Our first goal is to construct a K3 surface automorphism $f : X \to X$ such that $\lambda(f) = \lambda$ and $S(x)$ is the characteristic polynomial of $f^*|H^2(X)$.

2. Let $B = \mathbb{Z}[y]/(S(y))$, and let K be the field of fractions of B. Let $U(x)$ be a unit in the subring of B generated by $x = y + y^{-1}$. We make B into a lattice by defining the inner product

$$\langle g_1, g_2 \rangle_{B(U)} = \mathrm{Tr}_{\mathbb{Q}}^{K} \left(\frac{U(x) g_1(y) g_2(y^{-1})}{R'(x)} \right), \tag{3.1}$$

where $R(x)$ is the minimal polynomial of $\lambda + \lambda^{-1}$. Assuming $|S(\pm 1)| = 1$ (that is, λ is *unramified*) and U is suitably chosen, the above inner product makes $B(U)$ into an even, unimodular lattice of signature (3,19).

3. Let $F : B \to B$ be multiplication by y. Then F is an isometry of the lattice $B(U)$.

   By the synthesis construction, Theorem 3.34, there is an automorphism $f : X \to X$ of a K3 surface X, marked by $B(U)$, such that the diagram:

$$\begin{array}{ccc} B(U) & \xrightarrow{F} & B(U) \\ \downarrow & & \downarrow \\ H^2(X) & \xrightarrow{f^*} & H^2(X) \end{array}$$

   commutes. Thus $\lambda(f) = \lambda$ and $\delta(f)$ is a particular conjugate of λ. The fixed-points of f are isolated because the only subvarieties of X are points.

4. Now suppose the trace of λ is -1. Thus f has a unique fixed-point $p \in X$, since its Lefschetz number is given by

$$L(f) = \mathrm{Tr}\, f^*|(H^0 \oplus H^2 \oplus H^4) = 1 - 1 + 1 = 1.$$

From the Atiyah-Bott fixed-point formula we also obtain:

$$\begin{aligned}\frac{1}{\det(I-Df_p)} &= \frac{1}{1-\operatorname{Tr} Df_p+\delta} = \sum_0^{s=2}(-1)^s \operatorname{Tr} f^*|H^{0,s}(X)\\ &= 1+\overline{\delta}.\end{aligned}$$

Using the fact that $|\delta|=1$, we then get:

$$\operatorname{Tr} Df_p = \frac{1+\delta+\delta^2}{1+\delta}.$$

We already know that $\det Df_p=\delta$. Thus the eigenvalues α, β of Df_p are determined by δ.

5. For suitable values of δ, Df_p is an irrational rotation. That is, its eigenvalues α,β lie on S^1 and are *multiplicatively independent*, meaning

$$\alpha^i=\beta^j \iff (i,j)=(0,0).$$

The eigenvalues lie on S^1 if $\tau=\delta+\delta^{-1}>1-2\sqrt{2}$, and they are multiplicatively independent if τ has a conjugate $\tau'<1-2\sqrt{2}$.

6. Assume now that the algebraic numbers α and β are multiplicatively independent. Then they are jointly Diophantine, by a result of Fel'dman. That is, there exist $C,M>0$ such that

$$|\alpha^i\beta^j-1|>C(|i|+|j|)^{-M}$$

for all $(i,j)\neq(0,0)$. The proof uses transcendence theory and the Gel'fond-Baker method.

7. By a result of Siegel and Sternberg, once the eigenvalues of Df_p are jointly Diophantine, f is locally linearizable. We conclude that f has a Siegel disk centered at p.

8. To complete the construction, we must exhibit unramified degree 22 Salem polynomials $S(x)$ of trace -1, and units $U(x)$, such that the root δ of $S(x)$ satisfies the bounds required in step 5. We note that Salem numbers with trace -1 are rather rare; there are only finitely many such numbers of degree 22, and there are no known Salem numbers of trace <-1. Explicit examples are found by a computer search.

Lattices and number rings. Here is a more complete explanation of the formula (3.1) for the inner product space $B(u)$.

Lattices occur naturally in number rings. For example, let K be a number field of degree d over $\mathbb{Q}$, and let $L = \mathcal{O}_K \cong \mathbb{Z}^d$ be its ring of integers. Then L becomes a lattice with the inner product

$$\langle x, y\rangle_L = \mathrm{Tr}_{\mathbb{Q}}^K(xy).$$

This lattice is *never* unimodular (unless $K = \mathbb{Q}$). Given a basis (x_i) for L, the quantity $D = \det \mathrm{Tr}(x_i x_j)$ is both the determinant of L and the *discriminant* of $K/\mathbb{Q}$.

Let $r(x) \in \mathbb{Z}[x]$ be a degree d irreducible monic polynomial with roots $(x_i)_1^d$ in $\mathbb{C}$. Let A be the integral domain $\mathbb{Z}[x]/r(x)$ and let k be its field of fractions. Define an inner product on A by

$$\langle f_1, f_2\rangle_A = \mathrm{Tr}_{\mathbb{Q}}^k\left(\frac{f_1(x)f_2(x)}{r'(x)}\right) = \sum_1^d \left(\frac{f_1(x_i)f_2(x_i)}{r'(x_i)}\right)$$

(where $r'(x) = dr/dx$).

As was known to Euler, this inner product takes values in $\mathbb{Z}$ and makes A into a unimodular lattice. To prove this, one can use the residue theorem to compute:

$$\begin{aligned}\langle 1, x^n\rangle_A &= \sum \mathrm{Res}(x^n\,dx/r(x), x_i) \\ &= -\mathrm{Res}(x^n\,dx/r(x), \infty) = \begin{cases} 0, & 0 \le n < \deg(r) - 1 \\ 1, & n = \deg(r) - 1; \end{cases}\end{aligned}$$

compare [Ser1, §III.6].

Invariant forms. Now suppose $x^2 - 4$ is not a square in k. Let $K = k(y)$ be the quadratic extension of k obtained by adjoining a root of the equation

$$y + \frac{1}{y} = x.$$

Let $s(y) \in \mathbb{Z}[y]$ be the degree $2d$ minimal polynomial for y over $\mathbb{Q}$. Regarding $K \cong \mathbb{Q}[y]/(s(y))$ as a space of polynomials in y, let $F : K \to K$ be the multiplication map

$$F(g(y)) = y \cdot g(y).$$

Then $s(y)$ is the characteristic polynomial for F as a linear endomorphism of $K/\mathbb{Q}$.

We will construct a lattice $B \subset K$ such that F is an *isometry* of B. For the underlying group, we take

$$B = \mathbb{Z}[y]/(s(y)) = A \oplus Ay \subset K.$$

Then $F(B) = B$.

The Galois group of K/k is generated by $\sigma(y) = 1/y$. Clearly $B^\sigma = B$, and the trace map

$$\mathrm{Tr}_k^K(g) = g + g^\sigma = g(y) + g(y^{-1})$$

sends B into A. We make B into a lattice by defining the inner product:

$$\langle g_1, g_2\rangle_B = \langle 1, \mathrm{Tr}_k^K(g_1 g_2^\sigma)\rangle_A = \mathrm{Tr}_{\mathbb{Q}}^K\left(\frac{g_1 g_2^\sigma}{r'(x)}\right).$$

Then $F : B \to B$ is an isometry, because

$$F(g_1)F(g_2)^\sigma = (yg_1(y))(yg_2(y))^\sigma = yg_1(y)y^{-1}g_2(y^{-1}) = g_1 g_2^\sigma.$$

Unimodularity. Our main concern is with automorphisms of unimodular lattices. Thus it is of interest to compute the discriminant of B.

Theorem 3.35 *The lattice B is even, with discriminant satisfying*

$$|\operatorname{disc}(B)| = |N_{\mathbb{Q}}^k(x^2 - 4)| = |r(2)r(-2)|.$$

Here $N_{\mathbb{Q}}^k : A \to \mathbb{Z}$ is the norm map, defined by $N_{\mathbb{Q}}^k(f) = \prod_1^d f(x_i)$.

Proof. The inner product on A makes A^2 into a unimodular lattice as well. Define $Q : A^2 \to A^2$ by

$$Q(a,b) = \begin{pmatrix} 2 & x \\ x & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Then for $a + by \in A \oplus Ay = B$ we have

$$(a + by)(a + by)^\sigma = a^2 + b^2 + abx,$$

and therefore

$$\langle a + by, a + by\rangle_B = 2\langle 1, a^2 + b^2 + abx\rangle_A = \langle Q(a,b), (a,b)\rangle_{A^2}.$$

Thus B is even, and we have

$$|\operatorname{disc}(B)| = |\det(Q)| = |N_{\mathbb{Q}}^k(4 - x^2)| = |N_{\mathbb{Q}}^k(2 - x)N_{\mathbb{Q}}^k(2 + x)| = |r(2)r(-2)|,$$

since $N_{\mathbb{Q}}^k(n - x) = r(n)$. ∎

Corollary 3.36 *The following conditions are equivalent: (a) B is an even, unimodular lattice; (b) $y+1$ and $y-1$ are units in B; (c) $x+2$ and $x-2$ are units in A; (d) $|r(\pm 2)| = 1$; (e) $|s(\pm 1)| = 1$.*

Proof. Use the fact that $N^K_{\mathbb{Q}}(y \pm 1) = N^k_{\mathbb{Q}}(x \pm 2)$. ∎

Twisting by a unit. Let $u \in A^\times$ be a *unit* in A. Then multiplication by u is a symmetric automorphism of A with determinant ± 1, so the lattice $A(u)$ with inner product

$$\langle f_1, f_2 \rangle_{A(u)} = \langle u f_1, f_2 \rangle_A$$

is still unimodular. Similarly, the lattice $B(u)$ with inner product

$$\langle g_1, g_2 \rangle_{B(u)} = \langle u g_1, g_2 \rangle_B$$

is still even, with $|\operatorname{disc}(B(u))| = |\operatorname{disc}(B)|$; and $F : B(u) \to B(u)$ is still an isometry, since $u^\sigma = u$.

Signature. In general the signature of $B(u)$ varies with the unit u. To calculate the signature, first observe that the lattice $B(u)$ determines a Hermitian inner product on $B(u) \otimes \mathbb{C}$. Using the fact that $F : B(u) \to B(u)$ is an isometry, we obtain an orthogonal, F-invariant decomposition

$$B(u) \otimes \mathbb{C} = \bigoplus_{r(\tau)=0} E(\tau),$$

where $E(\tau) = \operatorname{Ker}(F + F^{-1} - \tau I)$ is 2-dimensional, and the eigenvalues λ, λ^{-1} of $F|E(\tau)$ satisfy $\lambda + \lambda^{-1} = \tau$.

Theorem 3.37 *Let τ be a zero of $r(x)$. For $\tau \in \mathbb{R}$, the subspace $E(\tau) \subset B(u) \otimes \mathbb{C}$ has signature*

$$\begin{array}{ll} (2,0) & \textit{if } |\tau| < 2 \textit{ and } u(\tau) r'(\tau) > 0; \\ (0,2) & \textit{if } |\tau| < 2 \textit{ and } u(\tau) r'(\tau) < 0; \textit{ and} \\ (1,1) & \textit{otherwise.} \end{array}$$

For $\tau \notin \mathbb{R}$ the subspace $E(\tau) \oplus E(\overline{\tau})$ has signature $(2,2)$.

Proof. First suppose $\tau \in \mathbb{R}$. Then using the isomorphism $B = A + Ay \cong A^2$ as in Theorem 3.35, we find the Hermitian inner product on $E(\tau) \cong \mathbb{C}^2$ comes from the complexification of the quadratic form

$$q(a, b) = 2u(\tau)(a^2 + b^2 + ab\tau)/r'(\tau)$$

on $\mathbb{R}^2$. The signature of the form $(a^2+b^2+ab\tau)$ is $(1,1)$ if $|\tau|>2$ and $(2,0)$ if $|\tau|<2$. The signature of $q(a,b)$ is the same, unless $u(\tau)/r'(\tau)<0$, in which case it is reversed.

Now suppose $\tau \notin \mathbb{R}$, and let $S \subset E(\tau) \oplus E(\overline{\tau})$ be the span of the $(\lambda, \overline{\lambda})$ eigenvectors for F, where $\lambda + \lambda^{-1} = \tau$. Since $\tau \notin \mathbb{R}$, $\overline{\lambda}$ is distinct from both λ and λ^{-1}; therefore S is a 2-dimensional isotropic subspace, and thus $E(\tau) \oplus E(\overline{\tau})$ has signature $(2,2)$.

■

Corollary 3.38 *The lattice $B(u)$ has signature $(d,d)+(p,-p)+(-q,q)$, where p is the number of roots of $r(x)$ in $[-2,2]$ satisfying $u(\tau)r'(\tau)>0$, and q is the number satisfying $u(\tau)r'(\tau)<0$.*

Dynamics from Salem polynomials. Recall that a monic irreducible polynomial $S(x) \in \mathbb{Z}[x]$ is *unramified* if $|S(\pm 1)| = 1$. A more complete development of the construction above, appealing to class field theory, leads to the following results [GM]:

Theorem 3.39 *Let $F \in \mathrm{SO}_{p,q}(\mathbb{R})$ be an orthogonal transformation with irreducible, unramified characteristic polynomial $S(x) \in \mathbb{Z}[x]$. If $p \equiv q \bmod 8$, then there is an even unimodular lattice $L \subset \mathbb{R}^{p+q}$ preserved by F.*

Corollary 3.40 *Let $S(x)$ be an unramified Salem polynomial of degree 22, and let $\delta \in S^1$ be a root of $S(x)$. Then there exists:*

- *A complex analytic K3 surface X, and an automorphism $f : X \to X$, such that*
- *$S(x) = \det(xI - f^*|H^2(X))$ and*
- *f^* acts on $H^{2,0}(X)$ by multiplication by δ.*

Corollary 3.41 *There are no unramified Salem numbers of degree 22 and trace less than -2.*

From Salem numbers to automorphisms. Consolidating the preceding results, we can now give

– a general construction of K3 surface automorphisms from unramified Salem numbers, and

– a criterion for the resulting automorphism to have a Siegel disk.

Theorem 3.42 *Let $R(x), U(x) \in \mathbb{Z}[x]$ be a pair such that:*

- *$R(x)$ is an unramified degree 11 Salem trace polynomial;*
- *$U(x)$ represents a unit in $\mathbb{Z}[x]/(R(x))$; and*
- *there is a unique root τ of $R(x)$ in $[-2, 2]$ such that $U(\tau)R'(\tau) > 0$.*

Then there exists a K3 surface automorphism $f : X \to X$ such that

- *$\delta(f) + \delta(f)^{-1} = \tau$, and*
- *$S(x) = \det(xI - f^*|H^2(X))$,*

where $S(x)$ is the degree 22 Salem polynomial associated to $R(x)$.

Note that $S(x) = x^{11}R(x + x^{-1})$.

Proof. The trace-form construction yields a lattice automorphism $F : B(U) \to B(U)$ with characteristic polynomial $S(x)$. Since $R(x)$ is unramified, $B(U)$ is an even, unimodular lattice, with signature $(3, 19)$ by Corollary 3.38.

Using the Torelli theorem, surjectivity of the period mapping, and uniqueness of the even unimodular (3,19) lattice, Theorem 3.34 (Synthesis) yields a K3 surface automorphism $f : X \to X$ and a marking $\iota : H^2(X, \mathbb{Z}) \to B(U)$ such that $F = \iota \circ f^* \circ \iota^{-1}$. Thus $S(x)$ is also the characteristic polynomial of $f^*|H^2(X)$.

By Theorem 3.37, the eigenspaces of $F + F^{-1}$ are 2-dimensional, and

$$E(\tau) = \mathrm{Ker}(F + F^{-1} - \tau I) \subset B(U) \otimes \mathbb{C}$$

is the unique eigenspace of $F+F^{-1}$ with signature $(2, 0)$. Similarly, $H^{2,0}(X) \oplus H^{0,2}(X) \subset H^2(X)$ is the unique eigenspace of $f^* + (f^*)^{-1}$ with signature $(2, 0)$. Thus $\iota^{-1}(E(\tau)) = H^{2,0}(X) \oplus H^{0,2}(X)$ and therefore $\delta(f)+\delta(f)^{-1} = \tau$. ■

Traces. The *trace* of a monic polynomial $P(x) = x^d + a_1x^{d-1} + \cdots + a_d$ is $-a_1$, the sum of its roots. The traces of $R(x)$ and $S(x)$ agree and coincide with $\mathrm{Tr}(f^*|H^2(X))$. Continuing from the previous result, the Atiyah-Bott fixed-point theorem plus results from transcendence theory imply:

Theorem 3.43 *Suppose in addition that $R(x)$ and τ satisfy:*

- *The trace of $R(x)$ is -1,*

- *$\tau > 1 - 2\sqrt{2} = -1.8284271\ldots$, and*
- *$R(x)$ has a root $\tau' < 1 - 2\sqrt{2}$.*

Then f has a unique fixed-point $p \in X$, and p is the center of a Siegel disk.

Detailed example. We conclude with a specific example of a K3 surface automorphism with a Siegel disk. This example was constructed using an intensive computer search.

Consider the degree 22 Salem number $\lambda \approx 1.37289$, whose irreducible polynomial is:

$$\begin{aligned} S(x) &= 1 + x - x^3 - 2\,x^4 - 3\,x^5 - 3\,x^6 - 2\,x^7 + 2\,x^9 + 4\,x^{10} + 5\,x^{11} \\ &\quad +4\,x^{12} + 2\,x^{13} - 2\,x^{15} - 3\,x^{16} - 3\,x^{17} - 2\,x^{18} - x^{19} + x^{21} + x^{22}. \end{aligned}$$

The corresponding Salem trace polynomial, satisfied by $\lambda + \lambda^{-1}$, is

$$\begin{aligned} R(x) &= -1 - 8\,x + 24\,x^2 + 42\,x^3 - 54\,x^4 - 66\,x^5 + 40\,x^6 + 42\,x^7 - 11\,x^8 \\ &\quad -11\,x^9 + x^{10} + x^{11}. \end{aligned}$$

Note that both polynomials have trace -1 and are unramified — for example, $R(\pm 2) = -1$. The graph of $R(x)$, displaying its 11 real roots, is shown in Figure 5. All roots except $\lambda + \lambda^{-1} \approx 2.10128$ lie in $[-2, 2]$, as required.

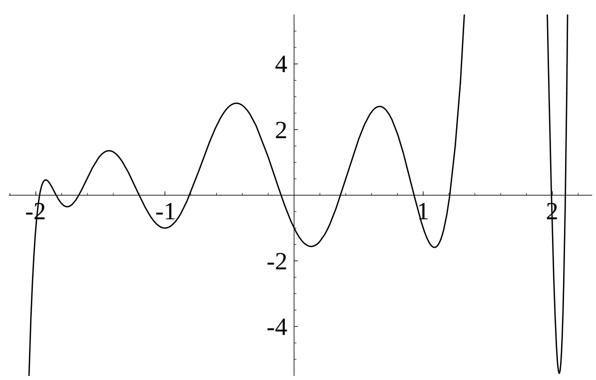

Figure 5. Graph of the degree 11 Salem trace polynomial $R(x)$.

A unit compatible with $R(x)$ is given by

$$U(x) = -2\,x + 6\,x^3 - 5\,x^5 + x^7.$$

To verify that $U(x) \in \mathbb{Z}[x]/(R(x))$ is a unit, one can check that $|\det U(M_R)| = 1$, where M_R is the 11×11 companion matrix of R.

There is a unique root $\tau \approx -1.66716$ of $R(x)$ in $[-2, 2]$ with $R'(\tau)U(\tau) > 0$.

Thus the lattice $B(U)$ has signature $(3, 19)$, and we obtain a corresponding K3 surface automorphism $f : X \to X$. By Theorem 3.43 above, the map f has a unique fixed-point p, and a Siegel disk centered at p.

Salem numbers of negative trace. We remark that Salem numbers with negative trace seem to be rather rare. Among the 630 degree 22 Salem polynomials with coefficients satisfying $|a_i| \leq 1$, 596 have trace 1, 33 have trace 0 and only one has trace -1, namely

$$\begin{aligned} S(x) &= 1 + x - x^3 - x^4 - x^5 - x^6 - x^7 - x^8 - x^9 - x^{10} - x^{11} \\ &\quad -x^{12} - x^{13} - x^{14} - x^{15} - x^{16} - x^{17} - x^{18} - x^{19} + x^{21} + x^{22}. \end{aligned}$$

On the other hand, Smyth has shown there exist infinitely many Salem numbers with trace -1 [Smy]. McKee and Smyth have recently (2001) constructed a Salem number of degree 1278 and trace -2; its value is $\lambda \approx 5 - 0.29514 \cdot 10^{-45}$. See [MRS] for related developments.

3.9 Appendix: Lattices

Definitions. A *lattice* $L \cong \mathbb{Z}^n$ is a finitely-generated free abelian group, equipped with a symmetric bilinear form or *inner product* $L \times L \to \mathbb{Z}$. We denote the inner product by $\langle x, y \rangle_L = x \cdot y$ and write $x^2 = x \cdot x$ for the associated quadratic form. Given a basis e_i for L, the *matrix* of the lattice is given by $a_{ij} = \langle e_i, e_j \rangle$. The *determinant* of L is $\det(L) = \det(a_{ij})$.

The inner product determines a natural map $L \to L^* = \mathrm{Hom}(L, \mathbb{Z})$. A lattice is *unimodular* if we have $L \cong L^*$. We have $|L^*/L| = |\det(L)|$, so unimodularity is equivalent to determinant ± 1. A lattice is *even* if its inner product assumes only even values; otherwise it is *odd*.

Cohomology. Let M^{2n} be a closed, oriented manifold, such that $L = H^n(M, \mathbb{Z})$ is torsion-free. Then by Poincaré duality, cup-product makes L into a unimodular lattice.

Odd unimodular lattices. The most basic odd lattice is $\mathbb{Z}^n$ with the usual inner product. More generally, we let $\mathbb{Z}^{p,q}$ denote the odd unimodular lattice of rank $n = p + q$ with $L = \mathbb{Z}^n$ and with quadratic form

$$x \cdot x = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

of signature (p, q).

Even unimodular lattices. The most basic even unimodular lattice is the 'hyperbolic plane' $H \cong \mathbb{Z}^2$ with quadratic form $(x, y)^2 = 2xy$. It has signature $(1, 1)$.

Here is procedure to construct an even unimodular lattice L', starting from the odd unimodular lattice $L = \mathbb{Z}^{p,q}$.

First, note that the map $L \to \mathbb{Z}/2$ given by $x \mapsto x^2$ is a homomorphism. Its kernel L_0 is a lattice of index two in L; thus $|\det(L_0)| = 4$, and $G = L_0^*/L_0$ is an abelian order 4.

It is not hard to check that G is $\mathbb{Z}/4$ when $n = p + q$ is odd and $(\mathbb{Z}/2)^2$ when n is even. When $G = \mathbb{Z}/4$, the image of L in G is the unique subgroup of order 2.

When $G = (\mathbb{Z}/2)^2$, there are 3 subgroups $L, L', L'' \subset L_0^*$ mapping to the three subgroups of index two in G. The subgroup L' can be taken to be generated by L_0 and the single vector $v = (1/2)^n = (1/2, \ldots, 1/2)$. We have $v \cdot v = (p - q)/4$. Thus L' is again a lattice when $p = q \bmod 4$. Moreover L' is unimodular, because both it and L contain L_0 with index two. Finally L' is *even* when $p = q \bmod 8$, and odd otherwise.

When one starts with the definite lattice $\mathbb{Z}^n$, resulting new lattices are usually denoted $D_n = L_0$ and $D_n^+ = L'$. The lattice D_n is defined for any n and consists of those $x \in \mathbb{Z}^n$ with $\sum x_i = 0 \bmod 2$. The lattice D_n^+ is defined when $n = 0 \bmod 4$, and is generated by D_n and the vector $(1/2)^n = (1/2, 1/2, \ldots, 1/2)$.

The lattice E_8. We have $D_4^+ \cong \mathbb{Z}^4$ as a lattice. However D_8^+ is a new lattice, usually denoted E_8. It is the *unique* even, unimodular lattice of rank 8.

The lattice D_{12}^+ is also interesting; it is an odd unimodular lattice whose shortest odd vector $x = (1/2)^{12}$ satisfies $x \cdot x = 3$, so it is distinct from $\mathbb{Z}^{12}$ and from $E_8 \oplus \mathbb{Z}^4$. Finally D_{16}^+ gives an even unimodular lattice of rank 16, distinct from E_8^2. (These lattices have the number of vectors of each given length, and hence the same θ-function. By considering the quotient tori $\mathbb{R}^{16}/L$, $L = D_{16}^+$ and $L = E_8$, Milnor constructed the first example of manifolds which are isospectral but not isometric.)

When starting with the indefinite lattice $\mathbb{Z}^{p,q}$, $p = q \bmod 8$, the resulting even unimodular lattice is denoted $\mathrm{II}_{p,q}$.

Classification of unimodular lattices.

Theorem 3.44 *There exists a definite even unimodular lattice of rank n iff $n = 0 \bmod 8$. There number of such lattices up to isomorphism is* $(1, 2, 24, \ldots)$ *in ranks* $(8, 16, 24, \ldots)$.

Theorem 3.45 *There is a unique odd indefinite unimodular lattice* $\mathrm{I}_{p,q}$ *with a given signature* (p, q).

Theorem 3.46 *There exists an even indefinite lattice* $\mathrm{II}_{p,q}$ *of signature* (p, q) *iff* $p = q \bmod 8$*, in which case it is unique.*

Example. There are two even, unimodular lattices of signature $(16, 0)$, but only one of signature $(17, 1)$. Thus we have

$$E_8 \oplus E_8 \oplus H \cong D_{16}^{+} \oplus H.$$

This isomorphism reflects the fact that the hyperbolic orbifold $\mathbb{H}^{17} / \mathrm{SO}(\mathrm{II}_{17,1})$ has two cusps.

Roots and reflections. A basic source of symmetries of a lattice L are its *roots*, i.e. those vectors $x \in L$ such that x^2 is ± 1 or ± 2. For any such vector, the map $\rho : L \to L$ given by

$$\rho(y) = y - \frac{2\langle x, y\rangle}{\langle x, x\rangle} x$$

is an isometry of order 2, given geometrically by reflection in the plane normal to x.

When L is definite, reflections in its roots generates a finite *Coxeter group.* The indefinite, unimodular lattices have infinitely many roots, which generate groups whose rich structure is only partly understood.

3.10 Exercises

1. Let C and D be a pair of ellipses in $\mathbb{RP}^2$, defined by $x^2 + y^2 = 1$ and $ax^2 + by^2 = 1$ respectively. Let μ be the invariant measure on C for Poncelet's iteration. Show the projection of μ to the real axis is proportional to:

$$\left.\frac{dx}{\sqrt{|1 - x^2| \cdot |ax^2 + b(1 - x^2) - 1|}}\right|_{[-1,1]}$$

(Hint: consider the meromorphic quadratic differential on C with poles at the 4 points $C \cap D$.)

2. Assume D is an ellipse enclosing C. Then we can regard C as a subset of the Klein model for the hyperbolic plane bounded by D. Show that the invariant measure on C is proportional to arclength on C.

3. Prove that a compact complex Lie subgroup of $\mathrm{PGL}_n(\mathbb{C})$ is finite.
4. Prove that a complex torus cannot arise as a smooth hypersurface in $\mathbb{P}^3$.
5. Let X be the surface resulting from the singular quadric $x^2+y^2=t^2$ in $\mathbb{P}^3$ by blowing up the double point at the origin. Show that X is a ruled surface isomorphic to the Hirzebruch surface $\Sigma_{-2}=\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-2))$.
6. Let $X=\mathrm{Jac}(C)$ where C is the surface of genus two given by $y^2=x^5-1$. Give a description of X as $\mathbb{C}^2/L$ for an explicit lattice $L\subset\mathbb{C}^2$. Find the sympletic form on L coming from the isomorphism $L\cong H_1(X,\mathbb{Z})$.
7. Let $f_A : X\to X$ be the positive entropy automorphism of a product of elliptic curves, $X=E\times E$, determined by a hyperbolic matrix $A\in\mathrm{SL}_2(\mathbb{Z})$. Show that f has a nonzero fixed-point in $\mathrm{Pic}(X)$. (Hint: consider the $\mathbb{Z}^3$ subgroup in $\mathrm{Pic}(X)$ spanned by $E\times 0$, $0\times E$ and the diagonal.)
8. Show that a positive entropy automorphism of a complex torus cannot have any invariant curve (satisfying $f(C)=C$).
9. Give an example of an elliptic curve such that $\mathrm{Pic}(E\times E)$ is isomorphic to $\mathbb{Z}^4$.
10. Let E be the elliptic curve $y^2=x^3-1$. Define a map $\phi : E\times E\to X\subset(\mathbb{P}^1)^3$ by $\phi(p,q)=(x(p),x(q),x(r))$, where r is the third point of intersection of the line $\overline{pq}$ with E.

    Show that X is a $(2,2,2)$-surface and find its equation. How is X related to the Kummer surface for $E\times E$?
11. Describe the Kähler cone $C_X\subset H^{1,1}(\mathbb{R})$ for a generic (non-algebraic) Kummer surface.
12. Prove that any fixed-point free automorphism f of a K3 surface acts on $H^{2,0}(X)$ by -1. Conclude that f^2 has a fixed-point.
13. Given an example of a homeomorphism $f : K\to K$ of a compact Hausdorff space consisting of more than one point, such that f has a unique fixed-point p and $f^n(x)\to p$ for all $x\in K$.
14. Give an example as above where $K\subset V$ is a compact convex subset of a topological vector space V, and where f comes from a linear

automorphism of V. (Hint: consider the space of measures on K from the previous question.)

15. Show that the moduli space of marked complex tori M of dimension two can be described as the homogeneous space $M = \mathrm{GL}_4(\mathbb{R})/\mathrm{GL}_2(\mathbb{C})$, and that the action of $\Gamma = \mathrm{GL}_4(\mathbb{Z})$ on M corresponds to a change of marking. Is $\Gamma\backslash M$ a reasonable space?

16. Show that the period domain for K3 surfaces can be described as the homogeneous space

$$\Omega = \mathrm{O}(3,19)/(\mathrm{SO}(2) \times \mathrm{O}(1,19)).$$

Describe the discrete group Γ that acts on Ω by change of marking.

17. What does the Atiyah–Bott fixed-point formula say for a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$? Prove this result, using the residue theorem.

18. Reflections through the roots of the lattice $\mathrm{I}_{1,n}$ generate discrete groups acting on $\mathbb{H}^n$, yielding hyperbolic orbifolds for each n. Identify these orbifolds for $n = 2, 3$.

19. Show that D_4^+ is isometric to $\mathbb{Z}^4$.

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