Amenability, Poincaré series and quasiconformal maps

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Summary. Any covering $Y \to X$ of a hyperbolic Riemann surface X of finite area determines an inclusion of Teichmüller spaces $\mathrm{Teich}(X) \hookrightarrow \mathrm{Teich}(Y)$. We show this map is an isometry for the Teichmüller metric if the covering is *amenable*, and contracting otherwise. In particular, we establish $\|\Theta\| < 1$ for classical Poincaré series (Kra's 'Theta conjecture').

The appendix develops the theory of geometric limits of quadratic differentials, used in this paper and a sequel.

1. Introduction: The Θ operator

Definitions. Let X be a Riemann surface. X is *hyperbolic* if it is covered by the unit disk Δ, and X is of *finite type* if it is obtained from a compact surface by deleting a finite number of points.

Let $Q(X)$ denote the space of holomorphic quadratic differentials $\phi(z) \, dz^2$ on X, such that

$$\|\phi\| = \int_X |\phi(z)| \, |dz|^2 < \infty.$$

With the above norm, $Q(X)$ is a Banach space. Let $B_X = \{\phi : \|\phi\| < 1\}$ denote the open unit ball in $Q(X)$.

Now suppose $f : Y \to X$ is a covering map. Then there is a natural operator

$$\Theta : Q(Y) \to Q(X)$$

(pull-back over branches of f^{-1} and sum). This operator is similar to push-forward of measures, except that variations in the phase of ϕ over different sheets may cause cancellation. Thus $\|\Theta\| \leq 1$.

In Teichmüller theory, $Q(X)$ is the contangent space to the Teichmüller space of X, the norm introduced above is the infinitesimal form of the

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Teichmüller cometric, and Θ is the coderivative of the map $\mathrm{Teich}(X) \rightarrow \mathrm{Teich}(Y)$ induced by the covering $Y \rightarrow X$ (see [Gar]).

Θ was introduced by Poincaré in the case of the universal covering of a hyperbolic Riemann surface of finite type [Poin].

We will show that $\|\Theta\| < 1$ for these classical Poincaré series; this is Kra's 'Theta conjecture'.

More generally, say a covering $Y \rightarrow X$ is *amenable* if there are large balls with small boundary in a graphic caricature of the covering (see §§ 2 and 3.) If X is of finite type, its universal cover is nonamenable.

Theorem 1.1. *Let $Y \rightarrow X$ be a covering of a hyperbolic Riemann surface X. Either:*
1. *The covering is amenable, and $\Theta(B_Y) = B_X$, or*
2. *The covering is nonamenable, and $\overline{\Theta(B_Y)}$ is contained in the interior of B_X.*

Corollary 1.2. *Suppose in addition X is of finite type. Either*
1. *The covering is amenable, $\|\Theta\| = 1$ and the induced map* $\mathrm{Teich}(X) \rightarrow \mathrm{Teich}(Y)$ *is a global isometry for the Teichmüller metric, or*
2. *The covering is nonamenable, $\|\Theta\| < 1$ and $\mathrm{Teich}(X) \rightarrow \mathrm{Teich}(Y)$ is contracting.*

The corollary is immediate from finite dimensionality of $Q(X)$, and the necessity and sufficiency of Hamilton's condition for an extremal quasiconformal map (see [Str 2]).

Contraction of the inclusion of $\mathrm{Teich}(X)$ into universal Teichmüller space can be restated as follows:

Corollary 1.3. *Let $f: X_1 \rightarrow X_2$ be a Teichmüller mapping between hyperbolic Riemann surfaces of finite type. Then the map $\tilde{f}: \Delta \rightarrow \Delta$ obtained by lifting f to the universal covers of X_1 and X_2 is not extremal among quasiconformal maps with the same boundary values (unless f is conformal).*

Dependence on moduli

It is useful to understand how the amount of contraction of the inclusion

$$\mathrm{Teich}(X) \hookrightarrow \mathrm{Teich}(Y)$$

depends on the geometry of X. Using an Euler characteristic argument we obtain:

Theorem 1.4 (Geometric Bound). *Let X be a hyperbolic Riemann surface of finite type, $Y \rightarrow X$ an infinite-sheeted covering space with finitely generated fundamental group. Then $\|\Theta_{Y/X}\| < c(n, L) < 1$, where $c(n, L)$ is a continous function of $n =$ the number of generators of $\pi_1(Y)$ and $L =$ the length of the shortest geodesic on X.*

Thus the amount of contraction is controlled by the location of X in *moduli* space. (However there is no global bound – see § 10 below.)

Application to 3-manifolds

A key tool in Thurston's construction of hyperbolic structures on 3-manifolds is the *skinning map*, an iteration on Teichmüller space whose fixed point yields a geometric structure [Th 1].

The skinning map factors through canonical maps between Teichmüller spaces whose contraction is controlled by a refinement of the geometric bound above (Theorem 11.1). In a sequel [Mc], this refinement is coupled with the theory of geometric limits of quadratic differentials (developed in the Appendix below) to give a new, analytic proof of the existence of a fixed point.

Outline of the paper

We begin with a sketch of the proof of Theorem 1.1 for a nonamenable covering.

Suppose $\Theta(\phi) = \psi$. To prevent loss of mass, the phase of ϕ must nearly agree with that of the pull-back of ψ to Y, at least over a region Y_0 which contains most of the mass of $|\phi|$. But agreement of phase implies the mass distribution of ϕ mimics that of ψ, which in the large is determined by the combinatorics of the covering $Y \to X$. For a nonamenable covering, most of the mass of Y_0 will be near its boundary, where the pairing is inefficient by a definite amount; this forces $\|\Theta\| < 1$.

§§ 2 and 3 develop the combinatorial theory of amenable and nonamenable graphs and coverings, and establish the global features of the argument above. Explicit 'rates of nonamenability' are estimated in § 4; these estimates provide many examples of nonamenable coverings and also play a role in the proof of Theorem 1.4.

§ 5 is a study of the shape of the unit ball in $Q(\Delta)$, which justifies the local part of the argument: quadratic differentials with nearly synchronized foliations have similar mass distributions.

In § 6, we formulate a 'fiberwise' version of the statement $\|\Theta\| < 1$; this is established in § 7, and we deduce the nonamenable half of Theorem 1.1 in § 8. § 9 treats the case of amenable coverings, completing the proof of Theorem 1.1.

In §§ 10 and 11 we analyze the proof further to obtain the dependence on moduli (Theorem 1.4 and its refinements). The discussion in § 10 draws on classical results in Teichmüller theory, while § 11 employs a geometric limit argument.

Finally § 12 sketches a generalization to Poincaré series for higher order differentials and quotients of bounded domains in $\mathbf{C}^n$.

The Appendix develops the theory of geometric limits of quadratic differentials, for use in § 11 and in a sequel on the skinning map [Mc].

Bibliographical remarks. Corollary 1.3 was checked for many examples by Strebel [Str 1]; see Strebel's recent survey [Str 4] for further background on extremal quasiconformal maps. The relation between $\|\Theta\|$ and lifts of Teichmüller mappings is mentioned in [Kra 2]. Arithmetical aspects of $\|\Theta\|$ were explored by

Parson and Sheingorn [PS 1], [PS 2]. Interactions between function theory, covering spaces and amenability also appear in papers of Brooks [Br] and Lyons and Sullivan [LS].

We have recently learned that Ohtake proved $\|\Theta\| = 1$ for abelian coverings [Oh].

2. Nonamenable graphs and slowly varying functions

In this section we will isolate some combinatorial properties of nonamenable groups, and more generally nonamenable *graphs*. The added generality will allow us to deal with irregular covers.

Definitions. Let G be a graph. For any set V of vertices of G, define $b(V)$, the *border* of V, to be the set of vertices at distance 1 from V. (A vertex at distance 1 is connected to V by an edge but does not itself lie in V).

Define the *expansion* γ of G as the infimum of $|b(V)|/|V|$ as V ranges over all finite sets of vertices. If the expansion is 0, G is *amenable*. Otherwise G is *nonamenable*, and the size of the border of any set of vertices is a definite fraction of the size of the set itself.

For example: an infinite tree in which every vertex has degree d is amenable for $d = 2$, but nonamenable, with expansion $\gamma = (d - 2)$, when d is 3 or more.

Now let $f: V \rightarrow \mathbf{R}$ be a positive integrable function on a subset V of the vertices of G (i.e. $\sum_{x \in V} f(x) < \infty$.) For x in the border $b(V)$, define

$$F(x) = \sup \{f(y): y \text{ is adjacent to } x.\}$$

Theorem 2.1. *Suppose that G is nonamenable with expansion γ, and f is slowly varying in the sense that, for some λ, $0 < \lambda < 1$,*

$$f(x) \geqq \lambda f(y)$$

whenever x and y are adjacent vertices of V. Then

$$\sum_{x \in b(V)} F(x) \geqq (\lambda \gamma + \lambda - 1) \sum_{x \in V} f(x).$$

Remarks.

1. Of course $f(x) \leqq \lambda^{-1} f(y)$ by symmetry.

2. If f varies sufficiently slowly, a definite fraction of its mass is picked up by F. This mass will give a definite contribution to the contraction of Θ in § 7.

3. Setting $f \equiv 1$ on V and $\lambda = 1$ gives $|b(V)| \geqq \gamma|V|$ as a special case.

Proof. For every integer n, let V_n denote the subset of V on which $f(x) \geqq \lambda^n$. Then V_n is an increasing sequence of finite sets and V_n is empty for n sufficiently negative.

Let $B_n = b(V_n) \cap b(V)$, and let v_n and b_n denote $|V_n|$ and $|B_n|$ respectively. By the slow variation of f, $b(V_n) \subset B_n \sqcup (V_{n+1} - V_n)$, so by nonamenability,

$$b_n + \varDelta v_n \geqq \gamma v_n,$$

where $\varDelta v_n = v_{n+1} - v_n$ is the usual combinatorial derivative. Applying this inequality after integrating by parts (note that $\varDelta \lambda^n < 0$), we find

$$\sum \lambda^{n+1} \varDelta b_n = -\sum b_n \varDelta \lambda^n \geqq \sum \varDelta v_n \varDelta \lambda^n - \gamma v_n \varDelta \lambda^n,$$

where we sum over all integers n. Integrating the second term on the right by parts, this can be rewritten

$$\sum \lambda^{n+1} \varDelta b_n \geqq (\lambda - 1 + \lambda \gamma) \sum \lambda^n \varDelta v_n.$$

Now $f(x)$ is bounded above by λ^n on $V_{n+1} - V_n$, so the sum appearing on the right is an upper bound for $\sum f(x)$; similarly the left hand sum is a lower bound for $\sum F(x)$, establishing the theorem. $\square$

In geometric situations, we will deal exclusively with graphs of bounded degree (there is a bound on the number of edges attached to each vertex). A set of vertices I is *independent* if no two elements of I are joined by an edge.

Proposition 2.2. *Let G be a graph of degree at most d, $f: V \to \mathbf{R}$ a non-negative integrable function on the vertices of G. Then there exists a set of independent vertices I such that*

$$\sum_I f(x) \geqq \frac{1}{d+1} \sum_V f(x).$$

Proof. Sort the vertices of G so $f(x)$ is descending ($f(x_0) \geqq f(x_1) \geqq \ldots$), and construct I by a greedy algorithm: recursively add to I the next vertex in sequence which is not adjacent to those already chosen. Then every vertex y not in I can be assigned to an adjacent vertex x in I so $f(x) \geqq f(y)$. This assignment is at most d-to-1, and the proposition follows. $\square$

3. From coverings to graphs

In this section we define (in several equivalent ways) the notion of an amenable, but possibly irregular covering. ([Gre] is a concise reference for the theory of amenable groups, of which this material is a generalization; see also [Pier].)

All manifolds considered are assumed smooth and connected. Let X be a manifold, $p: Y \to X$ a covering, $p_*: \pi_1(Y) \subset \pi_1(X)$ the induced inclusion of fundamental groups (for our considerations basepoints will be unimportant).

Definitions. The covering $Y \to X$ is *amenable* if there is a linear functional

$$m: L^\infty(\pi_1(X)/\pi_1(Y)) \to \mathbf{R},$$

invariant under the left-action of $\pi_1(X)$, such that $\inf(f) \leq m(f) \leq \sup(f)$ for all f.

Such a functional m is an invariant mean for the group action.

A *group* G is amenable if $L^\infty(G)$ admits a G-invariant mean. Thus, amenability of a regular covering is equivalent to amenability of its group of deck transformations.

Examples.

1. Solvable groups are amenable, as are subgroups and quotient groups of amenable groups.

2. The free group on 2 or more generators is nonamenable.

3. Any nonelementary Fuchsian group contains a free group on 2 generators, and hence is nonamenable.

4. In particular, the fundamental group of hyperbolic Riemann surface of finite type is nonamenable.

5. Subexponential growth implies amenability, but there are amenable groups with exponential growth – such as the solvable group $\langle x, y : x y = y^2 x \rangle$.

Using graphs, one can give a more combinatorial characterization of an amenable covering.

Assume for the moment that $\pi_1(X)$ has a finite generating set $\mathcal{G}$. From this data, construct the *coset graph* as follows: take the cosets $\pi_1(X)/\pi_1(Y)$ as vertices, and connect $x\pi_1(Y)$ and $y\pi_1(Y)$ by an edge if there exists a g in $\mathcal{G}$ such that $g x \pi_1(Y) = y \pi_1(Y)$. This graph is combinatorial model for the covering.

It will be useful to introduce finer models, as follows. A *net* $\mathcal{E}$ on X is a collection of simply connected open sets, whose union U is connected, such that $E \cap E'$ is either empty or connected for any two members of $\mathcal{E}$.

To $\mathcal{E}$ we associate its *Čech graph*, whose vertices are the elements of $\mathcal{E}$ and whose edges join E and E' iff $E \cap E' \neq \emptyset$. The Čech graph is the same as the 1-skeleton of the nerve of $\mathcal{E}$.

There is an induced net on Y, which we denote by $p^*(\mathcal{E})$, whose elements are the *components* of $p^{-1}(E)$ for each E in $\mathcal{E}$.

Our second combinatorial model for the covering $p: Y \to X$ is the Čech graph of $p^* \mathcal{E}$.

Proposition 3.1. *A covering $p: Y \to X$ is amenable iff one of the following equivalent conditions is satisfied:*

1. *The coset graph of $K/(K \cap \pi_1(Y))$ is amenable for every finitely generated subgroup K of $\pi_1(X)$.*

2. *The pullback of the covering to X_0 is amenable for every submanifold of X_0 of X with finitely generated fundamental group.*

3. *The Čech graph of $p^* \mathcal{E}$ is amenable for every finite net $\mathcal{E}$ on X.*

Equivalence of the three conditions is easy to check. That the first condition is equivalent to the existence of an invariant mean on the coset space is essentially Rosenblatt's generalization of Følner's condition [Ros].

As we will never make use of the invariant mean, the reader may also take these combinatorial conditions as the definition of amenability.

4. Explicit expansion constants

Let $Y \to X$ be a covering of surfaces, $H \hookrightarrow G$ the corresponding inclusion of fundamental groups. Assume both H and G are finitely generated. We will give an estimate for expansion constant γ of the coset graph G/H which depends only on Euler characteristics.

This estimate provides many examples of *irregular* nonamenable coverings, and will also be used to obtain bounds on $\|\Theta\|$ which depend only on the location of X in moduli space (§ 10).

Of course the expansion constant depends on a choice of generating set $\mathscr{G}$ for G. Any surface group may be presented as:

$$G = \langle a_1, \ldots, a_n \rangle \text{ or}$$
$$G = \langle a_1, b_1, \ldots, a_g, b_g : [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle$$

and we will use these generators for $\mathscr{G}$.

The Euler characteristics of G and H are the same as the topological Euler characteristics of X and Y. Assume G is nontrivial; then $\chi(G) \leq 0$.

Lemma 4.1. *For any finite set of vertices* $V \neq \emptyset$ *in the coset graph* G/H,

$$|b(V)| \geq 2(\chi(H) - |V| \chi(G)).$$

Proof. The presentation given for G corresponds to a cell complex C which is either a bouquet of circles or a compact surface. There is a covering $\tilde{C}$ corresponding to H, whose 1-skeleton is the coset graph.

Let $V \subset \tilde{C}$ be a finite set of vertices, and let D be the largest subcomplex of $\tilde{C}$ whose 0-skeleton is $V \cup b(V)$. For the purposes of the estimate, we may assume no component of $\tilde{C} - D$ is a 2-disk, since otherwise we may include its vertices in V and increase $|V|$ while decreasing $b(V)$. With this assumption, $\chi(D) \geq \chi(\tilde{C}) = \chi(H)$.

Let C_i, D_i denote the number of i-cells in C and D. Noting that the degree of the coset graph (the number of edges incident to each vertex) is $2C_1$, we find:

$$D_0 = |V| + |b(V)| = C_0 |V| + |b(V)|$$
$$D_1 = \tfrac{1}{2} \sum_{D_0} \deg(v) \geq \tfrac{1}{2}(2C_1 |V| + |b(V)|)$$
$$D_2 \leq |V| C_2$$

from which the theorem follows. □

Corollary 4.2 (Uniform expansion). *Assume* $[G:H] = \infty$. *Then the expansion of the coset graph satisfies*

$$\gamma \geq \begin{cases} 2|\chi(G)| & \text{if } \chi(H) \geq 0, \\ |\chi(G)/\chi(H)| & \text{otherwise.} \end{cases}$$

In particular, if $\chi(G) < 0$ *the coset graph is nonamenable with expansion bounded below independent of the embedding of* H *into* G.

Proof. Combine the preceding Theorem with the estimate $b(V) \geq 1$, which holds so long as the coset graph is infinite. □

Example. Let X be a hyperbolic Riemann surface of finite type, and let $S \subset X$ be a proper incompressible subsurface. (*Incompressible* means $\pi_1(S)$ injects into $\pi_1(X)$; *proper* means the image is a proper subgroup.) Corresponding to the subgroup $\pi_1(S)$ there is an infinite-sheeted covering $Y \to X$, which is nonamenable
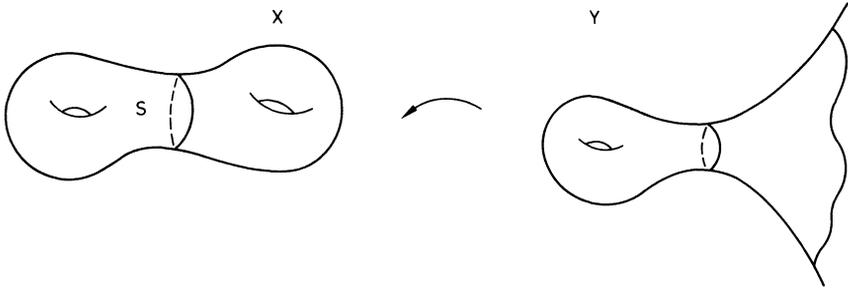

Fig. 1. The covering space of a subsurface

by the result above. Furthermore the expansion is bounded below independent of the choice of S, since $\chi(X) < \chi(S) \leqq 0$.

Coverings induced by subsurfaces are said to be *geometric*. Any covering $Y \to X$ with $\pi_1(Y)$ finitely generated is virtually geometric, i.e. it becomes geometric when pulled back to a finite covering of X [Scott].

5. Flex of the unit sphere in $Q(\Delta)$

This section lays the analytic groundwork for the proof of Theorem 1.1.

Let $Q(X)$ be the Banach space of quadratic differentials on a Riemann surface X, introduced in the § 1. There is a natural pairing between $Q(X)$ and $M(X)$, the space of measurable Beltrami differentials $\mu(z) \, d\bar{z}/dz$ with

$$\|\mu\| = \sup_X |\mu(z)| < \infty,$$

namely

$$\langle \phi, \mu \rangle = \operatorname{Re} \int_X \phi(z) \, \mu(z) \, |dz|^2.$$

In fact the dual of $Q(X)$ is the quotient of $M(X)$ by the subspace annihilated by Q.

Given any ϕ of norm 1, $\mu = \bar{\phi}/|\phi|$ is the unique element of norm 1 in $M(X)$ such that $\langle \phi, \mu \rangle = 1$. Geometrically, this means the unit sphere in $Q(X)$ has no sharp corners – at each point there is a unique supporting hyperplane.

On the unit disk Δ, consider the quadratic differential $\psi(z) = dz^2/\pi$. Then ψ is a point on the unit sphere in $Q(\Delta)$.

Theorem 5.1. *Let ϕ be an element of the unit sphere in $Q(\Delta)$, such that*

$$\langle \phi, \bar{\psi}/|\psi| \rangle = 1 - \varepsilon.$$

Then $\|\psi - \phi\| < O(\varepsilon^{\alpha})$ for some fixed exponent α.

Remark. The pairing between ϕ and $\bar{\psi}/|\psi|$ gives the average, with respect to $|\phi|$-measure, of the cosine of twice the angle between the foliations of ϕ and ψ. (The foliation of ψ is just the foliation of the disk by horizontal lines.) The theorem says that if the foliations are nearly synchronized, then ϕ is close to ψ.

Proof. Writing $\|\cdot\|$ and $\langle , \rangle$ in coordinates, our task is to show that

$$\int_{\Delta} |\phi(z)| = 1$$

and

$$\int_{\Delta} \operatorname{Re} \phi(z) = 1 - \varepsilon$$

together imply

$$\int_{\Delta} |1 - \phi(z)| = O(\varepsilon^{\alpha}),$$

all integrals being with respect to normalized area measure on the disk.

In any measure space, the first two equations imply

$$\int_{\Delta} |\operatorname{Im} \phi(z)| = O(\varepsilon^{1/2}).$$

Since $\operatorname{Im} \phi(z)$ is harmonic, it is controlled by its average over a ball and we obtain

$$|\operatorname{Im} \phi(z)| = O(\varepsilon^{1/2}/d(z)^2),$$

where $d(z) = d(z, \partial \Delta)$ denotes Euclidean distance from z to the boundary of the disk. Thus ϕ maps $B(0, 1-d)$ into the strip $\{z : |\operatorname{Im} z| < O(\varepsilon^{1/2}/d^2)\}$.

Similarly $\operatorname{Re} \phi$ is harmonic, so its average over the disk gives the value at the origin: $\operatorname{Re} \phi(0) = 1 - \varepsilon$. By the Schwarz lemma, most of $B(0, 1-d)$ maps near $\phi(0)$, since the image of $B(0, 1-d/2)$ is confined to a narrow strip. Making this precise, one obtains

$$|1 - \phi(z)| = O(\varepsilon + \varepsilon^{1/2} \log(1/d(z))/d(z)^2).$$

Finally we control $|\phi|$ near the boundary of the disk: since $|\phi|$ is subharmonic,

$$1 - \varepsilon \leqq |\phi(0)| \leqq \frac{1}{(1-d)^2} \int_{B(0, 1-d)} |\phi|$$

and since the total integral of $|\phi|$ is 1, this implies

$$\int_{\{1-d < |z| < 1\}} |\phi| = O(d + \varepsilon).$$

Combining this with the uniform bound for $|1-\phi(z)|$ on $B(0, 1-d)$, one obtains

$$\int_{\Delta} |1-\phi| = O(\varepsilon + d + \varepsilon^{1/2} \log(1/d)/d^2) = O(\varepsilon^{1/6} \log(1/\varepsilon))$$

if we set $d = \varepsilon^{1/6}$. Thus the theorem is true for any exponent $\alpha < 1/6$. □

Remarks.

1. In terms of the geometry of Banach spaces, the theorem states that a point of the unit sphere in $Q(X)$ which is close to the supporting hyperplane for ψ is close to ψ itself; intuitively, the sphere is 'flexed' at ψ. If $Q(X)$ were a Hilpert space with its usual norm, the theorem would be true with $\alpha = 1/2$.

2. Conversely, say a point p of the unit sphere S of a Banach space is *flat* if for any hyperplane H supporting S at p, there exist $\langle p_n \rangle$ of norm 1 whose distance from H tends to 0 but $\|p - p_n\| > \varepsilon > 0$. Then many points of the unit sphere $Q(X)$ are flat.

For example, let $\langle z_i \rangle$ be a sequence of points in the disk such that the hyperbolic balls $B(z_i, r_i)$ are disjoint and $r_i \to \infty$. Let $M_i : \Delta \to \Delta$ be a sequence of Möbius transformations such that $M_i(z_i) = 0$, and consider the differential

$$\alpha = c \sum_{i=1}^{\infty} 2^{-n} M_i^*(dz)^2,$$

where the constant c is chosen to give this expression norm in 1 in $Q(\Delta)$. It is not hard to check that $\langle M_i^*(dz^2/\pi), \bar{\alpha}/|\alpha| \rangle \to 1$ but these differentials do not approach α in the norm topology.

3. There is a close relationship between flexing of the unit sphere and reflexivity. For example, a theorem of Milman asserts that a uniformly convex Banach space is reflexive [Mil]; see also [Pet]. $Q(\Delta)$ is not reflexive, but its unit ball *is* uniformly convex at certain points, such as ψ.

6. The nonamenable case: selecting the net

Fix the following data:
● A Riemann surface X;
● A quadratic differential ψ in $Q(X)$ with $\|\psi\| = 1$; and
● A nonamenable covering $p: Y \to X$, with induced relative Poincaré operator

$$\Theta: Q(Y) \to Q(X).$$

For this data, we will establish the following 'fiberwise' version of the Theta Conjecture:

Theorem 6.1 (Θ contracts fibers). *Let $Q_\psi(Y)$ denote the closed subspace of $Q(Y)$ consisting of differentials ϕ such that $\Theta(\phi)$ is a multiple of ψ. Then $\|\Theta | Q_\psi(Y)\| < 1$.*

To set up the proof, we first construct a net $\mathscr{E}$ on X such that $p^*\mathscr{E}$ detects the nonamenability of the covering.

Recall that ψ gives rises to a flat conformal metric on X with quadratic form $|\psi(z)|\,|dz|^2$ (see e.g. [Str 3]). At the zeros of ψ the metric has cone-like singularities; these will not concern us.

If $\psi(x)\neq 0$, there is a neighborhood V of x and a conformal isomorphism $g: (V,0)\rightarrow(\Delta_r,0)$ such that $\psi|V=g^*(dz^2)$ (here Δ_r denotes the disk of radius r centered at the origin in $\mathbf{C}$). The map g is an isometry from the ψ-metric to the Euclidean metric, and V is an *embedded* r-ball in the ψ-metric.

Let $r(x)$ denote the *injectivity radius* of ψ at x, i.e. the radius of the largest embedded r-ball centered at x.

Now choose a finite net $\mathscr{E}$ on X such that, for some fixed $r>0$,

(a) Each element of $\mathscr{E}$ is an embedded r-ball in the ψ-metric; and

(b) The Čech graph of $p^*\mathscr{E}$ is nonamenable.

This is easily done, for example as follows. By nonamenability, there is a finitely generated subgroup K of $\pi_1(X)$ whose coset graph is nonamenable. Choose a basepoint at which $\psi\neq 0$ and smooth loops generating this subgroup. The loops may also be chosen to avoid the zeros of ψ. By compactness, the injectivity radius of ψ on the loops is bounded below by some $s>0$. Let $\mathscr{E}$ be a finite covering of the union of the loops by embedded r-balls, where $r=s/2$. Any two balls which meet are contained in an embedded s-ball, so their intersection is connected, just as in Euclidean space; thus $\mathscr{E}$ is a net. Finally the subgroup K is carried by $\mathscr{E}$, so the Čech graph of $p^*\mathscr{E}$ is nonamenable.
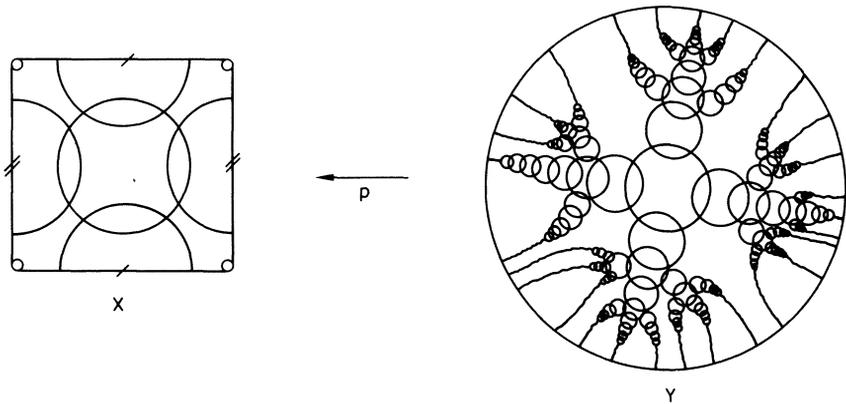

Fig. 2. A net for the punctured torus

For an example in the case of the universal covering of a punctured torus, see Fig. 2. Our bound on $\|\Theta\|$ will depend on the following quantities:

1. The expansion constant of the Čech graph of the induced net $\mathscr{F}=p^*\mathscr{E}$;
2. The degree of this graph (which is at most $|\mathscr{E}|$);
3. The ψ-area of $\bigcup\mathscr{E}$; and
4. The *minimum overlap* between elements of $\mathscr{E}$. This is the minimum fraction of the ψ-area of an element E which is shared with an overlapping element E'.

7. Capturing $\|\Theta\|$ fiberwise

Let ϕ be an element of $Q_\psi(Y)$ such that $\|\phi\| = 1$. To establish Theorem 6.1, we will bound $\|\Theta(\phi)\|$ away from 1 by an amount which is independent of ϕ. In this section, any bounds written using $O(\cdot)$ will involve constants which are independent of ϕ.

$\Theta(\phi) = t\psi$ for some constant t. Replacing ϕ by $t\phi/|t|$ if necessary, we many assume $t \geq 0$.

Let $\mu = \bar{\psi}/|\psi|$ denote the unique element of norm 1 in $M(X)$ such that $\langle \psi, \mu \rangle = 1$. For F a measurable subset of Y, define the *mass* and *inefficiency* of F by:

$$m(F) = \int_F |\phi|,$$

$$i(F) = 1 - \frac{1}{m(F)} \operatorname{Re} \int_F \phi \, p^* \mu.$$

Since $m(Y) = 1$,

$$1 - i(Y) = \langle \phi, p^* \mu \rangle = \langle \Theta(\phi), \mu \rangle = \langle t\psi, \mu \rangle = t,$$

so $i(Y) = 1 - \|\Theta(\phi)\|$. Our goal is to bound the inefficiency of Y from below.

Mass and inefficiency are always non-negative, so

$$i(Y) = m(W) \, i(W) + m(Y - W) \, i(Y - W) \geq m(W) \, i(W),$$

and hence we need only find a set $W \subset Y$ which has definite mass and inefficiency.

Lemma 7.1. *For any F in $\mathscr{F}$, there is a constant c such that*

$$\int_F |c\phi - p^*\psi| \leq O(i(F)^\alpha)$$

for some fixed exponent α.

Proof. Choose a coordinate z so that F can be identified with the unit disk Δ. After an appropriate rotation, $p^*\psi = r^2 \, dz^2$ so $p^*\mu = d\bar{z}/dz$. By Theorem 5.1 on the flex of the sphere in $Q(\Delta)$,

$$\int_F \left| \frac{\phi}{m(F)} - \frac{dz^2}{\pi} \right| \leq O(i(F)^\alpha),$$

and multiplying through by πr^2 we obtain the Lemma. □

Corollary 7.2. *For all intersecting F and F' in $\mathscr{F}$, and for all i_0 sufficiently small (independent of ϕ),*

1. *If the inefficiency $i(F)$ is less than i_0, then $m(F') \geq O(m(F))$;*
2. *If $i(F)$ and $i(F')$ are both less than i_0, then $m(F') \geq \lambda m(F)$, where $\lambda = 1 - O(i_0^\alpha)$.*

In other words, efficient neighbors have nearly the same mass, and a definite amount of mass is inherited from an efficient neighbor.

Proof. Assume both inefficiencies are near 0. By the Lemma above, there are constants c and c' such that $c\phi$ and $c'\phi$ are L^1-close to $p^*\psi$ on F and F' respectively. Thus $m(F)$ and $m(F')$ are given approximately by $\pi r^2/c$ and $\pi r^2/c'$.

Now F and F' overlap a definite amount with respect to $p^*\psi$-area, since their overlap is bounded below by the minimum overlap in $\mathscr{E}$. It follows that the constants c and c' must be close, else there would be a definite deviation from $p^*\psi$ on the overlap. Making this precise yields the bound given.

If only $i(F)$ is near zero, it still follows that the ϕ-mass of the overlap is a definite fraction of $m(F)$. The overlapping mass also contributes to $m(F')$, so $m(F') \geqq O(m(F))$. $\square$

Lemma 7.3. *For any subset $\mathscr{F}'$ of $\mathscr{F}$,*

$$m\left(\bigcup \mathscr{F}'\right) \geqq \frac{1}{|\mathscr{E}|} \sum_{\mathscr{F}'} m(F)$$

and

$$i\left(\bigcup \mathscr{F}'\right) \geqq \frac{\inf\{i(F): F \in \mathscr{F}'\}}{|\mathscr{E}|}.$$

Proof. The degree of the Čech graph of $\mathscr{F}$ is at most $|\mathscr{E}|-1$, so Proposition 2.2 applies to give an independent subset $\mathscr{F}'' \subset \mathscr{F}'$ which carries at least $1/|\mathscr{E}|$ the sum of $m(\cdot)$ over $\mathscr{F}'$. Then $\bigcup \mathscr{F}''$ is a disjoint union (this is what independence means), so the mass is additive, establishing the first inequality.

By disjointness, $i(\bigcup \mathscr{F}'')$ is bounded below by the minimum inefficiency of members of $\mathscr{F}'$. Estimating $i(\bigcup \mathscr{F}')$ by the weighted average of the inefficiency of $\bigcup \mathscr{F}''$ and its complement gives the second inequality. $\square$

Proof of Theorem 6.1 (Θ contracts fibers). Let $\gamma > 0$ denote the expansion of the (nonamenable) Čech graph of $\mathscr{F}$. Choose $\lambda < 1$ such that $\lambda\gamma + \lambda - 1 \geqq \gamma/2$. Let

$$\mathscr{V} = \{F \in \mathscr{F}: i(F) < i_0\},$$

where $i_0 > 0$ is chosen sufficiently small that we may apply Corollary 7.2 to conclude:

(I) $m(F') \geqq O(m(F))$ whenever F' is adjacent to an element $F \in \mathscr{V}$; and

(II) $m(F') \geqq \lambda m(F)$ whenever F and F' are adjacent elements of $\mathscr{V}$.

Let

$$\mathscr{W} = \{F \in \mathscr{F}: i(F) \geqq i_0\}.$$

Then $(\mathscr{V}, \mathscr{W})$ forms a partition of $\mathscr{F}$. Let $W = \bigcup \mathscr{W}$, $V = \bigcup \mathscr{V}$.

By Lemma 7.3, $i(W) \geqq i_0/|\mathscr{E}|$, so

$$1 - \|\Theta(\phi)\| = i(Y) \geqq m(W) \, i(W) \geqq i_0 \, m(W)/|\mathscr{E}|$$

and hence to bound $\|\Theta(\phi)\|$ away from 1, it suffices to bound $m(W)$ from below.

Assume $\|\Theta(\phi)\| \geq 1/2$, since otherwise we are done. Then

$$m(W) + m(V) = \int_{p^{-1}(\bigcup \mathscr{E})} |\phi| \geq \|\Theta(\phi)\| \int_{\bigcup \mathscr{E}} |\psi| = O(1),$$

so to complete the proof it suffices to show the mass of W is not too much smaller than the mass of V.

For F in the border $b(\mathscr{V}) \subset \mathscr{W}$, define

$$M(F) = \sup\{m(F') : F' \in \mathscr{V} \text{ and } F' \text{ is adjacent to } F.\}$$

By (I), $m(F) \geq O(M(F))$, so by Lemma 7.3 the mass of W is estimated from below in terms of the sum of $M(F)$ over the border of $\mathscr{V}$. We will show this sum is comparable to the mass of V.

By (II), $m(\cdot)$ is a slowly varying function on the vertices $\mathscr{V}$ of the Čech graph; applying Theorem 2.1,

$$\sum_{F \in b(\mathscr{V})} M(F) \geq (\lambda\gamma + \lambda - 1) \sum_{F \in \mathscr{V}} m(F) \geq \frac{\gamma}{2} m(V),$$

establishing the theorem. $\square$

8. Completion of the nonamenable case

It is known that the operator Θ is always surjective; this fundamental property is called *completeness of Poincaré series*. By the open mapping theorem, the image of the unit ball in $Q(Y)$ contains a neighborhood of the origin in $Q(X)$. This can be made effective:

Theorem 8.1. *Let $Y \to X$ be a covering of a hyperbolic Riemann surface X. Then*

$$\Theta(B_Y) \supset \tfrac{1}{3} B_X,$$

i.e. the image of the unit ball in $Q(Y)$ contains the ball of radius $1/3$ in $Q(X)$.

This result is due to Ahlfors and Bers; for a proof, see [Kra 1, Theorem 3.3], or [Gar, § 4.3].

Proof of Theorem 1.1 (nonamenable case). Let $Y \to X$ be a nonamenable covering of a hyperbolic Riemann surface X. Suppose ψ is an element of norm 1 in $\overline{\Theta(B_Y)}$. Then there is a sequence ϕ_n in B_Y such that $\Theta(\phi_n) \to \psi$. By the Theorem above, there exist $\phi_n' \to 0$ with $\Theta(\phi_n') = \psi - \Theta(\phi_n)$. Then $\Theta(\phi_n + \phi_n') = \psi$ and $\|\phi_n + \phi_n'\| \to 1$, contradicting Theorem 6.1 (Θ contracts fibers).

Therefore the closure of $\Theta(B_Y)$ is contained in the interior of the unit ball in $Q(X)$. $\square$

9. The amenable case

Let $p: Y \to X$ be an amenable covering of a hyperbolic Riemann surface X.

Identifying the unit disk Δ with the universal cover of X, we may realize the fundamental groups of X and Y as Fuchsian groups $\Gamma_Y \subset \Gamma_X$ acting on the disk.

By amenability, there exist finite sets of cosets $\mathscr{C}_n \subset \Gamma_X / \Gamma_Y$ such that for any finite set $F \subset \Gamma_X$,

$$\frac{|\{C : F \cdot C \subset \mathscr{C}_n\}|}{|\mathscr{C}_n|} \to 1$$

as $n \to \infty$. Here $F \cdot \mathscr{C}$ denotes the set of cosets obtained by translating C on the left by elements of F.

It will be useful to consider the Θ operator on the space of *measurable* integrable quadratic differentials, which we denote by $L^1(\cdot)$, and which contains $Q(\cdot)$ as a closed subspace. Corresponding to the factorization of the universal covering $u: \Delta \to X$ through Y, we have a factorization of

$$\Theta_{\Delta/X}: L^1(\Delta) \to L^1(X)$$

as $\Theta_{\Delta/X} = \Theta_{Y/X} \circ \Theta_{\Delta/Y}$.

For each $\mathscr{C}_n$, define an averaging operator $A_n: L^1(\Delta) \to L^1(Y)$ as follows:

$$A_n(\xi) = \frac{1}{|\mathscr{C}_n|} \sum_{\{g\Gamma_Y\} \in \mathscr{C}_n} \Theta_{\Delta/Y}(g^* \xi).$$

Since $\Theta_{\Delta/Y}(h^* g^* \xi) = \Theta_{\Delta/Y}(g^* \xi)$ whenever h is in Γ_Y, this expression does not depend on what g we choose to represent a given coset.

The following properties are easily verified:

- $\|A_n\| \leq 1$.
- A_n carries $Q(\Delta)$ into $Q(Y)$.
- $\Theta_{Y/X}(A_n(\xi)) = \Theta_{\Delta/X}(\xi)$.

Theorem 9.1. *As $n \to \infty$,*

$$\|A_n(\xi)\| \to \|\Theta_{\Delta/X}(\xi)\|.$$

Proof. Let $\Omega \subset \Delta$ denote a measurable fundamental domain for the action of Γ_X on the disk. We may assume ξ is an integrable function whose support is contained in $F^*\Omega$, the inverse images of Ω under a finite set $F \subset \Gamma_X$, since such ξ are dense in $L^1(\Delta)$.

The function $\psi = \Theta_{\Delta/X}(\xi)$ is obtained by collecting together the finitely many tiles which support ξ, superimposing them and adding up the corresponding values of ξ.

The tiling of the disk by translates of Ω descends to a tiling of Y, such that the pre-image of a tile on Y is $C^*\Omega$ for some coset C of Γ_X / Γ_Y. If $F^{-1} \cdot C \subset \mathscr{C}_n$, call the corresponding tile of Y a *filled tile*.

On a filled tile,

$$A_n(\xi) = p^*(\psi)/|\mathscr{C}_n|,$$

since the averaging operation moves every tile of the support of ξ into $C^*\Omega$. The integral of $|p^*\psi|$ over a single tile of Y is the same as the integral of $|\psi|$ over X, so as $n \to \infty$,

$$\int_{Y'} |A_n(\xi)| = \frac{|\{C : F^{-1} \cdot C \subset \mathscr{C}_n\}|}{|\mathscr{C}_n|} \|\psi\| \to \|\psi\|,$$

where Y' denotes the union of the filled tiles of Y.

A coset C corresponds to an unfilled tile in the support of $A_n(\xi)$ exactly when $F^{-1} \cdot C$ meets, but is not contained in, $\mathscr{C}_n$. Thus the number of such tiles is at most

$$|F| \times |\{C : F^{-1} \cdot C \not\subset \mathscr{C}_n\}|$$

which is small compared to $|\mathscr{C}_n|$ as $n \to \infty$. Over an unfilled tile the integral of $|A_n(\xi)|$ is bounded by $\|\xi\|/|\mathscr{C}_n|$, so the contributions to $\|A_n(\xi)\|$ coming from the unfilled tiles of Y tend to 0, establishing the theorem. $\quad\square$

As a corollary, we may complete the:

Proof of Theorem 1.1 (amenable case). Let ψ be an element of norm <1 in $Q(X)$. By completeness of Poincaré series, there exists a ξ in $Q(\Delta)$ such that $\Theta_{\Delta/X}(\xi) = \psi$. Setting $\phi_n = A_n(\xi)$, we have $\Theta_{Y/X}(\phi_n) = \psi$ and by the theorem above, $\|\phi_n\|$ is also eventually less than 1. Therefore the image of the open unit ball B_Y contains the open unit ball B_X (and equality holds because $\|\Theta_{Y/X}\| \leqq 1$). $\quad\square$

10. Dependence on moduli

In this section we use a result of Reich and Strebel, and the explicit expansion constants obtained in §4, to study the behavior of $\|\Theta\|$ as a function of the conformal structures on X and Y.

Fix a covering of Riemann surfaces $Y_0 \to X_0$. For any $X \in \mathrm{Teich}(X_0)$, there is a corresponding covering $Y \to X$.

Theorem 10.1. $\|\Theta_{Y/X}\|$ *is uniformly continuous with respect to the Teichmüller metric on* $\mathrm{Teich}(X_0)$.

Assuming this for the moment, we specialize to the case where X is of finite type. Fix a finite generating set $\mathscr{G}$ for $\pi_1(X)$. As remarked in §6, the covering Y enters only combinatorially into the estimate for $\|\Theta\|$:

Theorem 10.2 (Combinatorial bound). *For any nonamenable covering* $Y \to X$,

$$\|\Theta\| < f(\gamma) < 1,$$

where γ is the expansion of the coset graph of $\pi_1(X)/\pi_1(Y)$ with respect to $\mathscr{G}$, and f is a function which depends only on X.

Proof. We simply analyze the dependence of the proof upon Y. Since $\pi_1(X)$ is finitely generated we may choose the net $\mathscr{E}$ so that $\bigcup \mathscr{E}$ carries the fundamental group of X. The expansion of $p^* \mathscr{E}$ is then bounded below in terms of the expansion of the coset graph, and so the corollary is true fiberwise. Using Theorem 8.1, one sees that a bound $\|\Theta \,|\, Q_\psi(Y)\| < 1 - c < 1$ provides a bound of $1 - c/2$ for all nearby ψ, and the corollary follows globally by compactness of $\overline{B_X}$. $\square$

Remark. A more careful analysis shows we may take $f(\gamma) = 1 - O(\gamma^\beta)$ for some exponent β.

   Now suppose $\pi_1(Y_0)$ is finitely generated, X_0 is a hyperbolic Riemann surface of genus g with n punctures, and $Y_0 \to X_0$ is an infinite-sheeted covering. Let $\mathscr{M}_{g,n}$ denote the moduli space of X_0, and define

$$N: \mathscr{M}_{g,n} \to [0, 1]$$

by

$$N(Z) = \sup\{\|\Theta_{Y/X}\| : X \in \text{Teich}(X), \text{ and } [X] = Z\},$$

where $[X]$ denotes the location of X in moduli space.

Theorem 10.3. *N is a uniformly continuous proper map from moduli space to the half-open interval $[0, 1)$. Moreover, on any compact subset of $\mathscr{M}_{g,n}$, N is bounded away from 1 by an amount which depends only on the number of generators of $\pi_1(Y)$.*

Remarks.

   1. Properness *means* $\|\Theta\| \to 1$ as Z tends to infinity in moduli space. Thus there is no uniform bound over all $\mathscr{M}_{g,n}$.

   2. Theorem 1.4 is an immediate corollary.

Proof. By Theorem 10.1, N is uniformly continuous in the Teichmüller metric on moduli space. The combinatorial bound of Theorem 10.2, combined with the lower bound for the expansion factor γ obtained in Corollary 4.2, bounds N away from 1 by an amount which depends only on the Euler characteristic of Y, not its position over X. Thus N maps moduli space into $[0, 1)$, and on any compact set there is a bound which depends only on the topological complexity of Y.

   To see N is proper, we must show $\|\Theta\| \to 1$ as $[X] \to \infty$ in moduli space. One knows in the end of moduli space the length L of some simple geodesic σ on X is near zero. We will show $N([X])$ is within $O(L)$ of 1.

   Let $W \to X$ be the covering space for a cyclic subgroup generated by σ. W may be identified with the annulus:

$$\{z : 1 < |z| < \exp(2\pi^2/L)\}.$$

Most of the mass of the quadratic differential $\phi = dz^2/z^2$ is concentrated in the thin part of W, which *injects* into the thin part of X. The remainder causes only a small amount of cancellation; one may estimate $\|\Theta_{W/X}\| > 1 - O(L)$. Since $\Delta \to W$ is an amenable covering, $\Theta(B_\Delta) = B_W$ by Theorem 1.1 and so

$$\|\Theta_{\Delta/X}\| = \|\Theta_{W/X}\| > 1 - O(L).$$

On the other hand $\Theta_{\Delta/X}$ factors through $\Theta_{Y/X}$, so the same lower bound holds for N. $\square$

Proof of Theorem 10.1. It does not appear that continuity of the norm is a formal consequence of its definition; to obtain continuity we appeal to a geometric interpretation.

Denote by p the induced map $\mathrm{Teich}(X_0) \to \mathrm{Teich}(Y_0)$. Letting Y and Y' denote $p(X)$ and $p(X')$ respectively, set

$$N(X, r) = \frac{\sup\{d(Y, Y'): d(X, X') < r\}}{r}.$$

$N(X, r)$ is the factor by which p compresses the Teichmüller radius of the ball of radius r at X.

It is known that $N(X, r) \to \|\Theta_{Y/X}\|$; in fact, the rate of convergence is *uniform* over all Teichmüller spaces.

More precisely, let $f: X \to X'$ be a quasiconformal map, μ the complex dilatation of its lift $\tilde{f}: Y \to Y'$, and ν the dilatation of an extremal map $g: Y \to Y'$ equivalent to $\tilde{f}$. By a result of Reich and Strebel ([RS, Theorem 9], cf. [Gar, § 6.4]):

$$\left| \frac{\|\nu\|_\infty}{1 - \|\nu\|_\infty^2} - \sup_{\phi \in B_{Y_0}} \mathrm{Re} \int \frac{\mu\phi}{1 - \|\mu\|_\infty^2} \right| \leqq \frac{\|\mu\|_\infty^2}{1 - \|\mu\|_\infty^2}.$$

This implies

$$|N(X, r) - \|\Theta_{Y/X}\|| < Cr$$

for a universal constant C.

Since p is distance non-increasing,

$$|N(X, r) - N(X', r)| \leqq 2s/r$$

where s is the Teichmüller distance from X to X'. Setting $r = \sqrt{s}$ and combining these inequalities, we find $\|\Theta_{Y/X}\|$ and $\|\Theta_{Y'/X'}\|$ differ by at most $C'\sqrt{s}$. $\square$

Remark. The proof gives uniform Hölder continuity of exponent 1/2.

11. Refinements

To get a more precise picture of the way in which the norm of Θ can tend to 1, in this section we study the fiberwise behavior.

Definitions. Let $p: Y_0 \to X_0$ denote an infinite-sheeted covering with $\pi_1(Y_0)$ finitely generated and X_0 of finite type.

Assume Y_0 is a geometric covering; i.e. there is a proper incompressible subsurface $S_0 \subset X_0$ such that Y_0 is the covering space for the subgroup $\pi_1(S_0)$.

Since surface groups are LERF [Scott], a general finitely generated covering becomes geometric after passing to a finite covering of X_0. Finite coverings induce isometries on Teichmüller space, so the study of contraction properties of $\mathrm{Teich}(X_0) \to \mathrm{Teich}(Y_0)$ can in principle be reduced to the case of geometric coverings. We restrict attention to the geometric case to get a clearly formulated result.

Fix any small $\varepsilon > 0$. The *thin part* X_{thin} of $X \in \text{Teich}(X_0)$ is the subset where the injectivity radius in the Poincaré metric is less than ε. For ε sufficiently small every component of X_{thin} has abelian (hence amenable) fundamental group.

For any $X \in \text{Teich}(X_0)$, there is subsurface S corresponding to S_0. The *liftable part* X_{lift} of X is the union of those components of X_{thin} and $X - X_{\text{thin}}$ which are isotopic into S. $X_{\text{thin}} \cup X_{\text{lift}}$ is the *amenable part* X_{am} of X.

By definition of a geometric cover, S lifts isomorphically to a unique subsurface of Y. Thus X_{lift} also has a unique isomorphic lift to Y. The *amenable part* Y_{am} of Y is the union of this lift and the total pre-image of X_{thin}.

Intuitively, Y_{am} is the part of Y which is converging to an amenable cover. Our next results asserts that the only way $\|\Theta\|$ can tend to 1 is by taking advantage of the part of Y which is becoming more amenable.

Theorem 11.1. *Let $\phi \in Q(Y)$, $\|\phi\| = 1$. If $\|\Theta_{Y/X}(\phi)\|$ is close to 1, then most of the mass of $|\phi|$ is in the amenable part of Y.*

More precisely, if unit norm ϕ_n on $Y_n \to X_n$ satisfy $\|\Theta_{Y_n/X_n}(\phi_n)\| \to 1$, then

$$\int\limits_{(Y_n)_{\text{am}}} |\phi_n| \to 1$$

as $n \to \infty$.

For each $X \in \text{Teich}(X_0)$ and nonzero $\psi \in Q(X)$, define

$$N(X, \psi) = \|\Theta_{Y/X}| Q_\psi(X)\|,$$

i.e. $N(X, \psi)$ is the fiberwise norm of Θ over ψ. The fiberwise norm only depends on the location of ψ in the compact projective space $\mathbf{P}Q(X)$.

Corollary 11.2. *If $N(X, \psi)$ is near 1, then most of the mass of $|\psi|$ is in the amenable part of X.*

Proof. The amenable part of Y lies over the amenable part of X. $\square$

Corollary 11.3. *If $N(X, \psi)$ is near 1 for the universal covering, then most of the mass of $|\psi|$ is in the thin part of X.*

Proof. In this case $X_{\text{am}} = X_{\text{thin}}$. $\square$

Proof of the Theorem 11.1. We will use the idea of geometric limits of quadratic differentials and their underlying Riemann surfaces, developed in the Appendix below.

Let $\psi_n = \Theta_{Y_n/X_n}(\phi_n)$ where $\|\phi_n\| = 1$, and assume $\|\psi_n\| \to 1$.

The portion of $|\phi_n|$ which lies over the thin part of X_n is already carried by the amenable part of Y_n, by definition. Any other portion lies over a component of the thick part of X_n.

Let Z_n be any component of the thick part of X_n, and fix $\delta > 0$. We will show that the limsup of the $|\phi_n|$-mass of $p^{-1}(Z_n) - (Y_n)_{\text{am}}$ is at most δ. Since δ is arbitrary and the number of thick components is bounded, this will complete the proof: the amenable part carries most of the mass of $|\phi_n|$ when n is large.

Since ϕ_n is pushing down more and more efficiently,

$$|\int_E |\psi_n| - \int_{p^{-1}(E)} |\phi_n|| \leq 1 - \|\psi_n\| \to 0$$

for any $E \subset X_n$. If

$$\int_{Z_n} |\psi_n| \to 0$$

then the $|\phi_n|$-mass of its entire pre-image tends to zero and there is nothing to prove.

So assume the $|\psi_n|$-mass of Z_n is bounded below by m. Choose a baseframe v_n over any point of Z_n. By compactness of the space $\mathbf{P}\mathscr{Q}_{g,n}$ of projective classes of quadratic differentials viewed at the scale of the injectivity radius (Theorem A 3.1 below), we may pass to a subsequence such that $(X_n, v_n, c_n \psi_n)$ converges geometrically to (X, v, ψ). Here c_n is a sequence of nonzero constants and $\psi \neq 0$. Since the $|\psi_n|$-mass of Z_n is between m and 1, we may assume $c_n = 1$.

The limiting Riemann surface X is hyperbolic and of finite type, since the baseframe was chosen in the thick part. Z_n converges geometrically to a component Z of the thick part of X.

Construct a finite net $\mathscr{E}$ on Z composed of embedded r-balls in the $|\psi|$-metric, as in § 6, such that $\mathscr{E}$ carries the fundamental group of Z and the $|\psi|$-mass of $\bigcup \mathscr{E}$ is within δ of that of Z. By taking r-balls in the $|\psi_n|$-metric about points converging geometrically to the centers of the balls in $\mathscr{E}$, we can construct nets $\mathscr{E}_n$ converging geometrically to $\mathscr{E}$ for all n sufficiently large. Then the geometric data concerning $\mathscr{E}_n$ (enumerated in § 6) on which the bound obtained in Theorem 6.1 (Θ contracts fibers) depends, converges to the data for $\mathscr{E}$.

Thus, if $\lim \|\psi_n\| = 1$, the combinatorial input to the fiberwise bound – the expansion γ_n of the Čech graph of $p^* \mathscr{E}_n$ – must tend to zero.

Z_n must belong to the liftable part of X for all large n; otherwise the Čech graph corresponds to a covering by a proper incompressible subsurface of Z_n, whose expansion is bounded below by Corollary 4.2 (Uniform expansion).

So assume that Z_n is isotopic into the subsurface of X_n defining the covering Y_n. Then the Čech graph of $p^* \mathscr{E}_n$ consists of countably many components, one of which is finite (and corresponds to the isomorphic lift of Z_n to $(Y_n)_{am}$), and the rest of which are infinite (and correspond to the universal cover of Z_n). The expansion γ is uniformly bounded below on the infinite components; thus the $|\phi_n|$-mass of $p^{-1}(\bigcup \mathscr{E}_n) - (Y_n)_{am}$ tends to zero. Since the $|\psi_n|$-mass of $Z_n - \bigcup \mathscr{E}_n$ is at most δ, the limsup of the $|\phi_n|$-mass of $p^{-1}(Z_n) - (Y_n)_{am}$ is also at most δ, as claimed.   □

12. Quotients of bounded domains

Poincaré series on the disk have a natural generalization in the setting of automorphic forms of weight $q \geq 2$ for the action of a discrete group Γ on a bounded domain $\Omega \subset \mathbf{C}^n$ (see e.g. [Ba]). The quotient space $X = \Omega/\Gamma$ has the structure of a normal complex analytic space, and automorphic forms of weight d correspond to holomorphic sections of the qth power of the canonical bundle of X; call such a section a q-canonical form.

Let $K(z, w) \, dz \, d\bar{w}$ denote the Bergman kernel on Ω, where $dz = dz_1 \ldots dz_n$ as usual. Then $|K(z, z)| \, |dz|^2$, the *Bergman density* is a natural volume form on Ω, invariant under any holomorphic automorphism. Using this volume element, one may define Banach spaces $H_q^p(\Omega)$ consisting of holomorphic q-canonical forms $\phi(z) \, dz^q$ such that

$$\| \phi \| = \int_\Omega |\phi(z)| \cdot |K(z, z)|^{1 - q/2} |dz|^2 < \infty.$$

One may define Banach spaces $H_q^p(X)$ in a similar fashion, since the Bergman density descends to quotients. These Banach spaces of automorphic forms have been considered by Earle and others [Earle].

If Y is a quotient intervening between X and Ω, there is a Poincaré series operator

$$\Theta_q \colon H_q^1(Y) \to H_q^1(X),$$

which satisfies $\| \Theta_q \| \leq 1$.

In general, the Poincaré series is *not* defined for elements of $H_q^p(Y)$ with $p > 1$, just as the sequence $a_n = 1/|n + 1|$ in $l^p(\mathbf{Z})$ (for $p > 1$) cannot be averaged to become $\mathbf{Z}$-invariant.

Theorem 1.1 may be partly generalized to this setting, as follows:

Theorem 12.1. *Let B_X, B_Y denote the unit balls in $H_q^1(Y)$ and $H_q^1(X)$ respectively.*

1. *If $Y \to X$ is nonamenable, then $\overline{\Theta_q(B_Y)}$ is contained in the interior of B_X.*
2. *If $Y \to X$ is amenable and Ω is a bounded symmetric domain, then $\Theta_q(B_Y) = B_X$.*

Sketch of the Proof. The amenable case is a straightforward generalization of the proof for quadratic differentials on the disk, once given surjectivity of Θ_q. Surjectivity seems to be unknown in general, but it is established for quotients of bounded symmetric domains by Earle and Resnikoff, see [Earle].

For the nonamenable case, the key points are the following.

First, the proof for quadratic differentials may be modified to avoid use of completeness of Poincaré series; in fact, the demonstration that Θ uniformly contracts the space $Q_\psi(Y)$ works just as well so long as $\Theta(\phi)$ is *close* to a multiple of ψ.

Secondly, away from its zeros any q-canonical form $\phi(z) \, dz^q$ is equal to dz^q for appropriate choice of chart.

Finally, in such a chart the ratio of the Bergman density to the volume element $|dz|^{2n}$ varies smoothly, so it is nearly constant on sufficiently small neighborhoods. The local problem of generalizing Theorem 5.1 so it applies in these neighborhoods is fairly straightforward. The global considerations are only combinatorial and carry over immediately. $\quad\square$

Appendix. Geometric limits of quadratic differentials

A.1. Introduction

In this appendix, we consider geometric limits for Riemann surfaces and quadratic differentials *viewed at the scale of the injectivity radius*. The central result

is a compactification of the space $\mathbf{P}\mathcal{Q}_{g,n}$ of projective integrable quadratic differentials on pointed Riemann surfaces of genus g with n punctures.

This compactification formalizes our intuition: the foliations of an integrable quadratic differential cannot swirl too much on a Riemann surface of finite volume and fixed topological type.

A.2. Riemann surfaces in the geometric topology

Definitions. For $\kappa \in [-1, 1]$, consider the metric

$$ds_\kappa = \frac{4|dz|}{(4 + \kappa |z|^2)}$$

of constant curvature κ on a domain U_κ which is the Riemann sphere if κ positive, the complex plane $\mathbf{C}$ if $\kappa = 0$ and the disk

$$U_\kappa = \{z : |z| < R\},$$

where $R = 2/\sqrt{-\kappa}$ if $\kappa < 0$. In this metric, U_κ is complete.

Let $\mathcal{X}$ denote the space of all pairs (U_κ, Γ) where Γ is a discrete group of Möbius transformations acting freely on U_κ. To this data we associate a Riemann surface $X = U_\kappa / \Gamma$ with base-frame $v \in T_1(X)$ by setting v to be the image of a unit vector at the origin pointing along the positive real axis. We require that the injectivity radius of X is $\geqq 1$ at v.

Conversely, a Riemann surface (X, v) with a complete metric of constant curvature $\kappa \in [-1, 1]$ and base-frame v over a point of injectivity radius $\geqq 1$ uniquely determines an element of $\mathcal{X}$.

Every Riemann surface X admits a complete metric of constant curvature, and so has many representatives in $\mathcal{X}$, corresponding to scaling the metric so its injectivity radius at a basepoint becomes at least 1. If X is hyperbolic, we will refer to its complete metric of constant curvature -1 as the *Poincaré metric* on X.

We give $\mathcal{X}$ the *geometric topology*: a sequence of pairs converges iff the curvatures κ converge and the groups Γ converge in the Hausdorff topology on closed subsets of $\mathrm{PSL}_2\mathbf{C}$.

Proposition A.2.1. *The space of Riemann surfaces $\mathcal{X}$ is compact in the geometric topology.*

Proof. The space of closed subgroups of a Lie group such as $\mathrm{PSL}_2\mathbf{C}$ is compact in the Hausdorff topology. The limiting groups are discrete by our condition on the injectivity radius.   □

This proposition is well-known and goes back to Chabauty, who introduced the geometric topology in the general setting of a locally compact topological group [Cha]; see also [Har], [Th 2].

Geometric limits of Riemannian manifolds with controlled curvature have been considered much more generally; see [Grom]. The geometric topology

coincides with the *quasiisometric topology*, in which a neighborhood of (X, v) consists of those (Y, w) admitting a smooth near-isometry taking v to w defined on a large compact subset of X. Nearby surfaces need not be homeomorphic.

Example. Let X be a hyperbolic Riemann surface with a puncture, v_n a sequence of base-frames tending to the omitted point. Rescale the Poincaré metric so the injectivity radius at v_n remains greater than one. The limiting Riemann surface is an infinite cylinder – or a plane, if we rescale so the injectivity radius tends to infinity.

Our main concern will be with holomorphic data on X, which is independent of the metric. Rather, the metric provides a convenient way to pass smoothly from hyperbolic to parabolic Riemann surfaces, as in the example.

A.2.1. Compact surfaces with punctures

A Riemann surface X is of *finite type* if it is obtained from a compact surface by removing a finite set of points; these points are the *punctures* of X. Let $\mathscr{X}_{g,n} \subset \mathscr{X}$ denote the space of finite type Riemann surfaces of genus g with n punctures.

We can compactify $\mathscr{X}_{g,n}$ by forming its closure in $\mathscr{X}$.

Proposition A.2.2. *For $n > 0$,*

$$\overline{\mathscr{X}_{g,n}} = \bigcup \{\mathscr{X}_{h,m} : 2h + m \leq 2g + n, 0 \leq h \leq g, 1 \leq m\}.$$

while for $n = 0$,

$$\overline{\mathscr{X}_{g,0}} = \mathscr{X}_{g,0} \cup \overline{\mathscr{X}_{g-1,2}},$$

if $g > 0$; and

$$\overline{\mathscr{X}_{0,0}} = \mathscr{X}_{0,0} \cup \mathscr{X}_{0,1}.$$

In particular, the space of hyperbolic surfaces of genus g with n punctures is compactified by the plane, the punctured plane and hyperbolic surfaces of smaller complexity.

Sketch of the proof. Genus and volume can only decrease in the geometric limit. When combined with Gauss-Bonnet and the fact that a limit of noncompact surfaces is noncompact, this implies the closure of $\mathscr{X}_{g,n}$ is no larger than stated. The reverse inclusion is established by surgery: joining surfaces at punctures, or gluing pairs of punctures together; and by obvious scaling limits, such as a sequence of larger and larger spheres converging to the plane. $\square$

A.2.2. The universal curve

Over $\mathscr{X}$ one can construct the *universal curve* $\mathscr{C}$, the space whose fiber over (X, v) is X itself. To define $\mathscr{C}$, start with the bundle of universal covers

$$\{(z, (U_\kappa, \Gamma)) : z \in U_\kappa\} \subset \hat{\mathbf{C}} \times \mathscr{X} \to \mathscr{X}$$

and form the quotient space $\mathscr{C}$ by collapsing the fiber over (U_κ, Γ) by the action of Γ. We give $\mathscr{C}$ the quotient topology; the quotient map is a local homeomorphism.

This construction provides a way to formalize many notions whose intuitive meaning is clear.

For example, we say a closed set $E \subset X$ is a geometric limit of $E_n \subset X_n$ if $E_n \to E$ in the Hausdorff topology on closed subsets of the universal curve. The convergence is *faithful* if any neighborhood of E contains E_n for all n sufficiently large; this means no points are lost in the limit by straying to infinity.

Similarly, say a sequence of continuous maps $f_n: X_n \to Z$ converges to $f: X \to Z$ in the geometric topology if their graphs converge in the Hausdorff topology on $\mathscr{C} \times Z$. This definition will be useful in describing the geometric topology on quadratic differentials and rational maps.

A.2.3. Quadratic differentials

Let $\mathscr{Q} \to \mathscr{X}$ denote the space of triples (X, v, ϕ) where $\phi(z)\, dz^2$ is a holomorphic quadratic differential on X. The canonical bundles on each (X, v) in $\mathscr{X}$ piece together to form a continuous complex line bundle $K \to \mathscr{C}$ on the universal curve, with holomorphic structure along each fiber. The topology on $\mathscr{Q}$ is simply the geometric topology on continuous maps whose target Z is total space of the bundle $K \otimes K$.

In down-to-earth terms, a sequence of quadratic differentials converges if the underlying Riemann surfaces converge and lifts of the differentials to the universal covers converge uniformly on compacta.

Note that we impose no growth constraints on ϕ and allow essential singularities; however those ϕ arising in the compactification of $\mathbf{P}\mathscr{Q}_{g,n}$ will have only finite order poles at punctures.

A.2.4. Covering spaces

Let $\mathscr{Y} \subset \mathscr{X} \times \mathscr{X}$ denote the 'universal covering space'. An element of $\mathscr{Y}$ is given by a pair of pointed Riemann surfaces $((Y, w), (X, v))$ admitting a covering map $p: Y \to X$ whose derivative sends w to v. The map p is redundant data; if it exists, it is unique. The geometric topology on $\mathscr{Y}$ is inherited from $\mathscr{X} \times \mathscr{X}$; it's easy to check that $\mathscr{Y}$ is a closed subspace, establishing:

Proposition A.2.3. *The space of coverings $\mathscr{Y}$ is compact in the geometric topology.*

One may also think of $\mathscr{Y}$ as the space of triples $(U_\kappa, \Gamma_X, \Gamma_Y)$ where Γ_Y is a subgroup of Γ_X.

A.2.5. Poincaré series

Let $p: (Y, w) \to (X, v)$ be a covering space, and let ϕ be a holomorphic quadratic differential on Y. The *push-forward* $p_*(\phi)$ is defined as follows: on a ball $B \subset X$,

$p_*(\phi)$ is the sum of $(p^{-1})^*(\phi)$ over all branches of $p^{-1}: B \to Y$. On the universal cover of X, p_* can be expressed as a relative Poincaré series:

$$p_* \phi = \sum_{[\gamma] \in \Gamma_X / \Gamma_Y} \gamma^* \phi.$$

The push-forward is a well-defined holomorphic differential so long as $\int_{p^{-1}(K)} |\phi| < \infty$ for every compact set K. In this case we say ϕ is *fiberwise integrable*.

How continuous is the Poincaré series as a function of its input data? Consider, for example, an integrable differential ϕ on the universal cover of a compact hyperbolic surface X. For any nontrivial element γ in the covering group, $\phi_n = (\gamma^n)^* \phi \to 0$ geometrically, while $p_*(\phi_n)$ remains constant, so $\lim p_*(\phi_n) \neq p_*(\lim \phi_n)$ in general. To obtain continuous variation of the push-forward, we must constrain the geometric topology on quadratic differentials so that significant contributions cannot disappear in the limit by straying to infinity.

To make this precise, consider a sequence of coverings $p_n: (Y_n, w_n) \to (X_n, v_n)$ and quadratic differentials ϕ_n on Y_n. Assume the coverings converge geometrically to $p: (Y, w) \to (X, v)$, $\phi_n \to \phi$ on Y, and ϕ_n and ϕ are integrable along the fiber.

Think of $|\phi_n|$ as a sequence of measures on the universal curve. We say the convergence is *faithful* if for every $\varepsilon > 0$ there is a compact set K on the universal curve whose complement has $|\phi_n|$-measure less than ε for all n sufficiently large. Since there is uniform convergence on K and the total mass of the remainder is small, it is easy to see:

Proposition A.2.4. *The Poincaré series varies continuously under faithful convergence.*

Faithful convergence is equivalent to geometric convergence coupled with the constraint that $\int_{Y_n} |\phi_n| \to \int_Y |\phi|$. Similarly, we may define *fiberwise* faithful convergence by requiring

$$\int_{p_n^{-1}(K_n)} |\phi_n| \to \int_{p^{-1}(K)} |\phi|$$

for any faithfully convergent sequence $K_n \to K$ of compact subsets of $X_n \to X$. The same reasoning justifies:

Proposition A.2.5. *If $\phi_n \to \phi$ faithfully fiberwise, then $(p_n)_*(\phi_n)$ converges geometrically to $p_*(\phi_n)$.*

A.2.6. Systems of disjoint simple loops

Let $\mathscr{S}$ denote the space of triples (X, v, S) where $S \neq \emptyset$ is a finite system of disjoint simple closed curves on X (up to isotopy), with no curve contractible and no pair isotopic.

To define the geometric topology, first assume X is hyperbolic; give X its Poincaré metric. To S we associate a compact subset $K(S) \subset X$ consisting of a component of the thin part for each peripheral curve or short geodesic in S, and a geodesic representative for the remaining elements of S. If X is not hyperbolic, we think of all of X as thin and set $K(S) = X$. Then (X_n, v_n, S_n) converges geometrically to (X, v, S) iff $K_n(S_n)$ converges *faithfully* to $K(S)$. (We do not allow curves to disappear in the limit by straying to infinity.)

Equivalently, S may be thought of as a subset of the fundamental group Γ of X consisting of a finite union of conjugacy classes; then the geometric topology is the topology of faithful convergence of closed subsets of $\mathrm{PSL}_2\mathbf{C}$.

Example. Let (X_n, v_n) be a sequence of punctured tori converging to a triply punctured sphere (X, v) by pinching a simple geodesic S_n. Then the limiting thin part has a two components, and the limit of S_n consists of *two* loops, one around each of the two new punctures. This phenomenon occurs whenever a *nonseparating* curve is pinched.

A.2.7. Maps to $\hat{\mathbf{C}}$ of bounded degree

Let Rat_d denote the space of rational maps $f\colon \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ of degree $\leq d$. Let us say $f_n \to f$ iff there is a finite set $E \subset \hat{\mathbf{C}}$ such that f_n converges to f uniformly on compact subsets of $\hat{\mathbf{C}} - E$. The degree of a map can decrease in the limit. With this topology, Rat_d is compact; the lower-degree maps provide a compactification for those of degree exactly d.

Let $\mathcal{R}_d$ denote the space of triples $[(X, v, f)]$, where $f\colon X \to \hat{\mathbf{C}}$ is a holomorphic map which is at most d-to-1. Motivated by the example of rational maps on the sphere, we define the geometric topology by:

$$(X_n, v_n, f_n) \to (X, v, f)$$

if the first two factors converge as before, and there exists a finite set $E \subset X$ such that $f_n \to f$ geometrically on compact subsets of $X - E$.

If f is not the constant map to ∞, it is a *meromorphic function*; f is *invertible* if it is also not identically 0.

Theorem A.2.6. *$\mathcal{R}_d$ is compact in the geometric topology. Moreover, for any sequence f_n of invertible meromorphic functions, there exist constants c_n such that $c_n f_n$ has a convergent subsequence whose limit is invertible.*

Proof. Let (X_n, v_n, f_n) be a sequence in $\mathcal{R}_d$; we may assume that the first two factors converge to a pair (X, v), and each f_n is nonconstant (otherwise the theorem is obvious).

Let $E_n \subset X_n$ denote the pre-images of 0, 1 and ∞ under f_n. Then $|E_n| \leq 3d$. Passing to a subsequence, we may assume E_n converges geometrically to a finite set $E \subset X$; some points of E_n may be lost by straying far from the basepoint, but this is irrelevant.

By Montel's theorem, $\langle f_n \rangle$ has a subsequence converging uniformly to a holomorphic map f outside of E. The limiting map is at most d-to-1, so any singularity at E is removable, and we have obtained a limiting $f : X \to \hat{\mathbf{C}}$.

The limiting map can be identically zero or infinity only if every point of E is a limit of a zero or a pole of f_n. Pick a sequence $x_n \in X_n - E_n$ converging to $x \in X - E$, and set $c_n = 1/f_n(x_n)$. For the new sequence $c_n f_n(x)$, x will be a limit of the pre-images of 1 but not a limit of zeros or poles, so f will be invertible. □

A.3. Compact spaces of quadratic differentials

A holomorphic quadratic differential ϕ on a Riemann surface X has a beautiful concrete picture: it is the combination of a foliation and a flat metric, with isolated singularities. One way to see the structure is to change coordinates locally so that $\phi = dz^2$; in these coordinates the metric is the usual Euclidean metric, and the foliation is by lines of constant imaginary part. Such a change of coordinates is always possible away from the zeros of ϕ, and is well-defined up to post-multiplication by -1 (thus the foliation is unoriented.)

Any real multiple of ϕ has the same foliation. In this section we establish compactness of a class of quadratic differentials determined only up to multiples. One consequence is that the foliation of a quadratic differential on a surface of type (g, n), with at most simple poles at the punctures, ranges through a compact set of pictures when viewed at the scale of the injectivity radius.
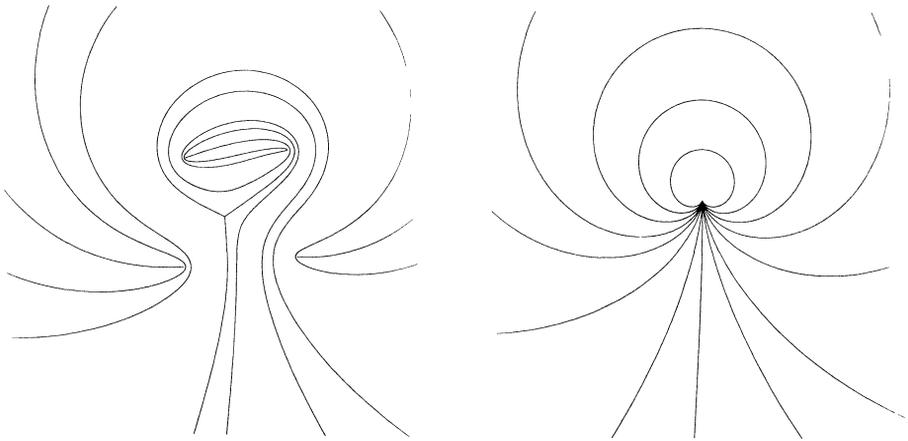

Fig. 3. The geometric limit is a triple pole

An example appears in Fig. 3. The first picture is the foliation of a quadratic differential in the plane with four simple poles (one prong singularities) and one zero (three pronged). This can be thought of as a holomorphic differential on the four-times punctured plane. Now imagine shrinking the picture towards

the zero, while holding a basepoint a fixed distance away. The limiting Riemann surface is the once-punctured plane, and the limiting differential has a triple pole at the single puncture consolidating what was once four punctures.

Definitions. As before, $\mathscr{Q}$ denotes the space of quadratic differentials with the geometric topology. Let $\mathbf{P}\mathscr{Q} = \mathscr{Q}^*/\mathbf{C}^*$ be the quotient of the set of nonzero quadratic differentials by the multiplicative action of complex scalars, with the quotient topology.

Now restrict attention to quadratic differentials ϕ on Riemann surfaces in $\mathscr{X}_{g,n}$, i.e. those with genus g and n punctures. Let $\mathscr{Q}_{g,n}$ denote those ϕ with at most simple poles at the punctures of X. The fiber of $\mathscr{Q}_{g,n}$ over $X \in \mathscr{X}_{g,n}$ is canonically identified with the cotangent space to Teichmüller space at X, once a marking is chosen.

Theorem A.3.1. $\mathbf{P}\mathscr{Q}_{g,n}$ *has compact closure in* $\mathbf{P}\mathscr{Q}$.

Remarks.

1. The quadratic differentials compactifying $\mathbf{P}\mathscr{Q}_{g,n}$ live on the Riemann surfaces compactifying $\mathscr{X}_{g,n}$, i.e. the plane, the punctured plane and hyperbolic surfaces of smaller complexity. The limiting differentials may have poles of high order at the punctures (although they certainly have no essential singularities.)

2. The fibers of $\mathbf{P}\mathscr{Q}_{g,n} \to \mathscr{X}_{g,n}$ are projective spaces; in general this is not true for the closure

$$\overline{\mathbf{P}\mathscr{Q}_{g,n}} \to \overline{\mathscr{X}_{g,n}}$$

as we will see in examples.

3. This compactification is related to, but quite distinct from, the compactification of moduli space by Riemann surfaces with nodes and the extension of the cotangent bundle to this completion, discussed in [DM], [Bers], [Masur], [EM] and elsewhere. One might further investigate the analytic or algebraic structure $\overline{\mathbf{P}\mathscr{Q}_{g,m}}$; however its topological structure will be sufficient for our applications.

4. $\mathbf{P}\mathscr{Q}$ itself is certainly not compact; for example, in the unit disk with basepoint at $z = 0$, the sequence $[z^n dz^2]$ has no limit in $\mathbf{P}\mathscr{Q}$.

A.3.1. Quadratic differentials from simple closed curves

To begin the proof, we construct continuous families of quadratic differentials using Poincaré series.

Let Y be a planar Riemann surface with fundamental group $\mathbf{Z}$. Choose an isomorphism

$$h: Y \to A(r, R) = \{z: r < |z| < R\} \subset \mathbf{C}^*,$$

for $0 \leq r < R \leq \infty$, and set $\phi(Y) = h^*(dz^2/z^2)$. One may check that $\phi(Y)$ is well-defined and varies continuously in $\mathscr{Q}$ as a function of (Y, w) in $\mathscr{X}$. In the $|\phi|$-metric, Y is a flat cylinder of radius 1. Up to a constant multiple, $\phi(Y)$ is the only quadratic differential on Y invariant under the automorphisms of Y.

Let $(X, v, S) \in \mathscr{S}$ be a Riemann surface together with a finite system of free homotopy classes

$$S = \{[\gamma_1], \ldots, [\gamma_s]\},$$

where $[\gamma_i]$ are conjugacy classes in $\pi_1(X)$. The cyclic subgroup generated by γ_i determines a covering $p_i \colon Y_i \to X$ where Y_i is a planar surface with fundamental group $\mathbf{Z}$. Define an associated quadratic differential on X by

$$\theta(X, v, S) = \sum_1^s (p_i)_* \, \phi(Y_i)$$

Proposition A.3.2. *The differential $\theta(X, v, S)$ is holomorphic, with at most double poles at the punctures of X. Moreover $\theta \neq 0$.*

Proof. If Y_i is an annulus, $|\phi(Y_i)|$ is integrable and hence its push-forward is holomorphic with at most simple poles. Otherwise Y_i is a punctured plane or punctured disk. Each puncture has a neighborhood mapping injectively to a neighborhood of a puncture on X, creating a double pole for θ; $|\phi|$ is integrable outside a neighborhood of the punctures, so its push-forward is holomorphic.

To check that $\theta \neq 0$, first suppose that X is a hyperbolic Riemann surface and S is a system of disjoint simple geodesics. Then Wolpert has shown the differentials $p_*(\phi_i)$ are linearly independent [Wol, Theorem 3.7]. In fact, there is a perfect pairing between these differentials, though of as elements of the cotangent space to Teichmüller space at X, and the Fenchel-Nielsen twist-vector fields for each curve $[\gamma_i]$. (Wolpert's proof is given for compact surfaces, but easily extends to the general situation.)

Otherwise, some element of S represents a peripheral curve around a puncture p of X. Then $\theta \neq 0$ because it has a double pole at p.

A.3.2. Variation of Poincaré series

Proposition A.3.3. *The mapping $\theta \colon \mathscr{S} \to \mathscr{Q}$ associating a quadratic differential to a system of simple closed curves is continuous.*

Proof. Let $(X_n, v_n, S_n) \to (X, v, S)$. It is easy to reduce to the case $|S_n| = 1$; then S consists of one or two curves, depending on whether or not a nonseparating curve is pinching off.

To start, assume $|S| = 1$.

Choose basepoints lying over $v_n \to v$ on the corresponding covering spaces, such that $(Y_n, w_n) \to (Y, w)$; then the corresponding quadratic differentials ϕ_n converge geometrically to ϕ on Y. To demonstrate that

$$\theta(X_n, v_n, S_n) \to \theta(X, v, S),$$

it is enough to show that the convergence of ϕ_n to ϕ is fiberwise faithful (by Proposition A.2.5).

We check this case by case.

1. $X = \mathbb{C}^*$.

(a) $X_n = \mathbb{C}^*$ for all n. This case is trivial.

(b) S_n are geodesics on hyperbolic surfaces.

In the flat $|\phi_n|$-metric, each Y_n is a finite cylinder of fixed circumference whose height is tending to infinity. In the Poincaré metric, the thick part of Y_n consists of two annuli at the ends of Y_n whose $|\phi_n|$-heights are uniformly bounded.
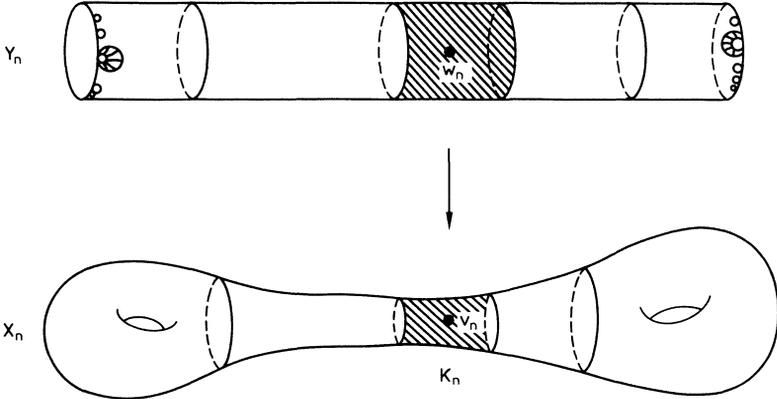

Fig. 4. A short geodesic converging to a cylinder

Let $K_n \to K$ be a faithfully convergence sequence of compact sets on $X_n \to X$. For n large, K_n is entirely contained in the component of the thin part corresponding to S_n, and its pre-image on Y_n consists of one component in the thin part (which persists in the limit) and other components in the thick part (near the ends of Y_n); see Fig. 4. The boundary of the thin part on Y_n lies over the boundary of the thin part on X_n, while the Poincaré distance from K_n to the boundary of the thin part tends to infinity; thus the lifts of K_n in the thick part of Y_n are confined to a narrow neighborhood of the ends of Y_n and their $|\phi_n|$-area tends to zero. Thus $\phi_n \to \phi$ faithfully fiberwise.

(c) S_n correspond to punctures on hyperbolic surfaces.

A similar argument applies, with half-infinite cylinders.

2. X is hyperbolic.

(a) S is a geodesic.

The Poincaré length $l(S) = \lim l(S_n)$, and $|\phi_n|$-area of Y_n is a continuous function of $l(S_n)$, so no mass is lost in the limit and the convergence is faithful.

(b) S is a puncture which is a limit of punctures.

Each Y_n is a half-infinite cylinder, such that the portion at fixed $|\phi_n|$-distance from the finite end injects into a neighborhood of the puncture. It follows that the convergence is fiberwise faithful.

(c) S is a puncture which is a limit of separating geodesics.

The thin part corresponding to S_n is bounded by a 'near end' and a 'far end'. The distance of a faithfully convergent sequence of compact sets K_n from the far end tends to infinity. Each Y_n is a finite cylinder, on which the lifts of K_n

to the near end persist in the limit, while those in the far end have small area by the same Poincaré metric argument as in case 1b. Thus the convergence is fiberwise faithful.

Finally, we treat the case $|S| = 2$, which we record as:

(d) S is a pair of punctures which are the limit of nonseparating geodesics. Now both ends of the thin part corresponding to S_n remain a bounded distance from the basepoint. Let (Y, w) and (Y', w') denote the two coverings of X corresponding to the two limiting punctures. We may choose basepoints w_n and w'_n near each of the two ends of Y_n, such that $(Y_n, w_n) \to (Y, w)$ and $(Y_n, w'_n) \to (Y', w')$. Then the pre-images of K_n near one end persists on (Y, w), while those in the other end persists on (Y', w'). Since $\theta(X, v, S)$ is defined as the sum of these two Poincaré series, $\theta(X_n, v_n, S_n) \to \theta(X, v, S)$ as claimed. $\quad \square$

A.3.3. The closure of $\mathbf{P}\mathcal{Q}_{g,n}$ is compact

Proof of Theorem A.3.1. Let $(X_n, v_n, [\phi_n])$ be a sequence in $\mathcal{Q}_{g,n}$. We may assume $(X_n, v_n) \to (X, v) \in \overline{\mathcal{X}_{g,n}}$.

Suppose X is a hyperbolic surface with a simple geodesic whose free homotopy class is S. Then we can find corresponding homotopy classes S_n on X_n such that $(X_n, v_n, S_n) \to (X, v, S)$, and hence

$$\theta_n = \theta(X_n, v_n, S_n) \to \theta = \theta(X, v, S).$$

Each of these differentials is nonzero with at most double poles.

Set $f_n = \phi_n/\theta_n$. Since ϕ_n has at most simple poles, f is a meromorphic function of degree at most $4g - 4 + 3n$. By Theorem A.2.6, there is a sequence of constants c_n such that $c_n f_n$ has a subsequence converging to an invertible function f uniformly on compact subsets of $X - E$, where E is a finite set. Since ϕ_n is holomorphic, $c_n \phi_n$ converges geometrically even at E, to $f\theta = \phi$. Thus $[\phi_n] \to [\phi]$ in $\mathbf{P}\mathcal{Q}$.

Now suppose X is the punctured plane or the triply punctured sphere. The same argument applies, with S a peripheral curve around each puncture. (It is here we use continuity of $\theta(X, v, S)$ when two punctures are sewn together.)

Finally suppose $X = (\mathbf{C}, 0)$. Rescale X_n so its injectivity radius at the basepoint is 1; then a subsequence has a limit (X', v', ϕ'), since X' cannot be a plane. It follows that the unrescaled differentials accumulate on points of the form $(\mathbf{C}, 0, [\phi(z) \, dz^2])$, where $\phi \neq 0$ is a polynomial whose degree is bounded by the order of zero of ϕ at the basepoint on X'. $\quad \square$

A.4. Low dimensional examples

We conclude with some examples of geometric limits.

1. Here is a complete description of the compactification of $\mathbf{P}\mathcal{Q}_{0,4}$, the quadratic differentials on the four-times punctured sphere. First, $\mathcal{X}_{0,4}$ is compactified by spheres with 1, 2 or 3 punctures. There is a unique projective class of differen-

tial with simple poles on a four-times punctured sphere; it has no zeros, so the same is true of the limit. The limiting differentials on a triply-punctured sphere have poles of orders (1, 1, 2); those on $\mathbf{C}^*$ have poles of orders (1, 3) and (2, 2); and the only limit on $\mathbf{C}$ is $[dz^2]$, with a fourth order pole at infinity. The orders of poles determine the projective class uniquely.

Since the poles can be distributed to various punctures, the fibers of the compactification over $\hat{\mathbf{C}} - \{0, 1, \infty\}$ and $\mathbf{C}^*$ consist of three points, hence are not projective spaces. Certain linear combinations of limiting differentials have zeros, hence do not occur.

2. Consider $\mathbf{P}\mathscr{Q}_{1,2}$, the quadratic differentials on twice-punctured tori. On a compact torus two pairs of points Z and P are the zeros and poles of a quadratic differential iff $Z - P = 0$ in the group law (a torus is its own Jacobian).

The four times punctured sphere arises as a limit of twice punctured tori by pinching a non-separating curve; normalize so the two new punctures are at 0 and ∞. Then any limiting differential has the form $\phi = f(z)\, dz^2/z^2$, where the zeros and poles of f now satisfy $Z/P = 1$ in the group law on $\mathbf{C}^*$ (assuming the divisor of f does not meet 0 or ∞.)

The condition $Z/P = 1$ is equivalent to the condition that the residues of ϕ at 0 and ∞ agree.

3. As a final example, we construct a family of differentials on compact surfaces of genus 2. Fix a torus with a holomorphic differential dz^2. Cut open two segments of the horizontal foliation, each of length L, and paste in a cylinder of height H and circumference $2L$ with its foliation by circles. The result is a quadratic differential on a surface of genus two, with four zeros at the endpoints of the slits.

Suppose the slits are the sides of a square and $L \to 0$ while H/L is fixed. For a basepoint on the cylinder, the limit differential lives on a punctured torus and has a fourth order pole. Further degenerations yield differentials with poles of order (4, 2, 2) on $\hat{\mathbf{C}} - \{0, 1, \infty\}$, of order (4, 4) on $\mathbf{C}^*$, and of order 8 on $\mathbf{C}$.

Remark. It would be interesting to have an explicit description of the closure of $\mathbf{P}\mathscr{Q}_{g,n}$ in $\mathbf{P}\mathscr{Q}$.

The local form of a quadratic differential at a point p is determined by the order N of zero or pole at p, and a complex residue R if N is even and ≤ -2 (see [Str 3]). An implicit description of the compactification can be given by 'regenerating' an element of $\mathbf{P}\mathscr{Q}_{g,n}$. The regeneration is specified by a quadratic differential ϕ on a possibly disconnected Riemann surface X, plus pairs of punctures (to be glued together) where the orders and residues of ϕ are compatible. It seems difficult, however, to describe which (X, ϕ) and gluing diagrams actually exist.

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