Teichmüller curves in genus two: The decagon and beyond

Curtis T. McMullen[*]

30 April, 2004

Contents

1 Introduction

Let $\mathcal{M}_g$ denote the moduli space of Riemann surfaces of genus g, and $\Omega\mathcal{M}_g \to \mathcal{M}_g$ the bundle of Abelian differentials. A point in $\Omega\mathcal{M}_g$ is specified by a pair (X, ω), where X is in $\mathcal{M}_g$ and where $\omega \in \Omega(X)$ a nonzero holomorphic 1-form on X.

There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on $\Omega\mathcal{M}_g$, giving moduli space a dynamical flavor. The projection of any orbit $\mathrm{SL}_2(\mathbb{R}) \cdot (X, \omega)$ yields a holomorphic *Teichmüller disk* $f : \mathbb{H} \to \mathcal{M}_g$, whose image is typically dense. On rare occasions, however, the stabilizer $\mathrm{SL}(X, \omega)$ of the given form is a lattice in $\mathrm{SL}_2(\mathbb{R})$; then the image of the quotient map

$$f : V = \mathbb{H}/\mathrm{SL}(X, \omega) \to \mathcal{M}_g$$

is an algebraic curve, isometrically embedded for the Teichmüller metric.

In this paper we address the classification of such *Teichmüller curves* in the case $g = 2$.

Billiards. Let $P \subset \mathbb{C}$ be a polygon with angles in $\pi\mathbb{Q}$. Via an unfolding construction, (P, dz) determines a holomorphic 1-form (X, ω) such that billiard trajectories in P correspond to geodesics on $(X, |\omega|)$.

We say P is a *lattice polygon* if $\mathrm{SL}(X, \omega)$ is a lattice; equivalently, if (X, ω) generates a Teichmüller curve. In this case, Veech showed that $\mathrm{SL}(X, \omega)$ allows one to renormalize the geodesic flow on $(X, |\omega|)$, and thereby establish optimal dynamical properties for billiards in P: for example,

1. Every billiard trajectory is either periodic or uniformly distributed; and

2. The number of types of closed trajectories of length $\leq L$ is asymptotic to $C_P \cdot L^2$ as $L \to \infty$.

Veech also showed that every regular polygon is a lattice polygon [V1]. In particular the regular pentagon, octagon and decagon give rise to Teichmüller curves in genus two.

Hilbert modular surfaces. The geometry of Teichmüller curves as above is best understood in the case of genus two: any such curve lies on a unique *Hilbert modular surface* H_D, $D > 0$ [Mc1]. More precisely, we have a commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\ f\ } & \mathcal{M}_2 \\
\downarrow & & \downarrow {\scriptstyle \mathrm{Jac}} \\
H_D & \longrightarrow & \mathcal{A}_2,
\end{array}
$$

where $H_D = (\mathbb{H} \times \mathbb{H})/\mathrm{SL}_2(\mathcal{O}_D)$ parameterizes the locus of Abelian surfaces $A \in \mathcal{A}_2$ with real multiplication by the quadratic order $\mathcal{O}_D \cong \mathbb{Z}[x]/(x^2 + bx + c)$, $D = b^2 - 4c$. We refer to D as the *discriminant* of the Teichmüller curve $f : V \to \mathcal{M}_2$.

Elliptic differentials. If a billiard table can be tiled by squares, then it is a lattice polygon. Similarly, for any covering

$$
p : X \to E = \mathbb{C}/\Lambda
$$

branched over torsion points on an elliptic curve E, if we set $\omega = p^*(dz)$ then $\mathrm{SL}(X, \omega)$ is a lattice, commensurable to $\mathrm{SL}_2(\mathbb{Z})$. Thus (X, ω) generates a Teichmüller curve, providing examples in every moduli space $\mathcal{M}_g$.

For $d > 1$, the surface H_{d^2} carries infinitely many Teichmüller curves, all inherited from genus one as above. Our main result shows the situation is radically different when $\sqrt{D}$ is irrational.

Theorem 1.1 (Finiteness) *If D is not a square, then there are only finitely many Teichmüller curves of discriminant D.*

We remark that H_D always carries infinitely many *Shimura curves*, covered by graphs of Möbius transformations $A : \mathbb{H} \to \mathbb{H}$ in $\widetilde{H_D}$. In contrast, the Teichmüller curves in H_D are covered by graphs of transcendental functions [Mc1, §10], and their abundance depends on the rationality of $\sqrt{D}$.

Primitive curves. A Teichmüller curve is *primitive* if it does not arise from a curve of lower genus via a branched covering construction. In genus two, a

Teichmüller curve of discriminant D is primitive if and only if $\sqrt{D}$ is irrational. Although the surface H_{d^2} carries infinitely many Teichmüller curves, none of them are primitive.

To construct primitive examples, let the *Weierstrass curve W_D* be the locus of those Riemann surfaces $X \in \mathcal{M}_2$ such that

   (i) $\mathrm{Jac}(X)$ admits real multiplication by $\mathcal{O}_D$, and

   (ii) X carries an eigenform ω with a double zero at one of the six Weierstrass points of X.

(Here $\omega \in \Omega(X)$ is an *eigenform* if $\mathcal{O}_D \cdot \omega \subset \mathbb{C} \cdot \omega$.)

It can be shown that W_D has either one or two irreducible components, each of which is a Teichmüller curve of discriminant D generated by billiards in an L-shaped table. Conversely, every Teichmüller curve generated by a form with a double zero belongs to some W_D [Mc1], [Mc3].

The regular decagon. The curves W_5 and W_8 are irreducible; they are exactly the Teichmüller curves generated by billiards in the regular pentagon and the regular octagon.

On the other hand, the vertices of the regular decagon fall into two equivalence classes when opposite edges are identified. It follows that the corresponding Teichmüller curve is generated by a form with a pair of simple zeros, rather than a single double zero. We suspect this is the only such example.

Conjecture 1.2 *The regular decagon gives the only primitive Teichmüller curve $V \to \mathcal{M}_2$ generated by a form with simple zeros.*

This conjecture implies:

   1. All Teichmüller curves generated by forms of genus two are already known: they come from branched covers of tori [GJ], billiards in L-shaped tables [Mc3], and billiards in the regular decagon [V1].

   2. For nonsquare discriminant $D > 5$, the only Teichmüller curves with discriminant D are the components of the Weierstrass curve W_D.

(We remark that the curves generated by the regular pentagon and octagon are also generated by suitable L-shaped tables.)

Algorithms. The proof of Theorem 1.1 is constructive, and it yields an effective algorithm to list all the Teichmüller curves of a given discriminant D. As evidence for Conjecture 1.2, in §9 we describe the proof of:

Theorem 1.3 *The conjecture above holds for all Teichmüller curves with discriminant $D \le 400$.*

In particular, in §8 we show:

Theorem 1.4 *There are only two Teichmüller curves of discriminant $D = 5$: one generated by the regular pentagon, and one by the regular decagon.*

3

Similarly, billiards in the regular octagon gives the unique Teichmüller curve with $D = 8$; the unique curve with $D = 12$ is generated by the polygon shown in Figure 1; and every other primitive Teichmüller curve with $D \leq 400$ comes from billiards in an L-shaped table, given explicitly in [Mc3, Cor. 1.3].
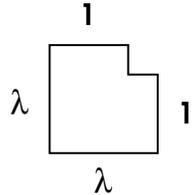

Figure 1. Billiard table for the unique Teichmüller curve of discriminant $D = 12$; $\lambda = (1 + \sqrt{3})/2$.

Proof of finiteness. We turn to a sketch of the proof of Theorem 1.1. Our goal is to show there are only finitely many Teichmüller curves of discriminant D, when $\sqrt{D}$ is irrational.

1. First some definitions. A *splitting prototype* of discriminant D is a set of integers $p = (a, b, c, e)$ such that

$$D = e^2 + 4bc, \quad 0 \leq a < \gcd(b, c), \quad c + e < b,$$
$$0 < b, \quad 0 < c, \quad \text{and} \quad \gcd(a, b, c, e) = 1.$$

   There are only finitely many prototypes for a given discriminant D.

2. Let $\lambda = (e + \sqrt{D})/2$ and let $I_t = [0, t\lambda]$, where $t \in (0, 1]$. Let $(E_i, \omega_i) = (\mathbb{C}/\Lambda_i, dz), i = 1, 2$ be the forms of genus one with period lattices $\Lambda_1 = \mathbb{Z}(\lambda, 0) \oplus \mathbb{Z}(0, \lambda)$ and $\Lambda_2 = \mathbb{Z}(b, 0) \oplus \mathbb{Z}(a, c)$.

   The *prototypical form* of *type p* and *width t* is given by the connected sum

$$(X_t, \omega_t) = (E_1, \omega_1) \underset{I_t}{\#} (E_2, \omega_2).$$

   The connected sum is obtained by slitting each torus open along the projection of the arc I_t, and then gluing corresponding edges (§3). The form (X_t, ω_t) has a double zero when $t = 1$, and otherwise a pair of simple zeros.

3. Let $f : V \to \mathcal{M}_2$ be a Teichmüller curve of discriminant D. In §3 we show that any such curve is generated by a prototypical form. Thus the search for Teichmüller curves is reduced, for each prototype p, to the study of suitable values of the width parameter t.

4. Any form of genus two can be presented as a connected sum in infinitely many ways. In §4 we show that any homology class

$$C \in H_1(E_1, \mathbb{Z}) \oplus H_1(E_2, \mathbb{Z}) \cong H_1(X_t, \mathbb{Z})$$

determines an open interval $U(C) \subset (0,1)$ such that for all $t \in U(C)$, we have a second splitting

$$(X_t, \omega_t) = (F_1, \eta_1) \#_J (F_2, \eta_2)$$

with $[J] - [I] = C$. We also construct a countable set $T(C) \subset \mathbb{R} \cup \{\infty\}$, projectively equivalent to $\mathbb{P}^1(\mathbb{Q})$, such that if $t \in U(C)$ and $\mathrm{SL}(X_t, \omega_t)$ is a lattice, then we also have $t \in T(C)$. (The set $T(C)$ is determined by the condition that the slope of J agrees with the slope of a period of (F_1, η_1).)

5. In §5 we show that for each prototype p, there are infinitely many homology classes C such that $U(C)$ contains a neighborhood of $t = 0$. When $\sqrt{D}$ is irrational, we can also insure that $T(C_1) \cap T(C_2)$ is finite for two such classes. Thus we can find a $t_0(D) > 0$ such that:

$$(*) \ t < t_0 \implies \mathrm{SL}(X_t, \omega_t) \text{ is not a lattice.}$$

6. Let $\Omega_1 E_D$ denote the set of eigenforms of discriminant D normalized by $\int_X |\omega|^2 = 1$. In [Mc5] we analyze the dynamics of $\mathrm{SL}_2(\mathbb{R})$ on forms of genus two, and show in particular that any orbit

$$Z = \mathrm{SL}_2(\mathbb{R}) \cdot (X, \omega) \subset \Omega_1 E_D$$

is either closed or dense. We also show the closed orbits in $\Omega_1 E_D$ are isolated: only finitely many meet any compact set.

7. Now suppose there are infinitely many Teichmüller curves V_i of discriminant D. Then there are infinitely many closed orbits $Z_i = \Omega_1 V_i \subset \Omega_1 E_D$. Each is generated by a prototypical form of type p_i and width t_i. Passing to a subsequence, we can assume $p_i = p$ is constant and $t_i \in (0, 1]$ is convergent. But the orbits Z_i are isolated, so the only possible limit of t_i is $t = 0$. This contradicts $(*)$, and therefore the number of Teichmüller curves of discriminant D is finite.

Converse Veech dichotomy. Are there billiards with optimal dynamics that cannot be analyzed via Teichmüller curves? In §7 we use the same methods to show the answer is *no* in genus two. Namely, we have:

Theorem 1.5 *Let (X, ω) be a holomorphic 1-form of genus two. Then the following conditions are equivalent:*

1. *The group $\mathrm{SL}(X, \omega)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$.*

2. *For every $s \in \mathbb{R} \cup \{\infty\}$, the foliation of $(X, |\omega|)$ by geodesics of slope s is either periodic or uniquely ergodic.*

The implication (1) $\implies$ (2) is the well-known Veech dichotomy, valid in any genus [V1]; the result above furnishes a converse in genus two.

Higher genus. In contrast to the case of genus two, at present only finitely many primitive Teichmüller curves are known for each genus $g \geq 3$. Of these, the Veech examples coming from regular polygons also arise in Mestre's construction of families of curves with real multiplication [Me], and Möller shows the Jacobian of X always has special endomorphisms when (X, ω) generates a Teichmüller curve [Mo]. Thus the theory of real multiplication may facilitate a classification in higher genus, as it does in genus two.

Notes and references. In [Mc5] we classify the orbit closures and ergodic invariant measures for the action of $\mathrm{SL}_2(\mathbb{R})$ on the space $\Omega\mathcal{M}_2$. The classification is explicit, apart from the issue of describing all Teichmüller curves in genus two. The present paper and [Mc3] undertake this description.

For additional background on Teichmüller curves, see [Th], [V1], [V2], [Vo], [Wa], [KS], [Pu], [GJ], [EO], [Lei] and [Lo]. Further results in genus two can be found in [EMS], [HL], [Ca], [Mc1] and [Mc2].

Added in proof: Conjecture 1.2 is established in [Mc4].

2 Elementary moves

A form (X, ω) of genus two splits, in infinitely many ways, as a connected sum of forms of genus one. In this section we define the intersection number of a pair of splittings, and show they are related by a Dehn twist when their intersection number is one.

Moduli spaces. We begin by recalling material from [Mc5]. Let $\Omega\mathcal{M}_g \to \mathcal{M}_g$ denote the bundle of holomorphic 1-forms (X, ω), $\omega \neq 0$, over the moduli space of Riemann surfaces of genus g. Within the space $\Omega\mathcal{M}_2$ of all forms of genus two, we let

- $\Omega\mathcal{M}_2(2)$ denote the closed stratum of forms with double zeros, and

- $\Omega\mathcal{M}_2(1, 1)$, the open stratum of forms with simple zeros.

Connected sums. Let $I = [0, v] = [0, 1] \cdot v$ be the segment from 0 to $v \neq 0$ in $\mathbb{C}$, and let $(E_i, \omega_i) = (\mathbb{C}/\Lambda_i, dz) \in \Omega\mathcal{M}_1$ be a pair of forms of genus one. When I maps to an embedded arc under each projection $\mathbb{C} \to E_i$, one can slit along these arcs and glue corresponding edges to obtains the *connected sum*

$$(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2). \tag{2.1}$$

The connect sum is a form of genus two with a pair of simple zeros, coming from the endpoints of the slits. To construct forms with double zeros, we also allow the case where I projects to a loop in one torus E_i and remains embedded in the other.

Splittings. Every form of genus two can be presented as a connected sum in infinitely many ways [Mc5, Thm. 1.7], each of which we regard as a *splitting* of (X, ω). The splitting (2.1) is uniquely determined by I, up to the ordering of its summands.

Saddle connections. Let $Z(\omega) \subset X$ denote the zero set of ω; it is invariant under the hyperelliptic involution $\eta : X \to X$. A *saddle connection* is a geodesic segment for the metric $|\omega|$, with endpoints in $Z(\omega)$ but with no zeros in its interior.

Given a splitting (2.1), the two sides of $\partial(E_i - I)$ determine a pair of saddle connections L, L' on X such that $\eta(L) = L'$. Conversely, by [Mc5, Thm. 7.3] we have:

Theorem 2.1 *Let $L \supset Z(\omega)$ be a saddle connection such that $L \neq L' = \eta(L)$. Then (X, ω) splits along $L \cup L'$ as a connected sum of tori.*

Suitably oriented, the saddle connections L and L' satisfy

$$\int_L \omega = \int_{L'} \omega = \int_I dz = v,$$

and we let

$$[I] = [L] = [L'] \in H_1(X, Z(\omega); \mathbb{Z})$$

denote the relative homology class they represent.

Intersection number. Given a pair of oriented saddle connections L, M on (X, ω), the *intersection number $L \cdot M$* is defined to be the algebraic number of transverse crossings between L and M outside the zeros of ω. Note that all crossings count with the same sign, and that $L \cdot M = -M \cdot L$.

We define the intersection number of a pair of distinct splittings

$$(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2) = (F_1, \eta_1) \underset{J}{\#} (F_2, \eta_2)$$

by

$$I \cdot J = L \cdot M + L' \cdot M = ((L + L') \cdot (M + M'))/2,$$

where (L, L') and (M, M') are the pairs of parallel saddle connections on (X, ω) corresponding to I and J respectively.
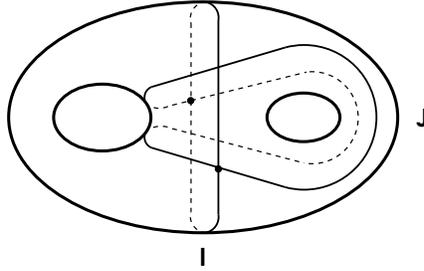

Figure 2. Elementary move. The two dots mark the zeros of ω.

In the case $I \cdot J = 1$, depicted in Figure 2, we can give a homological formula relating the two splittings.

Theorem 2.2 *Suppose* $(X, \omega) \in \Omega \mathcal{M}_2(1,1)$ *admits a pair of splittings*

$$(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2) = (F_1, \eta_1) \underset{J}{\#} (F_2, \eta_2)$$

satisfying $\partial[I] = \partial[J]$ *and* $I \cdot J = 1$. *Choose symplectic bases* (A_i, B_i) *for* $H_1(E_i, \mathbb{Z})$ *such that*

$$[J] = [I] + B_1 + B_2$$

in $H_1(X, Z(\omega); \mathbb{Z})$. *We then have*

$$
\begin{aligned}
H_1(F_1, \mathbb{Z}) &= \mathbb{Z}(A_1 + B_2) \oplus \mathbb{Z}B_1, \\
H_1(F_2, \mathbb{Z}) &= \mathbb{Z}(A_2 + B_1) \oplus \mathbb{Z}B_2.
\end{aligned}
$$

(Here by a *symplectic basis* we mean $A_i \cdot B_i = 1$.)

Proof. Let (L, M) be a pair of oriented saddle connections on (X, ω) representing $([I], [J])$ in relative homology and crossing positively at exactly one point $p \notin Z(\omega)$. The pair (L, M) is unique up to the action of the hyperelliptic involution η.

We can regard $E_i - I$ as a subsurface of X, whose closure T_i is a torus with boundary $L \cup \eta(L)$. Let $K \subset X - Z(\omega)$ be a smooth, simple loop such that $\eta(K) = K$, L and K cross transversally at p and nowhere else, and $[K] = B_1 + B_2$ in $H_1(X, \mathbb{Z})$ when suitably oriented. Then $K \cap T_i$ and $M \cap T_i$ both represent the class $B_i \in H_1(T_i, \partial T_i)$, $i = 1, 2$.

Let $\mathrm{tw}_K : (X, Z(\omega)) \to (X, Z(\omega))$ denote a left Dehn twist around K, fixing $Z(\omega)$. It is then straightforward to verify that the arcs $\mathrm{tw}_K(L)$ and M are isotopic, relative to their endpoints, using the fact that homologous loops on a torus are isotopic. It follows that X splits along $M \cup \eta(M)$ into a pair of tori F_1, F_2 satisfying

$$H_1(F_i) = \mathrm{tw}_K(H_1(E_i)).$$

Since the action of tw_K on $H_1(X, \mathbb{Z})$ is given by

$$\mathrm{tw}_K(x) = x + (x \cdot K)K,$$

and $A_i \cdot (B_1 + B_2) = 1$, a basis for $H_1(F_i)$ is as indicated above. ∎

See [Mc5, §9] for another occurrence of this elementary move.

3 Prototypical splittings

In this section we briefly summarize the relationship between Teichmüller curves, eigenforms and splittings in genus two. We then show every periodic splitting of an eigenform is equivalent to a unique, concretely described model.

Teichmüller curves. Recall there is a natural action of $\mathrm{GL}_2^+(\mathbb{R})$ on the space of holomorphic 1-forms $\Omega \mathcal{M}_g$, and that the stabilizer $\mathrm{SL}(X, \omega)$ of a given form

is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. The group $\mathrm{SL}(X,\omega)$ is a lattice if and only if (X,ω) generates a Teichmüller curve

$$f : V = \mathbb{H}/\mathrm{SL}(X,\omega) \to \mathcal{M}_g.$$

See e.g. [V1], [Mc5, Thm. 3.4].

Periodicity. Given $(X,\omega) \in \Omega\mathcal{M}_g$ and $s \in \mathbb{P}^1(\mathbb{R})$, we have a foliation $\mathcal{F}_s(\omega)$ of $(X,|\omega|)$ by geodesics of slope s. The tangent space $\mathrm{T}\mathcal{F}_s(\omega) \subset \mathrm{T}X$ coincides with the kernel of the harmonic 1-form $\rho = \mathrm{Re}(x+iy)\omega$, $s = x/y$. If all the leaves of $\mathcal{F}_s(\omega)$ are closed, we say $\mathcal{F}_s(\omega)$ is *periodic*. By [V1, 2.4,2.11], we have:

Theorem 3.1 (Veech dichotomy) *Suppose* $\mathrm{SL}(X,\omega)$ *is a lattice. Then for any slope s, the foliation $\mathcal{F}_s(\omega)$ of X is either periodic or uniquely ergodic.*

(In the uniquely ergodic case, one also knows there are no saddle connections of slope s.)

Genus two. We say a splitting $(X,\omega) = (E_1,\omega_1)\underset{I}{\#}(E_2,\omega_2)$ is *periodic* if the following equivalent conditions hold:

1. The foliation of $(X,|\omega|)$ by geodesics parallel to I is periodic;

2. I lies along a closed geodesic in each torus E_i; and

3. $t_i v \in \Lambda_i$ for some $t_i > 0$, $i = 1, 2$.

Theorem 3.2 *Let (X,ω) be a form of genus two such that $\mathrm{SL}(X,\omega)$ is a lattice. Then every splitting of (X,ω) is periodic.*

Proof. The foliation of $(X,|\omega|)$ by geodesics parallel to I has no leaves that are dense in both E_1 and E_2, so it cannot be uniquely ergodic; by the Veech dichotomy, it must be periodic. ∎

We also observe:

Theorem 3.3 *If (X,ω) has two different periodic splittings, then the relative and absolute periods of ω span the same rational subspace of $\mathbb{C}$.*

Eigenforms. Within the space $\Omega\mathcal{M}_2$ of all forms of genus two, let

- ΩE_D denote the eigenforms for real multiplication by $\mathcal{O}_D$, and

- $\Omega E_D(2)$ and $\Omega E_D(1,1)$, the eigenforms with one double and two simple zeros respectively.

Each space above is invariant under the natural action of $\mathrm{GL}_2^+(\mathbb{R})$. The importance of eigenforms for the classification of Teichmüller curves comes from [Mc5, Cor. 5.9]:

Theorem 3.4 *If (X, ω) is a form of genus two and $\mathrm{SL}(X, \omega)$ is a lattice, then $(X, \omega) \in \Omega E_D$ for a unique discriminant D.*

Real multiplication and isogeny. We recall two further properties of eigenforms. First, if (X, ω) is an eigenform for real multiplication by $\mathcal{O}_D$, and D is not a square, then

$$\mathrm{Per}(\omega) \otimes \mathbb{Q} \subset \mathbb{R}^2$$

is a 2-dimensional vector space over $\mathcal{O}_D \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{D})$. For the second property, let us say that forms (E_1, ω_1) and (E_2, ω_2) of genus one are *isogenous* if there is a surjective holomorphic map $p : E_1 \to E_2$ such that $p^*(\omega_2) = t\omega_1$ for some $t \in \mathbb{R}$. Then by [Mc5, Thm. 1.8] we have:

Theorem 3.5 *A form $(X, \omega) \in \Omega \mathcal{M}_2$ is an eigenform for real multiplication iff every splitting of (X, ω) has isogenous summands.*

Prototypes. We can now describe the periodic splittings of eigenforms (cf. [Mc3, §3]).

Let us say a quadruple of integers (a, b, c, e) is a *splitting prototype*, of discriminant D, if it satisfies the conditions

$$\begin{aligned} D = e^2 + 4bc, \qquad & 0 \le a < \gcd(b, c), \qquad c + e < b, \\ 0 < b, \qquad & 0 < c, \quad \text{and} \qquad \gcd(a, b, c, e) = 1. \end{aligned}$$

The *prototypical splitting* of *type* (a, b, c, e) and *width* $t \in (0, 1]$ is given by

$$(X_t, \omega_t) = (E_1, \omega_1) \mathbin{\#_{I_t}} (E_2, \omega_2)$$

where $I_t = [0, t\lambda]$, $(E_i, \omega_i) = (\mathbb{C}/\Lambda_i, dz)$,

$$\Lambda_1 = \mathbb{Z}(\lambda, 0) \oplus \mathbb{Z}(0, \lambda), \quad \Lambda_2 = \mathbb{Z}(b, 0) \oplus \mathbb{Z}(a, c),$$

and $\lambda = (e + \sqrt{D})/2$. The condition $c + e < b$ in the definition of a prototype is equivalent to $\lambda < b$, which insures that I projects an embedded arc in E_2.
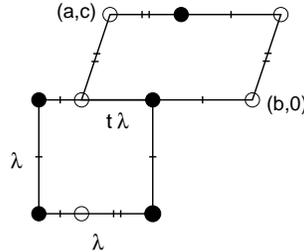

Figure 3. Prototypical splitting.

The prototypical splitting can be expressed in geometric terms as $(X_t, \omega_t) = (P, dz)/\sim$, where $P \subset \mathbb{C}$ is a polygon built from the period parallelograms for Λ_1 and Λ_2 as shown in Figure 3. The two parallelograms overlap along an interval of length $t\lambda$ corresponding to I. The equivalence relation identifies parallel edges of P. As indicated in the figure, for $0 < t < 1$ the vertices of P fall into two classes, corresponding to the two zeros of ω, while for $t = 1$ all vertices of P are equivalent, and ω has a double zero.

We refer to (X_t, ω_t) itself as the *prototypical form* of type (a, b, c, e) and width t.

Orbits. Let ΩE_D^s denote the *splitting space*, consisting of triples (X, ω, I) such that $(X, \omega) \in \Omega E_D$ splits as a connect sum

$$(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2).$$

There is a natural action of $\mathrm{GL}_2^+(\mathbb{R})$ on ΩE_D^s, and an equivariant projection

$$\Omega E_D^s \to \Omega E_D,$$

which is a local homeomorphism but not a covering map.

We can now state:

Theorem 3.6 *Let $D > 0$ be a discriminant that is not a square. Then every periodic splitting in ΩE_D^s is equivalent, under the action of $\mathrm{GL}_2^+(\mathbb{R})$, to a unique prototypical splitting.*

Proof. Let $(X, \omega, I) \in \Omega E_D^s$ be a periodic splitting, with isogenous summands $(E_i, \omega_i) = (\mathbb{C}/\Lambda_i, dz)$, $i = 1, 2$, and with $I = [0, v]$. By periodicity, for $i = 1, 2$ there is a unique primitive vector $e_i \in \Lambda_i$ and $t_i \in (0, 1]$ such that $v = t_i e_i$. Since the lattices Λ_1, Λ_2 rationally generate $\mathrm{Per}(\omega) \otimes \mathbb{Q} \subset \mathbb{R}^2$, and the latter is a vector space over $\mathbb{Q}(\sqrt{D})$, the ratio t_1/t_2 must be irrational; in particular, we can order the summands so that $|e_1| < |e_2|$.

Now let $(Y, \eta) = (E_1, \omega_1) \underset{J}{\#} (E_2, \omega_2)$, where $J = [0, e_1]$. Since the absolute periods of η and ω agree, (Y, η) is also an eigenform [Mc5, Cor 5.6]. But now J maps to a loop in E_1, so η has a double zero. By [Mc3, Thm. 3.3], there is a unique prototypical splitting in the $\mathrm{GL}_2^+(\mathbb{R})$-orbit of (Y, η, J), of type (a, b, c, e) and width $t = 1$. Consequently the orbit of (X, ω, I) also contains a unique prototypical splitting, namely that of type (a, b, c, e) and width $t = t_1$. $\blacksquare$

Corollary 3.7 *Every Teichmüller curve generated by an Abelian differential of genus two is also generated by a prototypical form.*

Proof. Let $f : V \to \mathcal{M}_2$ be a Teichmüller curve generated by (X, ω); then (X, ω) is an eigenform and all its splittings are periodic, by Theorems 3.2 and 3.4. By the preceding result, the orbit of (X, ω) under $\mathrm{GL}_2^+(\mathbb{R})$ contains a prototypical form. $\blacksquare$

11

4 Sifting

As we have just seen, the search for Teichmüller curves of genus two can be reduced to the study of prototypical forms. But if $\mathrm{SL}(X_t, \omega_t)$ is to be a lattice, then *all* of its splittings must be periodic. In this section we will see such periodicity imposes stringent conditions on the width $t \in (0, 1)$.

Slopes. Fix a non-square discriminant $D > 0$, and let

$$K = \mathbb{Q}(\sqrt{D}) \subset \mathbb{R}.$$

We emphasize that K is a real quadratic field with a *fixed* embedding into $\mathbb{R}$.

Let $(X, \omega) \in \Omega E_D$ be an eigenform for real multiplication by $\mathcal{O}_D$. Then $K \cong \mathcal{O}_D \otimes \mathbb{Q}$ acts on the rational homology of X, satisfying

$$\int_{k \cdot C} \omega = k \int_C \omega \tag{4.1}$$

for all $C \in H_1(X, \mathbb{Q})$ and all $k \in K$. This action makes $H_1(X, \mathbb{Q})$ into a vector space over K, which we will denote simply by $H_1(X) \cong K^2$. The K-linear function $I_\omega(C) = \int_C \omega$ sends $H_1(X)$ isomorphically to a dense subset of $\mathbb{C}$.

Recall that the action of K on $H_1(X)$ is self-adjoint with respect to the intersection pairing; that is, it satisfies $(kC) \cdot D = C \cdot kD$. In particular, every 1-dimensional subspace $K \cdot C \subset H_1(X)$ is Lagrangian. The collection of all such subspaces forms a projective line

$$\mathbb{P}H_1(X) \quad = \quad (H^1(X, \mathbb{Q}) - \{0\})/K^* \cong \mathbb{P}^1(K).$$

Given any set $A \subset H_1(X)$, we define $\mathbb{P}A \subset \mathbb{P}H_1(X)$ by

$$\mathbb{P}A = \{K^* \cdot C \; : \; C \in A - \{0\}\}.$$

Periods and periodicity. Any geometric splitting of an eigenform

$$(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2),$$

$I = [0, v]$, gives an algebraic splitting

$$H_1(X) = H_1(E_1) \oplus H_1(E_2), \tag{4.2}$$

where $H_1(E_i) = H_1(E_i, \mathbb{Q})$. Since the summands above are symplectically orthogonal, there is a $k \in K$ such that $kH_1(E_1) = H_1(E_2)$ [Mc5, Lem. 6.3]. Taking the quotient of (4.2) by the action of K^*, we obtain

$$\mathbb{P}H_1(E_1) = \mathbb{P}H_1(E_2) \subset \mathbb{P}H_1(X). \tag{4.3}$$

Note that $\mathbb{P}H_1(E_1)$ is isomorphic to a copy of $\mathbb{P}^1(\mathbb{Q})$ inside $\mathbb{P}H_1(X) \cong \mathbb{P}^1(K)$.

Now assume that the relative and absolute periods of ω span the same rational vector space in $\mathbb{C}$. Then there is a unique class $\langle I \rangle \in H_1(X)$ such that

$$\int_{\langle I \rangle} \omega = \int_I dz = v.$$

Theorem 4.1 *A splitting $(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2)$ of an eigenform is periodic if and only if we have $\mathbb{P}\langle I \rangle \in \mathbb{P}H_1(E_1)$.*

Proof. As we saw in §3, periodicity is equivalent to the condition that $v = t_i e_i$ for some $t_i > 0$ and $e_i \in \mathrm{Per}(\omega_i)$, $i = 1, 2$. This in turn is equivalent to the condition $\langle I \rangle = t_i e_i$ in $H_1(E_i)$, since integration against ω maps $H_1(X)$ injectively to $\mathbb{C}$. But the condition $\langle I \rangle = t_i e_i$ is equivalent to $\mathbb{P}\langle I \rangle \in \mathbb{P}H_1(E_i)$, and since $\mathbb{P}H_1(E_1) = \mathbb{P}H_1(E_2)$, we need only check this condition for $i = 1$. ∎

Prototypical setup. Now let (a, b, c, e) be one of the finitely many splitting prototypes of discriminant D, let $\lambda = (e + \sqrt{D})/2$, and let

$$(X_t, \omega_t) = (E_1, \omega_1) \underset{I_t}{\#} (E_2, \omega_2)$$

be the prototypical splitting of type (a, b, c, e) and width $t \in (0, 1)$. Since the summands in the algebraic splitting

$$H_1(X_t) = H_1(E_1) \oplus H_1(E_2)$$

are independent of t, the vector spaces $H_1(X_t)$ are canonically identified as t varies. Using this isomorphism, we obtain a symplectic basis $(a_i, b_i)_{i=1}^2$ for $H_1(X_t)$ from the bases

$$
\begin{aligned}
\Lambda_1 &= \mathbb{Z}(\lambda, 0) \oplus \mathbb{Z}(0, \lambda) &=& \mathbb{Z}a_1 \oplus \mathbb{Z}b_1, \\
\Lambda_2 &= \mathbb{Z}(b, 0) \oplus \mathbb{Z}(a, c) &=& \mathbb{Z}a_2 \oplus \mathbb{Z}b_2
\end{aligned}
\tag{4.4}
$$

for the period lattices $\Lambda_i = \mathrm{Per}(\omega_i) \cong H_1(E_i, \mathbb{Z})$.

Elementary moves. Consider a homology class $C \in H_1(X, \mathbb{Z})$ that decomposes as $C = B_1 + B_2$ with $B_i \in H_1(E_i, \mathbb{Z})$. We assume:

B_i *is primitive and* $a_i \cdot B_i > 0$ *for* $i = 1, 2$.

For each such class $C = B_1 + B_2$, we will obtain a rationality condition that must be satisfied by t if $\mathrm{SL}(X_t, \omega_t)$ is to be a lattice.

To formulate this condition, choose classes $A_i \in H_1(E_i, \mathbb{Z})$ satisfying $A_i \cdot B_i = 1$, so that the homology of X splits as a symplectic direct sum

$$
\begin{aligned}
H_1(X, \mathbb{Z}) &= L_1 \oplus L_2, \quad \text{with} \\
L_1 &= \mathbb{Z}(A_1 + B_2) \oplus \mathbb{Z}B_1, \\
L_2 &= \mathbb{Z}(A_2 + B_1) \oplus \mathbb{Z}B_2.
\end{aligned}
$$

As in (4.3), we have $\mathbb{P}L_1 = \mathbb{P}L_2$. Next, let

$$
\begin{aligned}
U(C) \;=\; &\{t \in (0, 1) \;:\; (X_t, \omega_t) \text{ splits along an interval } J \text{ satisfying} \\
&\quad \partial[I_t] = \partial[J], \; I_t \cdot J = 1, \text{ and } [J] = [I_t] + C\},
\end{aligned}
$$

13

as in the statement of Theorem 2.2. It is easy to see that $U(C)$ is a (possibly empty) open interval in $(0, 1)$. Finally, let

$$T(C) = \{t \in \mathbb{P}^1(K) \ : \ \mathbb{P}(ta_1 + C) \in \mathbb{P}L_1\}.$$

Note that $T(C) = A(\mathbb{P}^1(\mathbb{Q}))$ for some $A \in \mathrm{PGL}_2(K)$. (We include $t = \infty$ in $T(C)$ if $a_1 \in \mathbb{P}L_1$.) We then have:

Theorem 4.2 *If $t \in U(C)$, then the corresponding splitting of (X_t, ω_t) is periodic if and only if $t \in T(C)$.*

Proof. Let $(X_t, \omega_t) = (F_1, \eta_1) \#_J (F_2, \eta_2)$ denote the splitting corresponding to $t \in U(C)$. If this splitting is periodic, then the relative and absolute periods of ω_t span the same vector space over $\mathbb{Q}$ (as remarked in Theorem 3.3), and thus $t \in K$. Moreover we have $H_1(F_i) = L_i$ by Theorem 2.2, and

$$\langle J \rangle = \langle I_t \rangle + C = ta_1 + C$$

by the definition of $U(C)$. By Theorem 4.1 we have $\mathbb{P}\langle J \rangle \in \mathbb{P}H_1(F_1)$, which is equivalent to $\mathbb{P}(ta_1 + C) \in \mathbb{P}L_1$, and therefore $t \in T(C)$. The converse is similar. ∎

Triples of points. Since $T(C)$ is a copy of $\mathbb{P}^1(\mathbb{Q})$, the result above sifts out all but countably many values of t in the interval $U(C)$.

Note that $T(C)$ is determined by any three distinct points (t_1, t_2, t_3) it contains; namely, it coincides with $A(\mathbb{P}^1(\mathbb{Q}))$ for the unique $A \in \mathrm{PGL}_2(K)$ sending $(0, 1, \infty)$ to (t_1, t_2, t_3). In particular we have:

Theorem 4.3 *If $T(C) \neq T(D)$, then $|T(C) \cap T(D)| \leq 2$.*

By varying the class C, the values of t for which $\mathrm{SL}(X_t, \omega_t)$ can be a lattice can often be reduced to a finite set.

We conclude with a concrete formula for $T(B_1 + B_2)$.

Theorem 4.4 *The set $T(B_1 + B_2)$ contains the three points $t_1, t_2, t_3 \in K$ determined by the conditions*

$$(t_1 a_1 - A_1) \wedge B_1 = (t_2 a_1 - A_2) \wedge B_2 = (t_3 a_1 + B_1) \wedge B_2 = 0,$$

where the wedge products are taken in $\bigwedge_K^2 H_1(X) \cong K$.

(Note that each solution t_i is unique, because $a_i \cdot B_i \neq 0$.)

Proof. To show $t_i \in T(B_1 + B_2)$, it suffices to show that $t_i a_1 + B_1 + B_2$ lies in $K^* \cdot L_1$, or equivalently in $K^* \cdot L_2$ (since $\mathbb{P}L_1 = \mathbb{P}L_2$). For $i = 3$ the vanishing of the wedge product above implies $t_3 a_1 + B_1 = k_3 B_2$ for some $k_3 \in K$, and thus we have

$$t_3 a_1 + B_1 + B_2 = (1 + k_3)B_2 \in K^* \cdot B_2 \subset K^* \cdot L_2$$

14

as claimed.

For $i = 1$ we note that a_1, A_1 and B_1 all lie in $H_1(E_1, \mathbb{Q}) \cong \mathbb{Q}^2$, and thus the vanishing of the wedge product implies $t_1 a_1 = k_1 B_1 + A_1$ with $t_1, k_1 \in \mathbb{Q}$. We then have

$$t_1 a_1 + B_1 + B_2 = (1 + k_1) B_1 + (A_1 + B_2) \in \mathbb{Q}^* \cdot L_1 \subset K^* \cdot L_1.$$

Similarly, when the wedge product vanishes for $i = 2$, we can use the fact that $a_1 = (\lambda/b) a_2$ to conclude that $t_2 a_1 = t_2' a_2 = A_2 + k_2 B_2$, with $t_2', k_2 \in \mathbb{Q}$. Then

$$t_2 a_1 + B_1 + B_2 = (1 + k_2) B_2 + (A_2 + B_1) \in \mathbb{Q}^* \cdot L_2 \subset K^* \cdot L_2$$

as desired. ∎

5 Aperiodicity

Let $D > 0$ be a non-square discriminant, let $p = (a, b, c, e)$ be a splitting prototype of discriminant D, and let

$$(X_t, \omega_t) = (E_1, \omega_1) \underset{I_t}{\#} (E_2, \omega_2)$$

denote the prototypical splitting of type (a, b, c, e) and width $t \in (0, 1)$. In this section we will show:

Theorem 5.1 *There is a $t_0(p) > 0$ such that (X_t, ω_t) admits an aperiodic splitting for all $t \in (0, t_0)$.*

Corollary 5.2 *There is a dense, full-measure, open set $U \subset \Omega E_D$ such that every form $(X, \omega) \in U$ admits an aperiodic splitting.*

To prove Theorem 5.1, we first show explicitly that (X_t, ω_t) has many different splittings when t is small.

Theorem 5.3 *Let $B_i \in H_1(E_i, \mathbb{Z})$ be primitive vectors satisfying $a_i \cdot B_i > 0$ and $K \cdot B_1 = K \cdot B_2$. Then we have $0 \in T(B_1 + B_2)$ and*

$$(0, s) \subset U(B_1 + B_2)$$

for some $s > 0$.

Proof. To show $U(B_1 + B_2)$ contains $(0, s)$, we will show (X_t, ω_t) splits along a suitable interval J for all t sufficiently small.

To construct J, let α_i be the unique geodesic path on $E_i = \mathbb{C}/\Lambda_i$ beginning and ending at $z = 0$ and representing $B_i \in H_1(E_i, \mathbb{Z})$. Since B_i is primitive, the interior of α_i is an embedded arc disjoint from the projection of $I = [0, t\lambda]$ when t is sufficiently small. The same is true for the nearby geodesic arc α_i' beginning

at $z = 0$ and terminating at the other endpoint $z = t\lambda$ of the projection of I. Thus the four arcs $\alpha_1, \alpha_1', \alpha_2, \alpha_2'$ bound a quadrilateral Q on the connected sum $(X_t, |\omega_t|)$, whose short diagonal is one of the oriented saddle connections L corresponding to I (Figure 4).

Since $K \cdot B_1 = K \cdot B_2$, the arcs α_1 and α_2 are parallel, and therefore Q is convex. Its long diagonal M then provides a second saddle connection, which when suitably oriented satisfies

$$
\begin{aligned}
\partial M &= \partial L, \\
[M] &= [L] + B_1 + B_2, \quad \text{and} \\
L \cdot M &= 1
\end{aligned}
$$

(since $a_i \cdot B_i > 0$). Moreover M is disjoint from $\eta(L)$, and thus $\eta(M) \neq M$. By Theorem 2.1, the form (X_t, ω_t) admits a second splitting along an interval $J = [0, w] \subset \mathbb{C}$ satisfying $[J] = [M]$. The conditions above imply $I \cdot J = 1$ and $[J] = [I] + B_1 + B_2$, and therefore $t \in U(B_1 + B_2)$ for all t sufficiently small.

Finally, the condition $K \cdot B_1 = K \cdot B_2$ implies $(ta_1 + B_1) \wedge B_2 = 0$ when $t = 0$, and thus $0 \in T(B_1 + B_2)$ by Theorem 4.4. ∎
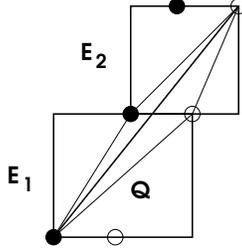

Figure 4. Construction of J when t is small.

Proof of Theorem 5.1. We will show there are classes $C_i \in H_1(X_t, \mathbb{Z})$, $i = 1, 2, 3$, such that $U(C_i) \supset (0, s_i)$, $s_i > 0$, and such that

$$
\begin{aligned}
0, 1 &\in T(C_1), \\
0, 1/2 &\in T(C_2), \quad \text{and} \\
0, b/\lambda &\in T(C_3).
\end{aligned}
$$

For $i = 1$, let $C_1 = B_1 + B_2$ where $B_1 = b_1$ and where B_2 is the unique primitive class in

$$
(K \cdot B_1) \cap H_1(E_2, \mathbb{Z}) \cong \mathbb{Z}
$$

such that $B_2 \cdot a_2 > 0$. Then $U(C_1)$ contains a neighborhood of zero by the previous result, and $0 \in T(C_1)$. Taking $A_1 = a_1$, we find $(a_1 - A_1) \wedge B_1 = 0$, and thus $1 \in T(C_1)$ by Theorem 4.4.

For $i = 2$, let $C_2 = B_1 + B_2$ where $B_1 = a_1 + 2b_1$ and where B_2 is determined by B_1 just as before. Then we have $U(C_2) \supset (0, s_2)$ and $0 \in T(C_2)$. Moreover, taking $A_1 = -b_1$ and $t = 1/2$, we have

$$(ta_1 - A_1) \wedge B_1 = (a_1/2 + b_1) \wedge (a_1 + 2b_1) = 0,$$

and therefore $1/2 \in T(C_2)$.

Finally for $i = 3$, let $C_3 = B_1 + B_2$ where $B_2 = b_1$ and where B_1 is the unique primitive vector in $(K \cdot B_2) \cap H_1(E_1, \mathbb{Z})$ satisfying $a_1 \cdot B_1 > 0$. Then we have $U(C_3) \supset (0, s_3)$ and $0 \in T(C_3)$. Moreover, taking $A_2 = a_2 = ta_1$ with $t = b/\lambda$, we have $(ta_1 - A_2) \wedge B_2 = 0$ and therefore $t \in T(C_3)$.

Now let $T_0 = T(C_1) \cap T(C_2) \cap T(C_3)$. Suppose $|T_0| \geq 3$. Then $T_0 = T(C_i)$ for all i, and in fact $T_0 = \mathbb{P}^1(\mathbb{Q})$, because it contains $0, 1$ and $1/2$. But $b/\lambda \in T(C_3)$ is irrational, so this is impossible.

Thus $|T_0| \leq 2$. Therefore we can choose $0 < t_0 < \min(s_1, s_2, s_3)$ such that $(0, t_0) \cap T_0 = \emptyset$. Then for any $t \in (0, t_0)$, there exists an i such that $t \notin T(C_i)$. On the other hand we have $t \in U(C_i)$, so the corresponding splitting of (X_t, ω_t) is aperiodic by Theorem 4.2. ∎

Width. To prove the Corollary, it is useful to introduce the function $\tau : \Omega \, \mathrm{E}_D^s \to \mathbb{R}$ given by

$$\tau(X, \omega, I) = \begin{cases} |I|/|\int_C \omega| & \text{if the splitting along } I \text{ is periodic, and} \\ 0 & \text{otherwise,} \end{cases}$$

where C is the shortest closed geodesic on $(X, |\omega|)$ with the same slope as I.

The function τ is $\mathrm{GL}_2^+(\mathbb{R})$-invariant, and satisfies $\tau(X_t, \omega_t, I_t) = t$ for a prototypical splitting; thus it extends the notion of width to general splittings. It is straightforward to check that τ is upper semicontinuous; that is,

$$\limsup \tau(X_n, \omega_n, I_n) \leq \tau(X, \omega, I)$$

if $(X_n, \omega_n, I_n) \to (X, \omega, I)$.

Proof of Corollary 5.2. Let P_D be the finite set of splitting prototypes $p = (a, b, c, e)$ of discriminant D. By the preceding Theorem, for each $p \in P_D$ there is a $t_0(p) > 0$ such that every prototypical form (X_t, ω_t) of type p with $t < t_0(p)$ has an aperiodic splitting. Let $\tau_0 = \min(t_0(p) : p \in P_D)$, and let

$$U = \{(X, \omega) \in \Omega \, \mathrm{E}_D(1,1) \, : \, \tau(X, \omega, I) < \tau_0 \text{ for some } I.\}$$

We claim that every (X, ω) in U has an aperiodic splitting. This is immediate if (X, ω) splits along an interval I with $\tau(X, \omega, I) = 0$. Otherwise, X has a periodic splitting with $0 < \tau(X, \omega, I) = t < \tau_0$. But then (X, ω) is $\mathrm{GL}_2^+(\mathbb{R})$ equivalent to a prototypical splitting (X_t, ω_t) of type p and width t. Since $t < \tau_0 \leq t_0(p)$, the form (X_t, ω_t) has an aperiodic splitting, so (X, ω) does as well.

17

The set U is nonempty because it contains the prototypical forms of small width, and it is open by semicontinuity of τ. Moreover U is $\mathrm{GL}_2^+(\mathbb{R})$-invariant. Since the action of $\mathrm{GL}_2^+(\mathbb{R})$ is ergodic [Mc5, Thm. 1.5], U is a dense open set of full measure in ΩE_D. ∎

6   Finiteness

In this section we establish:

Theorem 6.1 *If D is not a square, then the set of $(X, \omega) \in \Omega E_D$ such that $\mathrm{SL}(X, \omega)$ is a lattice falls into finitely many orbits under the action of $\mathrm{GL}_2^+(\mathbb{R})$.*

Corollary 6.2 *There are only finitely many Teichmüller curves in $\mathcal{M}_2$ with a given non-square discriminant D.*

For the proof we recall:

Lemma 6.3 *Let Y be a single orbit in a closed, $\mathrm{GL}_2^+(\mathbb{R})$-invariant set $\overline{Z} \subset \Omega E_D$. If Y is not open in $\overline{Z}$, then $\mathrm{int}(\overline{Z}) \neq \emptyset$.*

See [Mc5, Lem. 12.3].

Proof of Theorem 6.1. Suppose to the contrary that we can find an infinite sequence of forms $(X_i, \omega_i) \in \Omega E_D$, $i = 1, 2, 3, \ldots$, such that $\mathrm{SL}(X_i, \omega_i)$ is a lattice and such that

$$Z = \bigcup_1^\infty \mathrm{GL}_2^+(\mathbb{R}) \cdot (X_i, \omega_i)$$

is a countable union of disjoint orbits.

Let $U \subset \Omega E_D$ be the open, dense set of forms with aperiodic splittings provided by Corollary 5.2. Since every splitting of every form (X_i, ω_i) is periodic, we have $Z \cap U = \emptyset$. Moreover we can normalize by the action of $\mathrm{GL}_2^+(\mathbb{R})$ so that (X_i, ω_i) is a prototypical form of type $p_i = (a_i, b_i, c_i, e_i)$ and width $t_i \in (0, 1]$. There are only finitely many prototypes of discriminant D, so after passing to a subsequence we can assume that $p_i = p$ is constant.

Next, note that $\inf t_i > 0$; otherwise one of the forms (X_i, ω_i) would have an aperiodic splitting, by Theorem 5.1. Thus upon passing to a further subsequence, we can assume $t_i \to t > 0$, and $t_i \neq t$ for all i. Let $Y \subset \overline{Z}$ be the $\mathrm{GL}_2^+(\mathbb{R})$ orbit of the prototypical form (X_t, ω_t) of type p and width t. Then Y is not open in $\overline{Z}$, and thus $\overline{Z}$ has nonempty interior by the Lemma above. But this is impossible, because Z is disjoint from the open, dense set U. ∎

7   Characterization of Teichmüller curves

In this section we establish:

Theorem 7.1 *Let (X, ω) be a holomorphic 1-form of genus two. Then the following are equivalent.*

1. $\mathrm{SL}(X, \omega)$ *is a lattice.*

2. *For every slope s, the foliation $\mathcal{F}_s(\omega)$ is either periodic or uniquely ergodic.*

3. *For every slope s, the foliation $\mathcal{F}_s(\omega)$ is either minimal or periodic.*

(Here $\mathcal{F}_s(\omega)$ is *minimal* if every leaf disjoint from the zeros of ω is dense.) The implication (2) $\implies$ (1) is a converse to the Veech dichotomy in genus two, while (3) $\implies$ (2) shows:

Corollary 7.2 *Let (X, ω) be a form of genus two. If every geodesic parallel to a saddle connection is closed, then every other geodesic is uniformly distributed.*

Cylinders and eigenforms. We begin with a dynamical characterization of eigenforms. Recall that a *cylinder* for $(X, |\omega|)$ is a maximal open annulus $C \subset X$ foliated by closed geodesics of constant slope s.

Theorem 7.3 *For any $(X, \omega) \in \Omega\mathcal{M}_2$, the following conditions are equivalent:*

1. (X, ω) *is an eigenform for real multiplication.*

2. *Whenever $(X, |\omega|)$ has a cylinder of slope s, the foliation $\mathcal{F}_s(\omega)$ is periodic.*

A similar resulted was announced in [Ca]. For the proof, we will use:

Lemma 7.4 *Let $I = [0, v] \subset E = \mathbb{R}^2/\mathbb{Z}^2$ be an embedded segment on the square torus. Then there are infinitely many slopes of simple closed geodesics $L \subset E - I$.*

Proof. If I has rational slope, then it is contained in a simple closed geodesic $M \subset E$, and for L we can take any simple geodesic passing through $p \in M - I$ with intersection number $|L \cdot M| \leq 1$. These geodesics represent infinitely many elements of $H_1(E, \mathbb{Z})$, and hence infinitely many slopes.

On the other hand, if $I = [0, v]$ has irrational slope, then for any $\epsilon > 0$ there is an $A \in \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{Diff}(E)$ such that $|A(I)| < \epsilon$. (This follows from the fact that $\mathrm{SL}_2(\mathbb{Z}) \cdot v$ is dense in $\mathbb{R}^2$.) Since A sends geodesics to geodesics, it follows that the number of slopes of geodesics disjoint from I is arbitrarily large. ∎

Proof of Theorem 7.3. We first show (1) implies (2). Suppose (X, ω) is an eigenform, and $C \subset X$ is a cylinder of slope s. By [Mc5, Thm. 7.4], there is a splitting $(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2)$ such that a loop of ∂C is contained in one of the two summands, say $E_1 - I$. It follows that the geodesics of slope s are closed on both (E_1, ω_1) and (E_2, ω_2), since the summands are isogenous. Therefore those leaves of $\mathcal{F}_s(\omega)$ that are contained entirely in one summand are also closed. On the other hand, any closed geodesic of slope s on E_1 that meets I does so only once, since there is a parallel geodesic disjoint from I. Therefore any leaf of $\mathcal{F}_s(\omega)$ that crosses from E_2 to E_1 immediately returns to E_2 in the same position. Hence these leaves also close up on X, and therefore $\mathcal{F}_s(\omega)$ is periodic.

To show (2) implies (1), consider any splitting $(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2)$. Order the summands so that $I = [0, v]$ projects to an embedded arc in E_1. Suppose a slope $s \in \mathbb{P}\operatorname{Per}(\omega_1)$ is represented by a closed geodesic on (E_1, ω_1) disjoint from I. Then (X, ω) has a cylinder C of slope s, carried by E_1. Assumption (2) implies all geodesics of slope s on (X, ω) are periodic, and thus $s \in \mathbb{P}\operatorname{Per}(\omega_2)$ as well.

Using the Lemma, we can obtain in this way infinitely many slopes shared by $\mathbb{P}\operatorname{Per}(\omega_1)$ and $\mathbb{P}\operatorname{Per}(\omega_2)$. But $\mathbb{P}\operatorname{Per}(\omega_1)$ and $\mathbb{P}\operatorname{Per}(\omega_2)$ are simply copies of $\mathbb{P}^1(\mathbb{Q})$ inside $\mathbb{P}^1(\mathbb{R})$, so once they share three points, they are equal. Their equality implies the given splitting of (X, ω) has isogenous summands; and since the splitting was arbitrary, Theorem 3.5 implies (X, ω) is an eigenform. ∎

Lattices and periodic splittings. Next we characterize eigenforms that generate Teichmüller curves.

Theorem 7.5 *Let (X, ω) be an eigenform. Then $\operatorname{SL}(X, \omega)$ is a lattice if and only if every splitting of (X, ω) is periodic.*

Proof. If $\operatorname{SL}(X, \omega)$ is a lattice, then every splitting of (X, ω) is periodic by the Veech dichotomy (Theorem 3.1).

Now assume (X, ω) is an eigenform of discriminant D and every splitting of (X, ω) is periodic.

Suppose D is not a square. Then there is an nonempty open set $U \subset \Omega E_D$ consisting of forms with aperiodic splittings (Corollary 5.2). Therefore

$$Z = \overline{\operatorname{GL}_2^+(\mathbb{R}) \cdot (X, \omega)} \neq \Omega E_D;$$

but this can only happen when $\operatorname{SL}(X, \omega)$ is a lattice, by the classification of orbit closures for the action of $\operatorname{SL}_2(\mathbb{R})$ on $\Omega_1 \mathcal{M}_2$ given in [Mc5, Thm. 1.2].

Finally, suppose $D = d^2$ is a square. Then $\Lambda = \operatorname{Per}(\omega)$ is a lattice in $\mathbb{C}$, and there is a degree d map $p : X \to E = \mathbb{C}/\Lambda$ such that $\omega = p^*(dz)$. Since every splitting of (X, ω) is periodic, the relative and absolute periods of ω span the same space over $\mathbb{Q}$ (Theorem 3.3), and therefore the difference $c_1 - c_2$ of the critical values of p is a torsion point on E. Hence by [GJ], $\operatorname{SL}(X, \omega)$ is a lattice in this case as well, namely one commensurable to $\operatorname{SL}_2(\mathbb{Z})$. ∎

Finally we characterize Teichmüller curves themselves.

Proof of Theorem 7.1. By the Veech dichotomy (Theorem 3.1 and the remark that follows), (1) implies (2) and (3).

Now assume (2). Then every cylindrical direction is periodic, and hence (X, ω) is an eigenform (Theorem 7.3). Moreover, every splitting of (X, ω) is periodic, since it cannot be uniquely ergodic (cf. Theorem 3.2). By the preceding result, this periodicity implies $\mathrm{SL}(X, \omega)$ is a lattice, and thus (2) implies (1).

The same reasoning applies with minimality replacing unique ergodicity, so (3) also implies (1). ∎

8 Curves with $D = 5$

The sifting technique introduced in §4 can be used to explicitly determine all the Teichmüller curves with a given discriminant. In this section we demonstrate the method by showing:

Theorem 8.1 *There are exactly two Teichmüller curves with discriminant $D = 5$: one generated by the regular pentagon, and one generated by the regular decagon.*
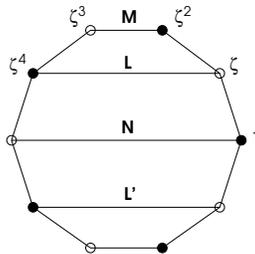
Figure 5. The regular decagon.

The regular decagon. Let $P \subset \mathbb{C}$ be the regular decagon with vertices $\{\zeta^i, 0 \le i \le 9\}$, where $\zeta = e^{2\pi i/10}$ is a primitive tenth root of unity (Figure 5). Identifying opposite sides of P, we obtain a holomorphic 1-form $(X, \omega) = (P, dz)/\sim$ of genus two. Note that the vertices of P fall in to two equivalence classes, and thus $(X, \omega) \in \Omega\mathcal{M}_2(1, 1)$. As shown by Veech, $\mathrm{SL}(X, \omega)$ is a triangle group of signature $(5, \infty, \infty)$; in particular, (X, ω) generates a Teichmüller curve in genus two [V1].

Since P has 10-fold symmetry, $\mathrm{Jac}(X)$ admits complex multiplication by $\mathbb{Z}[\zeta]$ with ω as an eigenform. Noting that $\zeta + \zeta^{-1} = \gamma$, where $\gamma = (1 + \sqrt{5})/2$ is the golden ratio, we see that $\mathbb{Z}[\zeta]$ contains the real subring $\mathcal{O}_D$ with $D = 5$. Thus (X, ω) belongs to ΩE_5.

Lemma 8.2 *The Teichmüller curve generated by the regular decagon is also generated by the prototypical form of type $(0, 1, 1, -1)$ and width $t = (2 + \gamma)/5$.*

Proof. The foliation of P by horizontal lines descends to a periodic foliation of $(X, |\omega|)$ by closed, horizontal geodesics, including four saddle connections labeled L, L', M and N in Figure 5. The hyperelliptic involution of X corresponds to a $180°$ rotation of P, and thus $\eta(L) = L'$. Therefore (X, ω) admits a periodic splitting

$$(X, \omega) = (E_1, \omega_1) \#_I (E_2, \omega_2)$$

along $L \cup L'$, where $I = [0, v]$ and $v = \int_L \omega = \zeta - \zeta^4$.

Since $p = (0, 1, 1, -1)$ is the only splitting prototype of discriminant $D = 5$, (X, ω, I) is $\mathrm{GL}_2^+(\mathbb{R})$-equivalent to a unique prototypical splitting (X_t, ω_t, I_t) of type $(0, 1, 1, -1)$ and width t. To compute the width, we note that t is simply the ratio of $|L|$ to the length of the shorter of the two closed horizontal geodesics on E_1 and E_2 respectively. These geodesics are represented by $L \cup N$ and $L \cup M$; since $|M| < |N|$, we have

$$t = \frac{|L|}{|L| + |M|} = \frac{\zeta - \zeta^4}{\zeta - \zeta^4 + \zeta^2 - \zeta^3} = \frac{2 + \gamma}{5}.$$

∎

Prototype $(0, 1, 1, -1)$		
C	$U(C)$	$T(C)$
$(0, 1, 0, 1)$	$(0, 1)$	$(0, 1, \gamma)$
$(1, 2, 1, 2)$	$(0, \gamma^{-1})$	$(0, 1/2, \gamma/2)$
$(0, 1, 1, 2)$	$(1/2, 1)$	$(1/2, 1, \gamma/2)$

Table 6. Splitting information for $D = 5$.

Proof of Theorem 8.1. Let (X_t, ω_t) be the prototypical form of type $(a, b, c, e) = (0, 1, 1, -1)$ and width $t \in (0, 1]$. Then $\lambda = (-1 + \sqrt{5})/2 = \gamma^{-1}$, where γ is the golden ratio.

By Corollary 3.7 and uniqueness of the splitting prototype, every Teichmüller curve with discriminant $D = 5$ is generated by (X_t, ω_t) for some t. The case $t = 1$ gives the regular pentagon (a form with a double zero), so we may assume $t < 1$.

The homology basis (a_1, b_1, a_2, b_2) given in (4.4) provides an isomorphism $H_1(X_t, \mathbb{Z}) \cong \mathbb{Z}^4$. Let

$$C_1 = (0, 1, 0, 1), \quad C_2 = (1, 2, 1, 2), \quad C_3 = (0, 1, 1, 2)$$

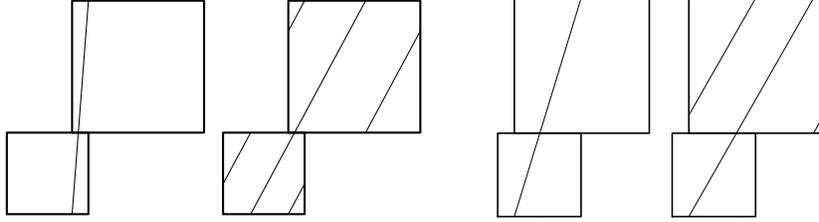be homology classes with respect to these coordinates.

Figure 7. Pairs of splittings, near $t = 0$ and $t = 1$

By Theorem 5.3, $U(C_1)$ and $U(C_2)$ each contains a neighborhood of $t = 0$, since they have the form $C_i = (p, q, p, q)$. In fact, it is straightforward to check that $U(C_1) = (0, 1)$, $U(C_2) = (0, \gamma^{-1})$ and $U(C_3) = (1/2, 1)$. Thus (X_t, ω_t) resplits along C_1 and C_2 when t is near 0, and along C_1 and C_3 when t is near 1 (Figure 7).

Theorem 4.4 allows one to explicitly compute three representative points in $T(C_i)$ for $i = 1, 2, 3$. The results are summarized in Table 6. The remainder of $T(C_i) \cong \mathbb{P}^1(\mathbb{Q})$ consists of those $x \in \mathbb{P}^1(\mathbb{R})$ having rational cross-ratio with respect to the three given points. Using this fact, we find:

$$
\begin{aligned}
T(C_1) \cap T(C_2) &= \{0, \infty\}, \quad \text{and} \\
T(C_1) \cap T(C_3) &= \{(2 + \gamma)/5, 1\}.
\end{aligned}
$$

Now suppose $t \in (0, 1)$ and $\mathrm{SL}(X_t, \omega_t)$ is a lattice. Then every splitting of (X_t, ω_t) is periodic. By Theorem 4.2, if t lies in $U(C_i)$ then it must lie in $T(C_i)$ as well.

Since $U(C_1) \cap U(C_2) = (0, \gamma^{-1})$ and $T(C_1) \cap T(C_2) \cap (0, 1) = \emptyset$, we must have $t \geq \gamma^{-1}$. But then we have $t \in U(C_1) \cap U(C_3) = (1/2, 1)$. Since $T(C_1) \cap T(C_3) \cap (0, 1) = \{(2 + \gamma)/5\}$, only one value of $t \in (0, 1)$ yields a lattice, namely the value corresponding to the regular decagon. ∎

9 Curves with $D \leq 400$

As mentioned in the Introduction, the following conjecture implies that all Teichmüller curves of genus two are known.

Conjecture 9.1 *Let $D > 5$ be a non-square discriminant, and let $(X, \omega) \in \Omega E_D(1, 1)$ be an eigenform of discriminant D with simple zeros. Then $\mathrm{SL}(X, \omega)$ is not a lattice.*

In this section we describe the proof of:

Theorem 9.2 *The conjecture above holds for all $D \leq 400$.*

23

The proof is based on an algorithm that effectively determines all the Teichmüller curves with a given non-square discriminant D.

The algorithm. For $\epsilon > 0$ we define (in the notation of §4)

$$T^\epsilon(C) = \{t \in K \; : \; ta_1 + C \in k(L_1 \cup L_2) \text{ for some } k \in K, |k| > \epsilon\}.$$

Since the lattices L_1 and L_2 are discrete, $T^\epsilon(C) \cap (0,1)$ is finite. The definition is arranged so that if $t \in U(C) \cap T(C)$, then $t \in T^\epsilon(C)$ if and only if the corresponding new splitting (X_t, ω_t, J) is equivalent to a prototypical splitting of width greater than ϵ.

To test the conjecture above for a given value of D, one may proceed as follows.

1. Begin by enumerating the finitely many splitting prototypes P_D of discriminant D.

2. For each $p \in P_D$, find a set of open intervals $I_i \subset (0,1)$ and homology classes C_i, D_i, $i = 1, \ldots, n_p$, such that

   (a) $\bigcup I_i$ contains a neighborhood of $t = 0$;

   (b) $I_i \subset U(C_i) \cap U(D_i)$ and

   (c) $T(C_i) \neq T(D_i)$.

   Here $U(C_i)$ and $T(C_i)$ are taken with respect to the prototypical splitting of type p. Let $T_p = \bigcup(I_i \cap T(C_i) \cap T(D_i))$, and let $K_p = (0,1) - \bigcup I_i$.

3. If T_p and K_p are empty for all $p \in P_D$, we are done — the conjecture is verified for D.

4. Otherwise, let $\epsilon = \inf(\bigcup T_p \cup K_p) > 0$. For each $p \in P_D$, find a finite set of classes C_i such that $K_p \subset \bigcup U(C_i)$. Then let S_p be the finite set

$$S_p = T_p \cup \bigcup(K_p \cap U(C_i) \cap T^\epsilon(C_i)).$$

5. If S_p is empty for all p, again we are done — the conjecture is true for D.

6. Otherwise, let $S = \bigcup(\{p\} \times S_p) \subset P_D \times (0,1)$. Consider $S \cup \{*\}$ as the vertex set of a finite graph G, initially with no edges.

   For each $(p, t) \in S$, construct one or more new splittings (X_t, ω_t, J) of the prototypical form of type p and width t. If the new splitting (X_t, ω_t, J) is aperiodic, add an edge to G connecting (p, t) to $\{*\}$. Otherwise, the new splitting is $\mathrm{GL}_2^+(\mathbb{R})$ equivalent to a prototypical splitting of type and width (q, s). If $(q, s) \in S$, connect it to (p, t) by an edge; otherwise, connect (p, t) to $\{*\}$.

7. If G is connected, the conjecture is verified.

8. Otherwise, choose one vertex from each component of G not containing the vertex $*$, and let their union be the finite set S'. Then any Teichmüller curve providing an exception to the conjecture is generated by a form of type and width $(p, t) \in S'$.

Justification. To explain the algorithm, we first recall that every Teichmüller curve is generated by a prototypical form (Corollary 3.7). Thus to verify the conjecture, it suffices to determine for each $p \in P_D$, the set of widths $t \in (0, 1)$ such that $\mathrm{SL}(X_t, \omega_t)$ is a lattice. At the conclusion of step (2), we know (by Theorem 4.2) that for $\mathrm{SL}(X_t, \omega_t)$ to be a lattice, we must have $t \in T_p \cup K_p$. (The proof of Theorem 5.1 shows how to insure that a neighborhood of $t = 0$ is covered by $\bigcup I_i$ in step 2.)

Often $\bigcup T_p \cup K_p$ is empty (see e.g. the case $D = 8$ below), so the conjecture is already verified by step (3). If not, we can at least conclude in step (4) that $\mathrm{SL}(X_t, \omega_t)$ is never a lattice when $t \leq \epsilon$. Thus we can replace $T(C_i)$ with $T^\epsilon(C_i)$, and thereby obtain a finite set S of pairs (p, t) accounting for every possible Teichmüller curve of discriminant D.

To check that remaining steps, just observe that

  (i) every splitting of a Teichmüller curve of discriminant D is equivalent to one of type and width $(p, t) \in S$; therefore

 (ii) the vertices (p, t) adjacent to $*$ in the graph G do not generate Teichmüller curves; and

(iii) vertices in the same component of G label the same $\mathrm{GL}_2^+(\mathbb{R})$-orbit, so if one generates a Teichmüller curve, they all do.

As we saw in the previous section, when $D = 5$ the algorithm actually locates the regular decagon and proves it generates the only exceptional Teichmüller curve for this discriminant. Here are two more examples.

Prototype $(0, 2, 1, 0)$		
C	$U(C)$	$T(C)$
$(0, 1, 0, 1)$	$(0, 1)$	$(0, 1, \sqrt{2})$
$(1, 1, 1, 2)$	$(0, 1)$	$(0, 1, \sqrt{2}/2)$

Prototype $(0, 1, 1, -2)$		
C	$U(C)$	$T(C)$
$(0, 1, 0, 1)$	$(0, 1)$	$(0, 1, 1 + \sqrt{2})$
$(1, 2, 1, 2)$	$(0, 2 - \sqrt{2})$	$(0, 1/2, (1 + \sqrt{2})/2)$
$(0, 1, 1, 2)$	$(1/2, 1)$	$(1/2, 1, (1 + \sqrt{2})/2)$

Table 8. Splitting information for $D = 8$.

$D = 8$. In this case there are only two splitting prototypes. With the classes chosen in Table 8, we obtain $T_p = K_p = \emptyset$ for both types, establishing the conjecture without recourse to the graph G. (As in §8, we use the basis (a_1, b_1, a_2, b_2) to describe classes $C \in H_1(X_t, \mathbb{Z})$, and we give three representative points to describe $T(C)$.)

Prototype $(0,1,3,-5)$		
C	$U(C)$	$T(C)$
$(0,1,0,1)$	$(0,1)$	$(0,1,(5+\sqrt{37})/6)$
$(1,2,3,2)$	$(0,(-5+\sqrt{37})/2)$	$(0,1/2,(5+\sqrt{37})/12)$
$(0,1,1,2)$	$(1/6,1)$	$(1/6,1,(5+\sqrt{37})/12)$

Prototype $(0,3,3,-1)$		
C	$U(C)$	$T(C)$
$(0,1,0,1)$	$(0,1)$	$(0,1,(1+\sqrt{37})/6)$
$(1,2,1,2)$	$(0,(3+\sqrt{37})/14)$	$(0,1/2,(1+\sqrt{37})/12)$

Table 9. Splitting information for $D = 37$.

$D = 37$. This case is more typical. There are nine splitting prototypes, of which two are of particular note (see Table 9).

For the prototype $p_1 = (0,1,3,-5)$, we obtain a complete covering of $(0,1)$ in step (2). However, if we use the classes $C_1 = (0,1,0,1)$ and $D_1 = (0,1,1,2)$ to handle a neighborhood of $t = 1$, we obtain $T_{p_1} = \{t_1\} \subset T(C_1) \cap T(D_1)$, where $t_1 = (9 + \sqrt{37})/22 \approx 0.68558$.

For the prototype $p_2 = (0,3,3,-1)$, T_{p_2} is empty but we have $K_{p_2} = [s,1)$ where $s = (3 + \sqrt{37})/14 \approx 0.648769$. This occurs because $C_1 = (0,1,0,1)$ is essentially the only class such that $U(C_1)$ contains a neighborhood of $t = 1$.

For every other $p \in P_D$, we have $T_p = K_p = \emptyset$.

Since $s < t_1$, we obtain $\epsilon = s$ in step (4). This leads to

$$S_{p_2} = T^{\epsilon}(C_1) = \{t_2, u_2\} = \{(9 + 3\sqrt{37})/14,\ (7 + 2\sqrt{37})/11\},$$

and finally to $S = \{(p_1, t_1), (p_2, t_2), (p_2, u_2)\}$. Luckily, it turns out that every $(p,t) \in S$ admits a splitting of type (q,s) with $q \notin \{p_1, p_2\}$. Thus all the vertices of G are connected to $*$, and we obtain a proof of the conjecture for $D = 37$.

$D \leq 400$. It is straightforward to automate the procedure just described, and obtain a detailed, if lengthy, proof of Theorem 9.2. To conclude, we comment on the general behavior of the algorithm for the 180 non-square discriminants with $5 \leq D \leq 400$.

First, the number of prototypes grows, albeit unevenly, as D does (Figure 10). (Note that $|P_D|$ is the same as the number of cusps of the Weierstrass curve W_D [Mc3, Cor 4.5].) The maximum number of prototypes, $|P_D| = 128$, is attained when $D = 385$. On the other hand, the coverings $\langle I_i, i = 1, \ldots n_p \rangle$ used in step (2) of the algorithm remain rather small; the average value of n_p is 1.58, and its maximum value is $n_p = 6$ (attained for $p = (8,9,9,e)$ with $e = -6, -7$ and -8).
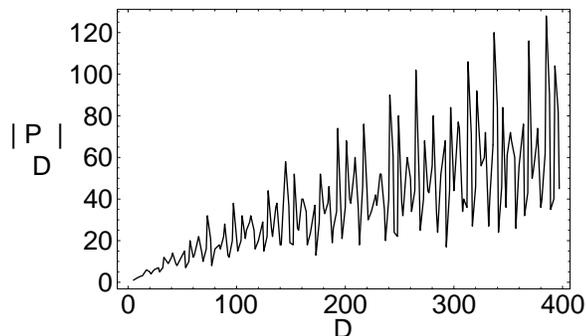
Figure 10. Number of splitting prototypes as a function of D.

Finally, we note that for 80 of the 180 values of D, the algorithm successfully terminates at step (3). The value of $|S|$ in step (4) is usually 1 and never more than 5; the maximum is attained when $D = 200$.

It would be interesting to have a conceptual explanation for the algorithm's success.

References

[Ca]   K. Calta. Veech surfaces and complete periodicity in genus 2. *J. Amer. Math. Soc.* **17**(2004), 871–908.

[EMS] A. Eskin, H. Masur, and M. Schmoll. Billiards in rectangles with barriers. *Duke Math. J.* **118**(2003), 427–463.

[EO]   A. Eskin and A. Okounkov. Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials. *Invent. math.* **145**(2001), 59–103.

[GJ]   E. Gutkin and C. Judge. Affine mappings of translation surfaces: geometry and arithmetic. *Duke Math. J.* **103**(2000), 191–213.

[HL]   P. Hubert and S. Lelièvre. Square-tiled surfaces in $\mathcal{H}(2)$. *Preprint, 2004.*

[KS]   R. Kenyon and J. Smillie. Billiards on rational-angled triangles. *Comment. Math. Helv.* **75**(2000), 65–108.

[Lei] C. J. Leininger. On groups generated by two positive multi-twists: Teichmüller curves and Lehmer's number. *Geom. Topol.* **8**(2004), 1301–1359.

[Lo]   P. Lochak. On arithmetic curves in the moduli space of curves. *Preprint, 2003.*

[Mc1] C. McMullen. Billiards and Teichmüller curves on Hilbert modular surfaces. *J. Amer. Math. Soc.* **16**(2003), 857–885.

[Mc2] C. McMullen. Teichmüller geodesics of infinite complexity. *Acta Math.* **191**(2003), 191–223.

[Mc3] C. McMullen. Teichmüller curves in genus two: Discriminant and spin. *Math. Ann.* **333**(2005), 87–130.

[Mc4] C. McMullen. Teichmüller curves in genus two: Torsion divisors and ratios of sines. *Invent. math.* **165**(2006), 651–672.

[Mc5] C. McMullen. Dynamics of $SL_2(\mathbf{R})$ over moduli space in genus two. *Annals of Math.* **165**(2007), 397–456.

[Me] J.-F. Mestre. Familles de courbes hyperelliptiques à multiplications réelles. In *Arithmetic Algebraic Geometry (Texel, 1989)*, volume 89 of *Progr. Math.*, pages 193–208. Birkhaüser Boston, 1991.

[Mo] M. Möller. Variations of Hodge structures of a Teichmüller curve. *J. Amer. Math. Soc.* **19**(2006), 327–344.

[Pu] J.-C. Puchta. On triangular billiards. *Comment. Math. Helv.* **76**(2001), 501–505.

[Th] W. P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc.* **19**(1988), 417–431.

[V1] W. Veech. Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Invent. math.* **97**(1989), 553–583.

[V2] W. Veech. The billiard in a regular polygon. *Geom. Funct. Anal.* **2**(1992), 341–379.

[Vo] Ya. B. Vorobets. Plane structures and billiards in rational polygons: the Veech alternative. *Russian Math. Surveys* **51**(1996), 779–817.

[Wa] C. C. Ward. Calculation of Fuchsian groups associated to billiards in a rational triangle. *Ergod. Th. & Dynam. Sys.* **18**(1998), 1019–1042.

MATHEMATICS DEPARTMENT
HARVARD UNIVERSITY
1 OXFORD ST
CAMBRIDGE, MA 02138-2901