Differential and algebraic geometry of earthquakes

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Abstract

Every orientation–preserving circle homeomorphism $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ extends to a unique right earthquake $F : \mathbb{H} \to \mathbb{H}$ shearing along the leaves of a measured lamination (Λ, m_Λ). The passage from f to F is global and nonlinear.

In this paper we obtain explicit formulas for recovering F, Λ and m_Λ directly from f, provided f is sufficiently smooth. For this purpose we introduce a two–variable version of the Schwarzian derivative, $S(f \times f)$, whose zero set and gradient encode Λ and m_Λ, and thereby determine F.

This perspective reveals new connections with algebraic geometry and analysis. Consequences include a characterization of the earthquake laminations $\Lambda \subset \mathbb{H}$ arising from sextic vector fields, and a sharp bilipschitz regularity theorem for earthquake extensions of $C^{1,1}$ diffeomorphisms.

Contents

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1 Introduction

Thurston showed that any orientation–preserving homeomorphism

$$f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$$

can be realized as the boundary values of a unique right earthquake

$$F : \mathbb{H} \to \mathbb{H},$$

itself geometrically determined by a unique measured lamination (Λ, m_Λ). The map F presents f as an amalgam of hyperbolic isometries indexed by the blocks of the convex partition of $\mathbb{H}$ determined by Λ (see §2).

The passage from f to the triple (F, Λ, m_Λ) is global, nonlinear, and somewhat mysterious.

Differential geometry. In this paper we obtain explicit formulas for recovering F, Λ and m_Λ directly from f when it is sufficiently smooth. To express these results succinctly, we introduce a two–variable version of the Schwarzian derivative, $S(f \times f)$, which lives on the space of geodesics in $\mathbb{H}$.

We initially focus on the case where f is real–analytic, since this guarantees that $T = \widehat{\mathbb{R}}/\Lambda$ is a finite tree. We show that the endpoints of T are controlled by the zeros and poles of the ordinary Schwarzian Sf, while its interior vertices correspond to tritangent planes of an associated space curve.

A parallel story is developed for the earthquake extension V of a vector field v on $\widehat{\mathbb{R}}$.

Algebraic maps. We then specialize to the algebraic setting, where $f(x) = P(x)/Q(x)$ is a *rational function*. In this case we show the topological complexity of Λ is explicitly controlled by the algebraic degree of f.

For sextic vector fields, the first case with moduli, we obtain a surprising *characterization* of the geometric regions

$$\mathbb{H} - \Lambda = \bigcup_1^m P_i$$

that can occur. The answer involves a subtle separation condition that can be expressed in terms of rational maps or potential theory.

Power maps. Returning to the analytic realm, we also study the earthquake extensions for the group of homeomorphisms

$$f_\alpha(x) = |x|^\alpha \operatorname{sign}(x),$$

$\alpha > 0$. These arise naturally in Teichmüller theory from extremal maps between cylinders.

Regularity. Suitably interpreted, the formulas for earthquakes in the real–analytic setting apply much more broadly. As one demonstration of this breadth, we conclude by establishing an optimal regularity theorem:

> *The earthquake $F : \mathbb{H} \to \mathbb{H}$ is bilipschitz whenever f is a $C^{1,1}$ diffeomorphism.*

We also show that $|\overline{\partial} F/\partial F|$ lies in $L^2(\mathbb{H})$.

1.1 Definitions

The hyperbolic plane. Let $\mathbb{H} \subset \mathbb{C}$ denote the upper halfplane model for the hyperbolic plane, endowed with the usual metric $\rho = |dz|/\operatorname{Im}(z)$ of constant curvature -1. Its boundary will be denoted by

$$\partial\mathbb{H} = \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\},$$

and its isometry group by

$$G = \operatorname{Isom}^+(\mathbb{H}) = \operatorname{PSL}_2(\mathbb{R}).$$

We let $\overline{\mathbb{H}} = \mathbb{H} \cup \widehat{\mathbb{R}}$.

The space of geodesics. The natural setting for our earthquake formulas is the space of geodesics

$$\mathbb{G} = \{(a, b) \in \widehat{\mathbb{R}}^2 \ : \ a \neq b\},$$

endowed with the G–invariant Lorentz metric

$$\xi = \frac{4da\, db}{(a-b)^2}.$$

A point $(a, b) \in \mathbb{G}$ corresponds to the geodesic $[a, b] \subset \mathbb{H}$ with endpoints $a, b \in \widehat{\mathbb{R}}$.

A set $L \subset \mathbb{G}$ is *timelike* if it determines a collection of disjoint geodesics in $\mathbb{H}$. The terminology comes from the fact that any two points in L can be joined by a timelike path in $\mathbb{G}$, up to changing (a, b) to (b, a).

Laminations. A *measured lamination* is a closed set $\Lambda \subset \mathbb{H}$, presented as a union of disjoint geodesics

$$\Lambda = \bigcup_{\alpha} [a_\alpha, b_\alpha],$$

and equipped with a transverse measure m_Λ.

A measured lamination can also be specified by an ordinary measure m_L on a closed timelike set $L \subset \mathbb{G}$, both invariant under $(a,b) \mapsto (b,a)$. The set L records the leaves of Λ, and m_L records its transverse measure. We refer to (L, m_L) as the *dual* of (Λ, m_Λ).

1.2 Statement of results

Our main results on earthquakes for real–analytic maps are Theorems 1.1 and 1.2 below (proved in §6).

The two–variable Schwarzian. To state these results, we define the *two–variable Schwarzian derivative* for smooth maps $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ by

$$S(f \times f) = (f \times f)^* \xi - \xi.$$

This symmetric two–form on $\mathbb{G}$ vanishes when $f \in G$, and its extension to the diagonal $a = b$ yields the usual one–variable Schwarzian Sf.

Theorem 1.1 *Let $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ be a real–analytic homeomorphism, and let (L, m_L) be the dual of its right earthquake lamination (Λ, m_Λ). Then:*

- *L is a finite union of arcs in the real–analytic subvariety Z of $\mathbb{G}$ defined by $S(f \times f) = 0$; and*
- *We have $m_L = |{*du}|/2$ on L, where $u = S(f \times f)/\xi$.*

Here $*du$ denotes the Hodge star on 1–forms determined by ξ; it satisfies $*da = da$ and $*db = -db$.

Formulas. More explicitly:

1. The locus Z is defined by the secant / derivative condition:
$$f'(a)f'(b) = \left(\frac{f(a) - f(b)}{a - b} \right)^2. \tag{1.1}$$
Equation (1.1) characterizes projective transformations $f \in G$.

2. The measure carried by L is given explicitly by
$$m_L = \left| \frac{du}{da} \right| \cdot |da| = \left| \frac{2}{a-b} - \frac{2f'(a)}{f(a) - f(b)} + \frac{f''(a)}{f'(a)} \right| \cdot |da|. \tag{1.2}$$
This formula simplifies to $|f''(a)\, da|$ if the 1–jets of f at a and b agree with the identity.

Geometry of F and Λ. A geodesic lamination Λ has *finite type* if

1. $\mathbb{H} - \Lambda = \bigcup_1^m P_i$ is a finite union of ideal polygons with distinct vertices, and
2. The only time two leaves of Λ share an endpoint is when they are adjacent edges of the same polygon P_i.

In this case, by identifying a and b whenever $[a, b]$ is a leaf of Λ, we obtain a finite *quotient tree*

$$T = \widehat{\mathbb{R}}/\Lambda.$$

An *endpoint* of T is a vertex of degree one. Every vertex v_i of T, other than its endpoints, has degree $d_i \geq 3$, and corresponds to one of the polygons P_i with d_i ideal vertices.

Theorem 1.2 *If f is real–analytic, and $\Lambda \neq \emptyset$, then:*

- *Λ has finite type;*
- *Each endpoint of $T = \widehat{\mathbb{R}}/\Lambda$ corresponds to a zero or pole of the Schwarzian derivative Sf in $\widehat{\mathbb{R}}$; and*
- *Each internal vertex of T corresponds to a tritangent plane for the space curve*

$$\mathrm{Gr}(f) = \{(x, f(x), xf(x)) : x \in \widehat{\mathbb{R}}\} \subset \mathbb{RP}^3.$$

Here a *tritangent plane* $H \subset \mathbb{RP}^3$ is a plane tangent to $\mathrm{Gr}(f)$ at 3 or more distinct points.

Corollary 1.3 *The associated earthquake $F : \mathbb{H} \to \mathbb{H}$ is a piecewise real–analytic, bilipschitz, area–preserving map.*

Corollary 1.4 *There exists an algorithm to find L, Λ and F from f.*

For the proof, see §7.

Example. Figure 1 shows the lamination $\Lambda \subset \mathbb{H}$ associated to the algebraic diffeomorphism $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ given by

$$f(x) = \frac{144x^5 + 6485x^3 + 216x}{(x^2 + 36)(36x^2 + 1)}.$$

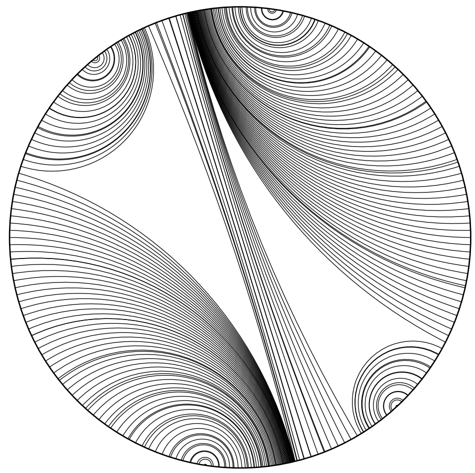

Figure 1. An earthquake $F : \mathbb{H} \to \mathbb{H}$ shearing along the leaves of a measured lamination Λ.

The drawing is made in the disk model for $\mathbb{H}$, using the isomorphism $\mathbb{H} \cong \Delta$ sending $(-1, i, 1)$ to $(-1, 0, 1)$. The complement of Λ in $\mathbb{H}$ consists of two ideal triangles.

Note. Mess introduced the use of space curves (and anti–de Sitter space) into the study of earthquakes [Me], and gave a new proof of Thurston's theorem by considering the convex hull of $\mathrm{Gr}(f)$.

Earthquakes and moderate smoothness. Theorems 1.1 and 1.2 can fail dramatically if f is only C^∞; for example, $\mathbb{H} - \Lambda$ may have infinitely many components, and L and Z may become Cantor sets. Nevertheless, many aspects of the real–analytic case persist when f is just differentiable (§3), moderately smooth (§5) or $C^{1,1}$ (§13).

Earthquakes from vector fields. In §8 we present a parallel story for earthquake extensions V of real–analytic vector fields v on $\widehat{\mathbb{R}}$.

Algebraic maps. We now turn to the case where the map $f(x)$ or the vector field $v(x)$ is a *rational function* of x. In both cases we will show that the topological complexity of Λ is controlled by the algebraic degree of f or v. To measure this complexity, define the Euler characteristic of an n–sided ideal polygon by $\chi(P) = 2 - n$. The Gauss–Bonnet formula gives:

$$(1/\pi)\,\mathrm{area}(\mathbb{H} - \Lambda) = \sum |\chi(P_i)|.$$

In §9 we establish:

Theorem 1.5 *Let Λ be the right earthquake lamination for a diffeomorphism $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ given by a rational map of algebraic degree $d > 1$. Then the components (P_i) of $\mathbb{H} - \Lambda$ satisfy*

$$\sum_1^m |\chi(P_i)| \leq 2d - 6.$$

In particular, Λ has at most $2d - 6$ complementary components.

Theorem 1.6 *Let v be a rational vector field of degree $d > 2$, and let $\mathbb{H} - \Lambda = \bigcup_1^m P_i$. Then*

$$\sum_1^m |\chi(P_i)| \leq 2d - 8.$$

Here the degree of v is the number of zeros it has as a vector field on $\widehat{\mathbb{C}}$.

These linear bounds are a substantial improvement over the $O(d^3)$ bounds obtained using the classical enumeration of tritangent planes; see §9. A bound linear in d is best possible; see Corollary 10.2.

1.3 Illustrative examples

Relatively few concrete examples of earthquakes appear in the literature. Leveraging the results above, in §10, §11 and §12 we present several illustrative families in detail, of interest in their own right.

1. Degrees 3 and 4, symmetry and singularity. In §10 we discuss maps f of degree 3 and vector fields v of degree 4. We will see that a quartic vector field is essentially unique, while cubic maps are related to elliptic curves and have a 1–dimensional moduli space. We also discuss a sequence of vector fields v_n with $\mathbb{Z}/n$ symmetry, and show that Z and L can be singular, even when v is rational.

2. Sextics. In §11 we analyze the first interesting class of rational vector fields, namely those of degree 6 — whose moduli space is 4–dimensional. Sextic space curves and their tritangents play a distinguished role in algebraic geometry, and for similar reasons sextic vector fields are especially approachable.

Our main results characterize, up to isometry, those subsets $\Lambda \subset \mathbb{H}$ that can arise from vector fields in degree 6.

Theorem 1.7 *The complement of the right earthquake lamination of a sextic vector field v consists of 0, 1 or 2 ideal triangles. All 3 possibilities occur.*

Note that Theorem 1.7 improves the upper bound on $\#(P_i)$ given in Theorem 1.6 from 4 to 2.

Rational separation. The next statement uses the following definition. Let A and B be disjoint sets in $\widehat{\mathbb{R}}$ with $|A| = |B| = n$, and let $R : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ be a rational map with divisor $(R) = A - B$; it is unique up to scale. We say A and B are *rationally separated* if there exist intervals $I \supset A$ and $J \supset B$ such that

$$|R(I)| \cap |R(J)| = \emptyset.$$

Theorem 1.8 *There exists a sextic vector field whose right earthquake lamination satisfies*

$$\mathbb{H} - \Lambda = P_1 \cup P_2$$

if and only if the vertices of the ideal triangles P_1 and P_2 are rationally separated.

We view this sharp synthesis result as a notable culmination of the preceding developments.

3. Power maps. In §12 we study earthquakes for the 1–parameter group of homeomorphisms given by

$$f_\alpha(x) = |x|^\alpha \operatorname{sign}(x), \tag{1.3}$$

$\alpha > 0$. These maps arise naturally from Teichmüller mappings between cylinders. Their earthquakes can be described in terms of the measured laminations (Λ_u, m_u) given, for $u > 0$, by

$$\Lambda_u = \bigcup_{t>0} [-t/u, tu] \quad \text{and} \quad m_u|[0,\infty] = dy/y. \tag{1.4}$$

Writing $s\Lambda_u$ as shorthand for (Λ_u, sm_u), we have:

Theorem 1.9 *The measured lamination for the right earthquake extension of $f_\alpha(x)$, $\alpha \neq 1$, is given by $\Lambda = s\Lambda_u$, where $u, s > 0$ satisfy*

$$\alpha(u + 1/u) = u^\alpha + 1/u^\alpha \quad \textit{and} \tag{1.5}$$

$$s(u + 1/u) = |(u^\alpha - 1/u^\alpha) - (u - 1/u)|. \tag{1.6}$$

We have $u > 1$ when $\alpha > 1$ and otherwise $u < 1$.

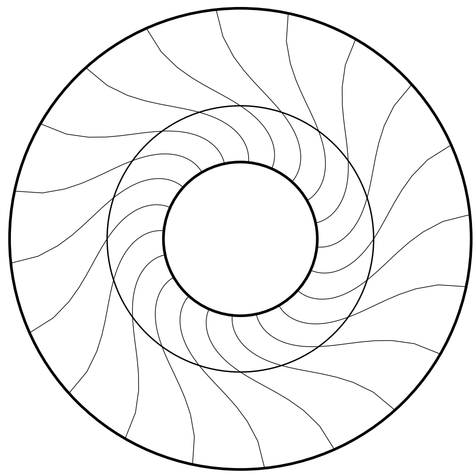

Figure 2. A small twist produces a radial stretch.

These conditions uniquely determine s and u.

Twists that stretch. Remarkably, although $f_\alpha(x)$ converges to the identity as $\alpha \to 1$ from above, the constant u does not tend to one; rather, its limiting value $u^* = 3.319050142\ldots$ is the unique solution to

$$\log(u^*) = \frac{u^* + 1/u^*}{u^* - 1/u^*} \quad \text{with } u^* > 1.$$

The lamination Λ_{u^*} makes an angle of θ^* with the imaginary axis, where $\tan(\theta^*/2) = u^*$.

Here is a consequence for complex deformations of cylinders. Fix $R > 1$ and let $A(R) \subset \mathbb{C}$ denote the annulus defined by $1 < |z| < R$. Note that $A(R)$ is conformally isomorphic to a Euclidean cylinder of radius 1 and height $\log R$.

For any $\alpha > 0$, the extremal quasiconformal map $\phi_\alpha : A(R) \to A(R^\alpha)$ is a *radial stretch* of the form $(r, \theta) \mapsto (r^\alpha, \theta)$. The map ϕ_α is also isotopic, rel boundary, to an earthquake map described by the results above. Taking the limit as $\alpha \to 1$, we obtain the following result.

Theorem 1.10 *Let Λ denote the foliation of $A(R)$ by geodesics making an angle of $\theta^* \approx 146°$ with the core curve $|z| = \sqrt{R}$. Then a small right twist on Λ moves the boundary components of $A(R)$ directly apart, with no rotation.*

The overall effect is reminiscent of a woven finger trap; see Figure 2.

1.4 Regularity

Finally, in §13 we relax our smoothness assumptions on $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$, and show:

Theorem 1.11 *Let F be the right earthquake extension of a $C^{1,1}$ diffeomorphism f. Then F is bilipschitz and $|\overline{\partial} F/\partial F|$ lies in $L^2(\mathbb{H})$.*

The L^2 bound above provides a bridge between earthquakes and the Weil–Petersson approach to universal Teichmüller space.

As a complement, we also show:

Theorem 1.12 *For every $0 < \epsilon < 1$, there exists a diffeomorphism $f \in C^{2-\epsilon}$ such that F is not even quasiconformal.*

These results demonstrate that the implications of Theorem 1.1 go well beyond the real–analytic setting.

Other highlights. This paper also provides:

- A new perspective on the construction of earthquakes (§2), emphasizing a partial ordering on $\mathrm{Homeo}^+(\widehat{\mathbb{R}})$ and the projective geometry of $\widehat{\mathbb{R}} \cong \mathbb{RP}^1$;
- A discussion of the *spine* Σ of a geodesic lamination Λ, which, in the case of an earthquake F, picks out the point of maximal distortion on each leaf of Λ (§4); and
- A proof, using the spine, of the formula

$$\int_{\mathbb{H}} \rho^2 \, \|\overline{\partial} F\|^2 = \frac{\pi}{8} \int_L (m_L/|ds|)^2 \, |ds|$$

  underlying the L^2 bound in Theorem 1.11; see Theorem 6.1.

Questions. We conclude by formulating two open problems.

1. Which trees $T = \widehat{\mathbb{R}}/\Lambda$ arise from rational maps or vector fields of degree d?
2. What is the best possible upper bound on the number of real zeros of $v'''(x)$, when $v(x) = P(x)/Q(x)$ is a rational function with no poles on $\mathbb{R}$ and $\deg(P) \leq \deg(Q) + 2 = d$?

These two questions are related, since each endpoint of T gives a real zero for the Schwarzian of the degree d vector field $v(x)\,d/dx$.

Acknowledgements. I would like to thank the IHES and SLMath for their support during the early stages of this project, which has its origins in Y. Wang's seminar in 2024.

Notes and references. The fundamentals of the theory of earthquakes are established in [Th2] and [Me, Prop. 22]. For many other perspectives, see e.g. [Ga], [GH], [ABB], [BB], [DS], and [Diaf].

A complex analogue of the two–variable Schwarzian $S(f \times f)$ occurs in the theory of univalent functions and the Grunsky inequalities; see e.g. [Pom, Ch. 3] and [SW, §2.2].

Many other useful references are collected in the bibliography and cited in context below.

Notation. As usual, $A = O(B)$ means $|A| \le C|B|$ for some implicit constant C. If, in addition, $B = O(A)$, we write $A \asymp B$.

2 Envelopes and earthquakes

In this section we sketch a coordinate–free, order–theoretic framework for Thurston's theory of earthquakes.

While traditional approaches rely on limits of finite laminations, or convex hulls in AdS_3, the present approach is indigenous to the real projective line, $\mathbb{RP}^1 \cong \widehat{\mathbb{R}}$. Its central players are the upper and lower *projective envelopes* of a homeomorphism $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$, which in turn are defined using a partial ordering on the group

$$H = \mathrm{Homeo}^+(\widehat{\mathbb{R}})$$

of orientation–preserving homeomorphisms of $\widehat{\mathbb{R}}$, and the distinguished subgroup of projective transformations $G \cong \mathrm{PSL}_2(\mathbb{R}) \subset H$.

Translation distance. The translation distance of $g \in G$ will be denoted by

$$|g| = \inf_{\mathbb{H}} d(z, g(z)).$$

For readability, we use the notation $|g - h|$ for $|gh^{-1}|$.

Order. There is a unique partial order $f_1 \ge f_2$ on H satisfying the following 3 axioms:

1. If $f_1 \ge f_2$, then $gf_1h \ge gf_2h$ for all $g, h \in H$.

2. If $f_1(\infty) = f_2(\infty) = \infty$, then $f_1 \geq f_2$ if and only if $f_1(x) \geq f_2(x)$ for all $x \in \mathbb{R}$.

3. If the graphs of f_1 and f_2 are disjoint, then f_1 and f_2 are incomparable.

Envelopes. We say $g_1 \in G$ is *minimal* if $g_1 \geq f$ and whenever $g_2 \in G$ satisfies $g_1 \geq g_2 \geq f$, we have $g_2 = g_1$. The set

$$\mathrm{Env}^+(f) = \{\text{all minimal } g \geq f\}$$

is the *upper projective envelope* of $f \in H$. (The lower envelope is defined similarly.)

Strata and laminations. Each $g \in \mathrm{Env}^+(f)$ determines a closed, convex subset of $\mathbb{H}$ by

$$S_g = \mathrm{hull}(\{x \in \widehat{\mathbb{R}} \, : \, f(x) = g(x)\}) \subset \mathbb{H}.$$

Here $\mathrm{hull}(E)$ is the smallest closed convex set in $\mathbb{H}$ containing all geodesics with endpoints in E.

The collection of sets $\mathcal{S} = \{S_g \, : \, S_g \neq \emptyset\}$ yields a *stratification* of $\mathbb{H}$. Each stratum $S \in \mathcal{S}$ is either a geodesic $[a, b]$ or a 2–dimensional convex set bounded by geodesics. Their union is $\mathbb{H}$, and when $S_g \neq S_h$, $S_g \cap S_h$ is either empty or a single geodesic.

Let

$$\Lambda = \bigcup \{[a, b] \, : \, [a, b] = S \text{ or } [a, b] \subset \partial S \text{ for some } S \in \mathcal{S}\}.$$

Including the second condition ensures that Λ is closed. The geodesics appearing on the right are disjoint, and therefore give Λ the structure of a geodesic lamination.

Earthquakes. The *right earthquake extension* of f is a bijection $F : \mathbb{H} \to \mathbb{H}$ that is almost characterized by the condition that

$$F|S = g|S$$

whenever $S = S_g$. The one nuance is that a given geodesic might have the form $[a, b] = S_g$ for many values of g. To make F canonical, we observe that all such g send $[a, b]$ to $[f(a), f(b)]$, and hence any two differ by translation along $[a, b]$. We specify F uniquely by taking the average of the maximum and minimum translations.

Osculation. The earthquake F determines a unique *osculation map*

$$\mathrm{osc}(F) : \mathbb{H} \to G,$$

almost characterized by the condition $\operatorname{osc}(F)|S_g = g$. The only issue here is that S_g may overlap with S_h for some $h \neq g$. In this case $S_g \cap S_h = [a,b]$ is a geodesic, there is a unique $k \in G$ such that $F|[a,b] = k|[a,b]$, and we define $\operatorname{osc}(F)|[a,b] = k$.

Measure. It remains only to define the measure m_Λ of an arc τ transverse to Λ. We may assume that $\tau = (z,w)$ is an open geodesic segment. Given $n > 0$, choose evenly spaced points $z = z_0, z_1, \ldots, z_{n+1} = w$ in $\overline{\tau}$, let $g_i = \operatorname{osc}(F)(z_i)$, and let

$$m_n(z,w) = |g_1 - g_2| + |g_2 - g_3| + \cdots + |g_{n-1} - g_n|.$$

(We deliberately omit the endpoints g_0 and g_{n+1}.) The transverse measure of τ is then given by

$$m_\Lambda(\tau) = \lim_{n\to\infty} m_n(z,w).$$

Atoms. When F is discontinuous along a leaf $[a,b]$ of Λ, the measure $m_\Lambda|\tau$ has an atom at $p = \tau \cap [a,b]$. Its mass μ is given by

$$\mu = \sup\{|g-h| \,:\, S_g = S_h = [a,b]\} = |g_+ - g_-|,$$

where g_+ and g_- are the limits of $\operatorname{osc}(F)(z)$ on the two sides of $[a,b]$. By defining m_Λ initially on open segments τ, we have avoided issues arising from atoms at its endpoints.

Properties of earthquakes. It is straightforward to verify:

1. The construction above is consistent with Thurston's definition; that is, (F,Λ) gives the unique right earthquake extension of f.

2. The earthquake map $F : \mathbb{H} \to \mathbb{H}$ is an area–preserving bijection.

3. The passage from f to (F,Λ) is natural; that is, for any $g,h \in G$, the earthquake data for gfh^{-1} is given by $(gFh^{-1}, h(\Lambda))$.

Left earthquakes can similarly be constructed from the lower envelope $\operatorname{Env}^-(f)$, defined by the condition $g \leq f$.

Example: An earthquake with a single fault line. A basic example is provided by the right earthquake extension of the piecewise–linear map $f(x) = 2x - |x|$, whose derivative is 3 for $x < 0$ and 1 for $x > 0$. Let

$\mathbb{H}_\pm = \{z : \pm \mathrm{Re}(z) \geq 0\}$. It is easy to verify that:

$$\begin{aligned}
\mathrm{Env}^+(f) &= \{g(x) = tx : 1 \leq t \leq 3\}, \\
\mathcal{S} &= \{\mathbb{H}_-, \mathbb{H}_+, [0,\infty]\}, \text{ and} \\
F(z) &= \begin{cases} 3z & \text{when } \mathrm{Re}(z) < 0, \\ 3^{1/2}z & \text{when } z \in [0,\infty], \text{ and} \\ z & \text{when } \mathrm{Re}(z) > 0. \end{cases}
\end{aligned}$$

The lamination Λ consists of the single, isolated leaf $[0,\infty]$, whose transverse measure is an atom of mass $\log(3)$. There are many $g \neq h$ in $\mathrm{Env}^+(f)$ such that $S_g = S_h$; indeed, $S_g = [0,\infty]$ whenever $g(x) = tx$ and $1 < t < 3$.

As can be seen here at $x = 0$, an atomic leaf $[a,b]$ forces f to be non–differentiable at a and b.

Continuity of F. On the other hand, if m_Λ has no atoms, then the earthquake F is a *homeomorphism.* We will exclusively be concerned with earthquakes of this type in the sequel.

3 Differentiable maps

In this section we record two results that only require differentiability of f. These results will play an important role in the sequel.

The first is a qualitative result in the direction of Theorem 1.2.

Theorem 3.1 *Suppose $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ is differentiable. Then:*

1. *m_Λ has no atoms;*
2. *Λ has no fans or isolated leaves;*
3. *For any two distinct elements $g, h \in \mathrm{Env}^+(f)$, the closures of S_g and S_h in $\overline{\mathbb{H}}$ are disjoint; and*
4. *Any two distinct components P_1, P_2 of $\mathbb{H} - \Lambda$ have disjoint closures in $\overline{\mathbb{H}}$.*

Here a *fan* is a collection of 3 or more leaves of Λ with a common endpoint.

Proof. (1) is immediate by differentiability. As for (3), if $\overline{S_g} \cap \overline{S_h}$ is nonempty, then it contains a point $a \in \widehat{\mathbb{R}}$. We may assume that $a \in \mathbb{R}$. Then $g(a) = f(a)$ and $g'(a) = f'(a)$ since $g \geq f$. The same is true for h. Interchanging g and h if necessary, we may assume that $h''(a) \geq g''(a)$. If

$h''(a) > g''(a)$ then $h > g$, so h and g cannot both belong to $\mathrm{Env}^+(f)$. Thus $h''(a) = g''(a)$ and hence $h = g$.

Note that (3) implies (4) because $\overline{P_1} = \overline{S_g}$ and $\overline{P_2} = \overline{S_h}$ for some $g, h \in \mathrm{Env}^+(f)$. It also implies Λ has no fans, and (1) rules out isolated leaves, establishing (2). ■

The second is the source of the definition of $Z \subset \mathbb{G}$ in Theorem 1.1.

Lemma 3.2 *If $[a, b]$ is a leaf of Λ, and $f(x)$ is differentiable at a and b, then the Schwarzian $S(f \times f)$ vanishes at (a, b).*

Proof. There is a unique $g \in G$ such that $F|[a, b] = g|[a, b]$, and we have $g \geq f$. These conditions imply that the 1–jets of f and g agree at a and b, and hence $S(f \times f) = S(g \times g) = 0$ at $(a, b) \in \mathbb{G}$. ■

Corollary 3.3 *If f is differentiable, then the dual L of Λ is contained in the zero set Z of $S(f \times f)$.*

In summary, differentiability of f already implies that Λ and L share some of the properties of laminations for real–analytic maps.

4 The spine of a geodesic foliation

This section develops the local differential geometry of a moderately smooth geodesic *foliation* Λ of an open set $U \subset \mathbb{H}$. In particular we define a canonical transversal to Λ, called its *spine*; an example is shown in Figure 3.

The spine will play a pivotal role in establishing L^2 bounds for $\|\overline{\partial} F\|$; see Theorem 6.1.

Assumptions on Λ. We will assume that Λ is *complete and moderately smooth*. This means its leaves $[a, b]$ are complete geodesics, given locally as the level sets of a C^2 function with nonzero gradient.

For convenience, we also choose an orientation of Λ; then its leaves are in *bijection* with its dual $L \subset \mathbb{G}$.

Stationary leaves. A leaf $[a, b]$ of Λ is *stationary* if the tangent space of L at (a, b) is lightlike (isotropic). In this case one endpoint of $[a, b]$ is stationary to first order.

Normal and spine. The hyperbolic metric $\rho = |dz|/y$ on $\mathbb{H}$ gives a natural norm on 1–forms and 2–forms. The *(co)normal 1–form* η for Λ is characterized by the property that, along each leaf $[a, b]$, we have

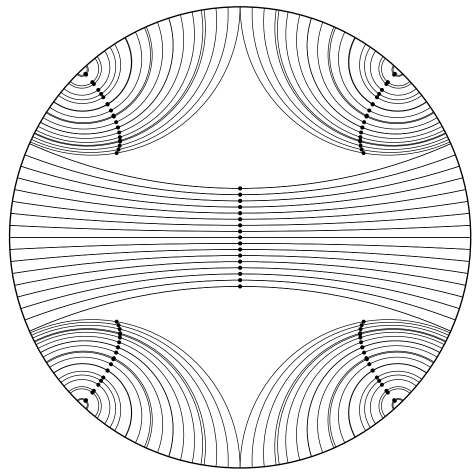

Figure 3. The spine Σ provides natural transversals to Λ.

- $\|\eta\| = 1$; $\eta|[a,b] = 0$; and
- $*\eta|[a,b] > 0$ with respect to the chosen orientation of Λ.

The *spine* of Λ is defined as the zero set

$$\Sigma = Z(d\eta).$$

We will show:

Theorem 4.1 *Provided Λ has no stationary leaves:*

1. *The spine Σ is a natural transversal to Λ. It picks out, on each leaf $[a,b]$, the point c closest to nearby leaves.*
2. *We have a natural isomorphism $(\Sigma, |\eta|) \cong (L, |ds|)$, where ds^2 is the Lorentz form on $\mathbb{G}$.*
3. *Along each leaf $[a,b]$ of Λ, we have $\|d\eta\| = |\tanh d(z,c)|$.*
4. *The Jacobi field of Λ along $[a,b]$ is given by $j(s) = \cosh s$, where $s(c) = 0$.*

The spine is analogous to the *line of striction* for a ruled surface [For, §226].

Jacobi fields. The behavior of Λ near a given leaf $[a,b]$ is described by its *Jacobi field* $j : [a,b] \to \mathbb{R}$. In our case j is positive and well defined up to multiplication by a scalar. One can picture nearby leaves as the graph of $y = \epsilon j(x)$ in Fermi coordinates near $[a, b]$. Equivalently, we have $d(z, [a', b']) \approx \epsilon j(z)$.

Since $\mathbb{H}$ has constant curvature -1, with respect to arclength s we have

$$j(s) = A\exp(s) + B\exp(-s)$$

for some $A, B \geq 0$. In the case of a stationary leaf, $j(s) = \exp(\pm s)$, and $\|d\eta\| = 1$ along $[a, b]$.

Geometry of the Lorentz metric. The Lorentz metric $ds^2 = 4da\,db/(a-b)^2$ on $\mathbb{G}$ has the following geometric interpretation: if $\gamma(t) = (a(t), b(t))$ is a smooth family of disjoint geodesics, and $D(t)$ is the *signed* hyperbolic distance from $\gamma(0)$ to $\gamma(t)$, then

$$|ds/dt|_{t=0} = |D'(0)|. \tag{4.1}$$

Proof of Theorem 4.1. Since $j(s) > 0$, the Jacobi field for a given leaf $[a, b]$ of Λ is, for a suitable choice of the arclength parameter s, proportional to $\exp(s)$, $\exp(-s)$ or $\cosh(s)$. But in the first two cases $[a, b]$ is a stationary leaf, contrary to assumption. In the third case, $\cosh(s)$ is minimized at the unique point c with $s(c) = 0$; this is the point where $[a, b]$ is closest to nearby leaves.

It is also the unique intersection of $[a, b]$ with the spine. Indeed, one can readily check that

$$\|d\eta\| = |(d/ds)\log j(s)| = |\tanh(s)|,$$

and thus $d\eta$ vanishes at c and nowhere else in $[a, b]$.

Since c is the closest point to nearby leaves, this distance is measured by $|\eta|$ when restricted to Σ. Since the same is true for $|ds|$ by equation (4.1), we have $(\Sigma, |\eta|) \cong (L, |ds|)$. ∎

Example. By symmetry considerations, the spine Σ of $\Lambda_u = \{[-t/u, tu] : t > 0\}$ is the imaginary axis, no matter what the value of $u > 0$ is. In particular Σ need not be orthogonal to Λ.

5 Smooth earthquakes

In this section we describe the local differential geometry of the right earthquake $F : \mathbb{H} \to \mathbb{H}$ attached to a homeomorphism $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$. For brevity we write

$$g = \operatorname{osc}(F) : \mathbb{H} \to G.$$

Measured foliations. Recall that any moderately smooth nowhere zero 1–form ω on an open set U determines a measured *foliation* $\mathcal{F}(\omega)$, with leaves tangent to $\mathrm{Ker}(\omega)$, and with the measure of a transversal τ given by $\int_\tau |\omega|$. By moderately smooth we mean ω is C^1.

Earthquakes. Let (Λ, m_Λ) be the measured lamination associated to a right earthquake F, with dual (L, m_L).

An open set $U \subset \mathbb{H}$ is *saturated* if it contains every leaf of Λ that it meets. We will study the case where there is a saturated open set U and a 1–form ω on U as above such that

$$\Lambda_0 = \Lambda | U$$

is isomorphic to $\mathcal{F}(\omega)$ as a measured foliation. Note that the 1–form provides each leaf of Λ_0 with an orientation. We let $L_0 \subset \mathbb{G}$ denote the smooth 1–manifold defined by these oriented leaves.

Shear. The *shear* of a linear map $A \in \mathrm{SL}_2(\mathbb{R})$ is given by

$$\mathrm{shear}(A) = |t| \iff A = k_1 \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} k_2$$

for some $k_1, k_2 \in K = \mathrm{SO}(2)$. If we regard A as a map on $\mathbb{R}^2 \cong \mathbb{C}$, then

$$\mathrm{shear}(A) = 2\|\overline{\partial} A\|, \tag{5.1}$$

as can be seen from the model $A(z) = z + t\,\mathrm{Im}(z)$. Similarly for an area–preserving map $F : \mathbb{H} \to \mathbb{H}$, we have

$$\mathrm{shear}(DF) = 2\|\overline{\partial} F\|. \tag{5.2}$$

Here we use the hyperbolic metric to identify the domain and range of

$$DF : \mathrm{T}_z\mathbb{H} \to \mathrm{T}_{F(z)}\mathbb{H}$$

isometrically with $\mathbb{R}^2$, and hence canonically associate to DF an element $[A] \in K\backslash G/K$ to which we can apply equation (5.1).

Lie algebras. Recall that $\mathrm{Lie}(G) \cong \mathrm{sl}_2(\mathbb{R})$ carries an invariant bilinear form,

$$\langle p, q \rangle = 2\,\mathrm{Tr}(p.q), \tag{5.3}$$

and that matrices in $\mathrm{sl}_2(\mathbb{R})$ can be identified with quadratic vector fields on $\widehat{\mathbb{R}}$, with

$$q = \begin{pmatrix} b/2 & c \\ -a & -b/2 \end{pmatrix} \iff (ax^2 + bx + c)(d/dx).$$

We have normalized $\langle q, q\rangle$ so it agrees with the discriminant $b^2 - 4ac$ of the polynomial on the right. The 1–parameter subgroup of G stabilizing a given geodesic $[a,b] \subset \mathbb{H}$ will be denoted by G_{ab}; its Lie algebra is spanned by $q = (x-a)(x-b)\, d/dx$.

Main results. We will show:

Theorem 5.1 *Throughout the open set U, we have*

$$\operatorname{shear}(DF) = m_\Lambda/|\eta| \quad \text{and} \quad |\omega|^2 = 2|\operatorname{Tr}(h.h)|,$$

where $h = g^{-1}dg$. Moreover the form $g^{-1}dg$ takes values in $\operatorname{Lie}(G_{ab})$ when restricted to any leaf $[a,b]$ of Λ_0.

Theorem 5.2 *Let $u = S(f \times f)/\xi$. Then $m_L = |*du|/2$ along the 1–manifold $L_0 \subset \mathbb{G}$.*

Theorem 5.3 *Suppose Λ_0 has no stationary leaves, and let Σ be its spine. Then the natural bijection*

$$\Sigma \leftrightarrow L_0$$

sends the function $\operatorname{shear}(DF)$ *to the function $m_L/|ds|$.*

Theorem 5.4 *Let $[a,b]$ be a leaf of Λ meeting the spine Σ at c. Then* $\operatorname{shear}(DF)$ *achieves its maximum at c, and*

$$\operatorname{shear}(DF)_z = \operatorname{shear}(DF)_c / \cosh d(z,c)$$

for all $z \in [a,b]$.

Corollary 5.5 *We have*

$$\sup_U \operatorname{shear}(DF) = \sup_\Sigma \operatorname{shear}(DF) = \sup_{L_0} m_L/|ds|$$

and

$$\int_U \rho^2 \operatorname{shear}(DF)^2 = \pi \int_{L_0} \left(\frac{m_L}{|ds|}\right)^2 |ds|,$$

provided Λ_0 has no stationary leaves.

Proof of Theorem 5.1. It suffices to verify these statements at a single point p on a leaf $[a,b]$ of Λ. Normalize coordinates on domain and range so that $p = i$, $[a,b] = [0,\infty]$, and $F|[a,b]$ is the identity (so $g|[a,b] = I$ as well).

Let $z = x + iy$ be the usual coordinates on $\mathbb{H}$. Since DF fixes the vertical line tangent to $[0, \infty]$, we have

$$DF(p) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad \text{and} \quad \text{shear}(DF(p)) = |t|$$

for some $t \in \mathbb{R}$; that is,

$$F(z) \approx z + itx$$

for z close to p. It follows that for small x, F is well–approximated near $i + x$ by the isometry g_x translating signed distance tx along the geodesic $[x, \infty]$. Since $|g_x| = |tx|$ we find $|\omega| = |t\,dx|$ at $z = i$, and

$$q = \frac{dg}{dx}(i) = \begin{pmatrix} t/2 & 0 \\ 0 & -t/2 \end{pmatrix} \in \text{Lie}(G_{0,\infty}).$$

Since $2\,\text{Tr}(q.q) = t^2$ we have verified that, for $h = g^{-1}dg$,

$$|\omega|^2 = 2|\,\text{Tr}(h.h)| = t^2\,|dx|^2.$$

Finally, since the hyperbolic metric $|dz|/y$ and the Euclidean metric $|dz|$ agree at $z = i$, we can take $\eta = dx$; then clearly

$$m_\Lambda/|\eta| = |\omega|/|dx| = |t| = \text{shear}(DF).$$

$\blacksquare$

Proof of Theorem 5.2. Since g is constant on leaves of Λ, it determines a function $g : L_0 \to G$. In view of Theorem 5.1, to compute m_L at a point $p \in L_0$ it suffices to compute $g^{-1}dg$. Continuing as in the proof above, we can arrange that $p = (0, \infty)$ and $g(p) = \text{id}$. Then, with (a, b) the usual coordinates on $\mathbb{G}$, we can take a as a local coordinate on L_0 (after swapping 0 and ∞ if $da(p) = 0$). Thus it suffices to compute dg/da at p.

Now for $(a, b) \in L_0$ near $[0, \infty]$, $g(a, b)$ is nearly in $G_{0,\infty}$, translating by signed distance $\log f'(a)$, and thus to first order we have

$$g(a, b) = \begin{pmatrix} f'(a)^{1/2} & 0 \\ 0 & f'(a)^{-1/2} \end{pmatrix} + O(a^2).$$

Since $f'(0) = 1$, this gives

$$q = \left.\frac{dg}{da}\right|_{a=0} = f''(0) \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix},$$

and hence

$$m_L/|da| = |2\,\mathrm{Tr}(q.q)|^{1/2} = |f''(0)|$$

at $a = 0$. But we also have $|du/da| = |f''(0)|$ by (a limiting form of) equation (1.2), using the fact that f fixes 0 and ∞ and $f'(0) = 1$; and hence

$$m_L = |du/da| \cdot |da| = |{*du}|/2$$

at p, completing the proof. ∎

Proof of Theorem 5.3. The bijection $\Sigma \leftrightarrow L_0$ sends m_Λ to m_L by definition, and it sends $|\eta|$ to $|ds|$ by Theorem 4.1; thus it sends $\mathrm{shear}(DF) = m_\Lambda/|\eta|$ to $m_L/|ds|$. ∎

Proof of Theorem 5.4. By Theorem 4.1, the Jacobi field of Λ along $[a, b]$ is given by $j(s) = \cosh(s)$, where $s(c) = 0$, while $\mathrm{shear}(DF) = m_\Lambda/|\eta|$. Since the numerator remains constant when transported along the leaves of Λ, while the denominator grows like $j(s)$, the Theorem follows. ∎

Proof of Corollary 5.5. The first chain of equalities is immediate. For the L^2 statement, we may assume that U is connected. Let $t : U \to \mathbb{R}$ give the signed distance to Σ along each leaf $[a, b]$ of Λ. We then have a bijection between U and $\Sigma \times \mathbb{R}$ that transports the hyperbolic measure ρ^2 to the measure $|\eta| \cdot |dt| \cosh(t)$, where the last factor comes from the divergence of nearby leaves passing through Σ. Using these coordinates, we find that

$$\int_U \rho^2 \,\mathrm{shear}(DF)^2 = \left(\int_\Sigma \left(\frac{m_\Lambda}{|\eta|}\right)^2 |\eta|\right) \cdot \left(\int_{\mathbb{R}} \frac{1}{\cosh^2(t)} \cosh(t)\,dt\right).$$

The integral over $\mathbb{R}$ gives π, and the ratio $m_\Lambda/|\eta|$ becomes $m_L/|ds|$ when transported to L_0. ∎

6 Earthquakes for real–analytic maps

Let $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ be a real–analytic homeomorphism with $f \notin G$, and let (L, m_L) be the dual to the measured lamination (Λ, m_Λ) for its right earthquake extension F.

In this section we prove Theorems 1.1 and 1.2 and Corollary 1.3. In particular, we show that Λ has finite type, that the components (P_i) of $\mathbb{H} - \Lambda$ correspond to tritangent planes for $\mathrm{Gr}(f)$, and that the endpoints of the tree $\widehat{\mathbb{R}}/\Lambda$ correspond to zeros and poles of the Schwarzian derivative Sf.

We will also prove:

Theorem 6.1 *The right earthquake extension F of a real–analytic map f satisfies*

$$\int_{\mathbb{H}} \rho^2 \, \|\overline{\partial} F\|^2 = \frac{\pi}{8} \int_L (m_L/|ds|)^2 \, |ds|.$$

This equation is relevant to the Weil–Petersson theory discussed in §13.

Geometry of tritangents. Tritangent planes connect earthquakes and algebraic geometry. Let us explain how they arise.

To set the stage, observe that the algebraic isomorphism

$$\phi : \widehat{\mathbb{R}} \times \widehat{\mathbb{R}} \to Q \subset \mathbb{RP}^3,$$

given in affine coordinates by $\phi(x, y) = (x, y, xy)$, sends the product space on the left to the smooth quadric surface Q defined by $x_1 x_2 = x_3$. In addition, if we let

$$\mathrm{gr}(f) = \{(x, f(x)) \ : \ x \in \widehat{\mathbb{R}}\} \subset \widehat{\mathbb{R}} \times \widehat{\mathbb{R}},$$

then ϕ sends $\mathrm{gr}(f)$ to $\mathrm{Gr}(f)$.

Linear systems. The map ϕ converts Möbius transformations into hyperplane sections, providing a bridge between hyperbolic and algebraic geometry.

In the language of algebraic geometry, ϕ is the projective embedding coming from the linear system of divisors $|\,\mathcal{O}(1,1)|$ on $\mathbb{P}^1 \times \mathbb{P}^1$. But a generic element of this linear system is nothing more than the graph $\mathrm{gr}(g)$ of a Möbius transformation $g \in G$. Since these divisors become hyperplane sections under ϕ, we have $\mathrm{Gr}(g) = Q \cap H_g$ for a unique hyperplane H_g.

Returning to earthquakes, recall that F acts isometrically on each leaf of Λ. Thus F determines a map $g : L \to G$, characterized by the condition that

$$F|[a,b] = g_{ab}|[a,b]$$

for each $(a, b) \in L$.

Typically $\mathrm{gr}(g_{ab})$ touches $\mathrm{gr}(f)$ in exactly two points, and hence it determines a unique leaf $[a, b]$ of Λ. However for each ideal polygon P_i complementary to Λ, there is also a unique $g_i \in G$ such that $F|P_i = g_i|P_i$. In this case $\mathrm{gr}(g_i)$ and $\mathrm{gr}(f)$ are tangent at each ideal vertex of P_i. Thus, writing $\mathrm{Gr}(g_i) = H_i \cap Q$, we find:

> *The images of the ideal vertices of P_i under the map $x \mapsto \phi(x, f(x))$ are points where H_i is tangent to* $\mathrm{Gr}(f)$.

In particular, H_i is a tritangent plane.

The Schwarzian derivative. We will use the following properties of the Schwarzian: $Sf = 0 \iff f \in G$; $S(f_1 \circ f_2) = Sf_2 + f_2^*(Sf_1)$; $Sf(x) = \infty \iff f'(x) = 0$; if $Sf(0)$ is finite, then there is a unique $g \in G$ such that

$$(g \circ f)(x) = x + Ax^3 + O(x^4),$$

where $A = Sf(0)/6$. For a geometric treatment of Sf, see [Th1].

The subvariety $\boldsymbol{Z}$. Recall that $Z \subset \mathbb{G}$ is defined by the vanishing of the 2–variable Schwarzian $S(f \times f)$, or equivalently by

$$f'(a)f'(b)(a-b)^2 = (f(a) - f(b))^2;$$

this shows $\overline{Z} \subset \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ is also an analytic subvariety. Yet another definition is:

$$Z = \left\{ (a,b) \ : \ \begin{array}{c} \text{for some } g \in G, \text{ the graphs of } f(x) \text{ and } g(x) \\ \text{are tangent at } x = a \text{ and } x = b. \end{array} \right\}$$

Let $\pi_1(a,b) = a$.

Lemma 6.2 *The projection $\pi_1 : \overline{Z} \to \widehat{\mathbb{R}}$ is finite.*

Proof. Otherwise we would have $\{a\} \times \widehat{\mathbb{R}} \subset \overline{Z}$ for some a. This is clearly impossible if $f'(a) = 0$. Assuming $f'(a) \neq 0$, we can normalize (replace f with $g_1 f g_2$) so that $a = f(a) = \infty$ and

$$f(x) = x + B/x + O(1/x^2) \quad \text{for } |x| \gg 1. \tag{6.1}$$

Then any $g \in G$ tangent to the graph of f at a must have the form $g(x) = x + m$ for some m. Thus $f'(b) = 1$ at any other point b where the graphs are tangent. But $f'(b) = 1$ at only finitely many points, since $f \notin G$, and hence $\pi_1^{-1}(a)$ is finite. ∎

Lemma 6.3 *For all $a \in \widehat{\mathbb{R}}$, if the Schwarzian $Sf(a)$ is neither zero nor ∞, then $[a,b]$ is a leaf of Λ for some $b \neq a$.*

Proof. Since Sf has no pole at a, $f'(a) \neq 0$, so we can again normalize so that $a = \infty$ and f takes the form given in equation (6.1) above. The statement $Sf(a) \neq 0$ is equivalent to the condition $B \neq 0$, which in turn implies that

$$m = \sup_{x \in \mathbb{R}} (f(x) - x)$$

is positive, and the maximum is achieved at some point $b \in \mathbb{R}$. Then $g(x) = x + m$ belongs to $\mathrm{Env}^+(f)$ and $[a,b]$ is a leaf of Λ. ∎

Tritangent points. We say $a \in \widehat{\mathbb{R}}$ is a *tritangent point* if it corresponds to a point of tangency between $\mathrm{Gr}(f)$ and a tritangent plane H_g. Since f is real–analytic, the set of such points is finite.

Lemma 6.4 *The locus L is a finite union of disjoint arcs.*

Proof. Let $E \subset \widehat{\mathbb{R}}$ be the finite set of *exceptional points*, consisting of the tritangent points, the zeros and poles of Sf, together with the finitely many $a \in \widehat{\mathbb{R}}$ such that $(a, a) \in \overline{Z}$.

Let $L_0 = \{(a, b) \in L : a \notin E\}$. Since $\overline{L} \subset \overline{Z}$ is compact, and every point of $\overline{L} - L$ lies over E, the map

$$\pi_1 : L_0 \to (\widehat{\mathbb{R}} - E)$$

given by $\pi_1(a, b) = a$ is proper. By Lemma 6.3, it is surjective, and since we have excluded tritangent points, by Theorem 3.1 it is also injective. Thus π_1 is a continuous proper bijection, and hence $\pi_1^{-1} : (\widehat{\mathbb{R}} - E) \to L_0$ is also continuous.

Since $\widehat{\mathbb{R}} - E$ is a finite union of arcs, so is L_0. By Theorem 3.1, L has no isolated points; hence $\overline{L}_0 = L$, and the Lemma follows. ■

Proof of Theorem 1.1. Since $f \notin G$, the locus Z defined by $S(f \times f) = 0$ is a 1–dimensional analytic subvariety of $\mathbb{G}$, and we have $L \subset Z$ by Corollary 3.3; indeed, this inclusion only requires differentiability of f. The locus L is timelike because the leaves of Λ are disjoint.

By the preceding Lemma, L is a finite union of real–analytic arcs, whose endpoints and singularities comprise a finite set E. The leaves of Λ corresponding to $L - E$ foliate an open set $U \subset \mathbb{H}$ and thus, by Theorem 5.2, we have

$$m_L = |{*du}|/2, \quad u = S(f \times f)/\xi,$$

along the smooth 1–manifold $L - E$. Since m_Λ has no atoms, $m_L(E) = 0$, and thus the formula above captures the full measure of L. ■

Proof of Theorem 1.2. Let P be a connected component of $\mathbb{H} - \Lambda$. Then $\overline{P} = S_g = \mathrm{hull}(F_g)$ for some $g \in G$, where $F_g \subset \widehat{\mathbb{R}}$ is the finite set defined by $g(x) = f(x)$. Thus $\overline{P}$ is an ideal polygon; and since $|F_g| \geq 3$, the corresponding plane $H_g \subset \mathbb{RP}^3$ is one of the finitely many tritangent planes for $\mathrm{Gr}(f)$. This shows that $\mathbb{H} - \Lambda = \bigcup_1^m P_i$ is indeed a finite union of ideal polygons. Theorem 3.1 then implies that Λ has finite type, and thus $T = \widehat{\mathbb{R}}/\Lambda$ is a finite tree. Each endpoint of T corresponds to a family of leaves of Λ nesting down to a single point $a \in \widehat{\mathbb{R}}$. Since no leaf of Λ terminates at a, it must be a zero or pole of Sf by Lemma 6.3. ■

Proof of Corollary 1.3. The main remaining point to check is that F is bilipschitz; equivalently, that

$$\sup_{\Lambda} \operatorname{shear}(DF) = \sup_{L} m_L/|ds| < \infty$$

(cf. Theorem 5.3). For this, we use the fact that $u|L$ is constant, and hence the pullback of du to L is zero. Therefore

$$m_L = |{*du}|/2 = |du/da|\,|da| = |du/db|\,|db|.$$

This gives

$$\frac{m_L}{|ds|} = \frac{|a-b|}{2}\sqrt{\left|\frac{du}{da}\frac{du}{db}\right|}.$$

Since u and its derivatives are bounded on any compact subset of $\mathbb{G}$, the only issue arises when L approaches a point (a, a) on the diagonal.

In this case we can normalize so that $a = 0$, $f(0) = 0$ and $f(x) = x^n h(x)$ for some odd $n \geq 1$, where $h(0) = 1$ and $h(x)$ is also real–analytic.

Suppose first that $h(x) = 1$. In this case $f(tx) = t^n f(x)$, so (L, m_L) is invariant under $z \mapsto tz$, $t > 0$, and hence $m_L/|ds|$ is actually constant (and L coincides with the line $a+b = 0$). For the general case observe that, under rescaling, $h(tx) \to 1$ in C^∞ as $t \to 0$, so a similar argument implies L is asymptotic to the line $a + b = 0$, $f(x)$ is asymptotically homogeneous, and hence $m_L/|ds|$ remains bounded. ■

We will revisit the earthquake extension of $f(x) = x^{2n+1}$ in §12.

Proof of Theorem 6.1. Combine Corollary 5.5 with equation (5.2), which states that $\operatorname{shear}(DF) = 2\|\overline{\partial}F\|$. This last equation changes the factor of π to $\pi/4$, which in turn becomes $\pi/8$ because every leaf of Λ gives two points in L. ■

Remark: Hamiltonians. For another perspective on (L, m_L), we note that $*du$ is dual to the Hamiltonian vector field generated by u with respect to the invariant area form $\omega = 4da \wedge db/(a-b)^2$ on $\mathbb{G}$.

7 Computing earthquake laminations

In this section we prove Corollary 1.4 by describing a theoretical algorithm to determine L, Λ and F from a real–analytic map f.

The underlying architecture is *analytic*, meaning the algorithm involves finding the zeros of equations determined explicitly from f. In practice the algorithm is robust for computing leaves of Λ, but locating tritangent points can be more challenging.

Note that the computation of $m_L = |{*du}|/2$ from formula (1.2) is immediate, but it is only useful in concert with L.

Computing $\boldsymbol{\Lambda}$. We begin with a procedure to find the leaf $[a, b]$ of Λ landing at a given point $a \in \mathbb{R}$.

1. Using equation (1.1) for Z, compute the finite set C of points $c \in \widehat{\mathbb{R}}$ such that $(a, c) \in Z$.
2. For each $c \in C$, compute the unique $g_c \in G$ whose 1–jet matches that of $f(x)$ at $x = a$, and which satisfies $g_c(c) = f(c)$.
3. Choose $c \in C$ maximizing $g_c''(a)$, and set $b = c$.
4. Then $[a, b]$ is the leaf of Λ through a.

The key point is that step (1) makes step (2) possible. The space G is 3–dimensional while the space of possible 1–jets of f at two given points is 4–dimensional. It is only when these 1–jets satisfy the secant / derivative condition (1.1) defining Z that they can be matched by an element of G.

Step (3) ensures that $g_c \in \mathrm{Env}^+(f)$, and hence $[a, b]$ is a leaf of Λ.

Special cases. How can this procedure fail? There are just two issues.

One arises when $C = \emptyset$. In this case $(a, a) \in \overline{Z}$, a gives an endpoint of $\widehat{\mathbb{R}}/\Lambda$, and there is no leaf of Λ through a.

A second arises when the b to be chosen in step (3) is not unique. In this case we have located a tritangent plane for $\mathrm{Gr}(f)$ and a complementary polygon P for Λ, whose vertices are defined by $g_b(x) = f(x)$. Then the points of C adjacent to a give the two leaves of Λ through a.

Computing $\boldsymbol{L}$. Taking into account these special cases, L can be computed as follows.

1. Determine the finite set of $g \in G$ such that $\mathrm{gr}(g)$ is tangent to $\mathrm{gr}(f)$ at 3 or more points. Let E_1 be the x–coordinates of these points of tangency.

2. Determine the finite set $E_2 = \{a \in \widehat{\mathbb{R}} \, : \, (a,a) \in \overline{Z}\}$.

3. Determine the finite set E_3 of $a_0 \in \widehat{\mathbb{R}}$ such that Z is tangent to the line $a = a_0$. (A singular point of Z automatically gives a tangency.)

4. Let $E = E_1 \cup E_2 \cup E_3$, and cut Z into finitely many arcs L_i by removing every point $(a,b) \in Z$ with a or b in E.

5. For each i, choose a point $(a_i, b_i) \in L_i$ and test if $[a_i, b_i]$ is a leaf of Λ using the procedure above. If not, discard L_i.

6. Then L is the union of the closures in $\mathbb{G}$ of the remaining arcs L_i.

Computing F. It is now straightforward to calculate $F(z)$.

1. First locate the tritangent points of f and use them to determine the polygons P_i complementary to Λ. There is a unique $g_i \in G$ that agrees with f on ∂P_i, and $F|P_i = g_i$.

2. For $z \not\in \bigcup P_i$, determine the finite set of $(a_i, b_i) \in Z$ such that $z \in [a_i, b_i]$.

3. Use the algorithm for computing Λ to select the unique leaf $[a,b]$ passing through z.

4. Compute the unique $g \in G$ whose 1–jets at a and b match those of f. Then $F(z) = g(z)$.

Ideally, one can instead aim to find a parameterization of each component L_i of L, and use it to describe F on the corresponding component of Λ.

Proof of Corollary 1.4. The correctness of the algorithm is readily justified using Theorems 1.1 and 1.2. ■

8 Earthquakes for vector fields

In this section we construct earthquake extensions V for continuous vector fields v on $\widehat{\mathbb{R}}$, and describe their main properties in the case where v is real–analytic.

In many ways the study of vector fields on $\widehat{\mathbb{R}}$, which form a linear space, is simpler and more geometric than the study of mappings. At the same time qualitative features of the earthquake for $v(x)$ often persist for the mapping

$f(x) = x + \epsilon v(x)$. For these reasons, we put vector fields in the foreground in §9, §10 and §11.

Most of the constructions, theorems and proofs already given for mappings can be translated directly into results for vector fields. We will formalize that translation and sketch the proof of:

Theorem 8.1 *The following statements hold when v is real–analytic.*

1. *L is a finite union of subarcs of the locus $Z \subset \mathbb{G}$ defined by $S(v \times v) = 0$, or equivalently, by*
$$\frac{v'(a) + v'(b)}{2} = \frac{v(a) - v(b)}{a - b}. \tag{8.1}$$
2. *Setting $u = S(v \times v)/\xi$, we have*
$$m_L = \left|\frac{*du}{2}\right| = \left|\frac{2(v(a) + v'(a)(b-a) - v(b))}{(a-b)^2} + v''(a)\right| \cdot |da|. \tag{8.2}$$
3. *Provided $v \notin \mathrm{Lie}(G)$, Λ has finite type; each endpoint of the tree $T = \widehat{\mathbb{R}}/\Lambda$ corresponds to a real zero or pole of Sv; and each internal vertex corresponds to a tritangent plane for $\mathrm{Gr}(v)$.*
4. *The earthquake vector field V is piecewise real–analytic and area–preserving ($\mathrm{div}(V) = 0$), with bounded metric distortion, meaning*
$$\sup \mathrm{shear}(DV) = 2 \sup |\overline{\partial} V| < \infty.$$
5. *Outside of finitely many leaves of Λ, we have*
$$\mathrm{shear}(DV) = \frac{m_\Lambda}{|\eta|}.$$
6. *If $\Lambda | U = \mathcal{F}(\omega)$, then $|\omega|^2 = 2|\mathrm{Tr}(dq.dq)|$, where $q = \mathrm{osc}(V)$.*
7. *We have*
$$\int_{\mathbb{H}} \rho^2 \, |\overline{\partial} V|^2 = \frac{\pi}{8} \int_L (m_L/|ds|)^2 \, |ds|. \tag{8.3}$$

Note that equation (8.1) characterizes $v \in \mathrm{Lie}(G)$, and that equation (8.2) reduces to $m_L = |v''(a)| \cdot |da|$ if the 1–jets of v at a and b vanish.

Vector fields. We begin by constructing the earthquake extension V of v using envelopes as in §2. A detailed account from a different perspective can be found in [Diaf].

Let $\mathrm{Vect}(\widehat{\mathbb{R}})$ denote the space of all continuous vector fields $v : \widehat{\mathbb{R}} \to \mathrm{T}\widehat{\mathbb{R}}$. One can informally regard $\mathrm{Vect}(\widehat{\mathbb{R}})$ as the Lie algebra of the group $H = \mathrm{Homeo}^+(\widehat{\mathbb{R}})$. Then, analogous to the inclusion $G \subset H$ that underpins the development of earthquakes in §2, we have the inclusion

$$\mathrm{Lie}(G) \subset `\,\mathrm{Lie}(H)' = \mathrm{Vect}(\widehat{\mathbb{R}}).$$

Concretely, $\mathrm{Vect}(\widehat{\mathbb{R}})$ consists of those continuous vector fields $v(x)\, d/dx$ on $\mathbb{R}$ such that

$$\lim_{|x|\to\infty} v(x)/x^2 \quad \text{exists;}$$

this condition ensures continuity at infinity. The subspace $\mathrm{Lie}(G)$ consists of the *quadratic* vector fields, given by $q = q(x)\, d/dx$ where

$$q(x) = a + bx + cx^2.$$

Note that the graph of $q(x)$ is a parabola.

Envelopes. Using the inclusion $\mathrm{Lie}(G) \subset \mathrm{Vect}(\widehat{\mathbb{R}})$, together with the partial ordering

$$v_1 \geq v_2 \iff v_1(x) \geq v_2(x)\ \forall x \in \mathbb{R},$$

we define, as in §2, the *upper parabolic envelope of* v by

$$\mathrm{Env}^+(v) = \{\text{all minimal } q \geq v\} \subset \mathrm{Lie}(G).$$

Here $q_1 \in \mathrm{Lie}(G)$ is *minimal* if $q_1 \geq v$ and any $q_2 \in \mathrm{Lie}(G)$ with $q_1 \geq q_2 \geq v$ satisfies $q_2 = q_1$.

Earthquakes. From the upper envelope of v we obtain a stratification $\mathcal{S}$ of $\mathbb{H}$ by convex sets of the form

$$S_q = \mathrm{hull}(\{x \in \widehat{\mathbb{R}} \,:\, q(x) = v(x)\}),$$

indexed by $q \in \mathrm{Env}^+(v)$. From $\mathcal{S}$ we obtain the geodesic lamination Λ, and then the earthquake extension V of v, which is characterized by

$$V|S_q \;=\; q|S_q$$

(up to resolving the values of V where it is discontinuous). The transverse measure on Λ is defined by

$$m_\Lambda(\tau) = \lim \sum |q_i - q_j|,$$

where $|q|^2 = |\langle q, q\rangle| = |b^2 - 4ac|$, consistent with equation (5.3). As usual the dual lamination in $\mathbb{G}$ will be denoted by (L, m_L).

A guide to translation. To pass from the language of maps to the language of vector fields, we make the following substitutions.

- The earthquake pair (f, F) is replaced by (v, V).
- The usual Schwarzian Sf is replaced by $Sv = v'''(x)\, dx^2$.
- The two–variable Schwarzian $S(f \times f)$ is replaced by the Lie derivative

$$S(v \times v) = \mathcal{L}_{v \times v}(\xi).$$

- The space curve $\mathrm{Gr}(f)$ is replaced by the curve

$$\mathrm{Gr}(v) = \left\{ \frac{(1 - x^2,\, 2x,\, 2v(x))}{1 + x^2} \ :\ x \in \widehat{\mathbb{R}} \right\} \subset \mathbb{RP}^3. \tag{8.4}$$

- The map $\mathrm{osc}(F) : \mathbb{H} \to G$ is replaced by the map

$$\mathrm{osc}(V) : \mathbb{H} \to \mathrm{Lie}(G)$$

  characterized by $\mathrm{osc}(V)|S = V|S$ for all $S \in \mathcal{S}$.

Shear. In addition, for vector fields, the definition of $\mathrm{shear}(A)$ is extended to $A \in \mathrm{Lie}(G) = \mathrm{sl}_2(\mathbb{R})$ by

$$\mathrm{shear}(A) = 2|\overline{\partial} A|.$$

Here we consider A as a map from $\mathbb{R}^2 \cong \mathbb{C}$ to itself. We then have

$$\mathrm{shear}(DV) = 2|\overline{\partial} V|$$

for a vector field on $\mathbb{H}$. The shear can also be characterized as the unique $\mathrm{Ad}\,\mathrm{SO}(2)$–invariant function on $\mathrm{sl}_2(\mathbb{R}) / \mathrm{so}_2$ satisfying

$$\mathrm{shear} \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = |t|.$$

A metatheorem. With this dictionary in place, we can state an informal principle:

> *Theorems for mappings translate, via the dictionary above, into results that are true for vector fields.*

Indeed, the study of v is essentially the same as the study of the map $f(x) = x + \epsilon v(x)$.

Proof of Theorem 8.1. These statements are translations of Theorems 1.1, 1.2, Corollary 1.3, Theorem 5.1 and Theorem 6.1; the proofs can be translated as well. ■

Remarks on the space curve $\mathrm{Gr}(v)$. The definition of $\mathrm{Gr}(v)$ is chosen so that hyperplane sections correspond to $q \in \mathrm{Lie}(G)$. Indeed, when $\mathrm{Gr}(q)$ lies on the hyperplane defined by $x_3 = Ax_1 + Bx_2 + C$, we have

$$2q(x) = A(1-x^2) + 2Bx + C(1+x^2).$$

The definition of $\mathrm{Gr}(v)$ is also compatible with the bijection between S^1 and $\widehat{\mathbb{R}}$ given by $x = \tan(\theta/2)$. If, under this change of coordinates, $v(x)\, d/dx$ becomes $v(\theta)\, d/d\theta$, then

$$v(\theta) = \frac{2v(x)}{1+x^2}, \tag{8.5}$$

and thus $\mathrm{Gr}(v)$ is also the image of S^1 under the map

$$\theta \mapsto (\cos\theta, \sin\theta, v(\theta)) \in \mathbb{R}^3.$$

Alternatively, if we write $v = izv(z)\, d/dz$ in the complex coordinate

$$z = \exp(i\theta) = (i-x)/(i+x),$$

then we also have $v(z) = 2v(x)/(1+x^2)$ as above. Whether viewed in the x or z coordinate, v remains a rational vector field of the same degree.

Formula (8.5) can also be checked using the fact that the usual metric $|d\theta|$ on S^1 becomes $2|dx|/(1+x^2)$ on $\widehat{\mathbb{R}}$.

9 Algebraic maps and vector fields

In this section we begin the study of the algebraic geometry of earthquakes. That is, we study the earthquake extensions of rational maps and vector fields on $\widehat{\mathbb{R}}$.

We have $\deg(f) = 1 \iff f \in G$ and $\deg(v) = 2 \iff v \in \mathrm{Lie}(G)$. In these two cases the associated earthquake is trivial, so henceforth we assume $\deg(f) \geq 3$ and $\deg(v) \geq 4$.

We begin with definitions, and then prove Theorems 1.5 and 1.6, showing that the complexity of $T = \widehat{\mathbb{R}}/\Lambda$ is controlled by the algebraic degree of f or v. In fact we will establish the following sharper result.

Theorem 9.1 *Let $\Lambda \neq \emptyset$ be the lamination determined by the right earthquake extension of a real–analytic diffeomorphism f or vector field v. Then the components (P_i) of $\mathbb{H} - \Lambda$ satisfy*

$$\sum |\chi(P_i)| \leq N/2 - 2,$$

where N is the number of real zeros of odd order *for Sf or Sv.*

Corollary 9.2 *The tree $T = \widehat{\mathbb{R}}/\Lambda$ has at most $N-2$ vertices.*

We conclude by comparing the resulting $O(d)$ bounds for maps and vector fields of degree d to the bounds available via the classical enumeration of tritangent planes over $\mathbb{C}$.

Remark: The 4 vertex theorem. Theorem 9.1 implies $N \geq 4$, giving, in the real–analytic case, a new proof of the four–vertex theorem for convex plane curves and of Ghys's theorem on the zeros of Sf. For these connections, see [OT, Ch. 4].

Definitions. Over the real numbers, a *rational function* is the quotient $f(x) = P(x)/Q(x)$ of a pair of relatively prime polynomials $P, Q \in \mathbb{R}[x]$. Its *algebraic degree* is given by

$$\deg(f) = \max(\deg P, \deg Q) = \#(\text{complex zeros of } f).$$

We will be interested in the case where $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ is an orientation–preserving homeomorphism. In this case $\deg(f)$ must be *odd*, and

$$\deg \operatorname{Gr}(f) = \deg(f) + 1.$$

A *rational vector field* is similarly given by $v = v(x)\, d/dx$, where $v(x) = P(x)/Q(x)$ is a real rational function. Its degree is defined by:

$$\deg(v) = \#(\text{complex zeros of } v),$$

counted with multiplicity.

We will be interested in the case where v gives a continuous vector field on $\widehat{\mathbb{R}}$. Continuity of v implies that $Q(x)$ has no real zeros, and that

$$\deg(v) = \deg(Q) + 2 \geq \deg(P).$$

In particular, $\deg(v)$ must be *even*, and we have

$$\deg(v) = \deg \operatorname{Gr}(v).$$

Schwarzians. It is elementary to control the complex zeros of Sf and Sv.

Lemma 9.3 *Counted with multiplicities, the number of complex zeros of the Schwarzian satisfies*

$$\#Z(Sf) \leq 4\deg(f) - 8 \quad \textit{and} \quad \#Z(Sv) \leq 4\deg(v) - 12.$$

Proof. The Schwarzian Sf is a meromorphic quadratic differential with a double pole at each critical point of f, regardless of its multiplicity. Since the total number of critical points is at most $2d-2$, $d = \deg(f)$, the number of poles of Sf is at most $4d - 4$. Since a quadratic differential has 4 more poles than zeros, this gives $\#Z(Sf) \leq 4d - 8$.

Similarly, the poles of the quadratic differential Sv come from the poles of v, of which there are $d-2$ counting multiplicity, $d = \deg(v)$. A pole of v of order m contributes a pole of order $m + 3$ to Sv, so the total number of poles of Sv is at most $4d - 8$. This gives $\#Z(Sv) \leq 4d - 12$. ■

Proof of Theorems 1.5 and 1.6 . Combining these bounds with Theorem 9.1 gives $\sum |\chi(P_i)| \leq 2\deg(f) - 6$ or $2\deg(v) - 8$. ■

Proof of Theorem 9.1. We first treat the case of a vector field v. Let Λ be the lamination determined by the right earthquake extension of v, and let e be the number of ends of $\widehat{\mathbb{R}}/\Lambda$. We have $\sum |\chi(P_i)| = e - 2$ since $\chi(T) = 1$, so it suffices to show that $e \leq N/2$.

Each end of T corresponds to a component of Λ whose leaves shrink down to a point $x_0 \in \widehat{\mathbb{R}}$. By the vector field version of Theorem 1.2 (see §8), $Sv(x_0) = 0$.

To make the meaning of the Schwarzian more concrete, consider any point $x_0 \in \widehat{\mathbb{R}}$. We can then change coordinates so that $x_0 = \infty$, and subtract a quadratic vector field $q \in \mathrm{Lie}(G)$ from $v(x)$, so that for $|x| \gg 0$ we have

$$v(x) = A/x + B/x^n + O(1/x^{n+1}), \tag{9.1}$$

where $B \neq 0$. Then $Sv(x_0) = 0 \iff A = 0$, in which case the order of vanishing of Sv at x_0 is $n - 1$.

Now assume, for simplicity, that all the real zeros of Sv are simple. This implies their total number N is even, and that the sign of Sv alternates between zeros. When a given zero x_0 is normalized as in equation (9.1), it has the form $v(x) = B/x^2 + O(1/x^3)$ with $B \neq 0$.

The sign of B is determined by the sign change of Sv at x_0 (from positive to negative or vice–versa).

We claim that if $B > 0$, then the leaves of Λ cannot nest down to x_0. Indeed, each leaf has the form $[a, b] = S_q$ for some quadratic vector field $q = q(x)\, d/dx \in \mathrm{Lie}(G)$. Since Λ comes from a right earthquake, q belongs to the upper parabolic envelope of v; thus $q(x) \geq v(x)$ for all x. Moreover, the 1–jet of q must agree with that of v at a and b. But if the leaf $[a, b]$ is close to $x_0 = \infty$, then we have $a \ll 0 \ll b$, and hence $q'(a) = v'(a) \approx -2B/a^3 > 0$;

similarly $v'(b) < 0$. It follows that the constant $q''(x)$ is strictly negative, and hence the graph of $q(x)$ is a downward parabola. But then $q(x) \ll v(x) \approx 0$ when x is large, contradicting the assumption that $q \geq v$.

Since the sign of B is determined by the sign of Sv, which alternates around $\widehat{\mathbb{R}}$, only half of the real zeros of Sv can contribute to the ends of T. Thus $e \leq N/2$, completing the proof in this case.

The same argument shows the bound $e \leq N/2$ also holds when all the real zeros of Sv have *odd order*.

Now suppose Sv vanishes to even order $2k$ at x_0. Then $v(x) = B/x^{2k+1} + O(1/x^{2k+2})$, with $B \neq 0$, when normalized so $x_0 = \infty$ as above. If $B > 0$ then for $a \ll 0 \ll b$ we have

$$v'(a) + v'(b) < 0 < 2(v(b) - v(a))/(b - a).$$

But if $[a, b]$ is a leaf of Λ then the expressions on the left and right must agree by equation (8.1). Thus x_0 cannot contribute an endpoint to $\widehat{\mathbb{R}}/\Lambda$. The same argument applies when $B < 0$.

Moreover, the sign of Sv does not change at a zero of even order, so the alternation argument giving $N/2$ continues to hold.

The argument with Sf in place of Sv follows very similar lines. ■

Remark: Algebraic geometry of space curves. One can also try to control the number of complementary polygons (P_i) in Theorem 1.5 by bounding the number of tritangent planes to $\mathrm{Gr}(f)$, a rational space curve of degree $D = d + 1$ where $d = \deg(f)$. Indeed, by a general result in algebraic geometry (de Jonquière's formula, [ACGH, §VIII.5]), we have:

$$\#\left\{ \begin{array}{c} \text{Complex tritangent planes to a smooth} \\ \text{rational space curve of degree } D \end{array} \right\} = 8\binom{D-3}{3}.$$

Since each polygon P_i contributes a tritangent plane, this classical result gives an $O(d^3)$ bound for the number of components of $\mathbb{H} - \Lambda$.

The $O(d)$ bound provided by Theorem 1.5 is thus a considerable improvement; it shows that the majority of the complex tritangents to $\mathrm{Gr}(f)$ do not create complementary polygons or contribute to $\sum |\chi(P_i)|$.

10 Degrees 3 and 4, symmetries and singularities

In this section we turn to the study of specific families of algebraic maps and vector fields.

We first define the moduli space of maps or vector fields of a fixed degree d. Next we discuss the simplest examples: maps of degree 3, vector fields of degree 4. We then describe a sequence of vector fields v_n with $\mathbb{Z}/n$ symmetry, and show

Theorem 10.1 *The lamination Λ associated to v_n has exactly one complementary region P_1, namely a regular ideal polygon with n sides.*

Corollary 10.2 *There exist instances of Theorem 1.6 with*

$$\sum |\chi(P_i)| \geq (\deg(v) - 4)/2.$$

(A similar lower bound of $(\deg(f) - 3)/2$ holds in Theorem 1.5.)

Finally we give algebraic examples showing that Z and L can be singular.

Moduli spaces. Let $F_d \subset \text{Homeo}^+(\widehat{\mathbb{R}})$ be the set of rational homeomorphisms of algebraic degree d. Note that $F_1 = G$, and $F_d = \emptyset$ when d is even.

Since the behaviors of the earthquake extensions of f and $g_1 f g_2$ are essentially the same, we define the *moduli space* of rational homeomorphisms of degree d by

$$\mathcal{MF}_d = G \backslash F_d / G.$$

It is a connected semialgebraic set of real dimension $2d - 5$, provided $d \geq 3$ (and d is odd).

Vector fields. Let $V_d \subset \text{Vect}(\widehat{\mathbb{R}})$ be the space of rational vector fields of degree d with no poles on $\widehat{\mathbb{R}}$. Note that $V_2 = \text{Lie}(G)$, and $V_d = \emptyset$ when d is odd.

Since the behaviors of the earthquakes for v and $g_*(tv + q)$, with $g \in G$, $t \neq 0$ and $q \in \text{Lie}(G)$, are essentially the same, we define the *moduli space* of vector fields of degree d by

$$\mathcal{MV}_d = (\mathbb{P}V_d / \operatorname{Lie}(G))/G.$$

It is a connected semialgebraic space of dimension $2d - 8$.

Cubic maps and elliptic curves. In the first interesting case, $d = 3$, one can readily verify:

Proposition 10.3 *The moduli space of cubic maps $\mathcal{MF}_3$ is 1–dimensional. There is a continuous bijection $[0, 1) \to \mathcal{MF}_3$ sending t to*

$$f_t(x) = \frac{x^3 + tx}{1 + tx^2}.$$

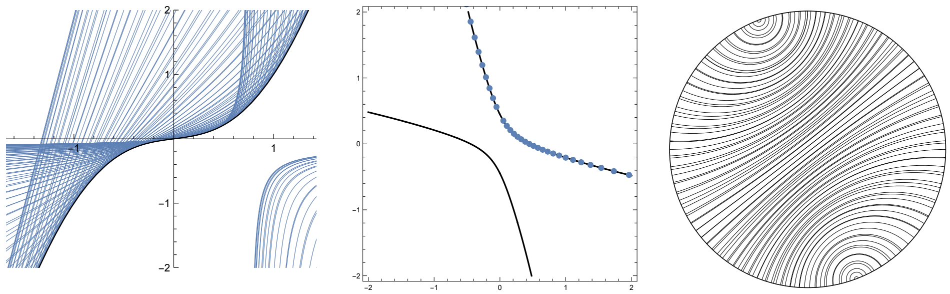

Figure 4. $\mathrm{Env}^+(f)$, $L \subset Z$, and Λ for the cubic rational map $f_{1/5}(x)$.

Note that $f_0(x) = x^3$. Setting this case aside, we observe that $f|\widehat{\mathbb{R}}$ has no critical points, and (using equation (1.1)) that $Z \subset \mathbb{G}$ is the elliptic curve defined by

$$a^2b^2t - a^2 - 4ab - b^2 + t = 0.$$

Another natural invariant of f is the elliptic curve E branched over its 4 (complex) critical points, defined by

$$y^2 = tx^4 + (3 - t^2)x^2 + t.$$

In fact Z and E are isomorphic; over the complex numbers, the degree two map $\pi_1 : Z(\mathbb{C}) \to \widehat{\mathbb{C}}$ is branched over the same four points.

Returning to the reals, the locus $Z \subset \mathbb{G}$ has two components, L and L'; these give the dual right and left earthquake laminations for f respectively. The 4 zeros of Sf are all real. The intersection of $\overline{L}$ with the diagonal singles out two of them, namely the roots of

$$x^2 + (\sqrt{6/t - 2})x - 1 = 0.$$

The leaves of Λ, which foliate $\mathbb{H}$, nest down to single points at the two roots of the equation above; they give the two endpoints of the tree $\widehat{\mathbb{R}}/\Lambda$, which is an interval.

Example. The case $t = 1/5$ is illustrated in Figure 4. The two distinguished zeros of Sf are x and $-1/x$, where $x = \sqrt{8} - \sqrt{7}$, and the dotted component of Z is L.

Quartic vector fields. For vector fields the first interesting case arises when $d = 4$. Again, one can readily check:

Proposition 10.4 *The moduli space $\mathcal{MV}_4$ is a single point, represented by*

$$v(x) = \frac{1}{1+x^2}\,\frac{d}{dx}.$$

It is straightforward to compute the right earthquake extension of the model quartic vector field v above. By symmetry, $\Lambda = \bigcup_{a>0}[-a,a]$, and its spine is the imaginary axis, $\Sigma = [0,\infty]$.

The locus Z has two components: the line L defined by $a+b=0$, dual to Λ; and the conic L' defined by $ab=1$, dual to the left earthquake lamination for v. We readily compute, using Theorem 8.1, that

$$m_L/|ds| = m(a) = \frac{8}{(a+1/a)^3}.$$

Since $m(a)$ achieves its maximum at $a=1$ and $[-1,1]\cap\Sigma = \{i\}$, the maximum distortion of V occurs when $z=i$, at which point we have

$$\operatorname{shear}(DV) = 2\max|\overline{\partial}V| = m(1) = 1.$$

We also have $|ds| = |da/a|$ along L which gives, by equation (8.3),

$$\int_{\mathbb{H}} \rho^2\,|\overline{\partial}V|^2 = \frac{\pi}{4}\int_0^\infty m(a)^2\,\frac{da}{a} = \frac{4\pi}{15}. \tag{10.1}$$

(The factor $\pi/4$ appears instead of $\pi/8$, because the last integral is only over half of L.)

One can also directly compute, for $z\in\mathbb{H}$, the formulas

$$V(z) = \frac{z^2(1+\overline{z}^2)}{(1+|z|^2)^2}\,\frac{d}{dz} \quad \text{and} \quad |\overline{\partial}V| = \frac{4y|z|^2}{(1+|z|^2)^3}.$$

The calculation of V uses the fact that $V|[-a,a] = q|[-a,a]$ where q is determined by $v(x)$ and $v'(x)$ at $x=\pm a$. The formula for $|\overline{\partial}V|$ above can be used to independently verify equation (10.1).

Vector fields with $\mathbb{Z}/n$ symmetry. Next we discuss the sequence of fields $v_n = v_n(x)\,d/dx$ of degree $2n$ with $\mathbb{Z}/n$ symmetry given by

$$v_n(x) = \operatorname{Re}(z^n)\cdot\frac{1+x^2}{2}, \quad \text{where } z = \frac{i-x}{i+x}\in S^1.$$

These examples are chosen so that $\operatorname{Gr}(v_n)$ coincides with the graph of

$$x_3 = \cos(n\theta)$$

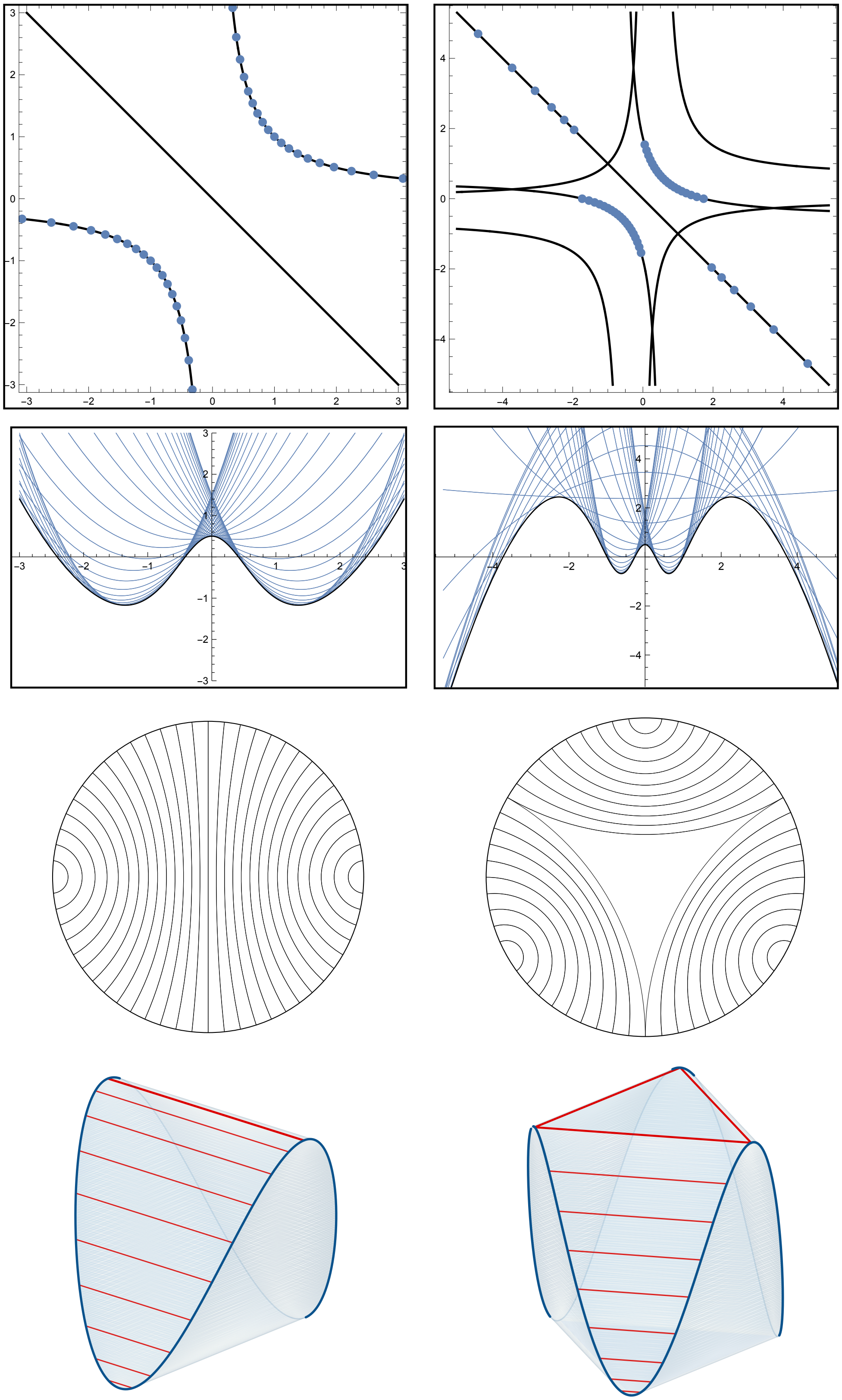

Figure 5. Earthquakes and space curves for the vector fields v_2 and v_3.

over the unit circle $x_1^2 + x_2^2 = 1$; see equation (8.5). For $n = 2$, the sets $L \subset Z$, $\mathrm{Env}^+(v_n)$, Λ and $\mathrm{hull}(\mathrm{Gr}(v_n))$ are illustrated in the left column of Figure 5; those for $n = 3$ are shown on the right. The dotted portions of Z correspond to the subset L.

Note that v_2 is a quartic vector field of the type just discussed. The space curve $X = \mathrm{Gr}(v_2)$ is shaped like the rim of a saddle. Since $\deg(X) = 4$, it has no tritangent planes; it is always unstable when placed on a tabletop, since it rests on just 2 points.

On the other hand, for $n = 3$ the sextic curve $X = \mathrm{Gr}(v_3)$ has two real tritangents, given by the planes $x_3 = \pm 1$ as can be seen in Figure 5.

The upper plane corresponds to the quadratic vector field $q = q(x)\, d/dx$ with

$$q(x) = \frac{1 + x^2}{2},$$

whose graph touches $y = v_3(x)$ from above at $x = 0$ and $x = \pm\sqrt{3}$. These three points are the vertices of the ideal triangle $P_1 = \mathbb{H} - \Lambda$.

The locus Z consists of 3 hyperbolas and a straight line, while $L \subset Z$ has 3 components. Despite appearances, there are dots along two different hyperbolas, indicating the parts of L coming from the right and left components of Λ. The dots along the line $a + b = 1$ correspond to the component of Λ at the top.

For general n the locus Z consists of n (1,1) curves in $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$, cyclically permuted by the $\mathbb{Z}/n$ symmetry group of v_n.

Proof of Theorem 10.1. Let Λ denote the right earthquake lamination for v_n. We will claim that $\mathbb{H} - \Lambda$ has just one component, namely a symmetric, ideal n–gon P_1. Indeed, the plane $x_3 = 1$, which is tangent to $\mathrm{Gr}(v_n)$ at the n points where $\cos(n\theta) = 1$, produces exactly one such polygon P_1 in $\mathbb{H} - \Lambda$. On the other hand, Sv_n has exactly $2n$ real zeros, all simple, coming from the zeros of the third derivative of $\cos(n\theta)$. It follows from Theorem 9.1 that $\sum |\chi(P_i)| \le n - 2 = |\chi(P_1)|$, so there are no other complementary components. ∎

Singularities of Z and L. To conclude we show that the 1–dimensional varieties Z and L can be singular.

For example, when

$$v(x) = \frac{x^3(1 - x)^4}{1 + x^6},$$

the locus Z has a cusp at $(a, b) = (0, 1)$, locally modeled on $3a^2 + 2(b-1)^3 = 0$; see Figure 6. (This local model can be computed from the Newton polygon for the polynomial defining Z coming from equation (8.1)).

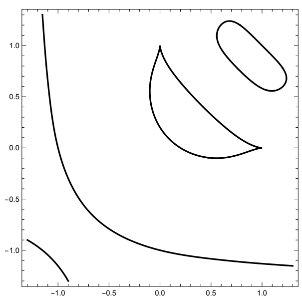

Figure 6. The locus Z has a cusp.

The singularity of Z at $(0,1)$ arises because $q(x)=0$ osculates $v(x)$ to high order at both 0 and 1. However $(0,1)\not\in L$ because $v(x)$ changes sign.

To remedy this, consider

$$v(x) = -\frac{x^4(1-x)^6}{1+x^8}.$$

Then $[0,1]$ is actually a leaf of Λ, and we obtain a (less visible) singularity in the interior of an arc of L itself, with the local model $4a^3+3(b-1)^5=0$. In particular, this shows:

> *The earthquake V for a rational vector field v need not be smooth in the interior of Λ.*

(It can be singular along finitely many leaves.)

Remark: Knotted space curves. All space curves of the form $\mathrm{Gr}(v)$ are unknotted, since they can be deformed to $\mathrm{Gr}(0)$. Freedman asked if every smooth knotted space curve has a tritangent plane [Fr, §5]; the answer is no [Mor], and in fact the counterexample can be taken as a rational space curve of degree 6 [RS].

11 Vector fields of degree 6

In this section we conclude our algebraic study of earthquakes by analyzing rational vector fields of degree 6. In particular we prove Theorems 1.7 and 1.8, which characterize the possibilities for $\Lambda\subset\mathbb{H}$ up to isometry.

Definitions. We begin with some terminology. Given $P\in\mathbb{R}[x]$ we write $P>0$ to mean $P(x)>0$ for all $x\in\mathbb{R}$. The expression $v=(P,Q)$ will denote a *lifting* of a rational vector field v to a pair of polynomials $P, Q \in \mathbb{R}[x]$ such that $v(x) = P(x)/Q(x)$, $\gcd(P, Q) = 1$ and $Q > 0$.

Types of tritangents. In this section alone, for clarity, we will distinguish between *geometric* and *algebraic* tritangent planes.

We say H_q is a *geometric* tritangent plane for v if H_q is tangent to $\mathrm{Gr}(v)$ at 3 or more *distinct* points. (This coincides with the definition of 'tritangent plane', given in the Introduction.)

We say H_q is an *algebraic* tritangent plane if, as a divisor over $\mathbb{C}$, we have

$$H_q \cap \mathrm{Gr}(v) = 2D + E,$$

where $\deg(D) \geq 3$. In other words, over $\mathbb{C}$, H_q is tangent to $\mathrm{Gr}(v)$ at 3 or more points *counted with multiplicity.*

The algebraic case. Let $v = (P, Q)$ be a lifting of a sextic vector field v. In the sextic case, $\deg(2D) = 6$ and hence $E = 0$. Thus H_q is an algebraic tritangent plane $\iff$ we can write

$$P(x) \pm C(x)^2 = q(x)Q(x) \tag{11.1}$$

for some polynomial $C \in \mathbb{R}[x]$. Note that, up to sign, C is determined by q and vice–versa. To make its sign unique, we adopt the convention that the leading coefficient of C is positive.

Generically we have $\deg(C) = 3$, so we will refer to C as a *supporting cubic* for $v = (P, Q)$. However lower degrees are allowed; these arise when $v - q$ has a zero at $x = \infty$.

There are two cases, upper and lower, depending on the choice of sign. When

$$P(x) + C(x)^2 = q(x)Q(x)$$

we say H_q is an *upper* algebraic tritangent plane, and C is an *upper* supporting cubic, since $q \geq v$. Otherwise we are in the lower case.

Geometric tritangents. An algebraic tritangent H_q is a geometric tritangent exactly when the divisor $2D = H_q \cap \mathrm{Gr}(v)$ satisfies

$$D = x_1 + x_2 + x_3$$

for 3 distinct points in $\widehat{\mathbb{R}}$. This means either $\deg(C) = 3$ and $C(x)$ has *3 distinct, real roots*; or C is a quadratic with 2 distinct real roots (and $v - q$ has a double zero at infinity).

The quotient ring determined by $\boldsymbol{Q}$. The supporting cubics for $v = (P, Q)$ can be efficiently analyzed using two facts about the ring

$$A = \mathbb{R}[x]/Q(x).$$

Namely, (i) the 4–dimensional vector space of cubic polynomials, $\mathbb{R}[x]_3$, maps isomorphically to A; and (ii) as a ring, we have

$$A \cong \mathbb{C}^2 \quad \text{or} \quad A \cong \mathbb{C}[\epsilon]/(\epsilon^2).$$

The second case arises when Q is a square, and the first when it is not. Indeed, if Q has distinct roots $u, \overline{u}, v, \overline{v}$, then

$$[P(x)] \mapsto (P(u), P(v)) \tag{11.2}$$

gives an isomorphism from A to $\mathbb{C}^2$; while if Q has double roots $u, \overline{u}$, then the map

$$[P(x)] \mapsto P(u) + \epsilon P'(u)$$

gives an isomorphism from A to $\mathbb{C}[\epsilon]/(\epsilon^2)$.

The 4 cubics of v. It is now easy to prove:

Theorem 11.1 *The space curve* $\mathrm{Gr}(v)$ *has either 2 or 4 distinct algebraic tritangent planes* H_q. *The first case arises when* v *has a double pole over* $\mathbb{C}$*; the second, when it does not.*

In either case, $\mathrm{Gr}(v)$ *has an equal number of upper and lower algebraic tritangent planes.*

Proof. Choose a lifting $v = (P, Q)$. It suffices to find all the supporting cubics $C \in \mathbb{R}[x]_3$. By equation (11.1), this is the same as solving the equation

$$[C^2] = [\pm P] \ \text{ in } A = \mathbb{R}[x]/(Q). \tag{11.3}$$

First suppose v has no double pole. Then Q has simple roots $(u, v, \overline{u}, \overline{v})$, and we can analyze equation (11.3) using the isomorphism $A \cong \mathbb{C}^2$ given by equation (11.2). Since $\gcd(P, Q) = 1$, neither coordinate of $(P(u), P(v))$ is zero. Each coordinate has two square roots in $\mathbb{C}$, and the same is true for $-P$, so we obtain 8 distinct solutions C, 4 of which have positive leading coefficient.

Now suppose v has a double pole. Then Q has double zeros at $(u, \overline{u})$, and $P(u) + \epsilon P'(u)$ has only the two square roots

$$\pm\sqrt{P(u)}\left(1 + \epsilon \frac{P'(u)}{2P(u)}\right)$$

in $\mathbb{C}[\epsilon]/(\epsilon^2)$, giving exactly 2 supporting cubics instead of 4.

In both cases half of the cubics arise from $[+P]$ and half from $[-P]$, so half of the corresponding planes H_q are upper and half lower. ∎

Proof of Theorem 1.7. We must show that $\mathbb{H} - \Lambda$ consists of 0, 1 or 2 ideal triangles, and all 3 possibilities arise.

Since $\deg(v) = 6$, every geometric tritangent plane H_q meets $\mathrm{Gr}(v)$ in exactly 3 distinct points, and hence every complementary component is an ideal triangle. Moreover for a right earthquake, H_q is also an upper algebraic tritangent plane; hence, by the preceding result, at most 2 triangles arise.

An example with no triangles is given by $v = 1/(1+x^2)^2\, d/dx$, as can be checked directly. An example with just 1 triangle is given by the vector field $v = v_3$ studied in §10; see Figure 5. Finally an example with 2 triangles is given by $v = (x^2-1)/(1+x^4)$; its lamination was used to produce Figure 3 in §4. ∎

Remark: Linear systems. Here is a consequence of Theorem 11.1 formulated in terms of divisors. Suppose we are in the typical case where $v = P/Q$ and Q is square–free. Let $Z(v)$ denote the degree 6 divisor defined by the complex zeros of v on $\widehat{\mathbb{C}}$. Then:

> *Exactly 4 divisors in the linear system $|Z(v + \mathrm{Lie}(G))|$ have the form $D = 2E$.*

Synthesis. We now turn to the inverse problem of *finding* a sextic vector field $v = (P, Q)$ with two given supporting cubics, C and D. We assume $\deg(C) = \deg(D) = 3$ and their leading coefficients are positive. (They need not be monic.)

To synthesize v, first observe that $C^2 + D^2 > 0$, provided $\gcd(C, D) = 1$. Hence we can factor this sum in $\mathbb{R}[x]$ (in more than one way) as

$$C^2 + D^2 = qQ,$$

where $\deg(q) = 2$, $\deg(Q) = 4$, $q > 0$ and $Q > 0$.

Now set $v = P/Q$ with $P = -C^2$. Then C and D are upper and lower supporting cubics for v, respectively, since

$$P + C^2 = 0 \cdot Q \quad \text{and} \quad P - D^2 = -(C^2 + D^2) = -qQ.$$

The same reasoning works provided $C^2 + D^2$ has no more than 2 real roots, counted with multiplicity, and we allow $q \geq 0$. This shows:

Theorem 11.2 *Any two cubics C, D with* $\deg \gcd(C, D) \leq 1$ *can be realized as the upper and lower supporting cubics of a sextic vector field $v = (P, Q)$.*

Corollary 11.3 *Any two ideal triangles P_1 and P_2 in $\mathbb{H}$ sharing at most one vertex can be realized, simultaneously, as complementary components of the right and left earthquake laminations of a sextic vector field.*

Upper and lower triangles sharing an edge cannot occur, since that would force $\mathrm{Gr}(v)$ to lie in a plane.

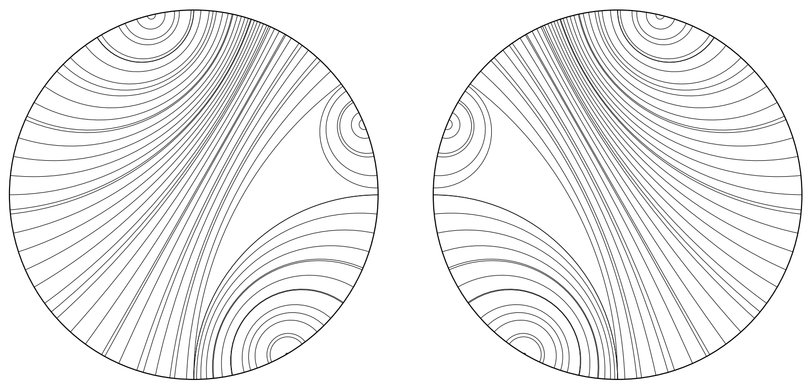

Figure 7. Realizing specified upper and lower ideal triangles.

Example. The ideal triangles with vertices $(0,1,2)$ and $(0,-1,-2)$ are realized by the right and left laminations of

$$v(x) = -\frac{C(x)^2}{Q(x)} = -\frac{(x(x-1)(x-2))^2}{x^4+13x^2+4};$$

see Figure 7. The six visible endpoints of the two laminations account for all the real zeros of Sv.

Configurations of ideal triangles. We now turn to the proof of Theorem 1.8.

The reasoning above also yields the following variant of Theorem 11.2:

Theorem 11.4 *Two cubics C and D with $\gcd(C,D)=1$ can be realized as the upper supporting cubics of a sextic vector field $\iff$ C^2-D^2 has at least 4 roots in $\mathbb{C}-\mathbb{R}$, counted with multiplicity.*

To make the condition on C^2-D^2 more geometric, we use the idea of rational separation.

Here is a slightly different perspective on the definition given in the Introduction. Given a pair of disjoint finite sets $A, B \subset \widehat{\mathbb{R}}$ with $|A|=|B|=n$, let $u=\log|R(x)|$, where R is a rational function with divisor $(R)=A-B$.

The *potential function* u is uniquely determined by A and B up to an additive constant. We say A and B are *rationally separated* if, for some $t \in \mathbb{R}$, A and B are each contained in a single connected component of $\widehat{\mathbb{R}} - u^{-1}(t)$. (This version can be generalized to $A, B \subset \widehat{\mathbb{C}}$.)

More concretely, if $A = (a_i)$ and $B = (b_i)$ with

$$a_1 < a_2 < \ldots < a_n < b_1 < b_2 < \cdots < b_n, \tag{11.4}$$

and we let

$$u(x) = \sum \log |x - a_i| - \sum \log |x - b_i|,$$

then A and B are rationally separated $\iff$

$$\sup_{[a_1, a_n]} u(x) \ < \ \inf_{[b_1, b_n]} u(x).$$

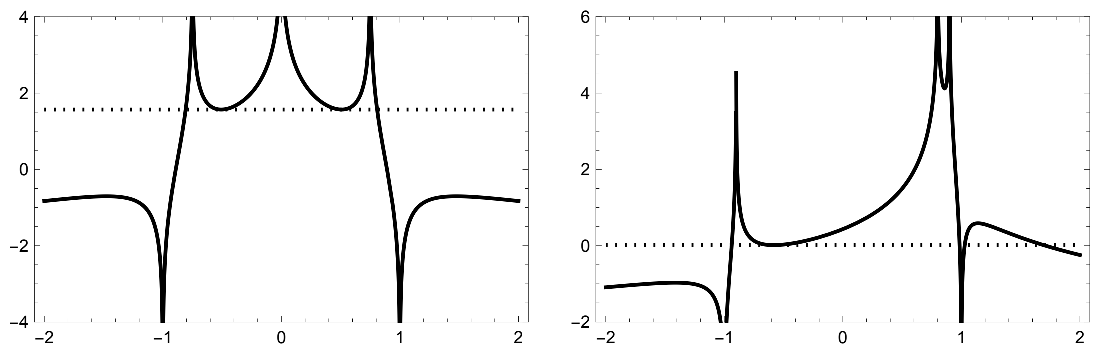

Figure 8. Potential functions for $A = (-1, 1, \infty)$ and $B = (-t, s, t)$, with $(s, t) = (0, 3/4)$ (left) and $(s, t) = (8/10, 9/10)$ (right).

Example. One can verify that $A = (-1, \infty, 1)$ and $B = (-t, 0, t)$ are rationally separated whenever $0 < t < 1$. On the other hand A and $B = (-t, s, t)$ are *not* rationally separated when t is close to 1 and s is close to t; see Figure 8. The height of the dotted line is the minimum of $u(x)$ over $[-t, t]$; separation fails on the right, because it crosses the graph of u over $[1, \infty)$.

Ideal triangles. Now let P_1 and P_2 be a pair of ideal triangles with vertex sets A and B respectively, such that $\overline{P}_1 \cap \overline{P}_2 = \emptyset$; equivalently, such that A and B lie in disjoint intervals in $\widehat{\mathbb{R}}$. We will show:

There exists a sextic vector field such that $\mathbb{H} - \Lambda = P_1 \cup P_2$ if and only if A and B are rationally separated.

This is Theorem 1.8.

Choose coordinates so that $A, B \subset \mathbb{R}$, and let C, D be the monic cubic polynomials with roots A and B respectively.

Lemma 11.5 *There exists a $t > 0$ such that $C^2 - t^2 D^2$ has 4 roots in $\mathbb{C} - \mathbb{R}$ $\iff$ A and B are rationally separated.*

Proof. Let I and J be the smallest pair of disjoint intervals containing A and B respectively, and let $R = C/D$. Note that $|R|$ maps each component of $\widehat{\mathbb{R}} - (I \cup J)$ bijectively to $(0, \infty)$. Thus the equation $|R| = t$ has 2 solutions outside of $(I \cup J)$, giving two real simple roots for $C^2 - t^2 D^2$. Consequently $C^2 - t^2 D^2$ has exactly 2 simple real roots $\iff$

$$\sup_I |R| < t < \inf_J |R|.$$

Thus there exists a t such that 4 of the roots of $C^2 - t^2 D^2$ are strictly complex $\iff$ A and B are rationally separated. ∎

Proof of Theorem 1.8. Combine the previous Lemma with Theorem 11.4. ∎

Example with 2 ideal triangles. To obtain ideal triangles P_1 and P_2 with vertices $(\pm\sqrt{3}, \infty)$ and $(\pm 1/\sqrt{3}, 0)$, take $C(x) = x^2 - 3$ and $D(x) = x(3x^2 - 1)$. (Here C is quadratic because one vertex is at infinity.) We then have

$$C^2(x) - D^2(x) = q(x)Q(x) = (1 - x^2)(9x^4 + 2x^2 + 9),$$

which gives $v(x) = -C(x)^2/Q(x)$. See Figure 3 in §4 for the corresponding lamination Λ.

Problem. Give a systematic method to produce sextic vector fields such that $\mathrm{Gr}(v)$ has exactly 4 geometric tritangent planes, the maximum possible.

Example. The sextic vector field $v = v(x)\, d/dx$ given by

$$v(x) = \frac{7x}{x^2 + 1} - \frac{7250x + 1700}{x^2 + 900}$$

has 4 geometric tritangent planes. Moreover, Sv has 10 simple real zeros, showing strict inequality can hold in Theorem 9.1; only 8 of these zeros are accounted for by endpoints of the trees $\widehat{\mathbb{R}}/\Lambda$ for the right and left earthquake laminations of v.

Remarks: Tritangent planes in algebraic geometry. The study of complex tritangent planes for sextic space curves over $\mathbb{C}$ has a long history; see e.g. [Cob, Ch. VI], [Dol, Ch. 5.4], [ACGH, Ch. 6], [Man].

It is well known that a smooth plane quartic curve X has exactly 28 bitangent lines L. Indeed, X is a canonically embedded curve of genus 3, so when $L \cap X = 2D$, the divisor D gives one of the 28 odd theta characteristics for X (square roots of its canonical bundle with $h^0(D) = 1$).

Similarly, a smooth sextic space curve X *of genus 4* has exactly 120 distinct complex tritangent planes, corresponding again to its odd theta characteristics. Such a curve is also canonically embedded, and hence it lies on a quadric surface Q, like our curves $\mathrm{Gr}(f)$ and $\mathrm{Gr}(v)$.

For work on real and rational sextics, see [Dye], [KNRS], [HL], [RS].

12 Power maps, twists and cylinders

As in §1, let u^* denote the unique solution to

$$\log(u) = \frac{u + 1/u}{u - 1/u} \quad \text{with } u > 1. \tag{12.1}$$

In this section we will prove Theorem 1.9. That is, we will find explicit s and u such that the lamination for the right earthquake extension of

$$f_\alpha(x) = |x|^\alpha \operatorname{sign}(x)$$

is given by $s\Lambda_u$. Throughout we assume $\alpha \neq 1$.

We will also prove:

Theorem 12.1 *The right earthquake lamination for the vector field $v(x) = x \log|x|$ is given by $s\Lambda_u$, where $u = u^*$ and $s = \log(u^*)$.*

The vector field v is the infinitesimal generator of the group $\{f_\alpha : \alpha > 0\}$.

We will also determine the (rather unexpected) boundary values for an earthquake along Λ_1.

Theorem 12.2 *A right earthquake along Λ_1, when suitably normalized, has boundary values*

$$f(x) = (1 + \sqrt{2})^{\operatorname{sign}(x)} |x|^{\sqrt{2}} \operatorname{sign}(x).$$

Finally we will discuss properties of the constant u^* and compare earthquake extensions to Teichmüller mappings.

Scale invariance. The measured lamination Λ_u with transverse measure m_u, defined in the Introduction, is characterized by the following properties:

1. It is invariant under the 1–parameter group $g_t(z) = e^t z$;
2. The geodesic $[-1/u, u]$ is a leaf of Λ_u; and
3. The transverse measure of an arc $\tau \subset i\mathbb{R}_+$ is its hyperbolic length; that is, $m_u|[0, \infty] = dy/y$.

The spine Σ of Λ_u is the imaginary axis, $i\mathbb{R}_+ = [0, \infty]$. The only other measured laminations satisfying (1) are those with a single leaf, $[0, \infty]$.

Now let Λ be the measured lamination for the right earthquake extension F of f_α. Since for all $t > 0$ we have

$$f_\alpha(tx) = t^\alpha f_\alpha(x), \tag{12.2}$$

its associated measured lamination Λ must also satisfy (1), and the case of a single leaf is ruled out because $f_\alpha|(0, \infty)$ is not a Möbius transformation. Thus $\Lambda = s\Lambda_u$ for some $s, u > 0$.

The value of u. Even though $f_\alpha(x)$ is not real–analytic globally, it *is* analytic at the endpoints of the leaf $[-1/u, u]$. Thus, if $\Lambda = s\Lambda_u$, we have

$$f'_\alpha(u) f'_\alpha(-1/u) = \frac{(f_\alpha(u) - f_\alpha(-1/u))^2}{(u + 1/u)^2}$$

by Lemma 3.2; equivalently,

$$\alpha(u + 1/u) = u^\alpha + 1/u^\alpha. \tag{12.3}$$

Setting $u = e^v$, another convenient form for this relation is:

$$\alpha \cosh(v) = \cosh(\alpha v).$$

This equation has exactly two real solutions, $\pm v$, as can be seen by differentiating. These values correspond to the right and left earthquakes for f_α, and give the two real solutions, u and $1/u$, to equation (12.3). The value for the *right* earthquake is characterized by

$$u > 1 \iff \alpha > 1.$$

This condition plus equation (12.3) uniquely determines u.

Symmetries. With this information in hand, we can now compute the right earthquake extension F of f_α.

To begin, we note that f_α commutes with $r(x) = -1/x$. By naturality, the same is true of F, and thus $F(i) = i$. Moreover F inherits homogeneity from f_α; that is, equation (12.2) implies that

$$F(tz) = t^\alpha F(z).$$

Since the geodesic $[-1/u, u]$ passes through $z = i$, the unique $g_1 \in G$ satisfying

$$F|[-1/u, u] = g_1|[-1/u, u]$$

lies in SO(2); it fixes i and further satisfies $g_1(u) = u^\alpha$ and $g_1(-1/u) = -1/u^\alpha$. These three conditions determine g_1 uniquely.

For the *right earthquake* we want $g_1 \geq f_\alpha$ and hence $g_1(0) > f_\alpha(0) = 0$; this inequality gives the condition $u > 1 \iff \alpha > 1$.

The earthquake F. From g_1 we obtain a formula for F: by homogeneity, for z on the leaf of Λ passing through iy, we have

$$F(z) = y^\alpha g_1(z/y) = g_y(z). \tag{12.4}$$

The transverse measure. There are now several ways to calculate the transverse measure m_Λ, which will determine the value of s.

The simplest is to apply our general formula (1.2), as justified by Theorem 5.2, together with the relation $b = -a/u^2$ defining L, to get

$$m_L = \frac{|(u - 1/u) - (u^\alpha - 1/u^\alpha)|}{u + 1/u}\, da/a.$$

Since da/a and dy/y give the same measure on L, we find

$$s = \frac{|(u - 1/u) - (u^\alpha - 1/u^\alpha)|}{u + 1/u}.$$

It is convenient to again write $u = e^v$ so we have

$$s = |\sinh(v) - \sinh(\alpha v)|/\cosh(v).$$

The denominator is exactly the ratio between $|da/a|$ and the Lorentz length element $|ds|$ along L, so we get

$$m_L/|ds| = m_\Lambda/|\eta| = \mathrm{shear}(DF)|\Sigma = |\sinh(v) - \sinh(\alpha v)|.$$

Proof of Theorem 1.9. The equations for u and s above agree with those in the statement of the Theorem. ∎

Remark. The value of s can also be recovered directly from the value of g_1, either by using (12.4) to compute the value of shear(DF) at $z = i$, or by using the formula for g_y in the same equation to compute $|2\,\mathrm{Tr}(h.h)|^{1/2}$, where $h = g_y^{-1} dg_y/dy$.

Proof of Theorem 12.1. Apply equations (8.1) and (8.2) to compute u and s, and then simplify the equation for s using the relation (12.1). ∎

General homogeneous maps. Suitably normalized, every homogeneous map $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ is given by $f(x) = B^{\mathrm{sign}(x)}|x|^\alpha \operatorname{sign}(x)$ for some $B, \alpha > 0$.

Its associated right earthquake lamination has the form $\Lambda = s\Lambda_u$, provided we require $B > 1$ when $\alpha = 1$. Writing $u = e^v$ and $B = e^\beta$, and applying equations (1.1) and (1.2), we find s and v are determined by the relations

$$\begin{aligned} \alpha\cosh(v) &= \cosh(\beta+\alpha v) \quad \text{and} \\ s &= |\sinh(v) - \sinh(\beta+\alpha v)|/\cosh(v). \end{aligned}$$

These equations generalize the case $B = 1$ just discussed above. The right earthquake solution is distinguished by $\beta + (\alpha-1)v > 0$.

Proof of Theorem 12.2. To obtain Λ_1 we need $s = 1$, $u = 1$ and hence $v = 0$. Then, since $B > 1$, we have $s = 1 = \sinh(\beta)$, which gives $B = e^\beta = 1 + \sqrt{2}$, and $\alpha = \cosh(\beta)$, which gives $\alpha = \sqrt{1+\sinh^2(\beta)} = \sqrt{2}$. ∎

Remarks on $f(x) = x^n$ and u^*. The results above describe, when $\alpha = n \geq 3$ is an odd integer, the earthquakes arising from the simplest polynomial maps, namely $f(x) = x^n$. At the same time they model the local behavior of earthquakes at critical points of f; cf. the proof of Corollary 1.3 in §6.

We note that $u = u(n)$ is an algebraic integer for all $n \geq 2$, by virtue of equation (12.3). Values for small n are listed below; we have included u^* as the limiting value for $n = 1$.

n	equation for u	value
1	$u = u^*$	3.3190
2	$u^4 - 2u^3 - 2u + 1 = 0$	2.2966
3	$u = \sqrt{2+\sqrt{3}}$	1.9318
4	$u^8 - 4u^5 - 4u^3 + 1 = 0$	1.7391
5	$u = (1+\sqrt{5})/2$	1.6180

The constant u^*. We now turn to the natural constant u^* that emerged from the study above. Let $\log(u^*) = v^*$. By the Lindemann–Weierstrass theorem, provided $x \neq 0$, x and e^x cannot both be algebraic; thus, from the relation

$$v^* \tanh(v^*) = 1,$$

we can conclude:

Both u^* and v^* are transcendental.

Note that the number v^* is the unique positive fixed point of $x \mapsto \coth(x)$.

Curiously, $L = v^* \operatorname{sech}(v^*)$ is the Laplace limit constant arising in the convergence theory of Kepler's equation; see [Sa, §5].

Earthquakes versus Teichmüller mappings. Finally we remark that, if we let V be the earthquake extension of $x \log|x|\, d/dx$ computed in Theorem 12.1, then

$$\sup 2|\overline{\partial} V| = \sup \operatorname{shear}(DV) = \frac{\cosh(v^*)^2}{\sinh(v^*)} = 2.171623\ldots$$

can be regarded as a measure of the *inefficiency* of the earthquake extension, since the quasiconformally optimal extension, given by $V_0 = z \log|z|\, d/dz$, satisfies $2|\overline{\partial} V_0| = 1$.

13 Earthquakes for $C^{1,1}$ diffeomorphisms

In this section we prove Theorems 1.11 and 1.12.

The results below are most naturally formulated in terms of the unit circle $S^1 \subset \mathbb{C}$ instead of $\widehat{\mathbb{R}}$. Let

$$\|h\|_{\mathrm{Lip}} = \sup|h(a)| + \sup_{a \neq b} \frac{|h(a) - h(b)|}{|a - b|}$$

denote the Lipschitz norm of a continuous function $h : S^1 \to \mathbb{R}$.

We will first establish the following strengthening of Theorem 1.11.

Theorem 13.1 *Let $f : S^1 \to S^1$ be an orientation–preserving C^1 diffeomorphism with*

$$\|\log f'\|_{\mathrm{Lip}(S^1)} < M < \infty.$$

Then its right earthquake extension F is a K–bilipschitz map satisfying

$$\int_{\mathbb{H}} \rho^2 \, \|\overline{\partial} F\|^2 \leq E,$$

where E and K depend only on M.

Here $f' = |df/d\theta|$ on S^1.

The strategy is to first give the proof when f is real–analytic, and then use compactness to pass to the limit. A parallel result holds for vector fields:

Theorem 13.2 *If v is C^1 and $\|v'\|_{\mathrm{Lip}(S^1)} \le M < \infty$, then the right earthquake extension V of v is Lipschitz,* $\sup |\overline{\partial} V| \le K(M)$ *and* $\int_{\mathbb{H}} \rho^2 |\overline{\partial} V|^2 \le E(M)$.

We will then prove Theorem 1.12, showing the result above is essentially sharp. In fact we will give two examples, the second showing:

Theorem 13.3 *For any $0 < \epsilon < 1$ there exists a diffeomorphism f and a vector field v in $C^{2-\epsilon}$ whose associated right earthquake laminations satisfy* $\mathrm{area}(\Lambda) = 0$.

Timelike sets. As usual, to pass between the two models for hyperbolic space given by $\mathbb{H} \cup \widehat{\mathbb{R}}$ and $\Delta \cup S^1$, we use the map

$$z \in \overline{\mathbb{H}} \mapsto \frac{i-z}{i+z} \in \overline{\Delta}.$$

Remarkably, this change of coordinates does not affect the formula for the Lorentz metric on $\mathbb{G}$; in particular, we have

$$|ds|^2 = \frac{4|da|\,|db|}{|a-b|^2}$$

on both $S^1 \times S^1$ and $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$.

One advantage of passing to $S^1 \times S^1$ is that we can easily formulate the following universal bound.

Theorem 13.4 *For any finite union of smooth arcs giving a timelike set $L \subset \mathbb{G} \subset S^1 \times S^1$, and any $\alpha > 1$, we have*

$$\int_L |a-b|^\alpha \, |ds| \ \le C_\alpha < \infty.$$

Proof. We will work with L in the space of unoriented geodesics, i.e. the quotient of $\mathbb{G}$ by $(a,b) \sim (b,a)$. This will only change our bound by a factor of two.

Any pair $(a,b) \in \mathbb{G}/\!\sim$ determines a natural interval $[a,b] \subset S^1$ of arclength $r(a,b) < \pi$, provided $b \neq -a$. We may neglect this case, since only one such point can occur in L. Note that $r(a,b) \asymp |a-b|$ for all $(a,b) \in L$.

The timelike condition means that any two intervals $[a,b]$, $[a',b']$ determined by points in L are nested or disjoint.

Fix a reference interval $[A,B] \subset S^1$ of arclength R. Let

$$L[A,B] = \{(a,b) \in L \;:\; [a,b] \subset [A,B] \text{ and } r(a,b) > R/2\}.$$

Since $[a,b]$ does not leave room in $[A,B]$ for another disjoint interval of length $> R/2$, the map $L[A,B] \to \mathbb{R}$ sending (a,b) to $r(a,b)$ is injective. Let $J \subset [R/2,R]$ denote its image. Since L is a finite union of smooth arcs, $J \subset \mathbb{R}$ is a finite union of disjoint intervals.

Now consider the inverse map $J \to L[A,B]$ sending r to $(a(r),b(r)) \in L[A,B]$. By nesting, $a(r)$ and $b(r)$ move in opposite directions as r increases; and since they span an arc of length r, we have $|a'(r)| + |b'(r)| = 1$. On the other hand we also have $|a(r) - b(r)| \asymp r$ and $r \geq R/2$. It follows that

$$\int_{L[A,B]} |ds| = \int_J 2\frac{\sqrt{|a'(r)| \cdot |b'(r)|}}{|a(r)-b(r)|}\, dr = O(|J|/R) = O(1).$$

Now it takes $O(1/R)$ sets of the form $L[A,B]$ to cover the subset $L_n \subset L$ where $r(a,b) \asymp R = 1/2^n$. Thus the Lorentz length of L_n is $O(2^n)$. Provided $\alpha > 1$, this gives

$$\int_L |a-b|^\alpha\, |ds| \asymp \sum_n (1/2^n)^\alpha \int_{L_n} |ds| = O\left(\sum_n 2^{n(1-\alpha)}\right) < \infty$$

as desired. ∎

Remarks. The proof shows we can take $C_\alpha = O(1/(\alpha-1))$ as $\alpha \to 1$.

Theorem 13.5 *Let (L, m_L) be the dual of the right earthquake lamination for a real–analytic diffeomorphism $f : S^1 \to S^1$. Then the inequality*

$$m_L/|ds| \leq K|a-b|$$

holds on L, where K only depends on $M = \|\log f'\|_{\mathrm{Lip}(S^1)}$.

Proof. In this case it is helpful to first work in the affine model, where $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ is normalized so that $f(\infty) = \infty$. Suppose we have, for $x \in [-1,1]$, the bounds

$$1/C < |f'(x)| < C \quad \text{and} \quad |f''(x)| \leq C. \tag{13.1}$$

To estimate $m_L/|ds|$ for $(a,b) \in [-1,1]^2$, first note that these bounds imply

$$\frac{f(a)-f(b)}{a-b} = f'(c) \asymp 1,$$

where c lies between a and b. (Here and below, all implicit constants depend only on C.) Now recall that L is contained in the locus $u = 1$, where

$$u = \frac{(f \times f)^* \xi}{\xi} = \frac{f'(a) f'(b) (a-b)^2}{(f(a)-f(b))^2}.$$

(This differs from the function u in Theorem 1.1 by the additive constant 1, which does not affect du.)

We have $u \asymp 1$ on $[-1,1]^2$. Taking the logarithmic derivative gives

$$\frac{du}{da} = 2u \cdot \left(\frac{f''(a)}{2f'(a)} + \frac{1}{a-b} - \frac{f'(a)}{f(a)-f(b)} \right).$$

The first term satisfies $f''(a)/f'(a) = O(1)$. The last two terms can be combined, using the fact that

$$f(b) = f(a) + f'(a)(b-a) + f''(c)(b-a)^2/2$$

for some $c \in [a,b]$ (Lagrange's remainder), to give

$$\frac{f(a)-f(b)-f'(a)(a-b)}{(a-b)(f(a)-f(b))} = \frac{-f''(c)(a-b)^2/2}{(a-b)(f(a)-f(b))} = -\frac{f''(c)(a-b)}{2(f(a)-f(b))} = O(1).$$

It follows that $|du/da| = O(1)$; and similarly, $|du/db| = O(1)$. Now along L, we have $m_L = |du/da|\,|da| = |du/db|\,|db|$; and since $|ds|^2 = 4|da|\,|db|/|a-b|^2$, this gives

$$(m_L/|ds|)^2 = O(|a-b|^2),$$

as desired.

To obtain the Theorem on S^1, use finitely many affine charts in $\mathbb{R}$ as above to cover all of $\mathbb{G}$, and observe that in each chart, a Lipschitz bound for $\log f'$ on S^1 gives bounds of the form (13.1) when f is transported to $\mathbb{R}$. ∎

Lemma 13.6 *Theorem 13.1 holds when f is real–analytic.*

Proof. By Theorem 13.5 we have

$$m_L/|ds| \leq K|a-b|$$

where K only depends on M. In particular, by Theorem 5.4 we have

$$\sup_{\mathbb{H}} \mathrm{shear}(DF) \leq \sup_L m_L/|ds| \leq 2K,$$

and thus F is bilipschitz with a bound only depending on M. In addition, Theorems 6.1 and 13.4 give

$$\int_{\mathbb{H}} \rho^2 \, \|\overline{\partial} F\|^2 = \frac{\pi}{8} \int_L (m_L/|ds|)^2 \, |ds| \quad \leq \quad \frac{K^2 \pi}{8} \int_L |a-b|^2 \, |ds| = O(K^2),$$

completing the proof. ∎

Proof of Theorem 13.1. Let (F, Λ) be the right earthquake extension of a mapping $f : S^1 \to S^1$ with $\| \log f' \|_{\mathrm{Lip}(S^1)} < M$. Choose a sequence of real–analytic mappings $f_n \to f$ in $C^1(S^1)$ (say by convolving f with a Gaussian on the universal cover), also with $\| \log f_n' \|_{\mathrm{Lip}(S^1)} < M$. Let (F_n, Λ_n) denote the right earthquake extension of f_n.

By the Lemma above, the mappings F_n are K–bilipschitz, where $K = K(M)$. In particular, they are uniformly quasiconformal on $\overline{\Delta}$. Hence after passing to a subsequence we can assume they converge uniformly on $\overline{\Delta}$ to a limit F_∞ that is also bilipschitz. Passing to a further subsequence, we can also assume that $\Lambda_n \to \Lambda_\infty$ as a sequence of geodesic laminations. (Formally, this means their duals satisfy $L_n \to L_\infty$ in the Hausdorff topology on closed subsets of $\mathbb{G}$.)

Since $\int \rho^2 \|\overline{\partial} F_n\|^2 \leq E(M)$ for all n, and $F_n \to F_\infty$ uniformly, we have $\overline{\partial} F_n \to \overline{\partial} F_\infty$ not only as a distribution but also weakly in L^2. It follows that

$$\int \rho^2 \|\overline{\partial} F_\infty\|^2 \leq \liminf \int \rho^2 \|\overline{\partial} F_n\|^2 \leq E(M),$$

as desired.

It remains only to show that $F = F_\infty$. Since F_n is a local isometry outside Λ_n, the same is true for $F_\infty | \mathbb{H} - \Lambda_\infty$ (since every compact set disjoint from Λ_∞ is also disjoint from Λ_n for all $n \gg 0$).

This shows that $\Lambda \subset \Lambda_\infty$ (although equality need not hold).

Now consider any leaf $[a, b]$ of Λ_∞. Then $[a, b]$ is a limit of leaves $[a_n, b_n]$ of Λ_n. There is an associated sequence $g_n \in G$ such that

$$F_n | [a_n, b_n] = g_n | [a_n, b_n].$$

Since $f_n \to f$ in the C^1 topology, and g_n is determined by the 1–jets of f_n at $a_n \to a$ and $b_n \to b$, there exists a $g \in G$ such that $g_n \to g$ and

$$F_\infty | [a, b] = g | [a, b].$$

We have $g_n \geq f_n$ so $g \geq f$, and since f is differentiable, g is minimal. Thus $g \in \mathrm{Env}^+(f)$ and hence

$$F | [a, b] = F_\infty | [a, b] = g | [a, b].$$

Since $[a, b]$ was arbitrary, the maps F and F_∞ agree on Λ_∞; and since they are locally isometric on its complement, we have $F = F_\infty$. ■

Proof of Theorem 1.11. Let $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ be a $C^{1,1}$ diffeomorphism. This means, after changing coordinates to S^1, $f'(z)$ is Lipschitz, and $f'(z)$ is nowhere vanishing since f is a diffeomorphism. It follows that $\log f'$ is also Lipschitz, so Theorem 13.1 applies to f. Consequently its right earthquake extension F is bilipschitz, and $\|\overline{\partial} F\|$ is in $L^2(\mathbb{H})$.

It follows that $\mu = \overline{\partial} F / \partial F$ is also in L^2. Indeed, since F is area–preserving, we have

$$\det DF = \|\partial F\|^2 - \|\overline{\partial} F\|^2 = 1;$$

in particular, $\|\partial F\| \geq 1$; and thus

$$|\mu| = \frac{\|\overline{\partial} F\|}{\|\partial F\|} \leq \|\overline{\partial} F\|.$$

We have just shown that $\|\overline{\partial} F\|$ is in $L^2(\mathbb{H})$, so the same is true for $|\mu|$. ■

Breakdown of quasiconformality. We now turn to sharpness. We first prove:

Theorem 13.7 *For any $0 < \epsilon < 1$, there exists a $C^{2-\epsilon}$ vector field v whose earthquake extension V is not quasiconformal.*

Proof. Let $\alpha = 2 - \epsilon$; then $1 < \alpha < 2$. Consider the vector field $v = v(x)\, d/dx$, where

$$v(x) = -|1 - x^2|^\alpha$$

on $[-1, 1]$, and $v(x) = 0$ elsewhere. Then $v \in C^\alpha$.

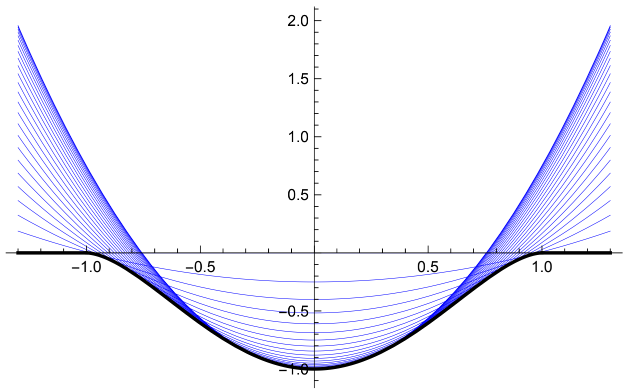

Figure 9. The upper envelope of $v(x) = -|1 - x^2|^{7/4}$.

Let (V, Λ) be its right earthquake extension, and write $V = V(z)\, d/dz$. We will show that the singularities of $v(x)$ at ± 1 cause shear(DV) to blow up at $z = i$.

It is easy to see that $[-a, a]$ is a leaf of Λ for $0 < a < 1$. Let

$$q = \operatorname{osc}(V) : \mathbb{H} \to \operatorname{Lie}(G),$$

and let q_a denote the value of q along the geodesic $[-a, a]$. A computation shows that, for $0 < a < 1$, we have

$$q_a = \left(1 - a^2\right)^{\alpha - 1} \left(\alpha x^2 + a^2(1 - \alpha) - 1\right) d/dx.$$

This vector field lies in $\operatorname{Env}^+(v)$; see Figure 9 for the case $\alpha = 7/4$.

By statements (5) and (6) of Theorem 8.1, for $|z| < 1$ we have, setting $h = dq_a/da$,

$$\operatorname{shear}(DV) = m_\Lambda/|\eta| = |2 \operatorname{Tr}(h.h)|^{1/2} \cdot \operatorname{Im}(z).$$

Setting $z = ia$, we compute

$$|2 \operatorname{Tr}(h.h)| = \frac{16a^4(\alpha - 1)^2\alpha^2}{(1 - a^2)^{2\epsilon}}. \tag{13.2}$$

Since $\epsilon > 0$, it follows that shear(DV) at $z = ia$ blows up as $a \to 1$ from below, so V cannot be quasiconformal. ∎

Proof of Theorem 1.12. A similar argument works for maps, using $f(x) = x + \delta v(x)$. ■

Singular laminations. In the preceding example we still have $\overline{\partial} F$ in L^2 for $0 < \epsilon < 1/2$, as can be seen from the denominator in equation (13.2). The next example is more extreme.

We will construct, for $0 < \epsilon < 1$, a vector field $v = v(x)\, d/dx$ in $C^{2-\epsilon}$ such that $v \notin \mathrm{Lie}(G)$ but the lamination Λ of its right earthquake extension V has

$$\mathrm{area}(\Lambda) = 0.$$

In fact we will arrange that there is a Cantor set $K \subset [0, 1]$, of Hausdorff dimension $1 - \epsilon$, such that

$$\Lambda = \{[-a, a] \,:\, a^2 \in K\}. \tag{13.3}$$

Since $v \notin \mathrm{Lie}(G)$, we have $V \notin \mathrm{Lie}(G)$, so certainly $\overline{\partial} V \neq 0$ as a distribution. At the same time, $\overline{\partial} V = 0$ on the full–measure set $\mathbb{H} - \Lambda$. It follows that V is not quasiconformal and that $\overline{\partial} V$ is not in L^2. Similarly, $m_\Lambda | \tau$ is singular with respect to linear measure on any smooth transversal τ.

Piecewise quadratic vector fields. To construct v we begin by choosing a probability measure μ on $\mathbb{R}$, supported on a Cantor set $K \subset [0, 1]$, such that for any interval I we have

$$\mu(I) = O(|I|^{1-\epsilon}). \tag{13.4}$$

(One can use a variant of the Cantor–middle third construction to build K and μ.)

Let $w : [0, \infty) \to \mathbb{R}$ be the unique C^1 function such that

$$w(0) = 0 \quad \text{and} \quad w'(t) = 1 - \mu[0, t]$$

for all $t \geq 0$. Then $w'(t)$ is decreasing and hence $w(t)$ is concave. Moreover $w(t)$ is linear on each component of $[0, \infty) - K$. By virtue of equation (13.4) we have $w' \in C^{1-\epsilon}$ and hence $w \in C^{2-\epsilon}$. Finally $w(t)$ is constant for $t > 1$.

Now let $v(x) = w(x^2)$. Then $v \in C^{2-\epsilon}$ as well, and $v(x)$ is constant for $|x| > 1$, so as a vector field it is smooth near $x = \infty$.

Given $a \geq 0$, let

$$y = \ell_a(t) = b(a) + c(a)t$$

be the tangent line to $y = w(t)$ at $t = a^2$. By concavity, we have $\ell_a(t) \geq w(t)$ for all $t \geq 0$, and $\ell_a(a^2) = w(a^2)$. Similarly,

$$y = q_a(x) = \ell_a(x^2)$$

defines a parabola with $q_a \geq v$, $q_a(a) = v(a)$ and $q_a(-a) = v(-a)$; thus $q_a \in \mathrm{Env}^+(v)$. It follows that every leaf of the right earthquake lamination Λ of v has the form $[-a, a]$.

Since $w(t)$ is linear outside of K, $v(x)$ is *quadratic* near $x = a$ whenever $a^2 \notin K$, and hence $[-a, a]$ is *not* a leaf of Λ. On the other hand $[-a, a]$ *is* in Λ when $a^2 \in K$, since K coincides with the support of μ. Equation (13.3) describing Λ now follows.

Proof of Theorem 13.3. For v use the vector field just constructed. Since $\mathrm{length}(K) = 0$, equation (13.3) gives $\mathrm{area}(\Lambda) = 0$. A similar method yields the desired diffeomorphism $f \in C^{2-\epsilon}(S^1)$. ■

Context: Weil–Petersson Teichmüller theory. A central player in universal Teichmüller theory is the group of *quasisymmetric* homeomorphisms $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$. Such a map f is characterized by the property that it admits a quasiconformal extension $F_0 : \mathbb{H} \to \mathbb{H}$.

To obtain an L^2 theory, one can impose the *additional* condition that

$$\int_{\mathbb{H}} \rho^2 \, |\overline{\partial} F_0 / \partial F_0|^2 < \infty.$$

In this case we say $f \in \mathrm{WP}(\widehat{\mathbb{R}})$, the space of *Weil–Petersson homeomorphisms.* For a systematic account of these two versions of universal Teichmüller space, see [GL, Ch. 16] and [TT].

It is known that, for any $0 < \epsilon < 1/2$, we have

$$C^{1,1}(\widehat{\mathbb{R}}) \subset C^{1,1/2+\epsilon}(\widehat{\mathbb{R}}) \subset \mathrm{WP}(\widehat{\mathbb{R}}) = \{f \;:\; \log f' \in H^{1/2}(\widehat{\mathbb{R}})\},$$

where the last equality is proved in [Sh]. Theorems 1.11 and 1.12 show that the *earthquake extension F of f* can be used to certify the inclusion $C^{1,1} \subset \mathrm{WP}(\widehat{\mathbb{R}})$, but not more.

Problem: Direct analysis. It is an interesting analytic problem to interpret and understand directly the set L and the measures it carries, when f is only a homeomorphism.

As a first step in this direction, we note that *any* closed timelike set $L \subset \mathbb{G}$ is rectifiable; that it carries a natural measure $|ds|$, derived from the Lorentz metric on $\mathbb{G}$; and that Theorem 13.4 continues to hold. One can then investigate the meaning and validity of formulas such as those stated in Theorem 1.1.

Notes and references. Continuity of the map from f to m_L, in the case of quasisymmetric maps, is established in [MS, Theorem 1]. For more on regularity of homeomorphisms of $\widehat{\mathbb{R}}$ and their extensions to $\mathbb{H}$, see [GHL], the survey [Hu] and [DMSST].

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