Unit 15: Double Integrals

Lecture

15.1. If \( f(x) \) is a continuous function, the Riemann integral \( \int_a^b f(x) \, dx \) is defined as the limit of the Riemann sums \( S_n f(x) = \frac{1}{n} \sum_{k/n \in [a,b]} f(k/n) \) for \( n \to \infty \). The derivative of \( f \) is the limit of difference quotients \( D_n f(x) = n[f(x + 1/n) - f(x)] \) as \( n \to \infty \). The integral \( \int_a^b f(x) \, dx \) is the signed area under the graph of \( f \) and above the \( x \)-axes, where “signed” indicates that area below the \( x \)-axes has negative sign. The function \( F(x) = \int_0^x f(y) \, dy \) is called an anti-derivative of \( f \). It is determined up a constant. The fundamental theorem of calculus states

\[
F'(x) = f(x), \quad \int_0^x f(x) = F(x) - F(0).
\]

It allows to compute integrals by inverting differentiation so that differentiation rules become integration rules: the product rule leads to integration by parts, the chain rule becomes partial integration.

Definition: If \( f(x, y) \) is continuous on a region \( R \), the integral \( \iint_R f(x, y) \, dx \, dy \) is defined as the limit of the Riemann sum

\[
\frac{1}{n^2} \sum_{(\frac{i}{n}, \frac{j}{n}) \in R} f(\frac{i}{n}, \frac{j}{n})
\]

when \( n \to \infty \). We write also \( \iint_R f(x, y) \, dA \), where \( dA = dx \, dy \) is a notation standing for “an area element”.

15.2. Fubini’s theorem allows to switch the order of integration over a rectangle if the function \( f \) is continuous:
Theorem: \( \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy. \)

Proof. For every \( n \), there is the "quantum Fubini identity"
\[
\sum_{\frac{i}{n} \in [a,b]} \sum_{\frac{j}{n} \in [c,d]} f(\frac{i}{n}, \frac{j}{n}) = \sum_{\frac{j}{n} \in [c,d]} \sum_{\frac{i}{n} \in [a,b]} f(\frac{i}{n}, \frac{j}{n})
\]
holding for all functions. Now divide both sides by \( n^2 \) and take the limit \( n \to \infty \). This is possible for continuous functions. Fubini’s theorem only holds for rectangles. We extend the class of regions now to so called Type I and Type II regions:

**Definition:** A type I region is of the form

\[ R = \{(x, y) \mid a \leq x \leq b, \ c(x) \leq y \leq d(x) \} . \]

An integral over a type I region is called a type I integral

\[
\int \int_R f \, dA = \int_a^b \int_{c(x)}^{d(x)} f(x, y) \, dy \, dx .
\]

A type II region is of the form

\[ R = \{(x, y) \mid c \leq y \leq d, \ a(y) \leq x \leq b(y) \} . \]

An integral over such a region is called a type II integral

\[
\int \int_R f \, dA = \int_c^d \int_{a(y)}^{b(y)} f(x, y) \, dx \, dy .
\]

15.3. Similarly as we could see in one dimensions an integral as a signed area, one can interpret \( \int \int_R f(x, y) \, dy \, dx \) as the **signed volume** of the solid below the graph of \( f \) and above \( R \) in the \( xy \)-plane. As in 1D integration, the volume of the solid below the \( xy \)-plane is counted negatively.

**Examples**

15.4. If we integrate \( f(x, y) = xy \) over the unit square we can sum up the Riemann sum for fixed \( y = j/n \) and get \( y/2 \). Now perform the integral over \( y \) to get \( 1/4 \). This example shows how to reduce double integrals to single variable integrals.

15.5. If \( f(x, y) = 1 \), then the integral is the **area** of the region \( R \). The integral is the limit \( L(n)/n^2 \), where \( L(n) \) is the number of lattice points \( (i/n, j/n) \) contained in \( R \).
15.6. The value \( \frac{\iint f(x, y) \, dA}{\iint 1 \, dA} \) is the **average** value of \( f \).

15.7. Integrate \( f(x, y) = x^2 \) over the region bounded above by \( \sin(x^3) \) and bounded below by the graph of \( -\sin(x^3) \) for \( 0 \leq x \leq \pi \). The value of this integral has a physical meaning. It is called **moment of inertia**.

\[
\int_0^{\pi/3} \int_{-\sin(x^3)}^{\sin(x^3)} x^2 \, dy \, dx = 2 \int_0^{\pi/3} \sin(x^3) x^2 \, dx
\]

We have now an integral, which we can solve by substitution

\[
-\frac{2}{3} \cos(x^3)|_0^{\pi/3} = \frac{4}{3}.
\]

15.8. Integrate \( f(x, y) = y^2 \) over the region bounded by the \( x \)-axes, the lines \( y = x + 1 \) and \( y = 1 - x \). The problem is best solved as a type I integral. As you can see from the picture, we would have to compute 2 different integrals as a type I integral. To do so, we have to write the bounds as a function of \( y \): they are \( x = y - 1 \) and \( x = 1 - y \)

\[
\int_0^1 \int_{x-1}^{1-x} y^2 \, dy \, dx = 1/6.
\]

15.9. Let \( R \) be the triangle \( 1 \geq x \geq 0, 0 \leq y \leq x \). What is

\[
\iint_R e^{-x^2} \, dxdy?
\]

The type II integral \( \int_0^1 \left[ \int_0^1 e^{-x^2} \, dx \right] dy \) can not be solved because \( e^{-x^2} \) has no antiderivative in terms of elementary functions.

The type I integral \( \int_0^1 \left[ \int_0^x e^{-x^2} \, dy \right] dx \) however can be solved:

\[
= \int_0^1 xe^{-x^2} \, dx = -\frac{e^{-x^2}}{2} \bigg|_0^1 = \frac{(1 - e^{-1})}{2} = 0.316... .
\]

15.10. The area of a disc of radius \( R \) is

\[
\int_{-\pi}^{\pi} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1 \, dy \, dx = \int_{-R}^{R} 2\sqrt{R^2 - x^2} \, dx.
\]

Substitute \( x = R \sin(u), dx = R \cos(u) \), to get

\[
\int_{-\pi/2}^{\pi/2} 2\sqrt{R^2 - R^2 \sin^2(u)} R \cos(u) \, du = \int_{-\pi/2}^{\pi/2} 2R^2 \cos^2(u) \, du = R^2 \pi.
\]
Homework

This homework is due on Tuesday, 7/21/2020.

Problem 15.1:  a) (4 points) Find the iterated integral
\[ \int_0^1 \int_0^2 6xy/\sqrt{x^2 + (y^2/2)} \, dy \, dx . \]
b) (4 points) Now compute
\[ \int_0^1 \int_0^2 6xy/\sqrt{x^2 + y^2/2} \, dx \, dy . \]
c) (2 points) Wouldn’t Fubini assure that a) and b) are the same? What change would be needed in b) to make the results agree?

Problem 15.2:  Find the area of the region
\[ R = \{ (x, y) \mid 0 \leq x \leq 2\pi, \sin(x) - 1 \leq y \leq \cos(x) + 2 \} \]
and use it to compute the average value \( \int \int_R f(x, y) \, dx\, dy / \text{area}(R) \) of \( f(x, y) = y \) over that region.

Problem 15.3:  Find the volume of the solid lying under the paraboloid \( z = 3x^2 + 3y^2 \) and above the rectangle \( R = [-2, 2] \times [-2, 3] = \{ (x, y) \mid -2 \leq x \leq 2, -2 \leq y \leq 3 \} \).

Problem 15.4:  Calculate the iterated integral \( \int_0^1 \int_{x^2}^{2-x} (x^2 - y) \, dy \, dx \).
Sketch the corresponding bottom to top region. Write this integral as integral over left to right region and compute the integral again.

Problem 15.5:  There is only one way to identify zombies: throw two difficult integrals at them and see whether they can solve them. Prove that you are not a zombie!
a) (6 points) Find the integral
\[ \int_0^1 \int_0^{y^2} \frac{3x^7}{\sqrt{x} - x^2} \, dx \, dy . \]
b) (4 points) Integrate
\[ \int_0^1 \int_0^{\sqrt{1-y^2}} 11(x^2 + y^2)^{10} \, dx \, dy . \]
You might want to “time travel” one lecture forward, where polar coordinates are known to solve this problem.