

THINKING LIKE ARCHIMEDES WITH A 3D PRINTER

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ABSTRACT. We illustrate Archimedes' method using models produced with 3D printers. This approach allowed us to create physical proofs of results known to Archimedes and to illustrate ideas of a mathematician who is known both for his mechanical inventions as well as his breakthroughs in geometry and new ideas leading to calculus. We use technology from the 21st century to trace intellectual achievements from the 3rd century BC. While we celebrate 2300 years of Archimedes (287-212 BC) in 2013, we also live in an exciting time, where 3D printing is becoming popular and affordable.

1. INTRODUCTION

Archimedes, whose 2300th birthday we celebrate this year or next was a mathematician and inventor who pursued mechanical methods to develop a pure theory. In admiration for his mathematical discoveries, which were often fueled by experiment, we follow his steps by building models produced with modern 3D printers. Archimedes was an early experimental mathematician [29] who would use practical problems and experiments as a heuristic tool [34]. He measured the center of mass and made comparisons between the volumes and surface areas of known and unknown objects by comparing their integrals, starting to build the initial ideas of calculus. He also tackled problems as a practical engineer, building winches, pumps, and catapults [19, 22, 24, 27].

In mathematics, Archimedes is credited for the invention of the exhaustion method, a concept that allowed him to compute the area of a disc, parabola segments, the volume of a sphere, cones and other quadrics, objects bound by cylinders and planes, like the hoof, or objects obtained by the intersection of two or three cylinders, or more general solids that we call now Archimedean spheres [46]. Archimedes' method was an important step towards calculus; his ideas would later be built upon by the Italian school like Cavalieri and Torricelli, and by Descartes and Fermat in France. Calculus would eventually be built by Leibniz and Newton to become the tool we know today [33]. Archimedes approach to research differed from that of Euclid. While appreciating Euclid's format in communicating proof, he did not follow a deductive method but pursued mechanical methods to develop a pure theory.

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FIGURE 1. The left figure shows an oil painting of Archimedes by Giuseppe Nogari from the mid-18th century. The oil canvas is located at the Pushkin Museum in Moscow. The right picture shows a portrait of Archimedes painted around 1620 by Domenico Fetti in Mantua. It is now located in the art museum “Alte Meister” in the German town of Dresden [44].

The life of Archimedes also led to a dramatic story in the history of mathematics [50]: the event was the discovery of the “palimpsest”, a Byzantine manuscript from the 10th century that had been reused to copy a prayer book three centuries later. It remained hidden until the mid-nineteenth century [42] when Johannes Heiberg [26], a philologist of Greek mathematics, consulted it in Istanbul. But only in the 21st century, with the help of modern computer technology, researchers finally were able to read the manuscript completely [25, 41]. This document revealed much about the thinking of Archimedes and gave historians insights about how he discovered mathematical results. It contains “The Method”, of which the palimpsest contains the only known copy.

We know the birth year 287 BC of Archimedes more accurately than from other contemporary mathematicians because of a biography about Archimedes written by Heraclides [12]. Therefore, the year 2013 or 2014 most likely celebrates the 2300th birth year of Archimedes. While we do not know how early historians counted years BC, historians today consider the Gregorian year -286 to be equal to the Julian year 287 BC [11, 40]. This arithmetic is more or less academic since the uncertainty about the birth of date is like any historical event BC of the order of magnitude of one year. The death year is known because his city was sacked in 212 BC, an event during which Archimedes was killed. We know much about Archimedes by the work of the historian John Tzetzes [53].

2. ARCHIMEDES MATHEMATICS

Archimedes achievements are covered in all textbooks dealing with the history of early geometry. Examples are [10, 38, 57, 4, 32]. While some of the main ideas in

calculus were pioneered during Archimedes' time, crucial ingredients like the concept of a function or a precise notion of limit were missing. Calculus needed to wait 1500 years to be completed [33]. Modern revelations have shown that Archimedes already seemed to have made an important step towards tackling the infinity in calculus [42, 41]. Only Newton and Leibniz, after the concept of a “function” was available, made calculus into the science we know today. Despite now having modern techniques available to compute, the method of Archimedes can still be admired today and influenced other techniques in analysis. His thinking contained the germ for understanding what a limit is. The mechanical approach Archimedes used to compute volumes consisted of a comparison method: cut a region or body into thin slices and then compare the area or volume of slices to different bodies of known area or volume. The method allowed him also to compute other integral quantities like the center of mass of bodies.

In his work “On the Sphere and the Cylinder” [3], Archimedes rigorously derived the surface area and volume of the unit sphere. He noticed that the ratio of the surface area $4\pi r^2$ to that of its circumscribing cylinder is $6\pi r^2$ is $2/3$ and that the ratio the volume of the sphere $4\pi r^3/3$ with the volume of the circumscribing cylinder $2\pi r^3$ is $2/3$ also. He announced generalizations of this: “Archimedean domes”, also known as “Archimedean globes”, have volume equal to $2/3$ of the prism in which they are inscribed. It was discovered only later that for Archimedean globes, the surface area is $2/3$ of the surface area of a circumscribing prism [2].

As reported by Plutarch [54], Archimedes was so proud of his sphere computation that he asked to have it inscribed in his tombstone. Heath concluded from this that Archimedes regarded it as his greatest achievement. It was Cicero who confirmed that “a sphere along with a cylinder had been set up on top of his grave” [51, 30]. Unfortunately, the tombstone is lost.

To cite [13]: *“But for all of these accomplishments, his undisputed masterpiece was an extensive, two-volume work titled ‘On the Sphere and the Cylinder’. Here with almost superhuman cleverness, he determined volumes and surface areas of spheres and related bodies, thereby achieving for three-dimensional solids what Measurement of a Circle had done for two-dimensional figures. It was a stunning triumph, one that Archimedes himself seems to have regarded as the apex of his career.”* The work “On the Sphere and the Cylinder” is considered an ideal continuation of Euclid’s elements [19, 16].

The work of Archimedes is now available in [23] and in annotated form in [25, 41]. Still, Archimedes can be difficult to read for a modern mathematicians. Lurje [38] writes on page 176: *“Archimedes is a very difficult author. He appears as such for us and must have so for ancient mathematicians. If Plutarch lauds the clarity of the proofs of Archimedes, then this only shows that Plutarch did not understand mathematics, that he never read Archimedes and only wanted to paint a picture of a genius.”* As it happens frequently for creative mathematicians, there could also be gaps like in Proposition 9 of “On the Sphere and the Cylinder” [39]. But shortcomings are frequent in works of pioneers. There is no doubt that the mathematics of Archimedes was astounding for the time. From [13]: *“Archimedes*

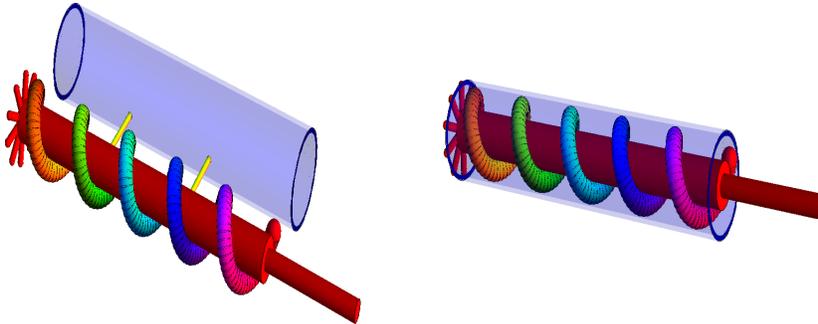


FIGURE 2. A model of the “cochlias” or Archimedes screw. The model is not a connected body, as it is printed with connectors, which are then broken off to assemble the final product.

was doing mathematics whose brilliance would be unmatched for centuries! Not until the development of calculus in the latter years of the seventeenth century did people advance the understanding of volumes and surface areas of solids beyond its Archimedean foundation. It is certain that, regardless of what future glories await the discipline of mathematics, no one will ever again be 2000 years ahead of his or her time.”

3. ARCHIMEDES ENGINEERING

The practical work of Archimedes is impressive. We dare to speculate that Archimedes, the engineer, would have liked modern 3D-prototyping technology. It is conceivable that he would have considered using it as a tool for experimentation or illustration. Instead of drawing winches, pulleys, or pumps and then building large-scale models, he could prototype smaller versions first to test the device.

Heath writes [23]: “Incidentally he made himself famous by a variety of ingenious mechanical inventions. These things were however merely the ‘diversions of geometry at play’ (Plutarch) and he attached no importance to them.” In [38] page 130, this statement is questioned. In Al Jallil-As-Siisi from the 10th to 11th century for example, one does not find this “contemptuousness with respect to practical mechanics”. Lurje claims that Plutarch obviously did not read the work of Archimedes and cites Tacquet, a geometer from the 17th century: “Archimedes is more praised than read, more lauded than understood.” The mechanical inventions include the Cochlias or the Archimedean screw, which is a machine to pump water, which he may have developed in his youth [27]. According to Diodorus, it has been used in Egypt for the irrigation of fields or in Spain in order to pump water out of mines. For the description we must rely on Vitruvius [12].

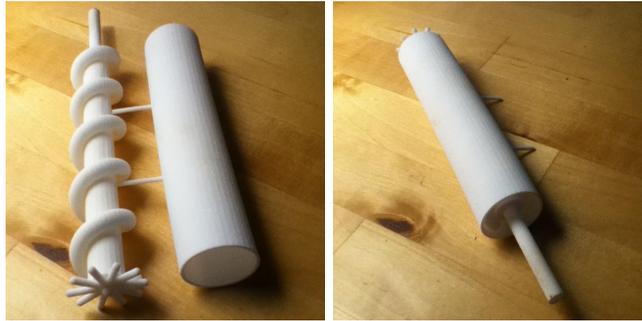


FIGURE 3. The printed model can be assembled and used by plugging it into a drilling machine.

Information revolutions		Industrial revolutions	
Gutenberg Press	1439	Steam Engine, Steel, Textile	1775
Electric programmable computer	1943	Automotive, Chemistry	1850
Personal computer, Cell phone	1973	Laser printer, Rapid prototyping	1969
2G cellular technology	1991	3D printer	1988

TABLE 1. Information and industrial revolutions

4. 3D PRINTING TECHNOLOGY

3D printing technology is a rapid prototyping process that allows the building of objects using various technologies. The industry is relatively young. Only 30 years ago, the first patents of ballistic particle manufacturing were filed [28, 7]. Jeremy Rifkin [47, 48, 14] considers the process part of the “third industrial revolution” which he describes as a time when “manufacturing becomes digital, personal, and affordable.”

The first commercially successful 3D-printing technology appeared in 1994 with printed wax material. More recently, with the technology of deposition of acrylic photopolymers, the costs have dropped into a range of consumer technology. Printing services now 3D print in color and high quality. 3D printing belongs to a larger class of construction methods called rapid prototyping [9, 7, 31], which started in the 1980s and include a larger class of additive manufacturing processes. It is related to other technologies like automated fabrication or computer numerical controlled machining [28]. The development of 3D printing belongs to an ongoing chain or evolution in industrialization and information. It is a trend created by different developments merging. Production, information technology, communication, and transportation are all linked closer than ever [7].

The attribute “additive” in the description of 3D technology is used because the models are built by adding material. In sculpting or edging techniques, by contrast, material is removed by chemical or mechanical methods. 3D printing is more flexible than molding [5]. In the simplest case, the model is built from a series of layers, which ultimately form the cross sections of the object. Other 3D technologies can

add material from more than one direction.

In the last ten years, the technology has exploded and printers are now available for hobbyists. We have experimented with our own “Up! StartPlus” printer which is small enough to be carried into a calculus classroom for demonstration. It allowed us for example to illustrate the concept of Riemann integral by illustrating the layer-by-layer slicing of a solid.

3D printing is now heavily used in the medical industry; the airplane industry; to prototype robots, art, and jewelry; nano structures; bicycles; ships; circuits; make copies of artworks; or even to print houses or decorate chocolate cakes. Last and not least, and this is the thesis project of the second author, it is now also more and more used in education. While many courses deal primarily with the technique and design for 3D printing, a thesis of the second author focus on its use in mathematics education [52].

5. ILLUSTRATIONS

Archimedes is credited for the first attempts to find the area and circumference of a circle. He uses an exhaustion argument using the constant π defined as the circumference divided the diameter. To compute the area, he would compute the area of an inscribed regular polygon with contains the circle. Each polygon is a union of n congruent triangles whose union exhausts the disk more and more. To get an upper bound, he would look at polygons containing the circle. The value of the circumference of the circle is so sandwiched between the circumferences of the inner and outer polygon which converge. Modern calculus shoots all this down with explicit “squeeze formulas” for the circumference $C(r)$ and area $A(r)$, which are $C_i(r) = nr \sin(\pi/n) \leq C(r) \leq nr \tan(\pi/n) = C_o(r)$ and $rC_i(r) \leq A(r) \leq rC_o(r)$. The calculus student of today sees that Archimedes was essentially battling the limit $\sin(x)/x$ for $x \rightarrow 0$. That the limit is 1 is sometimes called the “fundamental theorem of trigonometry” because one can use addition formulas to deduce the formulas the derivatives of the trigonometry functions in general. We mention the modern point of view because it shows that Archimedes worked at the core of calculus. He battled the notion of limit by comparing lower and upper bounds. The exhaustion argument also works for spheres, where it can be used that the volume $V(r)$ and surface area $A(r)$ of a sphere are related by $V = Ar/3$. We illustrated this with a 3D model.

Archimedes’ idea of comparing areas of cross sections of two different bodies is today also called the **Cavalieri principle** [33]. In calculus classrooms, it is related to the “washer method”: if we slice a body G perpendicularly along a line and if the area of a slice is $A(z)$ and we use a coordinate system, where $z = 0$ is the floor and $z = 1$ the roof,

$$\int_0^1 A(z) dz$$

is the volume of the solid. Consequently, if two bodies can be sliced so that the area functions $A(z)$ agree, then their volumes agree. Archimedes understood the

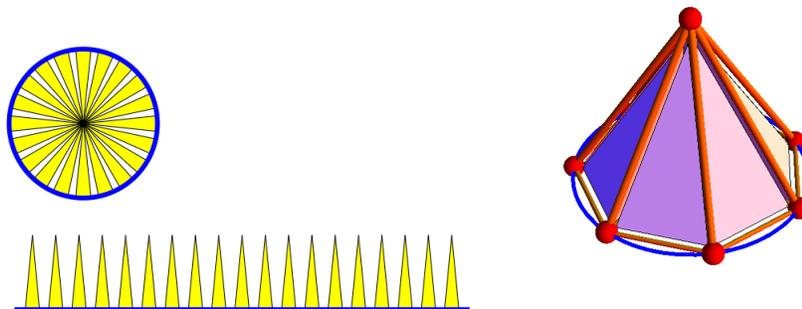


FIGURE 4. The area of the circle as half the circumference times the radius is illustrated by this figure of Archimedes. The argument can then be used to get the volume of a cone as $V = hA/3$ because each individual triangle of area A of the triangularization of the circle produces a tetrahedron of volume $Ah/3$.

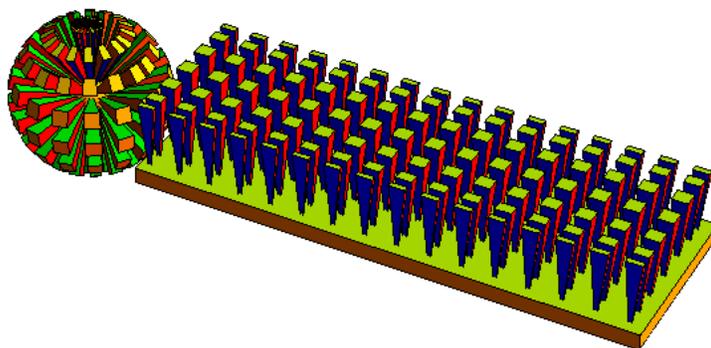


FIGURE 5. We propose an Archimedes-style proof computing the volume of the sphere, assuming that the surface area A is known. The formula $V = Ar/3$ can be seen by cutting up the sphere into many small tetrahedra of volume $dAr/3$. When summing this over the sphere, we get $Ar/3$.

integral as a sum. Indeed, the Riemann sum with equal spacing

$$\frac{1}{n} \sum_{k=1}^n A\left(\frac{k}{n}\right)$$

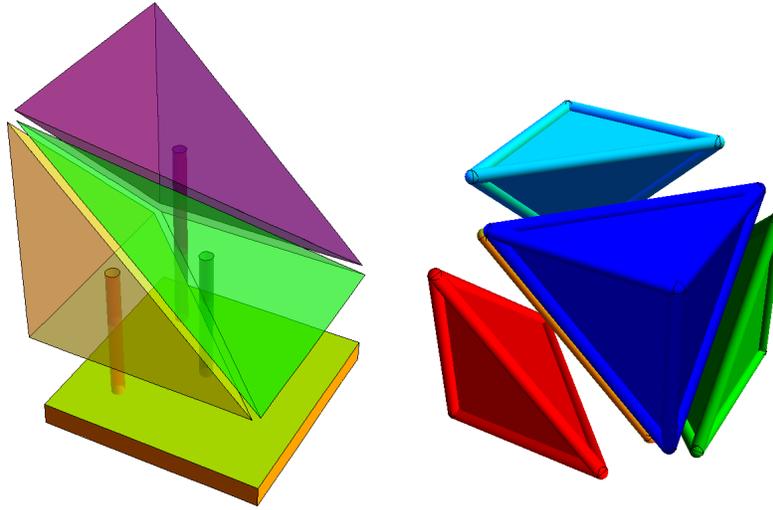


FIGURE 6. Two attempts to visualize cutting a parallelepiped into 6 pieces of equal volume.

is sometimes called **Archimedes sum** [1]. For a piecewise continuous cross-section area $A(z)$, it is equivalent to the Riemann sum. The Cavalieri principle is a fundamental idea that perpetuates to other ideas of mathematics.

Of “Archimedean style” [57], for example, is the Pappus Centroid theorem, which determines the volume of a solid generated by rotating a two-dimensional region in the xz plane around the z axis. Rotating two regions of the same area around produces the same volume. A mechanical proof is given in [36]. More general formulas can be obtained by taking a tubular neighborhood of a curve in space or a tubular neighborhood of a surface. The book [20] has made it an important tool in differential geometry.

Greek mathematicians knew that the volume of a tetrahedron is one third of the product of the area of the base times the height [15]. One can see this by cutting a parallelepiped with twice the base area into six pieces.

This fact can be difficult to visualize on the blackboard. We used a 3D printer to produce pieces that add up to a cube.

Given a two-dimensional region G in the xy plane and a point P the convex hull defines a pyramid. The cross section at height z has area $A(z) = A(0)(1 - (z/h)^2)$. Integrating this from 0 to h gives $A(0)h/3$. To see this geometrically without calculus, one can triangulate the region and connect each triangle D to P . This produces tetrahedra which each have volume $Dh/3$. Summing over all triangles gives the approximate area of the base and summing all the tetrahedron volumes leads to the volume of the cone.

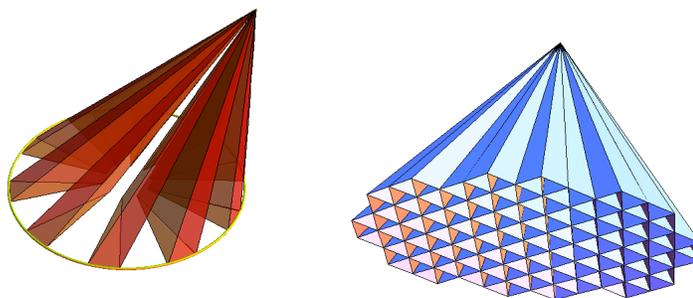


FIGURE 7. A cone cut into small pieces reduces the volume computation of the cone to the volume computation of the pyramid. The latter can be computed by cutting a cube into 6 pieces of equal volume. The volume formula $V = Ah/3$ for the pyramid with square base was already known to the Greeks. The dissection method verifies it for general cones.

An impressive example of the comparison method is the computation of the volume of the sphere. Archimedes used the same principle to compute the volume of Archimedean globes or domes (half globes) [2]. Archimedes compared the volume of a sphere with the differences of a cylinder and a cone. For the sphere, the area of a cross section is a disc of radius $\sqrt{1 - z^2}$ which leads to an area $\pi(1 - z^2)$. This is also the area of the ring obtained by cutting the complement of a cone in a cylinder.

To illustrate this idea, we printed a half-sphere bowl that drains into the complement of a cone in a cylinder (see Figure 5). To demonstrate the proof, we fill in the top with water and let it drain into the base. We can now drink the proof. A 3D model that illustrates this can be seen in Figure 5. In the case of the Archimedean dome, the comparison is done with the complement of a polygonal pyramid inside a circumscribing prism verifies also here that the volume is $2/3$ of the prism. In the limit, when the polygon approaches a circle, one gets again that the volume of the sphere is $2/3$ times the volume of the cylinder in which the sphere is inscribed.

A classical problem asks for the volume of the intersection of two cylinders intersecting perpendicularly leading to a “Himmelsgewölbe”. This object has been studied already by Archimedes [25] and is an example of an Archimedean globe. The problem to compute its volume or the surface area appears in every calculus book. Another problem is the intersection of three cylinders, an integration problem that is also abundant in calculus books.

The first one is a solid which projects onto a square or two discs. The latter is a body which is not a sphere and projects on to three discs.

Another solid that fits into this category is the **Archimedean hoof**. It is the solid bound by a cylinder and two planes. The hoof plays an important role in the understanding of Archimedes thinking [42].

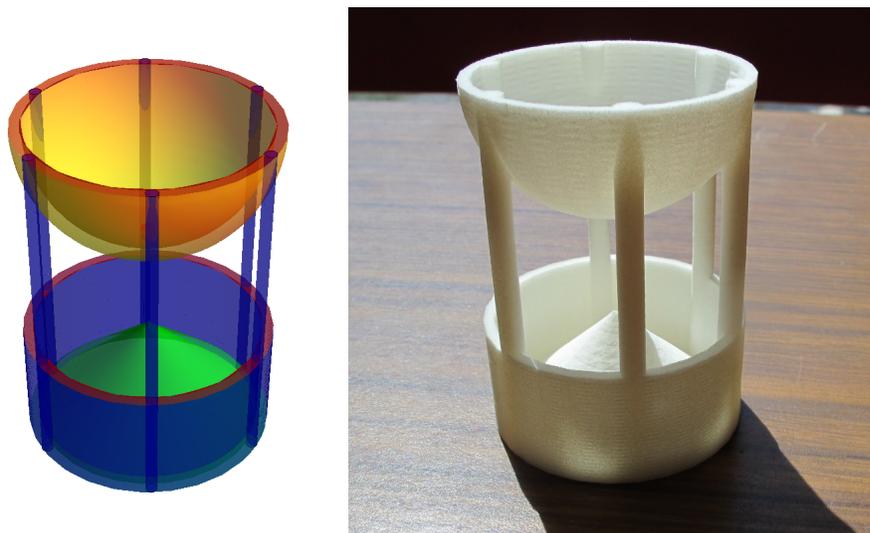


FIGURE 8. The drinkable Archimedes Proof. The Mathematica model and the printout. As a demonstration, one can fill the spherical reservoir on top with water. After it has dripped down, it fills the complement of a cone in a cylinder. The volumes match.

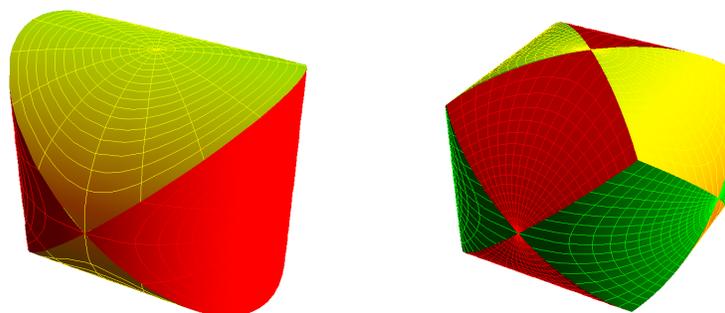


FIGURE 9. Archimedes studied the solid obtained by intersecting 2 cylinders. This as well as the intersection of 3 cylinders is now a standard geometry problem in calculus textbooks.

The **Archimedes hoof** or cylindrical wedge [23] is the solid between the planes $z = x$ and $z = 0$ inside the cylinder $x^2 + y^2 \leq 1$. This problem is treated in “the Method”. It reappeared in Kepler’s “Nova Stereometria Doliorum vinarorum” in 1615, where Kepler computed volumes of wine barrels and other bodies [55]. Of course since “the Method” had still been hidden, Kepler did not know that Archimedes has already done that. The problem is now contained in any calculus

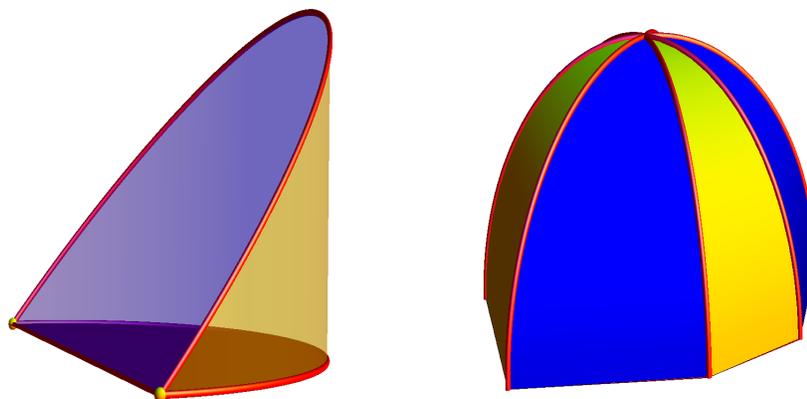


FIGURE 10. The hoof of Archimedes and the Archimedean dome are solids for which Archimedes could compute the volume with comparative integration methods [2]. The hoof is also an object where Archimedes had to use a limiting sum, probably the first in the history of humankind [42].

textbook as an application in polar integration

$$\int_{-\pi/2}^{\pi/2} \int_0^1 r \cos(\theta) r \, dr d\theta = \frac{4}{3},$$

which Archimedes knew already to be $1/6$ of the volume 8 of the cube of side length 2 containing the hoof. Archimedes figured out the volume differently. He noticed that the slices $y = c$ lead to triangles which are all similar and have area $1 - x^2$. Integrating this up from $x = -1$ to $x = 1$ is the area under the parabola $z = 1 - x^2$ and $z = 0$, a result which Archimedes already knew. The hoof might actually be the cradle of infinitesimal calculus as we know it today.

Finding the volume of the cylindrical wedge is related to computing the area defined by conic sections and especially the parabola. The mathematicians of the time period had been obsessed with “quadratures” and Archimedes was the first to do the quadrature of a segment to the parabola. To cite [55]: *“It constitutes a tremendous discovery! Let us remember that Hippocrates had squared a certain crescent and a few similar ones and thus demonstrated that there exist curvilinear figures which can be squared. After that, from 450 to 260 BC, some of the greatest minds, and many lesser ones as well, had tried to square other curvilinear figures, most of all the full circle, or the ellipse, or segments of the hyperbola—always in vain! And here comes young Archimedes - for it seems that this was his first discovery - and shows that any segment of a parabola is equal to $4/3$ of a certain inscribed triangle. This, in itself, is already an epoch-making event.”*

6. OTHER BODIES

Martin Gardner mentions as part of nine problems in [18] the question to find a volume of a body that fits snugly into a circle, square or triangle. Gardner hints

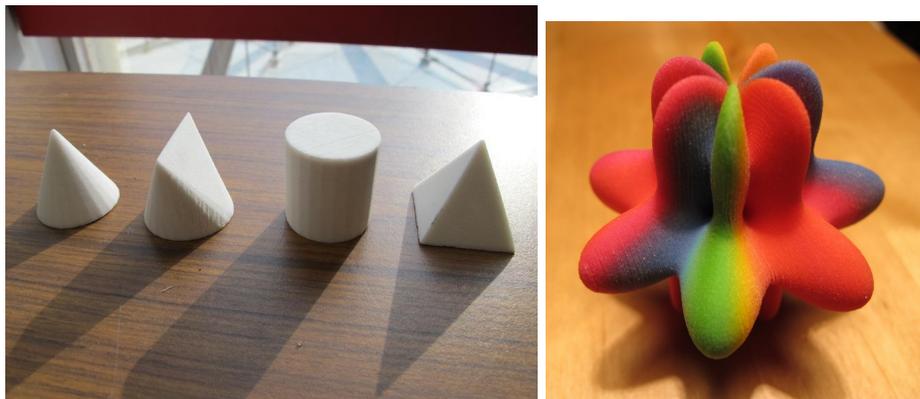


FIGURE 11. Printing related to a problem of Martin Gardner motivated by [21] who has used rapid prototyping to print this before. With printed version one can modify the problem in many ways like finding a solid where two cross sections are circles and one cross section is an equilateral triangle, etc. To the right we see a color printout of a surface built in Mathematica.

that the volume is easy to determine without calculus.

Indeed, this problem is in the Archimedes style because the volume is half of the cylinder as can be seen that when slicing it in the direction where one sees the triangle: at every cross section the triangle has half the area of the rectangle. Taking the rectangle instead of the triangle produces the cylinder of area π . The cork therefore has volume $\pi/2$.

Finally, we illustrate another theme of circles and spheres which Archimedes would have liked but did not know. The Steiner chain and its three dimensional versions illustrate a more sophisticated comparison method. Applying a Möbius transformation in the flat case leads to chains where the spheres have different radius. If one extends the circles to spheres, one obtains Soddy's multiplets [43] These are shapes that are interesting to plot.

7. 3D PRINTING AND EDUCATION

We illustrated in this project also that this technology can lead to new perspectives when looking at the history of mathematics and engineering. The new and still cutting-edge technology allows everybody to build models for the classroom.

Physical models are important for hands-on active learning. This has led to a repository of 3D printable models for education [37].

The principles of integration appear in a natural way in 3D printing, where a three dimensional object is built up automatically, like layer by layer. Printing objects has become a fancy activity as printers have become more affordable. One of us (E.S.) purchased a 3D printer to experiment with the technology to explore the technology for the classroom. There are also online services like “Shapeways”



FIGURE 12. Printing a three-dimensional Soddy multiplet. This solid illustrates that 3D printers are capable of printing fairly complicated shapes.

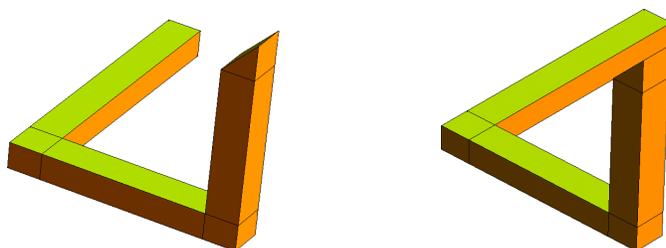


FIGURE 13. Printing the Penrose triangle. The solid was created by Oscar Reutersvard and popularized by Roger Penrose. See [17]. A Mathematica implementation has first appeared in [56].

or “Sculpteo” that print an object and deliver it. Using 3D printing technology in education is hardly new [37]. It has even been considered for [45] sustainable development in developing countries using sand in the desert.

3D printing technology has been used K-12 education in STEM projects [35], and elementary mathematics education [49]. There is optimism that it will have a large impact in education [8]. [6] write: “As fabrication tools become increasingly accessible, students will be able to learn about engineering design and experience the thrill of seeing their ideas realized in physical form”. We believe the same is true in mathematics.

In [58], Timothy Jump, who teaches at Benilde St. Margaret's Highschool is quoted as, "3D printing stimulates a student's mechanical-spatial awareness in ways that textbooks cannot." The article concludes with, "The use of 3D printing technology in education is growing quickly because educators at all levels and from many areas of expertise can see how the technology helps students learn in very real, visual, and tangible ways."

The technology is fantastic to illustrate concepts in calculus. One of us (E.S.) has taught part of a lecture at Harvard using this technology. We have brought our printer into the classroom to illustrate the concept with single variable calculus students.

It has also been used in a multivariable calculus courses at Harvard as a final project. Funding from the Elson Family Arts initiative is currently being used to print the student projects.

8. TIPS FOR PRINTING IN 3D

We conclude this article with some tips and remarks for anybody who wants to explore the subject more. First of all, the technology needs patience. Things might not work immediately, especially when printing on a personal printer. It takes both time and ability to overcome obstacles. Printer parts can fail, calibration can be subtle, software glitches occur frequently. Commercially available printing services can refuse to print certain parts. It is still an area for enthusiasts. In addition, printers costing tens of thousands dollars can break. We have seen this in local art shops or when submitting to professional printing services. It can happen for example that objects pass initial tests but cannot be printed in the end. The sphere volume proof, for example, was rejected several times, even after multiple optimization attempts. Another source for difficulties can be the size of the object, as it can affect the printing cost.

During the time of this project, the prices have fallen and the printers have become easier to work with. This is good news for anybody who wants to experiment with mathematical shapes beyond pen, paper, and computers.

The technology of 3D printing is exciting because it develops fast. The prices have dropped to as little as 500 dollars. Some of the insights we gained when working with printing could be useful also in the future, when printers will be a standard in many households. Here are few tips when dealing with 3D printing in education:

- To print more complicated objects or objects which have movable parts, add thin connections to be removed before use. An example is the Archimedes cochlias which contains a spiral moving inside a cylinder.
- Most printers use millimeters for units, so it is best to design using a millimeter template. The same units work best for importing into a printing program or printing service.
- It is important to check the tolerances of the printer and the materials before designing a shape.

- Walls must be designed with a certain thickness. Materials and printers require at least 1mm thickness, and sometimes up to 3mm, and this is another reason to design in millimeters and not inches.
- Connecting cylinders that provide stability cannot be too narrow or they break.
- Hollow shapes are less expensive to print than solids.
- Solid parts need to be closed to print correctly.
- The nozzle and platform of the printer need to be heated to the correct temperature.
- Ensure that the printout is in the correct STL format, either ASCII or binary. Software like “Meshlab” or “Netfabb” allows to work on the STL format (unfortunately, they both do not import WRL format). Mathematica only exports in binary, whereas the UP! printer requires ASCII.
- Not all modeling programs are equal. While Google SketchUp is easy to use, it is difficult to make accurate models. Computer algebra systems are better suited in this respect.
- The design program should export STL or 3DS format. “SketchUp” requires a plugin to export to STL. Mathematica can export both to STL and WRL, the latter even in color. For some reason, only `Export[“file.wrl”,Graphics]` works but `Export[“file.wrl”,Graphics,“WRL”]` does not.
- Printing in color works in principle, but not always. Since the export and import filters are still young, most color outputs do not print in color.
- Especially for small printouts, the material is crucial. It is possible now to print not only in plastic type materials, but with alumide, gypsum, sandstone, ceramics, metal or acrylic plastics which can even be translucent.
- The orientation can matter. To print a cylinder for example, it can be better to print along the axes.
- Printing yourself can take hours, and using professional printing services also requires patience. Typically we obtained the objects within a week.
- There are many materials already available. Plastic, ceramics, sandstone, metal and even chocolate is possible. Different materials have different tolerances.
- The price depends very much on the size of the model.
- Sometimes, the object is accepted by a printing service and the service only discovers days later that it is not possible to print.
- When printing in color from Mathematica, export in WRL (Virtual **W**orld **R**eality **M**odeling **L**anguage) format. (VRML currently does not print.)



FIGURE 14. Printouts illustrating the importance of the material. Too thin structures can break if the wrong material is picked. Some of the printouts can have high quality and be detailed even so the objects are small.

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