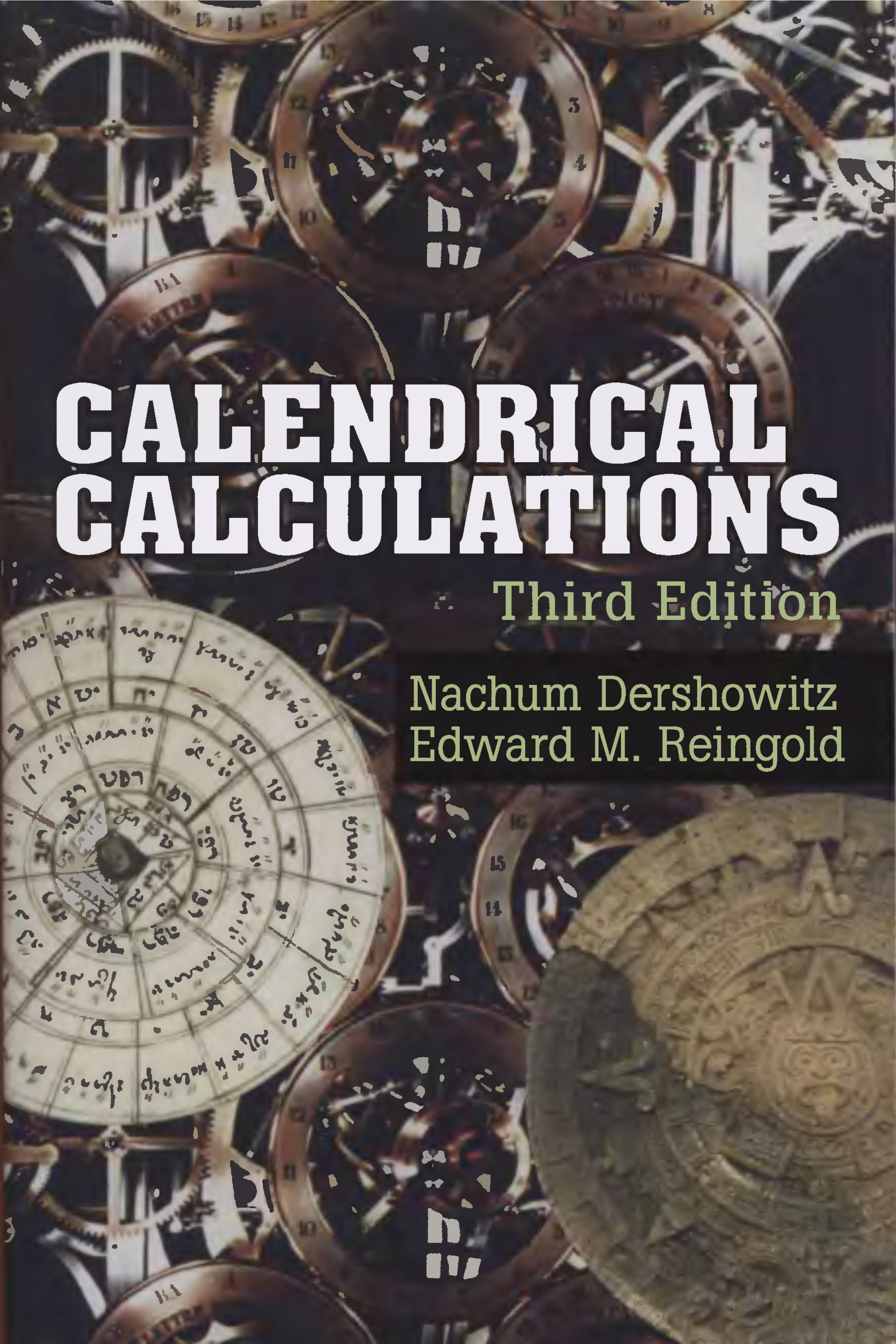

CALENDRICAL CALCULATIONS

Third Edition

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1

Calendar Basics

> A learned man once asked me regarding the eras used by different nations, and regarding the difference of their roots, that is, the epochs where they begin, and of their branches, that is, the months and years, on which they are based; further regarding the causes which led to such difference, and the famous festivals and commemoration-days for certain times and events, and regarding whatever else one nation practices differently from another. He urged me to give an explanation, the clearest possible, of all this, so as to be easily intelligible to the mind of the reader, and to free him from the necessity of wading through widely scattered books, and of consulting their authors. Now I was quite aware that this was a task difficult to handle, an object not easily to be attained or managed by anyone, who wants to treat it as a matter of logical sequence, regarding which the mind of the student is not agitated by doubt.
>
> —Abū-Raiḥān Muḥammad ibn 'Aḥmad al-Bīrūnī: *Al-Āthār al-Bāqiyah 'an al-Qurūn al-Khāliyah* (1000)

Calendrical calculations are ubiquitous. Banks need to calculate interest on a daily basis. Corporations issue paychecks on weekly, biweekly, or monthly schedules. Bills and statements must be generated periodically. Computer operating systems need to switch to and from daylight saving time. Dates of secular and religious holidays must be computed for consideration in planning events. Most of these calculations are not difficult because the rules of our civil calendar (the Gregorian calendar) are straightforward.

Complications begin when we need to know the day of the week on which a given date falls or when various religious holidays based on other calendars occur. These complications lead to difficult programming tasks—not often difficult in an algorithmic sense but difficult because it can be extremely tedious to delve, for example, into the complexities of the Hebrew calendar and its relation to the civil calendar.

The purpose of this book is to present, in a unified, completely algorithmic form, a description of thirty calendars and how they relate to one another: the present civil calendar (Gregorian); the recent ISO commercial calendar; the old civil calendar (Julian); the ancient Egyptian calendar (and its Armenian equivalent); the Coptic and the (virtually identical) Ethiopic calendars; the Islamic (Moslem) calendar (both the arithmetical version and one based on calculated observability);

the (modern) Persian calendar (both astronomical and arithmetic forms); the Bahá'í calendar (both present Western and future forms); the Hebrew (Jewish) calendar, both its present arithmetical form and a speculative observational form; the three Mayan calendars and two (virtually identical) Aztec calendars; the Pawukon calendar from Bali; the French Revolutionary calendar (both astronomical and arithmetic forms); the Chinese calendar and (virtually identical) Japanese, Korean, and Vietnamese calendars; both the old (mean) and new (true) Hindu (Indian) solar and lunisolar calendars; and the Tibetan calendar. Information that is sufficiently detailed to allow computer implementation is difficult to find for most of these calendars because the published material is often inaccessible, ecclesiastically oriented, incomplete, inaccurate, based on extensive tables, overburdened with extraneous material, focused on shortcuts for hand calculation to avoid complicated arithmetic or to check results, or difficult to find in English. Most existing computer programs are proprietary, incomplete, or inaccurate.

The need for such a secular, widely available presentation was made clear to us when we (primarily E.M.R., with contributions by N.D.), in implementing a calendar/diary feature for GNU Emacs [29], found difficulty in gathering and interpreting appropriate source materials that describe the interrelationships among the various calendars and the determination of the dates of holidays. Some of the calendars (Chinese, Japanese, Korean, Vietnamese, Hindu, and Tibetan) never had full algorithmic descriptions published in English.

The calendar algorithms in this book are presented as mathematical function definitions in standard mathematical format; Appendix A gives the types (ranges and domains) of all functions and constants we use. To ensure correctness, all calendar functions were automatically typeset[1] directly from the working Common Lisp [31] functions listed in Appendix B.[2] In Appendix C we tabulate results of the calendar calculations for 33 sample dates; this will aid those who develop their own implemenations of our calendar functions.

We chose mathematical notation as the vehicle for presentation because of its universality and easy convertibility to any programming language. We have endeavored to simplify the calculations as much as possible without obscuring the intuition. Many of the algorithms we provide are considerably more concise than previously published ones; this is particularly true of the arithmetic Persian, Hebrew, and old Hindu calendars.

We chose Lisp as the vehicle for implementation because it encourages functional programming and has a trivial syntax, nearly self-evident semantics, historical durability, and wide distribution; moreover, Lisp was amenable to translation into ordinary mathematical notation. Except for a few short macros, the code uses

[1] This has meant some sacrifice in the typography of the book; we hope readers sympathize with our decision.

[2] We provide these Lisp functions through a Cambridge University Press web site `http://www.cambridge.org/us/9780521702386` under the terms of the License Agreements and Limited Warranty on page xxviii Any errata are available over the World Wide Web at `http://www.calendarists.com`

only a very simple, side-effect-free subset of Lisp. We emphasize that our choice of Lisp should be considered irrelevant to readers, whom we expect to follow the mathematical notation used in the text, not to delve into Appendix B.

It is not the purpose of this book to give a detailed historical treatment of the material, nor, for that matter, a mathematical one; our goal is to give a logical, thorough *computational* treatment. Thus, although we give much historical, religious, mathematical, and astronomical data to leaven the discussion, the focus of the presentation is algorithmic. Full historical/religious details and mathematical/astronomical underpinnings of the calendars can be pursued in the references.

In the remainder of this chapter, we describe the underlying unifying theme of all the calculations along with some useful mathematical facts. The details of specific calendars are presented in subsequent chapters. Historically, the oldest of the calendars that we consider is the Egyptian (more than 3000 years old). The Chinese and Mayan calendars also derive from millennia-old calendars. Next are the classical (observation-based) Hebrew, the Julian (the roots of which date back to the ancient Roman empire), the Coptic and Ethiopic (third century), the current Hebrew (fourth century) and the old Hindu (fifth century), followed by the Islamic calendar (seventh century), the newer Hindu calendars (tenth century), the Persian and Tibetan calendars (eleventh century), the Gregorian modification to the Julian calendar (sixteenth century), the French Revolutionary calendar (eighteenth century), and the Bahá'í calendar (nineteenth century). Finally, the International Organization for Standardization's ISO calendar and the arithmetic Persian calendar are of twentieth-century origin.

For expository purposes, however, we present the Gregorian calendar first, in Part I, because it is the most popular calendar currently in use. Because the Julian calendar is so close in substance to the Gregorian, we present it next, followed by the very similar Coptic and Ethiopic calendars. Then we give the ISO calendar, which is trivial to implement and depends wholly on the Gregorian. The arithmetic Islamic calendar, which because of its simplicity is easy to implement, follows. Next, we present the Hebrew calendar, one of the more complicated and difficult calendars to implement, followed by a chapter on the computation of Easter, which is lunisolar like the Hebrew calendar. The ancient Hindu solar and lunisolar calendars are described next; these are simple versions of the modern Hindu solar and lunisolar calendars described in Part II. Next, the Mayan (and similar Aztec) calendars of historical interest, have several unique computational aspects, followed by the Balinese Pawukon calendar. All of the calendars described in Part I are "arithmetical" in that they operate by straightforward integer-based rules. We conclude Part I with a chapter describing generic arithmetic calendar schemata that apply to many of the calendars in this part.

In Part II we present calendars that are controlled by irregular astronomical events (or close approximations to them), although these calendars may have an arithmetical component as well. Because the calendars in Part II require some understanding of astronomical events such as solstices and new moons, we begin Part II with a chapter introducing the needed topics and algorithms. We then give the modern Persian calendar in its astronomical and arithmetic forms followed by the Bahá'í calendar, also in two versions: the Western, which depends wholly on the Gregorian, and the future version, which is astronomical. Next we describe the original (astronomical) and modified (arithmetic) forms of the French Revolutionary calendar. All of these calendars are computationally simple, provided certain astronomical values are available. Next is the Chinese lunisolar calendar and its Japanese, Korean, and Vietnamese versions. Then we describe the modern Hindu calendars, which are by far the most complicated of the calendars in this book. The Tibetan calendars are presented next. We conclude with some astronomical calendars based on the moon: the observational Islamic calendar, the classical Hebrew calendar, and the proposed astronomical calculation of Easter.

As each calendar is discussed, we also provide algorithms for computing holidays based on it. In this regard we take the ethnocentric view that our task is to compute the dates of holidays in a given *Gregorian year*; there is clearly little difficulty in finding the dates of, say, Islamic New Year in a given Islamic year! In general we have tried to mention significant holidays on the calendars we cover, but have not attempted to be exhaustive and include all variations. The interested reader can find extensive holiday definitions in [12], [13], and [14].

The selection of calendars we present was chosen with two purposes: to include all common modern calendars and to cover all calendrical techniques. We do not give all variants of the calendars we discuss, but we have given enough details to make any calendar easy to implement.

1.1 Calendar Units and Taxonomy

> Teach us to number our days, that we may attain a wise heart.
> —Psalms 90.12

The sun moves from east to west, and night follows day with predictable regularity. This apparent motion of the sun as viewed by an earthbound observer provided the earliest time-keeping standard for humankind. The day is, accordingly, the basic unit of time underlying all calendars, but various calendars use different conventions to structure days into larger units: weeks, months, years, and cycles of years. Different calendars also begin their day at different times: the French Revolutionary day, for example, begins at true (apparent) midnight; the Islamic, Bahá'í, and Hebrew days begin at sunset; the Hindu day begins at sunrise. The various definitions of *day* are surveyed in Section 13.3.

The purpose of a calendar is to give a name to each day. The mathematically simplest naming convention would be to assign an integer to each day; fixing day 1 would determine the whole calendar. The Babylonians had such a day count (in base 60). Such *diurnal* calendars are used by astronomers (see Section 13.3) and by calendarists (see, for example, Section 9.1); we use a day numbering in this book as an intermediate device for converting from one calendar to another (see the following section). Day-numbering schemes can be complicated by using a mixed-radix system in which the day number is given as a sequence of numbers or names. The Mayans, for example, utilized such a method (see Section 10.1).[3]

Calendar day names are generally distinct, but this is not always the case. For example, the day of the week is a calendar, in a trivial sense, with infinitely many days having the same day name (see Section 1.10). A 7-day week is almost universal today. In many cultures, the days of the week were named after the 7 "wandering stars" (or after the gods associated with those heavenly bodies), namely, the Sun, the Moon, and the five planets visible to the naked eye—Mercury, Venus, Mars, Jupiter, and Saturn. In some languages—Arabic, Lithuanian, Portuguese, and Hebrew are examples—some or all of the days of the week are numbered, not named. In the Armenian calendar, for example, the days of the week are named as follows [12, vol. 3, p. 70]:

Sunday	Kiraki (or Miashabathi)
Monday	Erkoushabathi
Tuesday	Erekhshabathi
Wednesday	Chorekhshabathi
Thursday	Hingshabathi
Friday	Urbath (or Vetsshabathi)
Saturday	Shabath

"Shabath" means "day of rest" (from the Hebrew), "Miashabathi" means the first day following the day of rest, "Erkoushabathi" is the second day following the day of rest, and so on. The Armenian Christian church later renamed "Vetsshabathi" as "Urbath," meaning "to get ready for the day of rest." Subsequently, they declared the first day of the week as "Kiraki" or "the Lord's day."

Other cycles of days have also been used, including 4-day weeks (in the Congo), 5-day weeks (in Africa, in Bali, and in Russia in 1929), 6-day weeks (Japan), 8-day weeks (in Africa and in the Roman Republic), and 10-day weeks (in ancient Egypt and in France at the end of the eighteenth century; see page 240). The mathematics of cycles of days are described in Section 1.10. Many calendars repeat after one or more years. In one of the Mayan calendars (see Section 10.2), and in many preliterate societies, day names are recycled every year. The Chinese calendar uses a repeating 60-name scheme for days and years, and at one time used it to name months.

An interesting variation in some calendars is the use of two or more cycles running simultaneously. For example, the Mayan tzolkin calendar (Section 10.2) combines a cycle of 13 names with a cycle of 20 numbers. The Chinese cycle of 60 names for years is actually composed of cycles of length 10 and 12 (see Section 17.4). The Balinese calendar takes this idea to an extreme; see Chapter 11. The mathematics of simultaneous cycles are described in Section 1.11.

[3] It has been claimed that in equatorial regions, where the tropical year is not of paramount agricultural importance, arbitrary year lengths are more prevalent, such as the 210-day Balinese Pawukon calandar (Chapter 11) and the 260-day Mayan divine year (Section 10.2)

The notions of "month" and "year," like the day, were originally based on observations of heavenly phenomena, namely the waxing and waning of the moon, and the cycle of seasons, respectively.[4] Some calendars begin each month with the new moon, when the crescent moon first becomes visible (as in the Hebrew calendar of classical times and in the religious calendar of the Moslems today—see Sections 20.2 and 20.4); others begin the month at full moon (in northern India, for example)—see page 128. For calendars in which the month begins with the observed new moon, beginning the day at sunset is natural.

Over the course of history, many different schemes have been devised for determining the start of the year. Some are astronomical, beginning at the autumnal or spring equinox, or at the winter or summer solstice. Solstices are more readily observable, either by observing when the midday shadow of a gnomon is longest (winter solstice in the northern hemisphere) or shortest (summer), or by noting the point in time when the sun rises or sets as far south at it does during the course of the year (which is winter in the northern hemisphere) or maximally north (summer). The ancient Egyptians began their year with the *heliacal rising* of Sirius—that is, on the day that the Dog Star Sirius (the brightest fixed star in the sky) can first be seen in the morning after a period during which the sun's proximity to Sirius makes the latter invisible to the naked eye. The Pleiades ("Seven Sisters") were used by the Maoris and other peoples for the same purpose. Various other natural phenomena have been used among North American tribes [3] to establish the onset of a new year such as harvests or the rutting seasons of certain animals.

Calendars have an integral number of days in a month and an integral number of months in a year. However, these astronomical periods—day, month, and year—are incommensurate: their periods do not form integral multiples of one another. The lunar month is about $29\frac{1}{2}$ days long, and the solar year is about $365\frac{1}{4}$ days long. (Chapter 13 has precise definitions and values.) How exactly one coordinates these time periods and the accuracy with which they approximate their astronomical values are what differentiate one calendar from another.

Broadly speaking, solar calendars—including the Egyptian, Armenian, Persian, Gregorian, Julian, Coptic, Ethiopic, ISO, French Revolutionary, and Bahá'í—are based on the yearly solar cycle, whereas lunar and lunisolar calendars—such as the Islamic, Hebrew, Hindu, Tibetan, and Chinese—take the monthly lunar cycle as their basic building block. Most solar calendars are divided into months, but these months are divorced from the lunar events; they are sometimes related to the movement of the sun through the 12 signs of the zodiac, notably in the Hindu solar calendars (see Chapter 18).

Because observational methods suffer from vagaries of weather and chance, they have for the most part been supplanted by calculations. The simplest option is to approximate the length of the year, of the month, or of both. Originally, the Babylonian solar calendar was based on 12 months of 30 days each, overestimating the length of the month and underestimating the year. Such a calendar is easy to calculate, but each month begins at a slightly later lunar phase than the previous, and the

[4] The lunar cycle formed the basis for palaeolithic marking of time; see [18] and [5].

seasons move forward slowly through the year. The ancient Egyptian calendar achieved greater accuracy by having 12 months of 30 days plus 5 extra days. Conversions for this calendar are illustrated in Section 1.9. To achieve better correlation with the motion of the moon, one can instead alternate months of 29 and 30 days. Twelve such months, however, amount to 354 days—more than 11 days short of the solar year.

Almost every calendar in this book, and virtually all other calendars, incorporate a notion of "leap" year to deal with the cumulative error caused by approximating a year by an integral number of days and months.[5] Solar calendars add a day every few years to keep up with the astronomical year. The calculations are simplest when the leap years are evenly distributed and the numbers involved are small; for instance, the Julian, Coptic, and Ethiopic calendars add 1 day every 4 years. Formulas for the evenly distributed case, such as when one has a leap year every fourth or fifth year, are derived in Section 1.12. The old Hindu solar calendar (Chapter 9) follows such a pattern; the arithmetical Persian calendar almost does (see Chapter 14). The Gregorian calendar, however, uses an uneven distribution of leap years but a relatively easy-to-remember rule (see Chapter 2). The modified French Revolutionary calendar (Chapter 16) included an even more accurate but uneven rule.

Most lunar calendars incorporate the notion of a year. Purely lunar calendars may approximate the solar year with 12 lunar months (as does the Islamic), though this is about 11 days short of the astronomical year. Lunisolar calendars invariably alternate 12- and 13-month years, according either to some fixed rule (as in the Hebrew calendar) or to an astronomically determined pattern (Chinese and modern Hindu). The so-called *Metonic cycle* is based on the observation that 19 solar years contain almost exactly 235 lunar months. This correspondence, named after the Athenian astronomer Meton (who published it in 432 B.C.E.) and known much earlier to ancient Babylonian and Chinese astronomers, makes a relatively simple and accurate fixed solar/lunar calendar feasible. The $235 = 12 \times 12 + 7 \times 13$ months in the cycle are divided into 12 years of 12 months and 7 leap years of 13 months. The Metonic cycle is used in the Hebrew calendar (Chapter 7) and for the ecclesiastical calculation of Easter (Chapter 8).

The more precise the mean year, the larger the underlying constants must be. For example the Metonic cycle is currently accurate to within 6.5 minutes a year, but other lunisolar cycles are conceivable: 3 solar years are approximately 37 lunar months with an error of 1 day per year; 8 years are approximately 99 months with an error of 5 hours per year; 11 years are approximately 136 months with an error of 3 hours per year; and 334 years are 4131 months with an error of 7.27 seconds per year. The old Hindu calendar is even more accurate, comprising 2,226,389 months in a cycle of 180,000 years (see Chapter 9) to which the leap-year formulas of Section 1.12 apply, and errs by fewer than 8 seconds per year.

The placement of leap years must make a trade-off between two conflicting requirements: Small constants defining a simple leap year rule of limited accuracy or greater accuracy at the expense of larger constants, as the examples in the last

[5] See [2, pp 677–678] for a discussion of the etymology of the term "leap."

paragraph suggest. The choice of the constants is aided by taking the continued fraction (see [16]) of the desired ratio and choosing among the convergents (where to stop in evaluating the fraction). In the case of lunisolar calendars, the solar year is about 365.24244 days, while the lunar month is about 29.53059 days, so we write

$$\frac{365.24244}{29.53059} = 12 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{18 + \cfrac{1}{3 + \cdots}}}}}}}.$$

By choosing further and further stopping points, we get better and better approximations to the true ratio. For example,

$$12 + \cfrac{1}{2 + \cfrac{1}{1}} = \frac{37}{3},$$

while

$$12 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}} = \frac{99}{8},$$

$$12 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1}}}} = \frac{136}{11},$$

and

$$12 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1}}}}} = \frac{235}{19};$$

these are the ratios of the previous paragraph. Not all approximations must come from continued fractions, however: 84 years are approximately 1039 lunar months with an error of 33 minutes per year, but this is not one of the convergents.

Continued fractions can be used to get approximations to solar calendars too. The number of days per solar year is about 365.242177 which we can write as

$$365.242177 = 365 + \cfrac{1}{4 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{5 + \cdots}}}}}}.$$

The convergents are $1/4$ (the basis of the Julian, Coptic, and Ethiopic calendars), $7/29$, $8/33$ (possibly used for an ancient Persian calendar), $23/95$, and $31/128$ (used in our implementation of the arithmetical Persian calendar—see Chapter 14).

Table 1.1 compares the values for the mean length of the year and month as implemented by the various solar, lunar, and lunisolar calendars in this book. The true values change over time, as explained in Chapter 13.

1.2 Fixed Day Numbers

May those who calculate a fixed date... perish.[6]
—Morris Braude: *Conscience on Trial: Three Public Religious Disputations between Christians and Jews in the Thirteenth and Fifteenth Centuries* (1952)

Over the centuries, human beings have devised an enormous variety of methods for specifying dates.[7] None are ideal computationally, however, because all have idiosyncrasies resulting from attempts to coordinate a convenient human labeling with lunar and solar phenomena.

For a computer implementation, the easiest way to reckon time is simply to count days: Fix an arbitrary starting point as day 1 and specify a date by giving a day number relative to that starting point; a single 32-bit integer allows the representation of more than 11.7 million years. Such a reckoning of time is, evidently, extremely awkward for human beings and is not in common use, except among astronomers, who use *julian day numbers* to specify dates (see Section 1.5), and calendarists, who use them to facilitate conversion among calendars—see equation (9.2) for the ancient Indian method; for a more modern example, see [27]. The day-count can

[6] This is a loose translation of a famous dictum from the Babylonian Talmud *Sanhedrin* 97b The omitted words from Braude's translation (page 112 of his book) are "for the coming of the Messiah." The exact Talmudic wording is "Blasted be the bones of those who calculate the end"

[7] The best reference is still Ginzel's monumental three-volume work [7], in German. An exceptional survey can be found in the *Encyclopædia of Religion and Ethics* [12, vol III, pp 61–141 and vol. V, pp. 835–894]. Useful, modern summaries are [2], [4], [26], and [30]; [2] and [26] have an extensive bibliographies. The incomparable tables of Schram [27] are the best available for converting dates by hand, whereas those of Parise [21] are best avoided because of an embarrassingly large numbers of errors

Table 1.1: *Length in days of mean year on solar and lunisolar calendars and length in days of mean lunar month on lunar and lunisolar calendars. Year length is given in italics when the sidereal, rather than tropical, value is intended. These may be compared with the astronomical values given for various millennial points*—in solar days current at the indicated time. *No values are given here for the Chinese, astronomical Persian, observational Islamic, future Bahá'í, and (original) French Revolutionary calendars because they are self-adjusting.*

	Calendar (or year)	Mean Year (days)	Mean Month (days)
Calendrical	Egyptian	365	
	Mayan (haab)	365	
	Julian/Coptic/Ethiopic	365.25	
	Hebrew	365.24682	29.530594
	Easter (Orthodox)	365.25	29.530851
	Islamic (Arithmetic)		29.530556
	Hindu (Arya)	*365 25868*	29.530582
	Hindu (*Sūrya*)	*365.25876*	29.530588
	Tibetan (*Phugpa*)	*365.27065*	29.530587
	Gregorian	365.2425	
	Easter (Gregorian)	365.2425	29 530587
	French (Arithmetic)	365.24225	
	Persian (Arithmetic)	365 24220	
Astronomical	Year −1000	365.24257	29.530598
	Year 0	365.24244	29.530595
	Year 1000	365 24231	29.530591
	Year 2000	365.24218	29.530588
	Year 3000	365.24204	29.530584

be augmented by a fractional part to give a specific moment during the day; for example, noon on a day i, an integer, would be specified by $i + 0.5$.

We have chosen midnight at the onset of Monday, January 1, 1 (Gregorian) as our fixed date 1, which we abbreviate as R.D.[8] 1, and count forward day-by-day from there. Of course, this is anachronistic because there was no year 1 on the Gregorian calendar—the Gregorian calendar was devised only in the sixteenth century—thus by January 1, 1 (Gregorian) we mean the day we get if we extrapolate backwards from the present; this day turns out to be Monday, January 3, 1 C.E.[9] (Julian); this too is anachronistic. We call an R.D. that has a fractional part giving the time of day a "moment."

The date Monday, January 1, 1 (Gregorian), though arbitrarily chosen as our starting point, has a desirable characteristic: It is early enough that almost all dates of interest are represented by positive integers of moderate size. We have been careful to write our functions in such a way that all dependencies on this choice of

[8] *Rata Die*, or fixed date We are indebted to Howard Jacobson for this coinage.

[9] Common Era, or A D

starting point are explicit. To change the origin of the calculations we have provided a function

$$\mathbf{rd}(t) \stackrel{\text{def}}{=} t - epoch \tag{1.1}$$

where

$$epoch = 0$$

that defines the origin, *epoch*. Changing this definition to $epoch = 710347$, for example, would make Monday, November 12, 1945 (Gregorian) the starting point.

We should thus think of the passage of time as a sequence of days numbered $\ldots, -2, -1, 0, 1, 2, 3, \ldots$, which the various human-oriented calendars label differently. For example, R.D. 710,347 is called

- Monday, November 12, 1945, on the Gregorian calendar.
- October 30, 1945 C.E., on the Julian calendar, which would be called *ante diem III Kalendas Novembris* in the Roman nomenclature.
- Julian day number 2,431,772 (at noon).
- Modified julian day number 31,771.
- Month 7, day 10, 2694, on the ancient Egyptian calendar.
- Trē 5, 1395, on the Armenian calendar.
- Day 1 of week 46 of year 1945, on the ISO calendar.
- Athōr 3, 1662, Era of the Martyrs, on the Coptic calendar (until sunset).
- Ḫedār 3, 1938, on the Ethiopic calendar (until sunset).
- Dhu al-Ḥijja 6, 1364, on the arithmetic and observational Islamic calendars (until sunset).
- Kislev 7, 5706, on the Hebrew calendar (until sunset).
- Kislev 6, 5706, on the observational Hebrew calendar (until sunset).
- 12.16.11.16.9 in the Mayan long count.
- 7 Zac on the Mayan haab calendar.
- 11 Muluc on the Mayan tzolkin calendar.
- Atlcahualo 11 on the Aztec xihuitl calendar.
- 9 Ozomatli on the Aztec tonalpohualli calendar.
- Lunag, Pepet, Pasah, Sri, Pon, Tungleh, Coma of Gumbreg, Ludra, Urungan, Pati on the Balinese Pawukon calendar.
- Tulā 29, 5046, Kali Yuga Era (elapsed) on the old Hindu solar calendar (after sunrise).
- Day 8 in the bright half of Kārtika, 5046, Kali Yuga Era (elapsed) on the old Hindu lunisolar calendar (after sunrise).
- Abān 21, 1324, on the modern Persian arithmetic and astronomical calendars.
- The day of Asmā', of the month of Qudrat, of the year Abad, of the sixth Vahid, of the first Kull-i-Shay on the Bahá'í calendar (until sunset).
- Décade III, Primidi de Brumaire de l'Année 154 de la République on the arithmetical and astronomical French Revolutionary calendars.
- Day 8 of the tenth month in the year Yĭyǒu on the Chinese calendar.

- Kārtika 27, 1867, Śaka Era (elapsed) on the modern and astronomical Hindu solar calendars (after sunrise).
- Day 7 in the bright half of Kārtika, 2002, Vikrama Era (elapsed) on the modern and astronomical Hindu lunisolar calendars (after sunrise).
- Day 7 of the tenth month, 2072 on the Tibetan calendar.

All that is required for calendrical conversion is to be able to convert each calendar to and from this fixed calendar. Because some calendars begin their day at midnight and others at sunrise or sunset,

We fix the time of day at which conversions are performed to be noon.

Figure 1.1 shows the relationships of various calendar's times for the beginning and ending of days.

We give, in subsequent chapters, functions to do the conversions for the thirty calendars. For each calendar x, we write a function **fixed-from-*x***(*x-date*) to convert a given date *x-date* on that calendar to the corresponding R.D. date, and a function ***x*-from-fixed**(*date*) to do the inverse operation, taking the R.D. *date* and computing its representation in calendar x. One direction is often much simpler to calculate than the other, and occasionally we resort to considering a range of possible dates on calendar x, searching for the one that converts to the given R.D. date (see Section 1.7). To convert from calendar x to calendar y, one need only compose the two functions:

$$\mathbf{y\text{-}from\text{-}x}(x\text{-}date) \stackrel{\text{def}}{=}$$

$$\mathbf{y\text{-}from\text{-}fixed}(\mathbf{fixed\text{-}from\text{-}x}(x\text{-}date))$$

Each calendar has an *epoch*, the first day of the first year of that calendar (see Section 1.4). We assign an integer R.D. date to an epoch, even if the calendar in question begins its days at a time other than midnight. Such assignment is done as per Figure 1.1. All of the algorithms given in this book give mathematically sensible results for dates prior to the calendar's epoch.

1.3 Negative Years

Quis enim potest intelligere dies et tempora et annos, nisi per numerum?
[Who can understand days and seasons and years, save by number?]
—attributed to the Venerable Bede· De Computo Dialogus

We cannot avoid dealing with dates before the common era. For example, the Hebrew calendar begins at sunset on Sunday, September 6, −3760 (Gregorian); scholarly literature is replete with such statements. Thus, to aid the reader, we now explain how years before the common era are conventionally handled. This convention is often a source of confusion, even among professional historians.

It is computationally convenient, and mathematically sensible, to label years with the sequence of integers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ so that year 0 precedes year 1; we do this when extrapolating backward on the Gregorian calendar so that the same leap-year rule will apply based on divisibility by 4, 100, and 400 (see

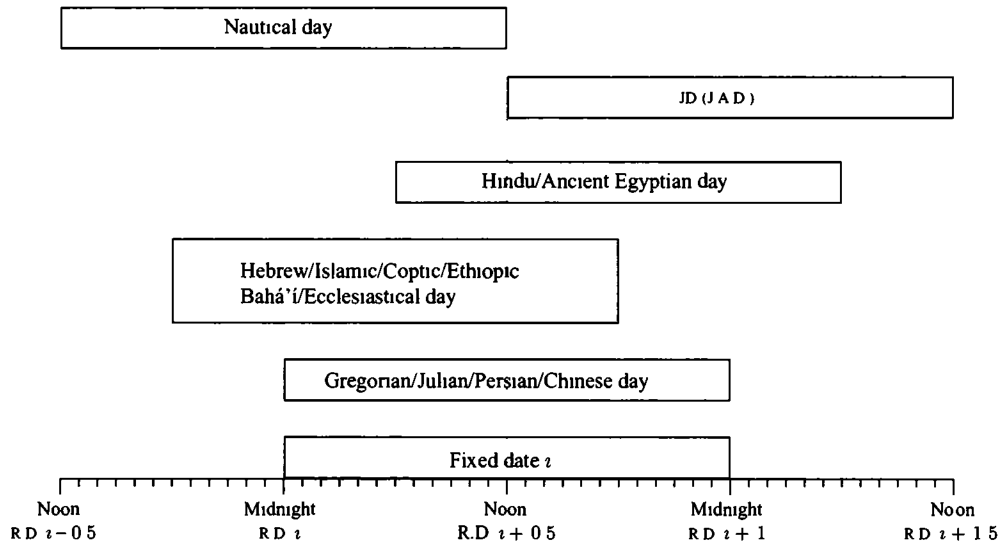

Figure 1.1: Meaning of "day" in various calendars. Conversion from a date on a calendar to an R.D. date is done as of noon; the rectangles indicate the day of a calendar that gets converted to R.D. i. For example, the Hebrew date corresponding to fixed date i is the Hebrew day that begins at sunset of the evening of fixed date $i-1$ and ends at sunset of the evening of fixed date i. Similarly, the Hindu date corresponding to fixed date i is the Hindu day that begins at sunrise in the morning of fixed i and ends at sunrise of the morning of fixed date $i+1$. The JD corresponding to fixed date i begins at noon of fixed date i and ends at noon of fixed date $i+1$.

Chapter 2). However, on the Julian calendar it is customary to refer to the year preceding 1 C.E. as 1 B.C.E.,[10] counting it as a leap year in accordance with the every-fourth-year leap-year rule of the Julian calendar. Thus, the beginning of the Hebrew calendar can alternatively be referred to as sunset October 6, 3761 B.C.E. (Julian). To highlight this asymmetry, in the *prose* of this book we append "B.C.E." *only* to Julian calendar years, reserving the minus sign for Gregorian calendar years. Care must therefore be taken when doing arithmetic with year numbers. For $n \geq 0$, the rough present-day alignment of the Julian and Gregorian calendars gives

$$\text{year } -n \text{ (Gregorian)} \approx \text{year } (n+1) \text{ B.C.E. (Julian)},$$

and, for $n \geq 1$,

$$\text{year } n \text{ (Gregorian)} \approx \text{year } n \text{ C.E. (Julian)}.$$

However, as an internal computer representation of B.C.E. Julian years in the Lisp functions in Appendix B and sample data in Appendix C, we also use negative numbers with the convention that year n B.C.E. (Julian) is represented as $-n$.

1.4 Epochs

Epochæ celebriores, astronomis, historicis, chronologis, Chataiorvm, Syro-Græcorvm Arabvm, Persarvm, Chorasmiorvm, usitatæ
[Famous epochs customarily in use by astronomers, historians, chronologists, Hittites, Syrian-Greeks, Arabs, Persians, and Chorasmians]
—Title of John Greaves' Latin/Persian edition (1650) of a work by the fourteenth-century Persian astronomer Ulugh Beg, grandson of Tamerlane

Every calendar has an *epoch* or starting date. This date is virtually never the date the calendar was adopted but rather a hypothetical starting point for the first day. For example, the Gregorian calendar was devised and adopted in the sixteenth century, but its epoch is January 1, 1. Because days begin at different hours on different calendars, we follow the convention that a calendar's epoch is the onset of the (civil) day containing the first noon (see Figure 1.1). For example, we take midnight at the onset of September 7, -3760 (Gregorian) as the epoch of the Hebrew calendar, which was codified in the fourth century, though the first Hebrew day began at sunset the preceding evening. For calendars like the Chinese or Balinese Pawukon, in which cycles are unnumbered, the choice of epoch is arbitrary; the first day of any cycle can be used.

Table 1.2 gives the epochs of the calendars discussed in this book. With the exception of Julian day number, we express the epochs of all the calendars as integer R.D. dates, that is, the integer R.D. day number at *noon* of the first day of the calendar (again, see Figure 1.1). Thus, the epoch for the Gregorian calendar is R.D. 1, and that for the Hebrew calendar is R.D. $-1,373,427$. Using this form of calendar epochs is convenient because

$$\text{R.D. } d = (d - \text{calendar epoch}) \text{ days since the start of that calendar.}$$

[10] Before the Common Era, or B.C.

Table 1.2: *Epochs for various calendars*

Calendar	Epoch (R.D.)	Equivalents
Julian day number	−1,721,424.5	Noon, November 24, −4713 (Gregorian) Noon, January 1, 4713 B.C.E. (Julian)
Hebrew	−1,373,427	September 7, −3760 (Gregorian) October 7, 3761 B.C.E. (Julian)
Mayan	−1,137,142	August 11, −3113 (Gregorian) September 6, 3114 B.C.E. (Julian)
Hindu (Kali Yuga)	−1,132,959	January 23, −3101 (Gregorian) February 18, 3102 B.C.E. (Julian)
Chinese	−963,099	February 15, −2636 (Gregorian) March 8, 2637 B.C.E. (Julian)
Egyptian	−272,787	February 18, −746 (Gregorian) February 26, 747 B.C.E. (Julian)
Tibetan	−46,410	December 7, −127 (Gregorian) December 10, 127 B.C.E. (Julian)
Julian	−1	December 30, 0 (Gregorian) January 1, 1 C.E. (Julian)
Gregorian	1	January 1, 1 (Gregorian) January 3, 1 C.E. (Julian)
ISO	1	January 1, 1 (Gregorian) January 3, 1 C.E. (Julian)
Ethiopic	2796	August 27, 8 (Gregorian) August 29, 8 C.E. (Julian)
Coptic	103,605	August 29, 284 (Gregorian) August 29, 284 C.E. (Julian)
Armenian	201,443	July 13, 552 (Gregorian) July 11, 552 C.E. (Julian)
Persian	226,896	March 22, 622 (Gregorian) March 19, 622 C.E. (Julian)
Islamic	227,015	July 19, 622 (Gregorian) July 16, 622 C.E. (Julian)
Zoroastrian	230,638	June 19, 632 (Gregorian) June 16, 632 C.E. (Julian)
French Revolutionary	654,415	September 22, 1792 (Gregorian) September 11, 1792 C.E. (Julian)
Bahá'í	673,222	March 21, 1844 (Gregorian) March 9, 1844 C.E. (Julian)
Modified julian day number	678,576	November 17, 1858 (Gregorian) November 5, 1858 C.E. (Julian)

For example,

$$\begin{aligned} 710{,}347 - (\text{Hebrew calendar epoch}) &= 710{,}347 - (-1{,}373{,}427) \\ &= 2{,}083{,}774, \end{aligned}$$

and hence

R.D. 710,347 = 2,083,774 days since the start of the Hebrew calendar.

Because, for the most part, our formulas depend on the number of days elapsed on some calendar, we often use the expression (d − calendar epoch) in our calendar formulas.

For many calendars, including the Gregorian, the same calendar rules were used with different eras and different month names at different times and in different places. In Taiwan, for instance, the Gregorian calendar is used with an era beginning with the founding of the republic in 1911. An often-encountered era from the second century B.C.E. until recent times—used with many calendars—was the Era of Alexander, or the Seleucid Era, in which year 1 corresponds to 312 B.C.E. In general, we avoid describing the details of such trivial variants of calendars.

1.5 Julian Day Numbers

Iulianam vocauimus: quia ad annum Iulianum dumtaxat accommodata est.
[I have called this the Julian period because it is fitted to the Julian year.]
—Joseph Justus Scaliger: *De Emendatione Temporum*,
end of introduction to Book V (1583)

Astronomers in recent centuries have avoided the confusing situation of date references on different calendars, each with its idiosyncrasies, by specifying moments in time by giving them in "julian days" or JD (sometimes "julian astronomical days" or J.A.D.). The "Julian period," published in 1583 by Joseph Justus Scaliger, was originally a counting of *years* in a repeating pattern 7980 years long, starting from 4713 B.C.E. (Julian). It is often claimed ([1, page 431], for example) that Scaliger named the period after his father, the Renaissance physician Julius Cæsar Scaliger, but this claim is not borne out by examination of Scaliger's great work, *De Emendatione Temporum*, from which the section quote above is taken. Grafton [8] gives a full history of *De Emendatione Temporum*. The details of the derivation for the value 7980 are given in [24]; the roots of the 7980-year cycle are much earlier than Scaliger, however, dating back to the twelfth century [23]. In the mid-nineteenth century, Herschel [15, page 532] adapted the system into a strict counting of *days* backward and forward from

$$\begin{aligned} \text{JD } 0 &= \text{Noon on Monday, January 1, 4713 B.C.E. (Julian)} \\ &= \text{Noon on Monday, November 24, } -4713 \text{ (Gregorian).} \end{aligned}$$

A fractional part of a julian[11] date gives the fraction of a day beyond noon; switching dates at noon makes sense for astronomers who work through the night. In this system, for example, sunset on the first day of the Hebrew calendar occurred at about JD 347,997.25 (local time), which is 1/4 of a day after noon. The literature on the Mayan calendar commonly specifies the beginning of the calendar in julian days. Because noon of R.D. 0 is JD 1,721,425, it follows that

$$\text{JD } n = \text{Noon on R.D. } (n - 1{,}721{,}425).$$

[11] We use lowercase here to avoid any confusion between a julian day number and a date on the Julian calendar

In other words,

$$\text{Midnight at the onset of R.D. } d = \text{JD } (d + 1{,}721{,}424.5). \qquad (1.2)$$

We do not use julian days directly, as suggested in [11], because we want our days to begin at civil midnight. *We also use fractional days when we need to calculate with time but begin each day at midnight.*

To distinguish clearly between the Julian calendar and julian days in our functions, we use the abbreviation "jd" instead of "julian." We have

$$\textbf{jd-epoch} \overset{\text{def}}{=} \text{R.D. } -1721424.5 \qquad (1.3)$$

$$\textbf{moment-from-jd}\,(jd) \overset{\text{def}}{=} jd + \textbf{jd-epoch} \qquad (1.4)$$

$$\textbf{jd-from-moment}\,(t) \overset{\text{def}}{=} t - \textbf{jd-epoch} \qquad (1.5)$$

where jd can be a fraction representing time as well as date. As used by historians, julian day numbers are defined as $jd + 0.5$ (see [19, vol. 3, p. 1064], for example). Thus our function **fixed-from-jd** gives the R.D. date intended by historians when they refer to julian dates.

For dates near the present, the julian day number is inconvenient because at least 7-digit accuracy is needed. Astronomers occasionally use *modified julian day numbers*, or MJD defined as

$$\text{Modified julian day number} = \text{julian day number} - 2400000.5,$$

which counts days from midnight, Wednesday, November 17, 1858 (Gregorian). This is equivalent to defining

$$\textbf{mjd-epoch} \overset{\text{def}}{=} \text{R.D. } 678576 \qquad (1.6)$$

$$\textbf{fixed-from-mjd}\,(mjd) \overset{\text{def}}{=} mjd + \textbf{mjd-epoch} \qquad (1.7)$$

$$\textbf{mjd-from-fixed}\,(date) \overset{\text{def}}{=} date - \textbf{mjd-epoch} \qquad (1.8)$$

We do not use modified julian days directly because we want positive numbers for dates within recent history.

1.6 Mathematical Notation

> The best notation is no notation.
> —Paul Halmos· *How to Write Mathematics* (1970)

We use the following mathematical notation (see [9]) when describing the calendar calculations: The *floor function*, $\lfloor x \rfloor$, gives the largest integer less than or equal to x. For example, $\lfloor \pi \rfloor = 3$. In general, $\lceil x \rceil = -\lfloor -x \rfloor$, so for example $\lfloor -\pi \rfloor = -4$. The similar *ceiling function*, $\lceil x \rceil$, gives the smallest integer greater than or equal to