0: Overview

(June 22, 2014, last update July 23, 2014)

The plan of this summer research project is to explore the geometry of graphs in the context of graph colorings. Graph coloring has a differential geometric flavor as we can see a graph as a **discrete geometric object** and the coloring as an element in the **fibre bundle** over the graph. In this more geometric setup, where graphs take the roles of **varieties**, we can see a coloring as a divisor. In a geometric setup, coloring has an algebraic aspect as it tells about the existence of special sections of finite-valued fields. In particular, there can be relations between cohomological obstructions and the chromatic number.

If we look at the coloring problem in a geometric setup, where we make assumptions about the unit spheres of graphs, the classical coloring story is modified. For d-dimensional graphs, where the unit spheres are $(d-1)$-dimensional discrete spheres, the coloring problem is a different interesting problem. For two-dimensional surfaces for example, we expect the geometric chromatic number always to be either 3 or 4 or 5. The case 3 is very interesting as a 3 coloring defines a discrete vector field on the graph which is a gradient field. The case 4 seems always to occur in the orientable case and in the non-orientable case, chromatic number 3,4,5 are possible. Note that the 7 coloring of the torus has little to do with the story explored here, as this graph is a 6 dimensional simplex. The discrete 2 torus should always be 4 colorable (with which we mean that the chromatic number satisfies $c(G) \leq 4$) and to explore such questions is exactly the point of this project. The two torus case could probably be dealt with similar as the sphere (planar) case by using the discrete Gauss-Bonnet theorem and do reductions. It is curious that curvature in the discrete has been introduced first in the context of the 4 color problem by Heesch, who looked at it as "charge". We have looked at the problem from various sides so far:

A) by looking at level surfaces, reducing the dimension by 1. This turned out to lead to difficulties similarly as the research of the 4 color problem has let to, even so it looks more manageable. This appears to be a new idea however. It is explained a bit below in the diary.
B) by doing reduction: assume there is a "minimal criminal" (=irreducable graphs). What properties does it have to have? Of course, we use Gauss-Bonnet, also for graphs with boundary. This is close to the proof ideas of the 4 color theorem but also not so easy. Basic ideas go back to George Birkhoff. We produced only problematic arguments here, fueled by wishful thinking. But we documented the failed searches.
C) by looking at graphs with small chromatic number and then look at the defects. This fundamental idea is probably due to Kempe, who introduced the notion of Kempe chains. These are essentially "flow lines" along which the vector field takes value 2. This idea appears in Kempe's flawed proof of the 4 color theorem. This is an interesting game already for concrete construction attempts: try to 3 color as well as possible, then fill in the cracks.
D) By looking at Poincare-Hopf and Gauss-Bonnet relations. While the Gauss-Bonnet relation has been introduced by Heesch to the problem, the Poincare-Hopf idea is new. We see that if we average the indices of vector fields obtained from colorings, we get curvature. Also this did not yet bring results yet.
E) by looking at cohomology and the fundamental group of the graph and dual graph. This is a new idea because we are especially interested in non-simply connected cases which are new cases. Cohomology matters because colorings define vector fields which are gradient fields, and

cohomological constraints occur when wanting to make a local coloring global. Similarly for the fundamental group, again because we need to continue a coloring along closed paths. This is fascinating because we have been surprised a couple of times here.

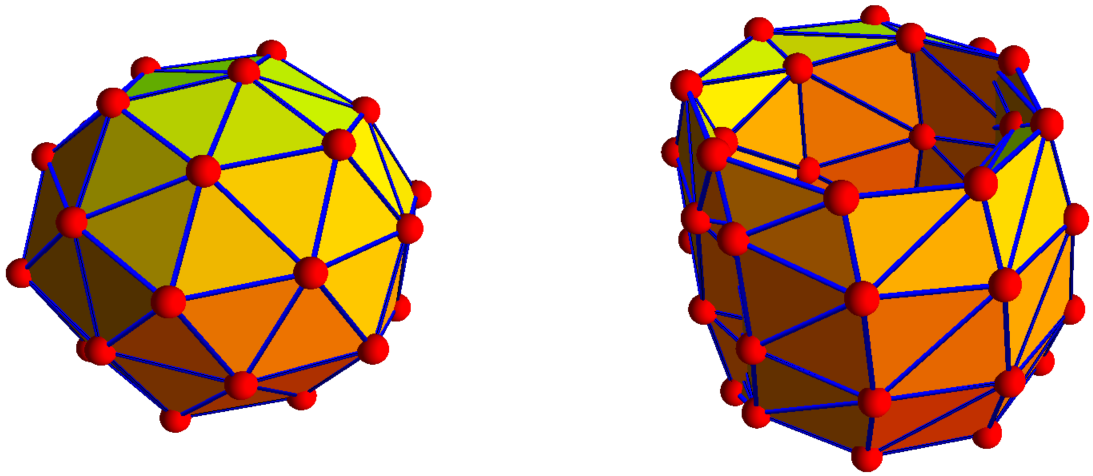


Figure 1: Two geometric graphs

What is the connection with vector fields? If we look at a coloring as a function f for which the exterior derivative $df = \nabla f$ produces a finite group-valued **vector field** without zeros and for which we can define level surfaces $\{f = c\}$. The indices of the gradient field produce a **Poincaré-Hopf theorem** and the average index is the **curvature**, where curvature is the analogue of Euler curvature in the continuum, a local quantity which sums up to the Euler characteristic like Gauss-Bonnet-Chern in the continuum. These relations do not need regularity and hold for any finite simple graph, but for graphs with more structure, we expect more parallels with the continuum. [July 17: we should mention that the picture of vector fields is hardly new. Tutte already in the 50ies looked at vector fields which are finite group valued and especially fields which satisfy the Kirchhoff condition $\text{div}(F) = 0$.]

It is in particular useful therefore to consider discrete manifolds: take a d-dimensional geometric graph and a coloring f, then define the surface $f = c$. This can be completed to be a geometric $(d-1)$-dimensional graph. This has been used already in the past to show that geometric graphs of odd dimension have zero Euler curvature at every vertex - the analogue of the fact that in the continuum, the Euler curvature is only defined for even dimensional manifolds.

Our work hopefully will confirm that the chromatic number is determined by global geometric connections. Indications to that are that some non-orientable two-dimensional manifolds have chromatic number 3-5 while orientable manifolds appear to have chromatic number 3-4. [July 17: we have now examples of projective planes with chromatic number 4 too but they need to be constructed carefully. A "generic" discrete projective plane has chromatic number 5.] [July 24: we also have a projective plane with chromatic number 3. This is a bit surprising since we believed that the bipartite property of the dual graph would follow from $c(G) = 3$.]

Our frame work is mathematical but also experimental as we explore an area of graph theory, which has not yet been studied. We consider a class of **geometric graphs**, graphs for which the unit sphere are "discrete spheres". This is the only assumption. We focus first on two-dimensions, where unit spheres are required to be circular graphs at every vertex. These graphs appear more approachable than planar graphs. This project is done in collaboration with Jenny Nitishinskaya during the summer of 2014. These notes are Oliver's. All errors are of course mine (Jenny had proven me wrong million of times already but also made already a few discoveries). Here is the main object:

Definition A **geometric two dimensional graph** is a graph for which every unit sphere is a circular graph with more than 3 vertices. A geometric two-dimensional graph with boundary is a graph for which every unit sphere is either a circular graph with more than 3 vertices (interior point) or a line graph with more than one vertices (boundary point) and for which the boundary is one-dimensional. A two-dimensional geometric graph is **orientable**, if there is an orientation of the triangles which is compatible on the intersections.

Examples:
1) The triangle K_3 is two dimensional but everything is boundary. It is not a geometric graph with boundary.
2) The wheel graph W_n with $n+1$ vertices is for $n > 3$ a geometric graph with boundary. The boundary is the circular graph C_n.
3) The octahedron and icosahedron are the two platonic solids which are two dimensional geometric graphs. They do not have any boundary.
4) The tetrahedron is a three dimensional graph. It is its own boundary.
5) The natural simplest triangularization of a torus with $2n^2$ triangles (first cut into n^2 squares and then put a diagonal in each) is a two dimensional graph without boundary.
6) In graph theory, geometric graphs are part of **"cubic maps"** as geometric graphs are triangulated. An example is the octahedron. But there are of course cubic maps, even planar ones which are not geometric: an example is the tetrahedron which is three dimensional. Complete graphs K_n have chromatic number n and are not geometric graphs because every point is a boundary point with unit sphere K_{n-1}. We have required for graphs with boundary to have the boundary have less dimensions.

The notion of "geometric graph" can be extended to any dimension. A one-dimensional graph is geometric if every unit sphere is a 2 point graph. A one-dimensional graph is a **one dimensional geometric graph with boundary** if every unit sphere $S(x)$ is a 1 point graph (where x is a boundary point) or 2 point graph (where x is an interior point) and the set of interior points generates a zero-dimensional graph. Since every line graph is two-colorable and every circular graph is either 2 or 3 colorable, we have:

A one-dimensional geometric graph has chromatic number 2 or 3.

The proof is trivial since every one dimensional geometric graph is a union of cyclic and interval graphs for which each can be colored with 2 or 3 colors (3 are only needed for odd order cyclic graphs).

One can look at more general one-dimensional graphs called "varieties", where we require the singular points (points for which the degree is different from 1 or 2) to form a discrete set. **Discrete varieties** can be defined inductively in any dimension by assuming that the singular set forms a lower dimensional variety. In this language we can say:

A one-dimensional variety has chromatic number 2 or 3.

Keeping the variety condition out makes the result false as there are triangle free graphs of arbitrary chromatic numbers. But these are not varieties (in the original constructions even all points are singular (meaning that the vertex degree is 3 or higher).

[June 25: As in classical mathematics, we want singularities of varieties to form a lower dimensional variety. The fact that the singularities must be isolated is necessary already in one-dimensions. Groetzsch has given examples of one-dimensional graphs - graphs for which every unit sphere is a discrete graph - for which the chromatic number is 4. And a construction of Mycielski has shown that the chromatic number of a one dimensional graph (a triangle free graph) can be arbitrarily large. But in all these cases, the singularities form a large part of the graph preventing it from being a variety in our sense. As in algebraic geometry, we do not want the singularities to be a substantial part of the geometry. In an analytic setting, this is automatic: singularity sets in a d-dimensional analytic variety can not be d-dimensional by analytic continuation. Our recursive definition of **discrete variety** captures that in the discrete.]

Before the project started, Oliver thought that **all** geometric two-dimensional graphs have chromatic number 3 or 4. This was even stated as a conjecture in the submitted proposal for this HCRP project. But it turned out to be false:

> **Jenny Nitishinskaya**: There are geometric graphs with the topology of the two-dimensional projective plane which have chromatic number 5. Projective planes also can have chromatic number 4. ([Jul 23: and even 3]).

We have therefore modified the conjecture (compare "Proofs and Refutations" of Lacatos):

> **HCRP 2014 conjecture:**
> **an orientable geometric two-dimensional graph with or without boundary has chromatic number** 3 **or** 4. **If it has a boundary, there is a coloring such that each boundary component has 3 colors only. Any geometric two-dimensional graph with or without boundary is 5 colorable (**$c(G) \leq 5$**).**

[**June 25:** it is likely that more generally all orientable varieties have chromatic number 3 or 4. Discrete varieties are two-dimensional finite simple graphs for which the singularities is a union of lower dimensional varieties. Singularities are points where the unit spheres are still one-dimensional but not interval, nor a single circular graph. An example is a discrete cone which has a vertex with unit sphere consisting of two circular graphs.]

[July 16: The general theory gives only trivial upper bounds in general. The maximal vertex degree of course can give a bound (Brooks theorem). There is also a bound in terms of arboricity $a(G)$ as Steve Butler once noticed $c(G) \leq 2a(G)$. Its a simple observation using the fact that every forest has chromatic number ≤ 2 but Butler seems the first to have noticed. There is a Nash-Williams formula for arboricity which together with Euler's formula and the geometric condition gives for the torus that $a(G) \leq 4$. For the torus, we have so only $c(G) \leq 8$ which is way off. We expect that $c(G) = 3, 4$ for the torus.]

The 4 color problem shows that this statement is true for subgraphs of triangularizations of the sphere. We hope to be able to give a proof of the same fact in special cases without referring to the 4 color theorem. We also still have to find an example of a planar graph which can not be made into a discrete manifold or variety by removing trivial one-dimensional parts, or removing tetrahedral nipples and filling up lakes with wheel graphs (Cayley). In other words, we want to

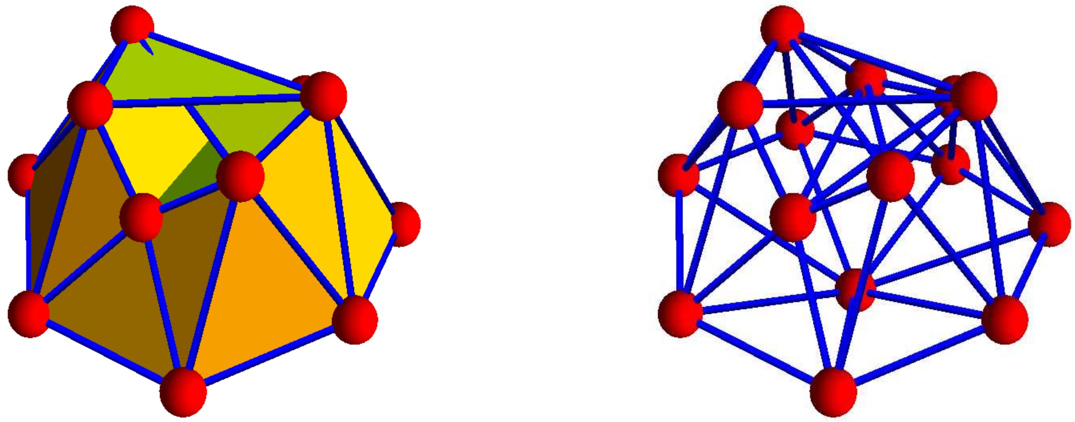


Figure 2: Jenny's discrete projective plane 5. It is a two-dimensional geometric graph but non-orientable. It has 15 vertices, 42 edges and 28 triangles and so the Euler characteristic is $15-42+28=1$. The Betti numbers are as in the continuum $b_0=1, b_1=0, b_2=0$, the non-orientability can be seen in the fact that Poincaré duality $b_2 \neq b_0$ fails. Unlike in the continuum, there is no implementation of the two-dimensional projective plane which has positive curvature everywhere (this follows from the fact that we can give a list of all positive curvature graphs). Jenny's particular implementation has curvatures $(0,0,0,1/6,1/6,0,0,1/6,0,1/6,0,1/6,1/6,1/6,-1/6)$. It appears to be one of the smallest implementations of the projective plane in the discrete. The chromatic number is 5 because the chromatic number of the Moebius strip is 4 and attaching the Moebius strip to a disc (thats what the construction did) will force an other color.

find a graph which is planar but which after removal of trivial tetrahedra is not a subgraph of a geometric graph or discrete variety.

It is important in our work that we do not want to refer to the 4 color theorem; as we believe the statements are easier for two-dimensional geometric graphs (or classes of such graphs) than for planar graphs, the reason being that these graphs are triangulizations of two-dimensional surfaces allowing tools from geometry. We hope of course that our result will confirm the 4 color theorem for a substantially large class of planar graphs but have an elementary or even constructive proof.

Note for example that if we have a geometric graph which is 4 colorable and add a lot of stuff and still have it 4 colorable. We can attach trees, we can attach tetrahedra (vertices at the centers of triangles and connect to the vertices of the triangle) or cut out any part of the graph and still have 4 colorability.

Bolder is the following conjecture for d-dimensional geometric graphs. These are graphs for which the unit sphere is a $(d-1)$-dimensional geometric graph which is a triangularization of the usual sphere.

Bold HCRP 2014 conjecture:
A geometric d-dimensional graph has chromatic number $d+1$ or $d+2$ if it is orientable and $d+1$,$d+2$ or $d+3$ if it is not orientable.

June 25: again we can be even bolder and ask whether all d-dimensional discrete varieties have chromatic number $d+2$ if orientable. First steps should give upper bounds.
June 28: we believe that if $H^d(G, Z_2)$ is trivial then an orientable d-dimensional variety even has the chromatic number $\leq (d+1)$. In the case $d=1, d=2$, this relies on known results in the graph coloring world.
July 10: There are other considerations to be explained later which might actually could render the chromatic number larger in dimension 3 or higher. But these same considerations suggest that these examples are only achievable by large graphs, definitely too large to be checked by computer.
July 22: Minimal colorings are possible in the nonorientable case. We expect also higher $d-$ dimensional geometric graphs of chromatic number $d+1$.
There is not much numerical evidence yet in the higher dimensional cases as the complexity of computing the chromatic number for larger graphs is large. We only have access to standard software to compute the chromatic number.

Why do we think the bold conjecture has some merit to be true? We can look at a function f on the vertices and define level surfaces $f=c$ which always are geometric graphs of dimension one lower. By induction we expect these level surfaces to be $(d+1)$-colorable. We expect that one can use the colorings of the edges given on two-dimensional subsurfaces of the upper line graph to build a coloring of the entire space using one color more. A **maximum principle for colorings** should be an other key: each boundary component of a geometric graph with boundary will have less colors than the chromatic number c. Again, once the coloring result is proven for graphs without boundary, we can by a Cayley argument get rid of the boundary components.

An other theme is the connection with the theorems of **Poincaré-Hopf** and **Gauss-Bonnet**. This is not surprising, as curvature plays an important role in any coloring problem. And talking about a coloring is nothing else than talking about a non-vanishing vector field over a finite ring. The point of looking at coloring problems is that we can deal with functions taking values in a small set.

Curvature is pivotal in the proof of the 4 color theorem. While we know that if we average the index $i_f(x)$ over all possible injective functions f on a general graph, then we get curvature, this seems to be true for functions taking much less values, functions which lead to gradient fields which nowhere vanish. Here is an other conjecture which we had in the proposal. It is confirmed by experiments so far and should not be too difficult to prove:

Definition Given a graph G with chromatic number c. For each coloring f, look at the **Poincaré-Hopf** index $i_f(x)$. The average $i_f(x)$ over all colorings with c colors is called the **color curvature.** As the sum over $i_f(x)$ is $\chi(G)$, also the sum over the color curvatures is $\chi(G)$.

Color curvature conjecture:
The color curvature of a graph agrees with the usual curvature.

We have checked this numerically for all connected graphs with 6 vertices or less and run a run with all graphs of 7 colors. For example, take the line graph L_3 which is a geometric one dimensional graph with boundary. There is one interior point. The graph is two colorable with 2 colorings $f_1 = (1,2,1)$ and $f_2 = (2,1,2)$. The index i_{f_1} is $(1,-1,1)$ and the index of i_{f_2} is $(0,1,0)$. The average is $(1/2, 0, 1/2)$.
June 28: if the chromatic number of a d-dimensional orientable variety is either $(d+1)$ or $(d+2)$, one needs to classify the graphs which need less colors. We understand the one-dimensional variety

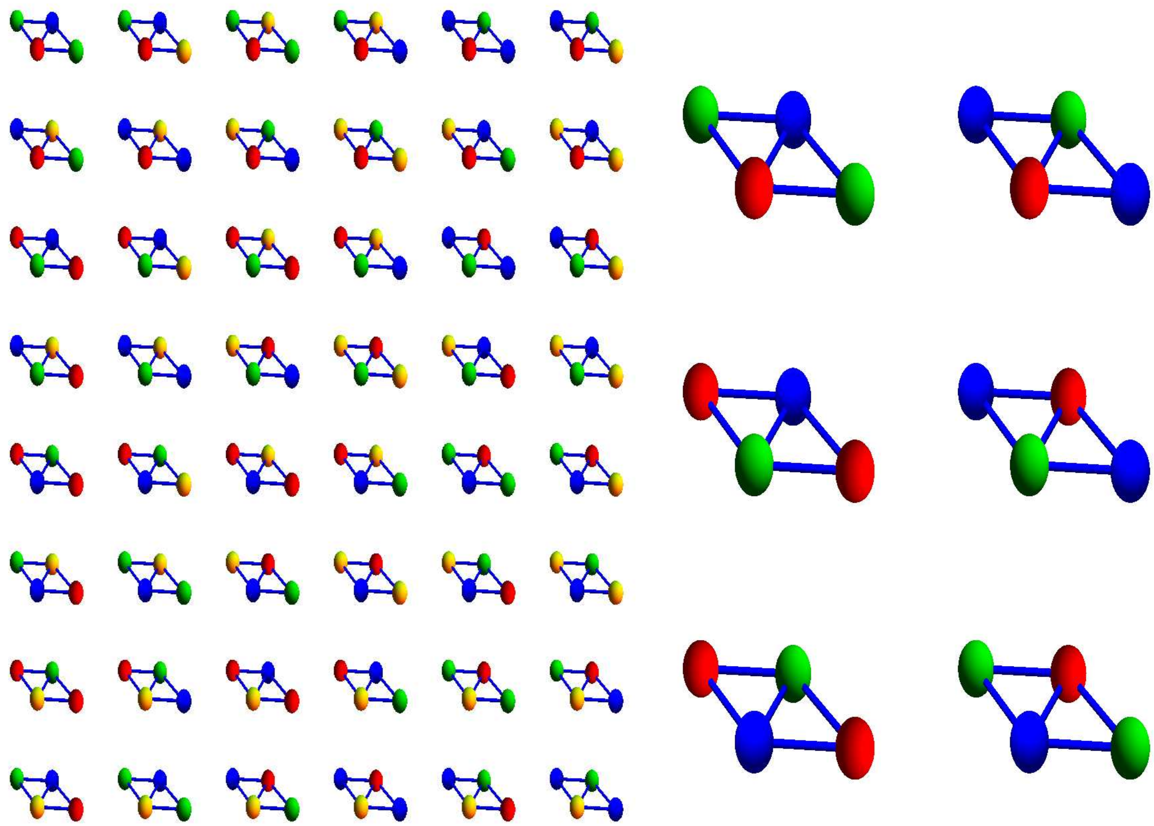


Figure 3: All 48 colorings of the diamond graph G with 3 or 4 colors. There are 6 colorings with 3 colors. If we average the indices we get the curvature. Here are the indices of the 6 colorings with minimal color $c = 3$: $\{(1,0,0,0),(1,0,0,0),(-1,1,0,1),(0,0,1,0),(0,1,-1,1),(0,0,1,0)\}$. The chromatic polynomial of G is $p(x) = x(x-1)(x-2)^2$ and indeed $p(3) = 6$ and $p(4) = 48$.

case very well: the obstruction is $H^1(G, Z_2)$: every closed loop must have even length. Note that in one-dimensions, we bypass with our definition the subtle topic of planarity: **Groetzsch's theorem** shows that one-dimensional planar graphs are 3 colorable but we know for much simpler reasons that one-dimensional graphs with isolated singularities are 3-colorable and that fact is much easier because we avoid planarity. Facts which are universally true are typically also easier to prove.

There would be many other interesting questions to be explored in the context of differential geometry and spectral theory. For example, a **theorem of Wilf** states that the chromatic number is bounded above by 1 plus the maximal eigenvalue of the adjacency matrix. How close are we in the geometric case? For the diamond graph for example, we have the adjacency matrix $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ which has the maximal eigenvalue $(1+\sqrt{17})/2 = 2.56155$. In this case Wilfs theorem has an overhang of 0.56155. For the octahedron graph, where the maximal eigenvalue is 4 and the chromatic number is 3, we have a huge overhang $5 - 3 = 2$. For a triangle, where the maximal eigenvalue is 2 we have no overhang as the chromatic number is 0. For an icosahedron again, where the maximal eigenvalue is 5, we have a huge overhang $6 - 4 = 2$. Since the overhang for geometric graphs without boundary is so large, this possibly can be improved a lot. We have not yet done many experiments, but all examples seen so far"

Spectral question:
The chromatic number of a two-dimensional geometric graph without boundary is bounded above by the maximal eigenvalue of the scalar Laplacian minus 3.

Maybe it is trivial. Maybe it is false.

Graph	Chromatic number	Spectra of the form Laplacians L_0, L_1, L_2
Diamond	3	(4,4,4) has boundary
Moebius strip	4	(6.25,6.25,5) has boundary
Octahedron	3	(6,6,6)
Icosahedron	4	(7.2,7.2,5)
Jenny Projective plane	5	(8.4,8.4,5.7)
Torus (4,4)	4	(8,8,6)
Torus (6,6)	3	(9,9,6)
Torus(7,7)	4	(8.69,6.69,6)
Groetsch	4	(6.73,6.73) one dimensional, not geometric
Cycle(4)	2	(4,4) one dimensional, geometric
Cycle(5)	3	(3.6,3.6) one dimensional, geometric
Cycle(7)	3	(3.8,3.8) one dimensional, geometric

As we will see, the **length spectrum**, the lengths of closed periodic geodesics might play a role. One knows in the manifold case some connections between the length spectrum and the spectrum of the Laplacian. Maybe there is a connection.

1: Geometric graphs

(June 10, 2014)

All graphs are finite simple graphs $G = (V, E)$, where V is the set of vertices and E is the set of edges. In other words, there are no multiple connections in the graph and also, there aren't any self loops present in the graph. The notion of geometric graphs is inductive. We start with zero dimensions:

Definition A zero dimensional graph is a graph without edges.

Definition A d-dimensional graph is a graph for which every unit sphere $S(x)$ is a $(d-1)$-dimensional graph.

A cyclic graph C_n is a 1-dimensional graph for $n \geq 4$. Any tree is a one dimensional graph. The figure 8 graph is a 1 dimensional graph. The graph K_n is $n-1$ dimensional. The icosahedron and octahedron are 2 dimensional graphs. The octahedron and cube are 1 dimensional graphs. The tetrahedron is a three dimensional graph.
We think of a d-dimensional graph as a model for a d dimensional variety.

Definition The **Euler characteristic** $\chi(X)$ of a graph G is defined as $\chi(X) = \sum_k (-1)^k v_k$, where v_k are the number of K_k subgraphs of G.

A tree has Euler characteristic 1. In general, if no triangles are present we have $\chi(G) = v - e$ where v is the number of vertices and e the number of edges. We can show by induction that for a connected graph without triangles $\chi(G) = 1 - g$ where g is the number of minimal closed loops. The number g is called the **genus** of the one dimensional graph.

Definition Given an injective function f on the vertices, the **index** $i_f(x)$ is defined as $1 - \chi(S_f^-(x))$ where $S_f^-(x) = \{y \in S(x) \mid f(y) < f(x)\}$.

Poincare-Hopf: $\sum_x i_f(x) = \chi(G)$.

Proof. Show by induction using $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \setminus B)$.

Definition A graph is called **contractible** if it admits a function f for which $S^-(x)$ is contractible or empty for all x.

Contractible graphs have $\chi(G) = 1$.

The graphs K_n are contractible. The graphs C_n are not contractible for $n \geq 4$. A tree is contractible. Any discretization of the sphere is not contractible.

Definition A vertex x is called a **critical point** of f if $S^-(x)$ is not contractible. Noncritical points are called **regular points**.

By definition, contractible graphs admit a function with only one critical point. Since the minimum is always a critical point, there is no other critical point. Also the following definition is inductive.

Definition A graph is called a d**-dimensional Reeb sphere**, if every unit sphere is a $(d-1)$-dimensional Reeb sphere and if it admits a function f with exactly 2 critical points.

A zero-dimensional Reeb sphere is a graph with two points and no edges. A one-dimensional Reeb sphere is a graph for which every unit sphere has exactly two disconnected points. It is a cyclic graph C_n with $n \geq 4$. A two dimensional Reeb sphere is graph for which every unit sphere is a cyclic graph and which admits a function with two critical points.

A d dimensional Reeb sphere has Euler characteristic $1 + (-1)^d$.

Proof: By induction, using Poincaré-Hopf. A one-dimenisonal graph for example has two critical points of index $1, -1$. A two dimensional graph has two critical points of index $1, 1-\chi(S)$ where S is a 1 dimensional sphere which by induction has $\chi(S) = 0$ so that $\chi(G) = 2$. A three dimensional graph has two critical points of index $1, 1-\chi(S) = -1$, since S is a two dimensional sphere leading to $\chi(G) = 0$.
The main object of investigation in this research project are geometric graphs:

Definition A d dimensional graph is called a d**-dimensional geometric graph**, if every unit sphere $S(x)$ is a $(d-1)$-dimensional Reeb sphere.

Definition A d-dimensional graph is called d**-dimensional geometric graph with boundary** if its vertices decomposes into interior and boundary, where the **interior** the set for which $S(x)$ is a Reeb sphere and where the boundary is the set where $S(x)$ is a contractible. We assume that the boundary is a $d-1$ dimensional geometric graph.

Here is a mini Poincare "conjecture".

Any simply connected two-dimensional geometric graph is a Reeb sphere.

Proof: Take a closed curve γ. Given an other vertex x, we can define what is inside and outside. The connectivity component of x is the inside. We can deform the curve to the point x. This produces a graph which is a two dimensional disc. Now deform the curve to the outside. This allows us to define a function f. Since we can always push the curve further, only possibility of a failure is that we reach a point x which has already been covered. Now take the shortest connection. This connection is not contractible.

Every Reeb sphere of dimension $d \geq 2$ is simply connected.

Proof. Use a gradient flow argument using the function f which has only two critical points to deform any closed curve to a point.
[P.S. We will talk about the fundamental group later in this project.]
Two dimensions: there exists now a function f with only one maximum and minimum. This injective function defines an ordering of the graph and there are no critical points. This implies that we can build up a graph and add circles after circles.
Geometric d-dimensional graphs play the role of manifolds.

Definition A **compact d-dimensional topological manifold** is a compact topological space M equipped with a finite cover U_j and homeomorphisms $\phi_j : U_j \to B$ where $B = \{|x| < 1 \mid x \in R^d\}$ is the open unit ball in R^d.

The finite cover is called an **atlas** of M. Its members U_j are called **coordinate charts**. Define $B_{ij} = \phi_i(U_i \cap U_j)$ and $B_{ji} = \phi_j(U_i \cap U_j)$. The functions $\phi_j \circ \phi_i^{-1} : B_{ij} \to B_{ij}$ are called coordinate change homeomorphisms.

Every d-dimensional geometric graph G is a triangularization of a compact d-dimensional topological manifold M. If G is a geometric sphere, then M is homeomorphic to the sphere.

Proof. We use induction with respect to dimension d. Let n denote the number of vertices. Given a vertex x, look at the unit ball $B(x)$. It contains of the vertex x and the sphere $S(x)$. By induction, the sphere $S(x)$ is a triangularization of a $d-1$ dimensional unit sphere S in R^d which is contained in the ball B of R^d. If K is a $(d-1)$-dimensional simplex in $S(x)$, then $K \cup \{x\}$ generates a d-dimensional simplex in G. We have realized the unit ball $B(x)$ in the graph as a triangularization of the unit ball B in R^d. Denote this ball by B_x. Let ϕ denote the map. We can now proceed as in the proof of the Whitney embedding theorem to find a compact topological manifold in R^{2n+1}.
The reverse is more difficult, as triangularizations of topological manifolds can be strange. We can say a triangularization of a d-dimensional topological manifold is nice if it is a d-dimensional geometric graph. This shows that we actually do have some topology: given a compact topological manifold with a nice triangularization. We can now study the graph itself and forget about the manifold.

2: Morse functions

(June 13, 2014)

Given an injective function on a geometric graph, we can not look at level curves $\{f = c\}$.

> **Definition** A vertex x is a **critical point** of f, if $\{f(y) < f(x)\}$ generates a contractible graph. If x is not critical, it is called **regular**.

The motivation is that if we have regular point, then the extension $\{f(y) < f(x)\} \to \{f(y) \leq f(x)\}$ is a homotopy step. This is an idea of Morse theory.
If you take a bagle and dump it into coffee, then at first a topological disc is immersed, once we reach the level where the coffee shore on the bagle is a figure 8 curve the topology changes. What has happened now is that there is a non-contractible loop on the submersed part of the bagle. It topologically has become a cylinder. There is an other such critical point after which we have two different non-contractible loops. The last critical point is when the entire bagle will be submersed. We see that there are 4 critical points. The Euler characteristic change at a critical point is called the **index**. It is 1 at the minimum where the disc is "born", then -1 at the next two critical points and finally 1 at the maximum.

> **Definition** Given a geometric graph G, let $m(G)$ denote the minimal number of critical points, an injective function can have.

For a sphere, we have $m(G) = 2$ by definition. For contractible graphs, we have $c(G) = 1$. For a discrete torus, we have $m(G) = 3$.

> **Definition** The **upper line graph** of a graph $G = (V, E)$ is the graph (E, F) where F consists of the pair of edges which are contained in a common triangle. The **lower line graph** is the set graph (E, F) where F consists of all pairs of edges which have a common vertex.

The lower line graph is what is what is traditionally called the line graph. It is different in general and in general contained in the first.
Take the diamond graph G for example, then then upper line graph is the fly graph consisting of two triangles with a common vertex and the lower line graph is the wheel graph W_4 with 5 vertices and where the central vertex corresponds to diagonal in the diamond.
For a graph C_4, the lower line graph is C_4 again while the upper line graph is the 4 point graph without edges. Take an octahedron containing the equator x, y, z, u, then xy and yz intersect but are not part of a common triangle.
How do we find natural functions on the graph? One possibility is to embed the graph into some Euclidean space, place it into general position and take $f(x) = x_1$.

An other possibility is to chose a vertex and define graphs G_k inductively as follows.

> **Definition** We define **wave fronts** G_k inductively. Take $G_0 = \{x\}$. Inductively, if G_k is known take the union of all unit balls of points in G_k and remove G_k.

Figure 1: The diamond graph to the left and the upper line graphs of the diamond graph to the right.

Figure 2: To the left the octahedron graph. To the right, the upper line graphs of the octahedron. Given any injective function f, we can look at $f = c$. It is the subgraph of the graph to the right, where the function changes sign. This is either empty, or a union of closed circular graphs.

Definition A wave front is regular if it is a $(d-1)$ dimensional geometric graph. Otherwise, it is critical

For an icosahedron, we have 2 regular wave fronts and 2 critical ones.
One could think that for sufficiently nice two dimensional geometric graphs, each wave front is either a union of cyclic graphs or a finite set of discrete points but already for discrete tori, we can have more complicated situations.

Remark: Here is the ultimate definition of what a Morse function is. It is designed so that the Morse inequalities hold true.

Definition An injective function on a geometric graph is a Morse function, if all indices are either -1 or 1 and if adding that point x to $G_x = \{f(y) < f(x)\}$ changes the Betti vector at one point by one.

For two dimensional geometric graphs, a function is a Morse function if and only if all indices of critical points are either 1 or -1.
Here is an example of a Monkey saddle. Take a wheel graph W_6 with center x_0 and spikes $x_1, ..., x_6$ and take $f(x_0) = 0$ and $f = (1, -2, 3, -4, 5, -6)$ on the boundary C_6. Now $S^-(x_0)$ consists of 3 points so that $i_f(x_0) = -2$. On the boundary, there are 3 local minima each having index 1. The other three points have index 0. We see that the sum of the indices is 1.
The most important definition for us in the coloring setup is the following:

Definition Given an injective function f on a geometric graph and let c be a value not taken. Define $f = c$ as the subgraph in the line graph of G containing the edges for which $f - c$ changes sign.

The level curve $\{f = c\}$ of a geometric two-dimensional graph is either empty or a one-dimensional graph without boundary.

Figure 3: The icosahedron to the left and the upper line graph of the icosahedron.

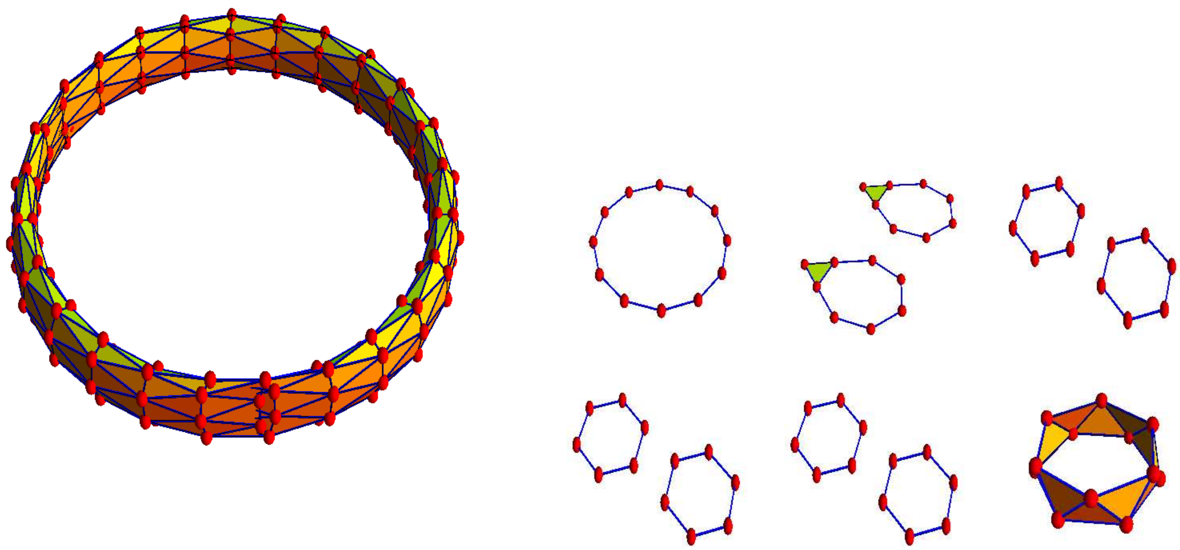


Figure 4: Wave fronts of a discrete torus. We see that the wave fronts can be two dimensional.

Proof: every edge is contained in exactly two triangles. In any triangle, if one edge is contained in the level curve there is a second edge in the triangle which also contains in the level curve. Together we see that every edge in the level curve has exactly two neighbors which also belong to the level curve.

For any geometric two dimensional graph $G = (V, E)$, there exists a function f and function values $c_1, \dots, c_k$ such that the union of the level curves $H_k = \{f = c_k\}$ is the entire edge set E and such that different H_k have no common edge or one common edge. In other words, we have a **foliation** of the edge set into one dimensional leaves.

Proof: start with an arbitrary vertex and assign it the value 0. Define the set $G_0 = \{x\}$. Now look at $G_1 = S(G_0)$ the set of all vertices connected to G_0 etc. The complements $G_k \setminus G_{k-1}$ are the wave fronts we have talked about. If every function value in $G_k \setminus G_{k-1}$ is larger than the function values in G_{k-1} then we can find a c value such that the level curve $f = c$ contains all connections from G_{k-1} to $G_k \setminus G_{k-1}$. Now continue like this until everything is covered.

Sometimes, this is not obvious.

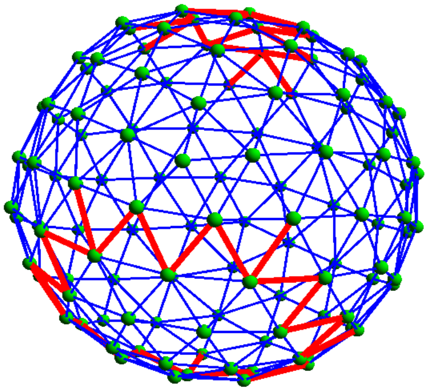


Figure 5: We see the level curve $H = \{f = c\}$ of a random injective function on a fullerene type graph G. The graph H is a subgraph of the upper line graph. The edges which are contained in the level curve $f = c$ are drawn out. In this case, they consist of two separate cyclic graphs. It is a general fact that for a geometric two dimensional graph the level set $f = c$ is always a union of cyclic graphs. We can do geometry in discrete two dimensional graphs and unlike in the continuum, never have singularities! Even in higher dimensions: the 'varieties' given as the common zero set of finitely many equations can always be completed to be a nice geometric graph. Of course, we can exploit that Morse theoretically and look at values c where the topology of these subgraphs changes.

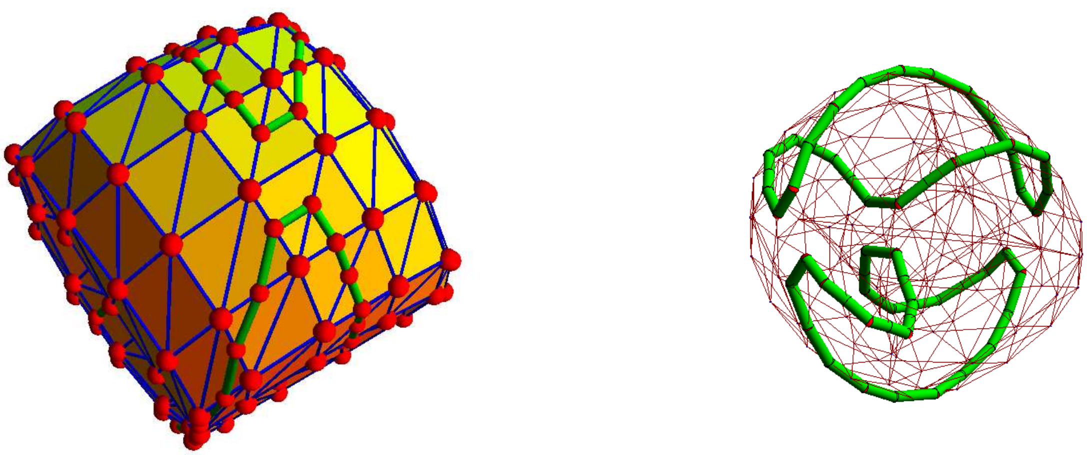


Figure 6: We see a discrete torus and the level curve $f = c$ of an injective function. What we actually see here are two seperate closed curves.

3: Graph coloring

(June 13, 2014)

Is every orientable geometric 2-dimensional graph 4 colorable?

> **Definition** Given a finite simple graph, the **chromatic number** is the minimal k for which the graph can be colored with k colors.

A graph with at least one edge has already chromatic number 2. The question which graphs have chromatic number 1 is trivial. They are the graphs without edges.
A tree has chromatic number 2. Especially, any path graph is 2 colorable.
A cyclic graph with $2n$ vertices has chromatic number 2.
A one dimensional graph is 2-colorable if every closed loop has even length. Otherwise it is 3 colorable.

> **Definition** A subset Y of vertices is an **independent** set if for all $x, y \in Y$ $S(x)$ and y are disjoint. The **chromatic number** $c(G)$ of a graph is the minimal number of disjoint independent sets. The collection $Y_1, ..., Y_k$ of independent sets is is called a **coloring**.

The question we explore in this project is whether $c(G) \leq d+2$ for d-dimensional geometric graphs. It is true for $d = 1$ and all examples we have seen. The first thing to look at is the case of two-dimensional graphs.
For the octahedron, we have $c(G) = 3$. For the icosahedron we have $c(G) = 4$.

> 1 and 2 colorability is a P-problem.

Proof. Which graphs have chromatic number 2? We first of all can not have triangles. Second, any closed loop must have an even number of edges. A graph without triangles for which every closed loop must have an even number of edges is 2 colorable. A graph which is two colorable is bipartite by definition.

Which graphs are 3-colorable. This turns out to be a difficult problem:

> The question whether a graph is 3-colorable is already NP-complete.

Proof. The problem is in NP. We can decide the question in 3^n steps. Given a solution of a 3 coloring, we can check it in Cn steps. To show that it is NP complete, we reduce it to 3-satisfiability, a special case of the satisfiability problem where every clause contains 3 variables x, y, z and possibly the negation x', y', z'. We take it for granted that 3-sat is NP complete (Cook 1971). Now build for any expression a graph with colors True,False,Red. Start with a red vertex and build a triangle t, f, r colored true,false, red. The red vertex r is the center. For each triple of variables x, y, z and clause $p(x, y, z)$ we attach vertices x, y, z, x', y', z' to r and connect x, x' and y, y' and z, z'. This encodes that x' is the negation of x etc. For the logical clause, build an other piece of the graph which is attached to t. For example, to encode $or(x, y)$, build a triangle and attach an edge with end point x to one corner, an edge with end point y at the other corner. Now the third corner is $or(x, y)$ because the only way that it is false is that the inputs were both false. Now every clause can be put into conjunctive normal form, where every clause inside contains only or expressions.

The octahedron is 3 colorable. The icosahedron is only 4 colorable.

Necessary for 3-colorability is that every unit sphere has even cardinality.

Known by the 4 color theorem and Haywood: a geometric two-dimensional planar graph is 4 colorable and 3 colorable if and only if every unit sphere has even length.

George Birkhoff introduced in 1912 the chromatic polynomial for planar graphs. It was later generalized by Hassler Whitney to general graphs.

Definition The **chromatic polynomial** $C(x)$ is the number of graph colorings with x colors.

For a finite simple graph, the chromatic polynomial is a monic polynomial of degree n, where n is the number of vertices. It stats with $t^n + ...$

Proof. By induction on the number of edges. Edge contraction with respect to an edge e $C_G(x) = C_{G-e}(x) - C_{G/e}(x)$ where $G - e$ is the graph without e and G/e is the graph for which the edge e is contracted.
For a tree $C(t) = t(t-1)^{n-1}$. For a complete graph $C(t) = t(t-1)(t-2)...(t-(n-1))$. For a cycle $C(t) = (t-1)^n + (-1)^n(t-1)$

4: The Moebius strip

(June 19, 2014)

The Moebius strip is important because it is a basic building block when building nonorientable surfaces. The classification of two dimensional geometric graphs is of course the same than the classification of two dimensional manifolds because every geometric graph defines a topological manifold (the wheel graphs produce the charts). As in the continuum, one can get the surfaces by taking a torus, cut out pairs of holes and identify their boundaries or cut out a hole and glue in a Moebius strip. The coloring of the Moebius strip is interesting already as we can have chromatic number 3 or 4. We only managed in July 23 to get a Moebius strip with chromatic number 3 and glue in a disc while still keeping the chromatic number 3.

From the classification of two dimensional orientable connected surfaces we know that the genus is the only topological invariant. We can try to prove the coloring result for larger and larger genus. For non orientable surfaces, we know that we can construct them by replacing a polygon with a Moebius strip.

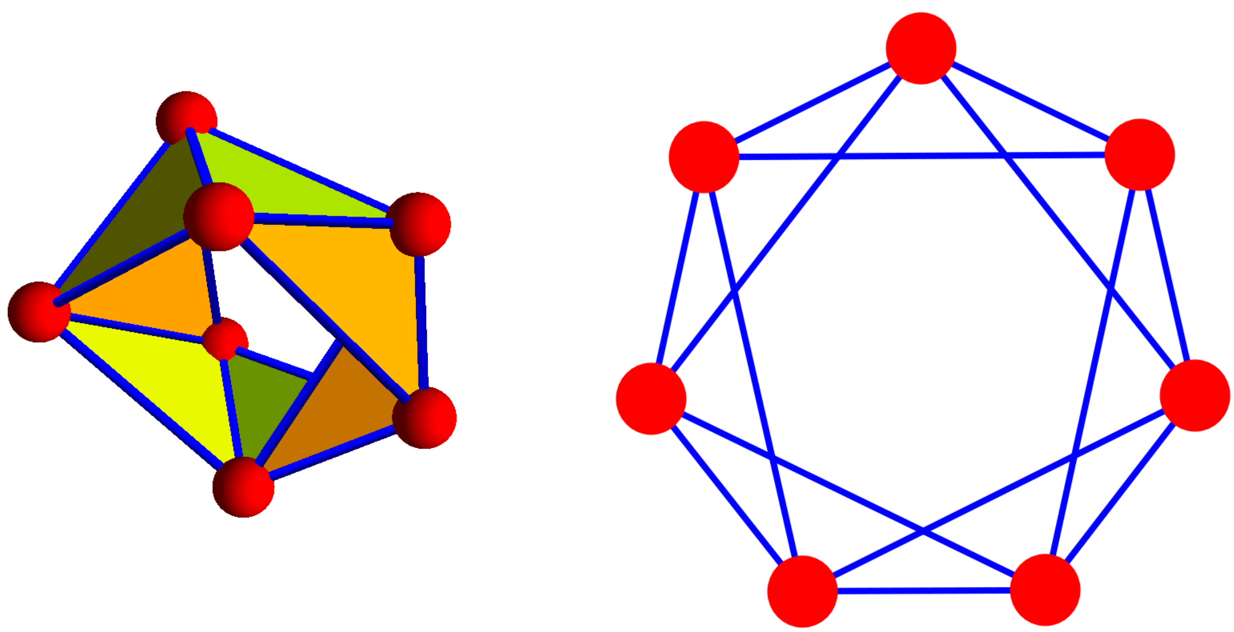


Figure 1: A discrete Moebius strip. It is the smallest implementation of the strip. It has $v_0 = 7$ vertices, $v_1 = 14$ edges and $v_2 = 7$ triangles so that the Euler characteristic is $7 - 14 + 7 = 0$ like the circle. Its chromatic number is 4 as one can even compute the chromatic polynomial. Its chromatic polynomial is $p(x) = x(x-1)(x-2)(x-3)(x^3 - 8x^2 + 25x - 29)$. There are $p(5) = 168$ colorings with 5 colors and $p(6) = 2520$ colorings with 6 colors.

Definition When circular graph $C_7 = \{0, ..., 6\}$ with edges $(x, x+1)$ is equipped with additional edges $(x, x+3)$ modulo Z_7 it is a discrete model for the Moebius strip. Having 7 vertices and 16 edges and 7 triangles, its Euler characteristic is $7 - 14 + 7 = 0$.
The Moebius strip is a two dimensional graph. It is not a geometric graph with boundary because the boundary is the graph itself.

If we add an other vertex v to C_7 and connect v to every point in C_7, we get a discrete disc, a wheel graph. which is a two dimensional geometric graph with boundary.

If we glue the wheel graph and the moebius strip together at C^7 we get a two-dimensional graph without boundary. It is the projective plane. Its chromatic number is 4.

Lets go back to the Möbius strip G and look a bit at the algebraic topology. Given an orientation of the 14 edges of G we have a 14×7 matrix d_0, the gradient. If a function f on the vertices is given, the linear transformation d_0 assigns to an edge (a, b) the value $f(b) - f(a)$. Here it is:

$$d_0 = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Given also an orientation of the triangles, we get a map, which assigns to a function on the edges a function on the triangles. This linear map d_1 is the curl:

$$d_1 = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix}$$

We can check that curl(grad(f) $= 0$. Lets look at the matrix d_0^*, the divergence. Now, as in calculus, div(grad(f) is the scalar Laplacian of G. It is a 7×7 matrix

$$L_0 = \begin{pmatrix} 4 & -1 & 0 & -1 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & 4 & -1 & 0 & -1 \\ -1 & -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & -1 & -1 & 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & -1 & 0 & -1 & 4 \end{pmatrix}$$

It agrees with $L = B - A$, where B is the diagonal matrix containing the degree and A is the adjacency Matrix. The eigenvalues of the Laplacian are 6.24698, 6.24698, 4.55496, 4.55496, 3.19806, 3.19806, 0 There is one eigenvalue 0, belonging to the constant function. The matrix $L_1 = d_0 d_0^* + d_1^* d_1$ is the Laplacian on 1 forms:

$$L_1 = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 3 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 3 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 3 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 3 \end{pmatrix}$$

It has one zero eigenvector $v = [1, -2, 2, -1, 1, 2, -1, -2, 2, -1, 1, -1, 2, 2]$ as well as $L_2 = d_1 d_1^*$

$$L_2 = \begin{pmatrix} 3 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 3 \end{pmatrix}$$

a matrix with no 0 eigenvalue.

The Betti numbers of the Moebius strip are the dimensions of the kernels of L_k which is $(1, 1, 0)$. The number $b_0 - b_1$ agrees with the Euler characteristic.

How can we distinguish the annulus with the Moebius strip? One possibility is to look at the cohomology $H^2(G, Z_2)$ and get Stiefel-Whitney class.

5: Reductions

(June 17, 2014)

Reductions are deformations of a graph which preserve the class of geometric graphs. We usually look at deformations which makes the graph smaller. A basic strategy which goes back to the early work of graph coloring problems is to prove a statement by looking at the class of graphs which do not satisfy the property and look for a **minimal criminal** = irreducable graph.

If one can reduce the minimal graph while still staying in the class of non-wanted graphs, then one has shown that it can not exist. One difficulty is to make sure the reduced graph still has the same properties and this can be a reason for pitfalls as we also have to consider situations, where the graph loses a property like to be geometric. An other basic tool is **curvature** as we know by Gauss-Bonnet what the sum of curvatures are. This can assure that some points with positive curvature exist, allowing reductions.

This works also at the boundary. We can for example force that the boundaries all have non-positive curvature.

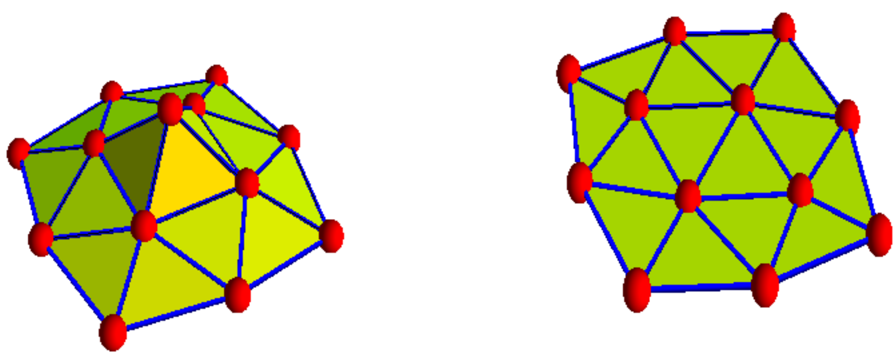


Figure 1: A geometric reduction of a degree 4 graph which makes the graph smaller. Since the total curvature remains 1 by Gauss-Bonnet, reducing the curvature on one graph will increase the curvature of others.

What are the reductions? They are homotopy steps at the boundary or edge collapses in the interior. We can get so from a wheel graph a fan graph, which reduces both the number of triangles as well as the number of vertices. If done right, then we can stay like this in the class of geometric graphs. Both deformations have one basic common theme: we remove peak curvatures. In some sense, we smooth out the graph.

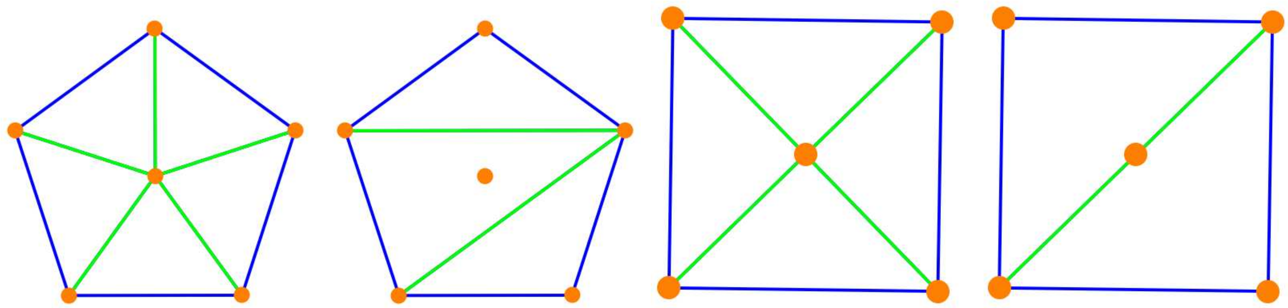


Figure 2: Geometric reductions in the interior.

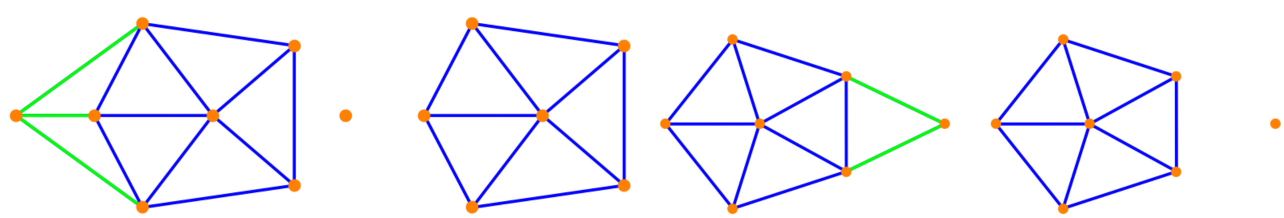


Figure 3: Geometric reductions of the boundary.

6: Maximum principle

(June 22-23, 2014)

In analogy to complex analysis we want to look at a **maximum principle** which appears to hold for colorings: in the class of coloring $V \to \{1, 2, 3, 4\}$, we can find an example, where the maximum is **not on the boundary**. This section consists of many wrong attempts in a hostile territory. The history of the 4 color theorem has shown how treacherous area it is. We of course know about the pitfalls of the past and also know that the 4 color theorem is true and implies the following proposition. While we know the following fact is true, we want to get a simple **proof**. With 4-colorable, we always mean that the chromatic number is ≤ 4 etc.

Proposition: Any two-dimensional disc with pentagonal boundary which is 4 colorable can be recolored so that the boundary is 3 colorable.

The result follows from the 4 color theorem as filling the hole at "infinity" produces a planar graph.

Definition Let $\mathcal{G}$ denote the class of 2-dimensional discs with pentagonal boundary which are 4 colorable but for which the boundary is not 3 colorable.

A variant of the following lemma is used in the classical proof and essentially a consequence of Gauss Bonnet and the possibility to remove quadrilaterals in the interior. In terms of curvature we can avoid points, where the curvature is 1/3.

Lemma: If a graph is in $\mathcal{G}$ then every vertex in the interior has degree 5 or higher.

We also can exclude positive curvature on the boundary. The reason is that we can cut a away triangles or diamonds at positive curvature parts.

Lemma: If a graph is in $\mathcal{G}$ then every boundary point has degree 4 or higher.

From Gauss-Bonnet, using the fact that the disc has Euler characteristic 1 and pentagons have curvature 1/6 and the fact that we just have excluded curvature 1/3 points one gets therefore:

Lemma: Every disc in $\mathcal{G}$ contains a vertex of degree 5.

This is a classical argument also in the usual 4 color theorem for planar graphs. Here is an other flawed attempt June 23 to show the maximum principle:
Proof. Take an element $G \in \mathcal{G}$. Pick two pentagonal vertices. Make the wheel to fan construction at x, y to get a graph H which has two vertices less. As H has less vertices and can not be in $\mathcal{G}$ any more. it can be recolored with 4 colors so that its outer boundary has 3 colors. Close off the outer boundary with a wheel gives a sphere S which is 4 colored. Now open up $S(x)$. We have a graph H with one opening at $S(x)$. No coloring of H can color both $S(x), S(y)$ with 3 colors because we would have a 4 coloring of the original graph G with 3 colors at the boundary. Assume $S(x)$ was not colorable with 3 colors. Then H would be in $\mathcal{G}$ and a smaller counter example. Therefore $S(x)$ is colorable with 3 colors. Close it with a wheel. We have a sphere which has 3 colors at $S(x)$ and 3 colors at the boundary which needs 4 colors at $S(y)$. Every coloring of the sphere requires 4 colors at $S(y)$.

This does not work! Lets try with a scissor-stone-paper type argument.

Definition Let $\mathcal{H}$ denote the class of two-dimensional cylinders G with two pentagonal circular boundaries such that G is 4-colorable but where both boundaries need 4 colors.

The **4 color theorem assures** that the two sets $\mathcal{G}$ and $\mathcal{H}$ are empty. But we do want to know that yet. The strategy is to see that there is no minimal criminal in either, showing therefore that both sets are empty.

Lemma: ?? If there exists a minimal disc G in $\mathcal{G}$ with n triangles then there is a minimal cylinder $H \in \mathcal{K}$ with $< 2n$ triangles.

Proof: Pick a pentagon in the inside of G and cut it out. We get a cylinder K which has a pentagonal boundary $S(x)$ with 3 colors and the other boundary with 4 colors. Take two copies and glue then along the new cut $S(x)$ to get a cylinder H which has 4 colors on both ends. We claim that we can not recolor H to have one end 3 colorable. If we could, then we would have a coloring where both ends are 3 colorable. We could remove in a symmetric way outer circles until exactly one part least needs 4 colors. If both parts need 4 colors we have a minimal cylinder. If only one part needs 4 colors, we have a smaller minimal disc and get both sides 3 colorable.

Lemma: ?? If there exists a minimal cylinder H in $\mathcal{H}$ with n triangles then there exists a minimal disc $G \in \mathcal{G}$ with $\leq n/2$ triangles.

Proof. Cut the cylinder in two. If both sides allow for a 3 coloring on one side we either have a smaller minimal disc or two examples which can be 3 colored on both sides leading to a 3 coloring of the boundaries of the original cylinder.

This also seems chasing gosts! Lets try a statistical approach using a result still to be proven:

If a graph has a c coloring, then the average of $i_f(x)$ over all $c!$ coloring permutations is the curvature $K(x)$ at x.

Assume we have a 4 colorable graph for which the boundary is not 4 colorable. Then we have a graph which is only 5 colorable for which only one global minimum exists and for which the boundary curvature is nonnegative.

The nonnegative curvature condition follows from minimality. Now look at the expectation of $i_f(x)$ for all 5 colorings and try to see that the statistics does not work and only works if we have can do a 4 coloring.

7: Varieties

(June 25, 2014)

A good advise from the late logician **Ernst Specker** is to avoid indirect proofs like the pest. Specker could explain in a humerous way that it Its easy to fool around with indirect arguments, somewhere make a mistake or forget a case, and bingo one has obtained a proof. The contradiction however had the flaw in the argument as the source. The entire castle which has carefully been built up, crumbles to dust. Virtually all flawed proofs are of this type. We will again look at constructive colorings for geometric graphs.

For now, lets look at the bigger picture again. The reason why we are interested in colorings is because every c-coloring defines a Z_c valued function f on the graph such that the gradient df is nowhere zero, where the gradient d is the incidence matrix defined in the Poincaré defined by an orientation of the vertices, an orientation which is irrelevant for most considerations. Its just that our functions take value now in a finite ring. Looking at colorings as functions for which the gradient is nonzero suggests:

c colorings of vertices are Z_c^* valued gradient fields on the graph given by Z_c valued functions f.

We believe (and have so far very encouraging numerical evidence and similar results where we integrate over all injective functions f):

If G is c colorable, then the expectation of the index function $i_f(x)$ averaged over all c colorings is the Euler curvature.

One of the interesting questions is to explore the relation of geometric graphs in the class of planar graphs. Assume G is a two-dimensional geometric graph which is 4 colored. Then we can color lots of other graphs and some of this has appeared very early on.

- We can take any subgraph: the coloring of a graph defines a covering of a subgraph.
- We add a one dimensional forest of graphs for which singularities are separated.
- We can place a pyramid over a triangle.
- Given a boundary cycle, we can place a wheel over it (this is Cayleys reduction to cubic graphs).

Our point of view is differential geometric.

- We look at one dimensional geometric graphs as **one dimensional manifolds**. or if singularities are isolated as **one dimensional varieties**.
- We look at two-dimensional geometric graphs without boundaries as **two dimensional manifolds without boundaries**.

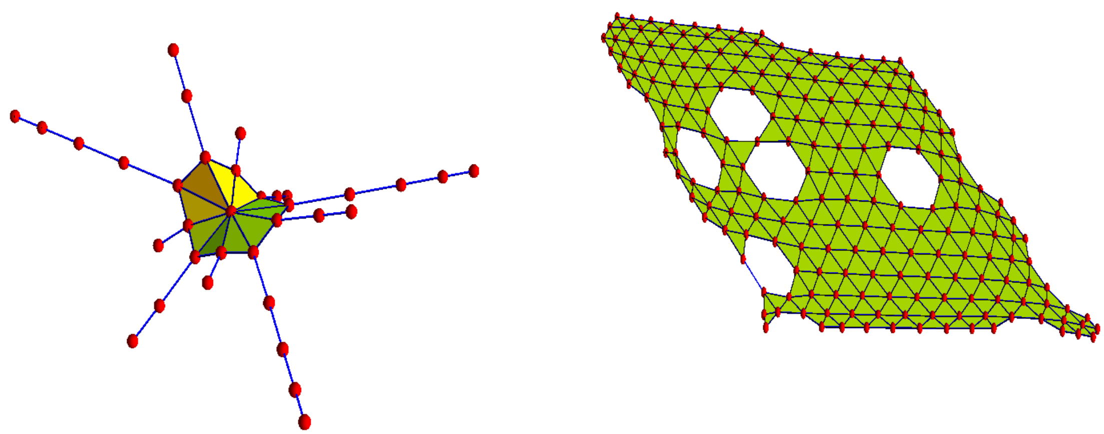


Figure 1: Modifications of geometric graphs: cutting out or filling in lakes. Or adding forests of one dimensional graphs where each graph is a one dimensional graph with isolated singularities (a discrete variety).

- We look at two dimensional geometric graphs with boundaries as **two dimensional manifolds with boundaries**.
- We look at a coloring of a graph as a divisor or section of a vector field.
- We look at a coloring of a graph as a scalar function which defines a gradient field, which then by Poincare-Hopf indices averages to curvature.

We know already that all one-dimensional varieties are 3 colorable. The original conjecture is that all orientable two-dimensional manifolds with boundary are 4 colorable. We know that there are non-orientable geometric graphs with chromatic number 5 (projective plane). [July 16: we know that there are also projective planes of chromatic number 4]. We know that there are one-dimensional graphs with arbitrary high chromatic number (Groetsch etc).

> **Definition** A graph is a **two dimensional**, if its dimension is 2 at every point meaning that at every point the unit sphere is one dimensional. A vertex is a **singularity** if the unit sphere is one-dimensional at every point and has Euler characteristic different from 0 or 1. A graph is a two-dimensional **variety** if it is two dimensional and the set of singularities is zero dimensional.

Of course, these definitions can be done in any dimension.

> **Definition** A graph is d-**dimensional**, if every unit sphere is a $(d-1)$-dimensional graph.

The definition of dimension is inductive. Recall that a graph is a d-dimensional geometric graph (= discrete manifold) if every unit sphere is a $d-1$ dimensional sphere. And we have defined what it means for a geometric graph to be a sphere.
Now we can look at d-dimensional graphs as analogues of d-dimensional varieties as long as singularities are not too bad ([June 26: = lower dimensional]). Here is a new definition suggestion: (the definition is inductive again):

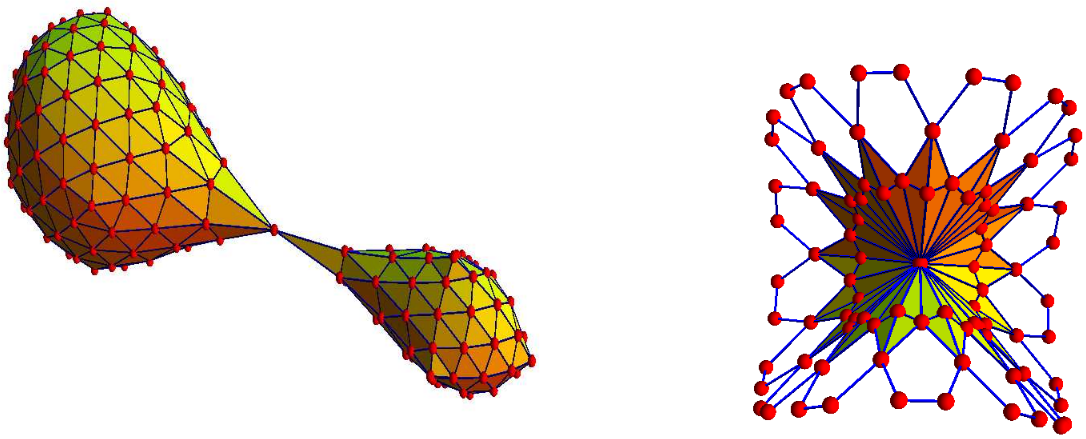


Figure 2: The first graph is a variety with one singularity. An example of a graph which is part of a geometric two dimensional graph.

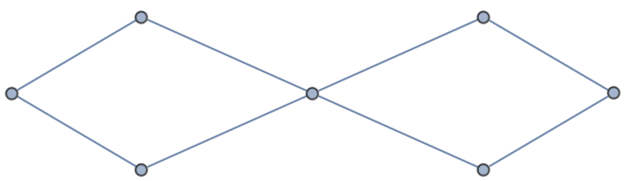


Figure 3: An example of a one dimensional variety is the figure 8 graph.

Definition A finite simple graph $G = (V, E)$ is called a d**-dimensional discrete variety**, if every unit sphere is a $(d-1)$-dimensional discrete variety for which the set of singularites is a union of smaller dimensional varieties. A **singularity** is a vertex for which the unit sphere is $(d-1)$-dimensional discrete variety but not a $(d-1)$ dimensional sphere nor a $(d-1)$-dimensional ball.

[June 26: we do not want to force singularities to be isolated points as we want to allow discretisations of $x^2 = y^2$ in R^3, where the singularity set is the z-axes. and the unit spheres on the z- axes are one dimensional varieties $x^2 + y^2 + z^1 = 1, x^2 - y^2 = 1$ which are 2-branched covers of the circle ramified at two points. The above inductive definition now mirrors this in the discrete.]

We will just say "variety" instead of "discrete variety" because we only deal with finite graphs. Geometric graphs with boundary are examples of varieties because every vertex is a regular point: every vertex is either a boundary point (the sphere is a $d-1$ dimensional ball) or an interior point (the sphere is a $d-1$ dimensional sphere). An other example of a geometric one dimensional variety is a graph (V, E) obtained from an arbitrary other graph without isolated points and subdividing every edge with an other vertex. The boundary points are points of degree 1, the singularities are points of degree larger than 2.

Again, we always mean with c-colorable that the chromatic number is $\leq c$.

HCRP Conjecture for varieties: every orientable d-dimensional variety is $(d+2)$-colorable.

It does not impossible: going from dimension d to dimension $d+1$ needs some geometric analysis of extending colorings along hypersurfaces. And going from manifolds to varieties is also not hopeless, as the parts between singularities are geometric singularities actually make the coloring easier as the unit spheres produce less constraints. The later statement so far has only annectotal evidence but take the figure 8 graph as an example of a one dimensional variety. It has one vertex as singularity. The two loops can each be 3 colored. It is easy then to modify the coloring on both sides so that at the singularity, there is no color conflict. Look at a star graph S_4, which actually is a one dimensional cone (the cross). The singularity at the origin decouples the coloring problems at the branches. Similarly for the two-dimensional double cone, where we have one singularity, where the unit sphere consists of two circles. Also there, we have two discs which are glued together at a point. Coloring the discs is easy. We can then recolor so that at the singularity, there is no color clash.

Why is it unlikely to get conflicts at singularities? Because there are so many colorings if there is one coloring. This insight is already due to George Birkhoff who defined the **chromatic polynomial** p which is of degree n if we have n vertices and gives large values for the smallest c for which $p(c)$ is nonzero. The only way that this number can be small is by Taylors theorem that many derivatives $p^{(k)}(x)$ are zero at $x = c - 1$. This is calculus intuition. In the geometric context we actually see that if we have one coloring, we can get lots of other colorings. And there might even exist a nice algebraic structure on all these colorings in a nontrivial way, trivial meaning color permutations or applying graph automorphisms.

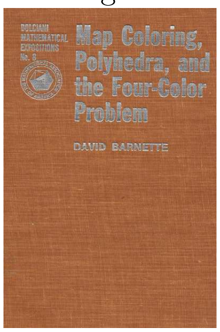
That said, it should explain why it is so important to first focus on two-dimensional geometric graphs. Of course, an exciting question is the following:

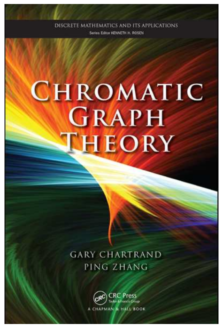
Find an example of a graph which is planar and which can not be reduced to a one-dimensional or two dimensional variety in trivial ways.

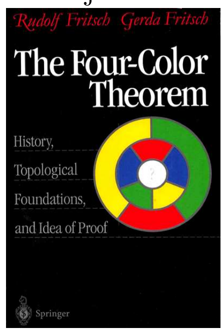
9: History

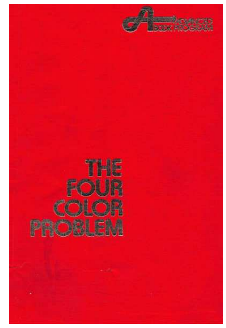
(July 3, 2014)

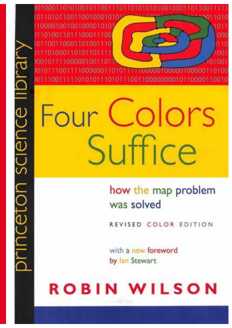
While checking the proof of the theorem on the chromatic number and can show that $(d+1)$-colorability of d dimensional graphs can be characterized, we look a bit at the history of the 4 color problem. The story has been written down in many places. Here are book covers of books we bought and scanned about the subject:



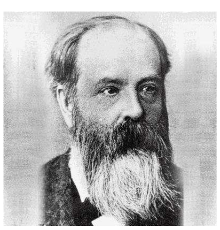
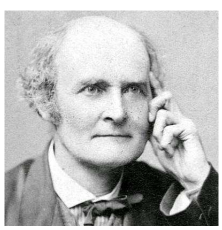
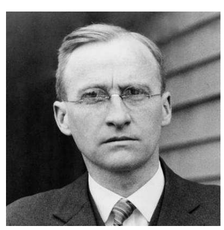
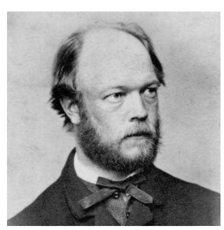
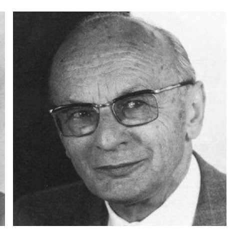
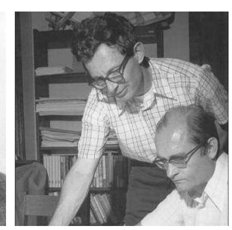




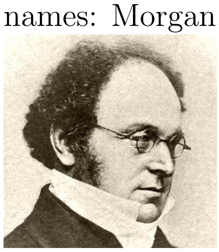
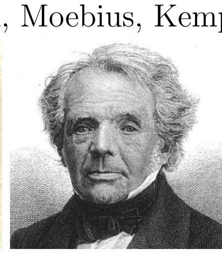
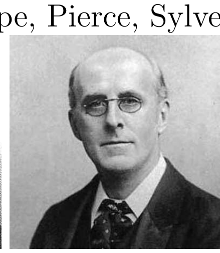
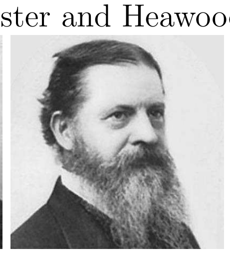
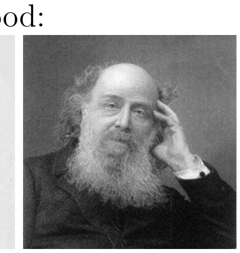




The story starts with Francis Guthrie in 1852. The first written references is by Arthur Cayley in 1878. Kempe's proof of 1879 and Tait's proof of 1880 turned out to be flawed still pushing the knowledge. Kempe proved that 5 colors suffice and introduced basic ideas like Kempe chains. Birkhoff introduced new key ideas in 1912 like reduction and chromatic polynomial. A computer assisted proof of Appel and Haken succeeded in 1977. They built on work (and competition) of other mathematitians, in particular Heesch who worked 50 years on this problem and who was the first to use computers to attack the problem and onto whom much of the work of Appel and Haken is based (Heesch lost the race a bit tragically due to funding problems). In 1995, Robertson, Sanders, Seymour, and Thomas rewrote a new program which is still accessible online. Haken died just about one year ago in New Hampshire. Here are Guthrie, Cayley, Birkhoff, Tait, Heesch, Appel and Haken.

The mathematics can be tricky: the problem saw some false proofs: Kempe and Tait or Charles Sanders Peirce (the son of Benjamin Peirce). Wilson's book mentions the story that Hermann Minkowski once stated arrogantly that the reason why the problem was not solved because "only third rank mathematicians have occupied themselves with it". He then tried to solve it himself and to give up. In Bigalke's book on Heinrich Heesch, the drama about the race of finding the solution. Heesch was the first to use computers for the problem in 1964. Here are some more names: Morgan, Moebius, Kempe, Pierce, Sylvester and Heawood:

10: The smallest chromatic number

(July 6-12, 2014)

The last couple of days, we have looked at the problem of characterizing geometric graphs G with minimal chromatic number $c(G) = d + 1$. We especially wanted characterize surfaces with chromatic number 3. In the simply connected case (the fundamental group is trivial rsp $H^1(G) = 0$), this is well understood:

> If G is a simply connected two dimensional graph then $c(G) = d+1$ if and only if the graph is Eulerian. In that case, if the graph is connected, the chromatic polynomial is of the form $p(t) = t(t-1)(t-2)q(t)$ with $q(3) = 1$.

Proof: if G is simply connected being Eulerian is equivalent to the dual graph being bipartite. This defines orientations on each triangle or a Z_3 valued vector field on the graph **which is nowhere zero**. This allows us to color the graph. Prescribing the color on one triangle (6 possibilites) determines the colors everywhere.

In the simply connected case, the non-vanishing vector field F has zero curl and by the simply connectedness, we have $F = df$ with some scalar function f. This scalar function f provides the coloring.

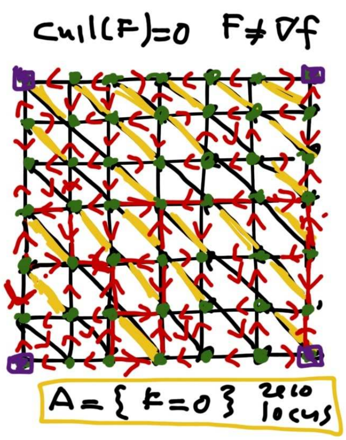
It was the vanishing of $H^1(G, Z_3)$ in the simply connected case which seduced us to look at cohomological conditions for characterizing $c(G) = 3$. This turns out to be a wrong path. Yes, a coloring f defines a vector field $F = \nabla f$ which is a gradient field but this does not prevent other vector fields to exist which have zero curl but are not gradient fields. Here is the thought process or analogy which was the reason for the seduction: look at the torus $G = T^2$. In the continuum, there is a basis of vector fields in $H^1(G)$ which are nonzero since every field of zero curl not being a gradient is of the form $F(x, y) = (a, b)$ with nonzero (a, b). In other words, in the continuum, harmonic vector fields do not vanish on the torus. Is this true also with discrete group valued fields? Look at a Z_3 valued irrotational vector field on the discrete torus and assume the graph can be 3-colored. If the vector field F is everywhere nonzero, it is a coloring, provided it is a gradient field. If it is zero somewhere, then the set where it is zero can not be in a simply connected region because otherwise, we just add a coloring vector field to make it zero where it has not been. And a vector field which has zero curl and is supported on a simply connected region must be a gradient field. We will look at at that below.

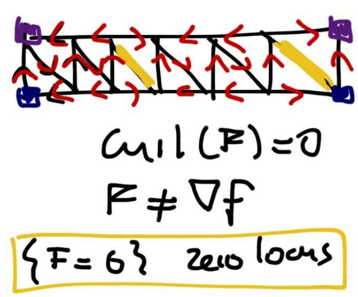
After having dismissed "all 2D geometric graphs have chromatic number ≤ 4") which was disproved by a projective plane, and ("Eulerian graphs have chromatic number 3") which is disproved by flat 4×4 tori or "Eulerian implies G' bipartite" which was contradicted by nonflat tori we tried "3 colorability implies trivial $H^1(G, Z_3)$" which was contradicted by examples of 3 colorable graphs for which $H^1(G, Z_3)$ is nonzero. The nontrivial cohomology classes contradict the intuition from the continuum:
here is this remark about the zero locus and nonzero locus of cohomology classes in that case of small chromatic number:

Let G be a 2D geometric graph with chromatic number 3 and let F be a field which has zero curl but which is not a gradient. Then the **zero locus set** A of triangles which touch an edge with $F(e) = 0$ and the set B of triangles which touch an edge $F(e) \neq 0$ both are not simply connected.

Proof. If B were simply connected, then we could cut off the graph in a neighborhood of the set B which is now simply connected and F has compact support and zero curl. This means that F is a gradient field since the cohomology is now trivial.
Now lets look at the zero locus set A. It can not be empty because otherwise, F were nonzero everywhere and be a gradient as it would define a coloring. Because G has chromatic number 3, there is a gradient field $H = df$ which is nowhere zero. There are only two such fields, H or $-H$ which belong to the two possible colorings of the triangles defined by the bipartite structure of the dual graph. Now either $G + H$ or $G - H$ has the property that it is zero on A and not a gradient field because both are equivalent modulo gradient fields. . As the set A has now switched to become the set where $G + H$ is nonzero, the same argument as before shows that A can not be simply connected.

Here is the example of Jenny which illustrates the picture. The first is a cohomology class in $H^1(G, Z_3)$ for a discrete torus (a vector field which has zero curl but is not a gradient). One can see that the zero locus set can not be simply connected. Since the graph itself has $c(G) = 3$, we can not find a nontrivial cohomology class for which the zero locus is simply connected. The second example is just the lowest part of the grraph. It is a thin cylinder of height 1. The zero locus is shown in both cases in yellow. These are of course places, where the 3 coloring fails when it comes from that vector field. Both graphs satisfy $c(G) = 3$. They admit vector fields which are nonzero everywhere, but these vector fields are then gradient fields.



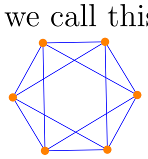


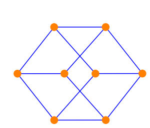
Definition An Eulerian geometric 2-dimensional graph is a graph for which every unit sphere is a circular graph with an even number of edges.

Definition The dual graph $\hat{G}$ of G is has the faces of G as vertices. Two vertices in the dual graph are connected if sharing a common edge in G.

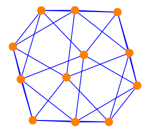
Remark: The dual graph is cubic: every vertex degree is 3. If $\hat{G}$ is completed with wheel graphs so that it becomes 2D geometric, we call this $\overline{G}$.

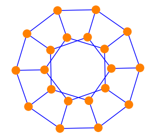
Octahedron: $c(O) = 3$.



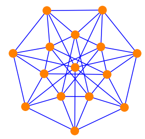


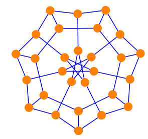
Icosahedron: $c(I) = 4$.



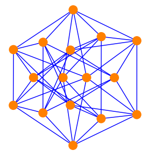


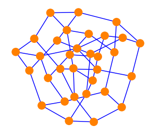
Projective: $c(P) = 5$.





Torus: $c(P) = 3$.



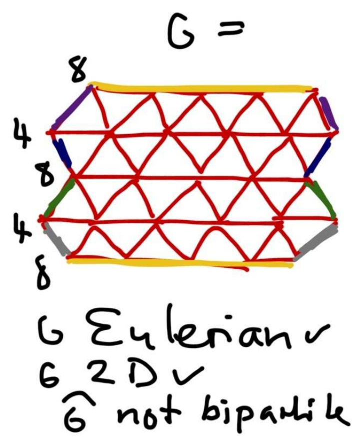


Poincaré duality: $\overline{G}$ is topologically equivalent to G. It has the same Betti numbers b_i and Euler characteristic.

Proof. Relate it to the classical theorem by seeing the graphs as triangularizations and note that the genus and orientability are complete invariants for 2D manifolds. $\chi(G) = v - e + f$ agree because duality switches $v \leftrightarrow f$ if we look at the polygonal faces at first and note that stellating adds a vertex, k edges and k triangles and removes the large face. The operation does not change $v - e + f$.
In higher dimensions d, we have $b_{n-i}(\overline{G}) = b_i(G)$ for the completed dual graph $\overline{G}$.
Here was the question:

Does every Eulerian geometric two dimensional graph have a bipartite cubic dual graph?

The answer is no. One could think that the fundamental group is in general generated by closed paths of even length. The sum of two paths of even length is even. So, every closed path has even length and the graph is bipartite. But this is only true locally, not globally. There can be generators of the fundamental group which have odd length. Jenny constructed a first example with geomags. Here is a simplified version:



The condition that $\hat{G}$ is bipartite is stronger than the property that G is Eulerian. Bipartite also takes care of **global holonomies**, while Eulerian only assures only that the graph locally has a bipartite structure. Here is a question:

For d dimensional geometric graphs: if $\hat{G}$ bipartite and the fundamental group generators have lengths which are multiples of $d+1$, is it true then that $c(G) = d + 1$?

We know that all conditions are sharp if they hold. $c(G) = d + 1$ implies that $\hat{G}$ is bipartite. We also need to show that $c(G) = d + 1$ implies the global holonomy condition. If $c(G) = d + 1$ we have a Z_{d+1} valued vector field satisfying $curl(F) = 0$.

Example. Lets look at the situation in three dimensions, where the highest dimensional simplices are tetrahedra. The condition $\hat{G}$ being bipartite means locally that at every edge of G there are an even number of tetrahedra attached and that globally, we have $H^1(\hat{G}, Z_2) = 0$ assuring that along every generator of the fundemantal group in the dual group has even length.
Placing tetrahedra onto each other does not really produce a coloring already. It tells us how the coloring changes if we go from one vertex to the next: it is a vector field which has zero curl. The condition $H^1(G, Z_4)$ assures that this vector field is a gradient field ∇f. The function f on vertices is the coloring.
We would like to find necessary and sufficient conditions for $c(G) = d + 1$. We especially ask whether for orientable graphs:

Is $c(G) = d+1$ equivalent to the fact that each generator of the fundamental group of G has length $(d+1)k$ and each generator of the dual graph $\hat{G}$ has length $2k$?

The first condition assures that a nonzero vector field F is actually a gradient field. The second condition assures that we can construct a vector field which is nowhere zero.

Having mentioned "simply connected" and this being a research diary, lets look at the fundamental group of two dimensional surfaces next.

11: The fundamental group

(July 11-12, 2014)

The definition of the fundamental group for graphs is the same than in the continuum. Since almost all literature in graph theory looks at graphs as one dimensional objects, where homotopy deformations are not possible, we look at graphs as discrete, possibly higher dimensional objects in which curves can be deformed as in the continuum and the same results as in the continuum apply. We do this however without referring to the continuum at all.

Definition A **path** in a graph $G = (V, E)$ is a finite sequence $x_0, \dots x_n$ of vertices such that $(x_{i-1}, x_i) \in E$ for all n. A path is **closed** if $x_0 = x_n$. The **length** of the above path is defined to be n.

While we assume that x_{i-1}, x_i are different vertices, we do not assume the graph to be simple. The path can visit the same vertex several times. In particular, we could for example have a path of length n which just visits 2 vertices, bouncing forward and backwards.

Now we need to define the analogue of homotopy deformation. Also for this, we do not need the path to be a geometric path:

Definition A **homotopy step** is a pair of paths $x = (x_0, \dots, x_n) \to y = (y_0, \dots, y_n)$ where y is obtained from x by applying steps A or B or A^{-1} or B^{-1}. $y = Ax$ is obtained replacing $(x_{i-1}, x_i, x_{i+1}$ in a triangle with x_{i-1}, x_{i+1}
$y = Bx$ is obtained by replacing $x_{i-1}, x_i, x_{i+1} = x_i$ with x_i. Two paths are called equivalent, if they can be transformed into each other by finitely many homotopy steps.

Definition The **sum** of two closed graphs x, y with $x_0 = x_n = y_0 = y_n$ is the concatenated path $x_0, \dots, x_n, y_1, \dots, y_n = x_0$.

If $x \sim u$ and $y \sim v$, then $x + y \sim u + v$. This implies that the equivalence classes of closed curves form a group.

Definition The equivalence classes of closed paths starting at x_0 form a group. It is called the **fundamental group** of the graph G.

Look at all the equivalence classes a_i, b_i of simple closed curves. There are only finitely many simple closed curves on a finite graph so that there only finitely many different classes. As in the surface, case, the fundamental group is a presented group with generators $a_1, b_1, \dots, a_g, b_g$ and relations $[a_i, b_i] = 1$. This is called a **surface group**. In the case $g = 0$ it is trivial. In the torus case $g = 1$, it is the Abelian group Z^2, for higher genus, it is non-abelian.

11: More on minimal colorings

(July 13-14, 2014)

We seem now to have a result characterizing geometric graphs with minimal colorings, at least for orientable graphs:

Definition Given a graph G and a field or ring k. A k-valued **vector field** F is a function on oriented edges of the graph. It is a form meaning that it satisfies $F((a,b)) = -F((b,a))$.

Definition The **line integral** of F along a path $C : x_0, x_1, .., x_n$ is the sum

$$\int_C F = \sum_{i=0}^{n-1} F((x_i, x_{i+1})) .$$

Definition The **curl** $\text{curl}(F)$ of F is a function on oriented triangles which is the line integral along the boundary of the triangle. We say F is **irrotational** if the curl of F is zero on every triangle of the graph.

Remarks.
1) The choice of the orientation is irrelevant for everything. It corresponds to a choice of a basis in linear algebra. Technically, a vector field is a one-form and the curl of F is the exterior derivative dF of F. The curl is a function on oriented triangles.
2) **Stokes theorem** tells that if we have a surface S given by a set of triangles in the graph, which have a boundary curve C, then $\int_C F\ ds = \int \int_S \text{curl}(F)\ dS$, where the later is a finite sum of $\text{curl}(F)(t_i)$ where t_i are the triangles in S. Stokes theorem is almost a tautology for finite graphs.
3) Given a vector field F which is irrotational. Two closed curves which are equivalent in the fundamental group have the same line integral. This can be seen by noticing that each deformation step in the homotopy equivalence relation of the curves does not change the line integral.

Definition A k-valued vector field F has a **stationary point** e if $F(e)$ is never zero for the edge e.

Definition We say that a vector field F satisfies a **monodromy condition** if $\int_\gamma F\ ds = 0$ for all γ in the fundamental group of G.

Here is our result: it characterizes orientable graphs with minimal chromatic number:

A d-dimensional orientable geometric graph G has minimal chromatic number $d+1$, if and only if the following three conditions are satisfied:
a) The dual graph of G is bipartite.
b) Every irrotational field F without stationary points satisfies the monodromy condition.

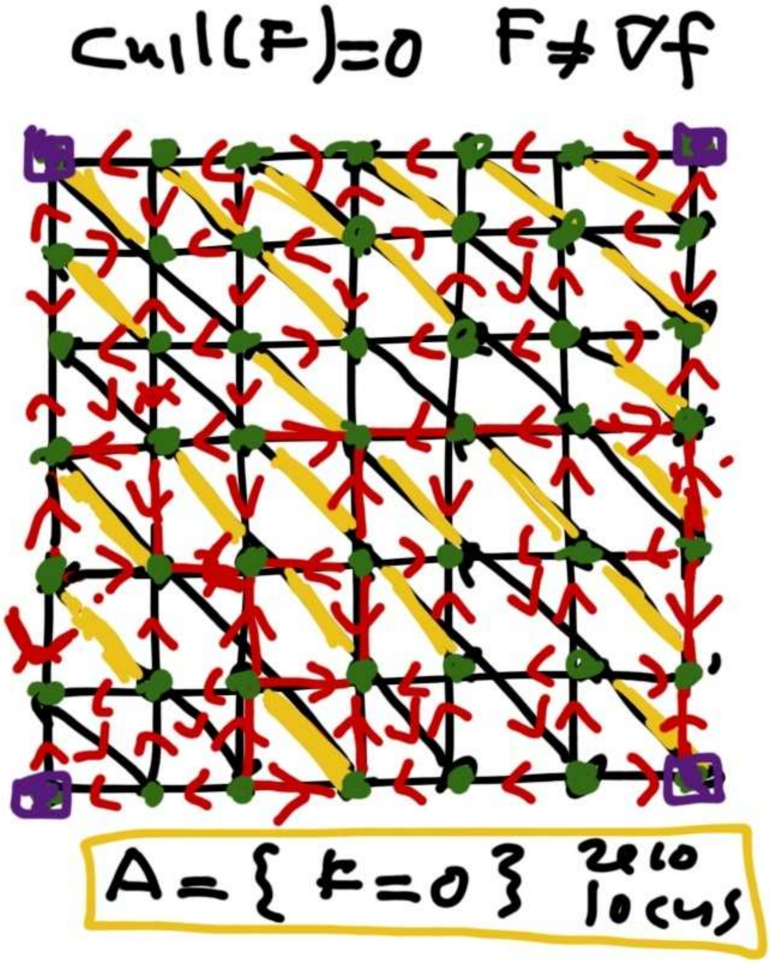
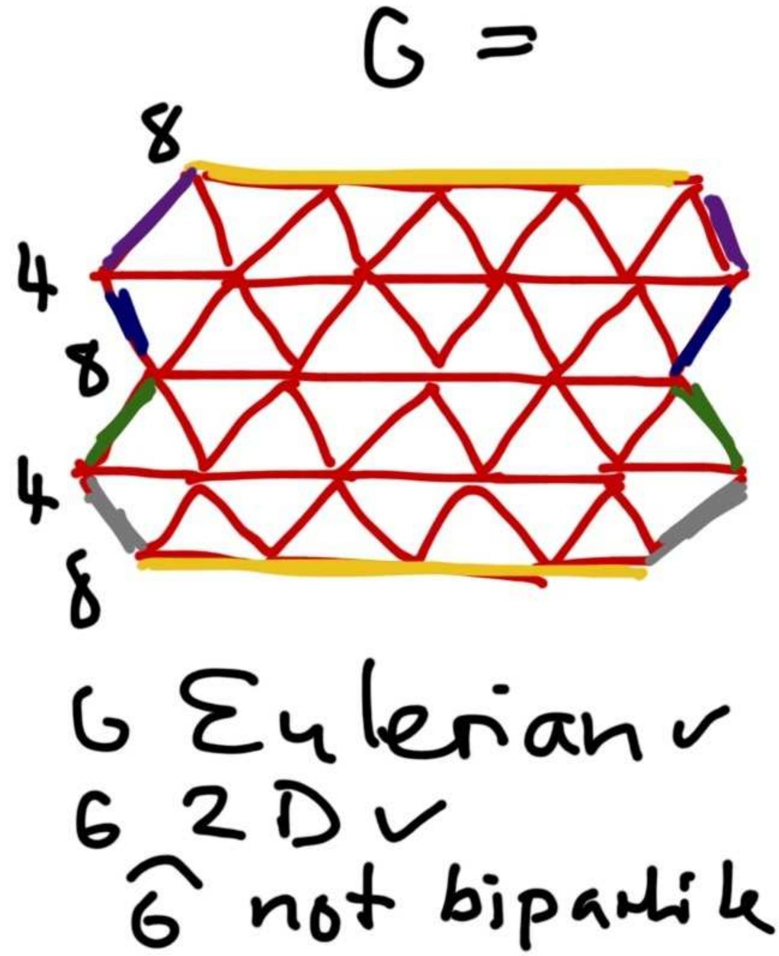


Figure 1: Jenny's example of a cohomology class on a two torus with chromatic number 3 which is nontrivial even so the graph has a minimal coloring.

[July 24: There are non-orientable graphs for which the dual graph of G is not bipartite.]
For some we had hoped that we could give a condition which is cohomological. It turned out that such conditions are too strong. The intuition from the continuum is a bit misleading: Lets look at the classical two torus in calculus. We know that all irrotational vector fields F without stationary points are constant fields $F(x, y) = \langle a, b \rangle$, which are harmonic fields by Hodge theory. And they form a two-dimensional space as the cohomology group $H^1(T^2, R)$ is two-dimensional ($b_1 = 2$). One could now think that also in the discrete, when looking at $k = Z_3$-valued vector fields, all nontrivial cohomology classes of $H^1(G, Z_3)$ have no stationary points if there is a minimal coloring. This is false (as Jenny pointed out after a half an hour "explanation" of Oliver's why every cohomology class should be without stationary points):
This forced us therefore to weaken the condition and only to assume that there are no nontrivial cohomology classes which have no stationary points. We also need orientability (which we first hoped to avoid):

Part 1) Lets see, why we have a coloring if the conditions $a), b)$ are satisfied.
The dual graph $\hat{G}$ being bipartite implies that G is Eulerian. There are examples of Jenny which are Eulerian but for which the dual graph is not bipartite. Here is a simplified version:



Having G Eulerian means that if we color triangles in the graph in an alternating way, then circling around a single vertex does not produce any incompatibility. This shows that we can locally color the triangles in an alternate way and so produce a coloring of the graph. This needs the orientation as a reference frame. We can only define a sign of a permutation if we have a reference ground orientation. If the graph were simply connected in the sense that the fundamental group were trivial, then this would already be enough to color the graph with 3 colors. Now note that a coloring f is a function on the vertices which produces a gradient field $F = \nabla f$ which has no stationary point. This is exactly what coloring means: two neighboring vertices have different colors. We also know that gradient fields are irrotational so that we can invoke condition b). This condition tells that the line integral of F along any closed curve is zero. This is exactly what we need to make the coloring global. If we go around a closed loop, then the line integral should be zero.

Part 2) Lets now turn things around and show that if there is a coloring f with $d+1$ colors then conditions a), b) hold.

Part a) is clear: we have a bipartite structure of the dual graph because the coloring defines a permutation which f defines on the vertices of every simplex and so an orientation which alternates for adjacent simplices.

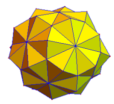
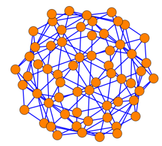
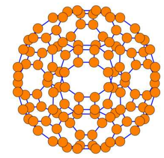
Bipartite does not imply that the graph is orientable as a Moebius strip with an even number of triangles shows. But the coloring produces not only a bipartite structure on the dual graph but also an orientation.

To see b), note that every irrotational field F with no stationary point locally defines a coloring

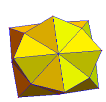
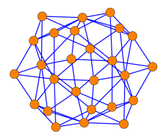
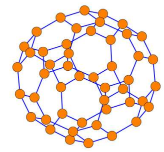
g of a tetrahedron. After some color permutation, this agrees with the coloring f on that simplex and F is up to a permutation equal to dg. Using the vector field F we can transport the coloring through the graph and have $g = \pi(f)$ globally. Therefore, the field F is a gradient field $F = dg$ and so satisfies the monodromy condition.

Here are the 4 Catalan solids which are geometric. We see the graph with colored faces, the planar graph and the dual graph. The PentakisDodecahedron has chromatic number 4, the others have minimal coloring.

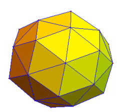
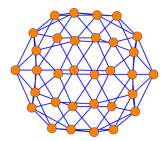
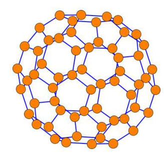
DisdyakisTriacontahedron:
$c(O) = 3$.

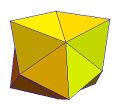
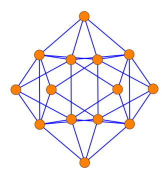
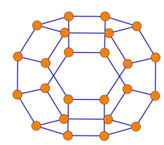
DisdyakisDodecahedron,:
$c(O) = 3$.

PentakisDodecahedron:
$c(O) = 4$.

TetrakisHexahedron:
$c(O) = 3$.

13: A projective plane with chromatic number 5

(July 17-24, 2014)

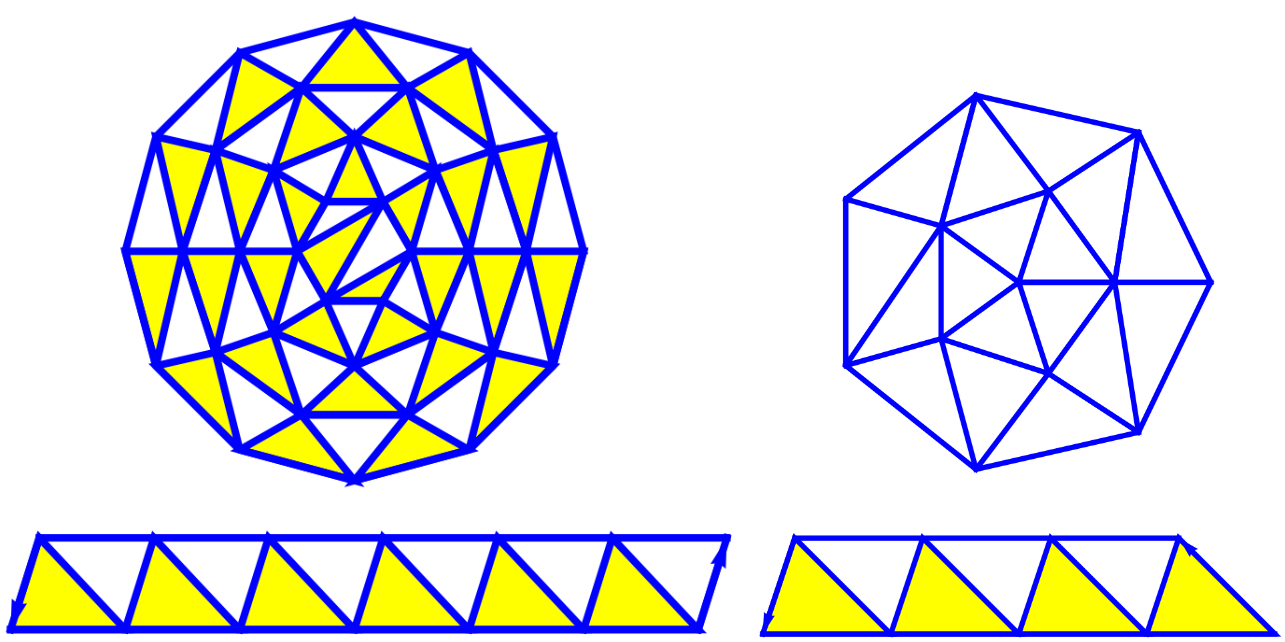
What was not clear yet was whether a projective plane can also be 4 colored or even 3 colored if done right: The answer is yes. This example shows that orientation, monodromy and bipartite conditions are independently necessary. The construction given below also shows:

a) For any $g \geq 0$, there is a discrete surface G which has genus g, and chromatic number 3. It must be orientable.
b) For any $g \geq 0$ there is a discrete surface G which has genus g and chromatic number 4.
c) For any $g \geq 0$, there is a non-orientable surface G which has genus g and chromatic number 5.
d) For any $g \geq 0$ there is a non-orientable surface G which has genus g and chromatic number 4.

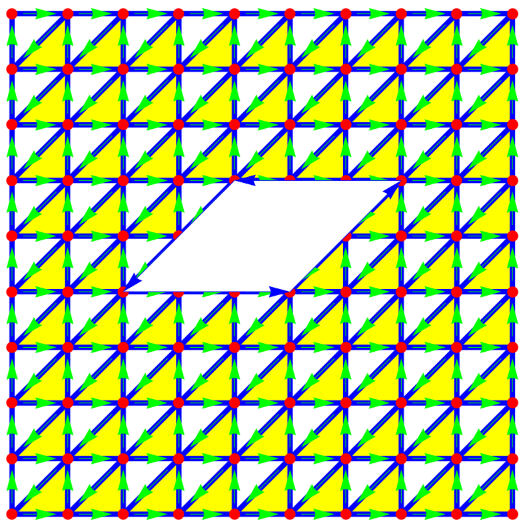
We believe these are all the cases: in particular, there should not exist any orientable surfaces with chromatic number 5 and there should not exist any surface at all with chromatic number 6 or higher.

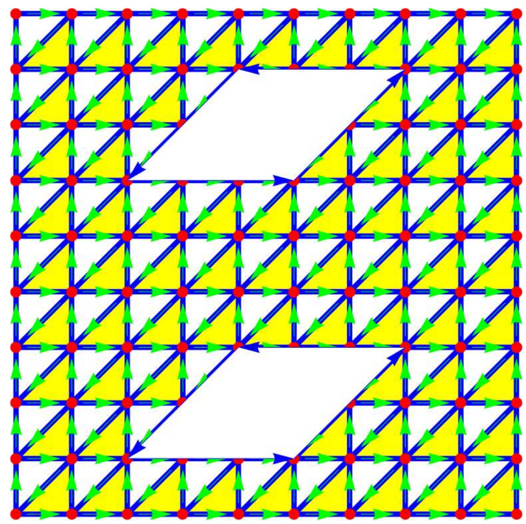
How do we get a projective plane with chromatic number 4? We must necessarily have that the embedded Moebius strip has chromatic number 4. This is only possible if the Moebius strip has a bipartite dual. We take therefore a Moebius strip for which the boundary has 12 points. It consists of 6 triangles. We now glue in an Eulerian disc.

As a comparison, we show in the following also the small projective plane which has chromatic number 4. We expect that most discrete projective planes have chromatic number 5.



What about other surfaces? Can we construct a non-orientable genus 1 surface of chromatic number $3-5$? It is no problem to construct this: we can for example dig out a hold from the surface of a 2 torus and glue in a Moebius strip. This produces a genus 1 surface which is non-orientable. We can also cut two holes into the torus and identify the boundaries. This increases the genus by 1. Depending on the geometry of the holes, we can get so examples of genus g surfaces which have genus 3 or 4 or 5.





13: The Tutte polynomial

(July 19, 2014)

The **dichromate invariant** $T(x, y)$ defined by Tutte in 1954 generalizes the chromatic polynomial. Nowadays it is called **Tutte-Whitney polynomial** or simply **Tutte polynomial**. It is a polynomial which also is defined for mathematical structures more general than graphs like **matroids**. For us, it is interesting since $xT(1-x, 0)$ is up to a sign the chromatic polynomial, and $T(2, 1)$ is the number of spanning forests (not rooted spanning forests) and $T(1, 1)$ the number of spanning trees (not rooted spanning trees). In the following, we follow Tutte, who ends his article with the following words. (I change notation and leave out a sentence, where Tutte gives a reference):
"The number C_G of spanning trees of a graph is important in the theory of electrical networks in which the conductance of each wire is unity. So so the theory of spanning trees provides a link between the theory of graph-colourings and the theory of electrical networks. The dichromate can be regarded as a generalization of C_G for we have $C_G = T_G(1, 1)$. The number C_G has a simple expression as a determinant, and its properties are well known. Perhaps some of them will suggest new properties of the dichromate and hence of the chromatic polynomials."

But lets start from the beginning and follow pretty much Tutte's 1954 article. Let $G = (V, E)$ be a finite simple graph. The edge set is equipped with a linear ordering (which Tutte calls "given enumeration of the edges".) Given a spanning tree, an edge e is **internal**, if it is part of the tree, otherwise is called **external**. The **fundamental cycle** of an external edge e is the set of edges f such that the spanning subgraph $(t \setminus \{f\}) \cup e$ is a tree. The **fundamental cycycle** of an internal edge e is the set of edges f such that the spanning subgraph $(t \setminus \{e\}) \cup \{f\}$ is a tree. If f is in the fundamental cycle of e, then e is in the fundamental cycle of f. An external edge is called **active** if it is minimal in its fundamental cycle. An internal edge is active if it is minimal in its fundamental cocycle. Denote by $i(t)$ the number of active internal edges and by $e(t)$ the number of external edges. The most natural definition of the Tutte polynomial is

$$T_G(x, y) = \sum_t x^{i(t)} y^{e(t)} ,$$

where the sum is over all spanning trees. This definition appears to depend at first on the given ordering of the edge set E at first, but Tutte has given other expressions which show that it is not. Let an edge be called a **bridge** if its removal changes the connectivity of the graph. The graph $G \setminus e$ is the graph with the edge e **deleted** and G/e is the graph with the edge e **contracted**. The first property also gives a recursive way to compute the polynomial by reducing it to two smaller graphs: and edge contracted and edge deleted version:

$T_G(x, y) = xT_{G/e}(x, y)$ if e is a bridge and $T_G(x, y) = T_{G\setminus e}(x, y) + T_{G/e}(x, y)$ else.

An other equivalent form is as follows. Assume c is the number of connected components of the graph and $|t|$ is the cardinality of a subset of E. If $t \subset E$ is given, it naturally defines a graph which has a connectivity $c(t)$.

$T_G(x, y) = \sum_t (x-1)^{c(t)-c}(y-1)^{c(t)+|t|-n}$ where $G = (E, V)$ with $|V| = n$ and the sum is over all spanning subgraphs $t \subset E$ of G.

Remark. While today the definition of the Tutte polynomial is often the later one, Tutte defined it the "natural way". Important definitions have to be simple. The sum formula over all spanning subgraphs is less natural. There are parallels with the notion of determinants for which Leibnitz's definition using the sum over all permutations is the most natural one. Textbooks started to replace it with Laplace expansions. While maybe more convenient at first (since one can avoid talking about permutations), simple properties like $\det(A) = \det(A^T)$ or multi-linearity become mysterious and the disadvantages later pop up with a revenge. The Leibniz definition is "unavoidable" as it can also be seen as a sum over all rooted spanning trees of the complete graph: by Cayley, there are n^{n-2} spanning trees, n^{n-1} rooted spanning trees and n^n paths in the graph, which possibly visit different vertices several times. Any path $x(1), ..., x(n)$ gives an expression $A_{1,x(1)}...A_{n,x(n)}$. When this is anti-symmetric, then only the $n!$ permutations $x(1)...x(n)$ survive and we get the determinant. The Kirchhoff theorem $\det(L)$="number of rooted spannning trees" for an adjacency matrix A can be seen as a special case when seen like that. The determinant of a matrix can be seen as the natural sum over the rooted spanning trees of the graph where $A_{i,j}$ is the weight attached to the edge (i, j). Also the Chebotarve-Shamis theorem telling that $\det(1+L)$ is the number of rooted spanning forests can be understood intuitively. Unlike for spanning trees, for spanning forests, we can "end a tree" and "start fresh with a new tree". It is the identity matrix 1 which allows such "fresh starts". Now, we see that the Fredholm version $\det(1 + L)$ is parallel to $\det(L)$. An advantage of $\det(1 + L) = p_L(1)$ over $\det(L) = p_L(0)$ is that the later goes over better to infinite dimensions like Fredholm determinants, which is important in statistical mechanics like when studying the Ising model or zeta functions in dynamical systems. It turns out that instead of $\det(x + L)$ which not well behaved at $x = 0$, it is better to look at

$$\det(1 + xA) = \sum_k x^k \operatorname{tr}(\Lambda^k A)$$

which is well behaved at $x = 0$ (where both sides are 1) since the later expression shows that it makes sense for trace class operators. Also, the shifted determinant $\det(1 + A)$ remains the multiplicative property $\det(1 + A)\det(1 + B)\det((1 + A)(1 + B))$ in infinite dimensions. Having mentioned zeta functions, this is an other reason, why Fredholm determinants are important. Because of the Taylor series $\log(1+x) = x-x^2/2+x^3/3-x^4/4$ .... expansion, Fredholm determinants are related to zeta functions by

$$\det(1 + xA) = \exp(\operatorname{tr}(\log(1 + xA))) = \exp(\sum_{n=1}^{\infty}(-1)^{n+1}x^n \operatorname{tr}(A^n)/n) .$$

This expression indicates already why trace class is enough as the original definition involved the traces of exterior powers $\Lambda^k A$. Before closing this rant, lets note that if L is the Laplacian of a graph $\det(1 + xL)$ is for $x = 1$ the number of spanning forests which is

$$\exp(\sum_{n=1}^{\infty}(-1)^{n+1}x^n \operatorname{tr}(L^n)/n) .$$

Since $\operatorname{tr}(L^n)$ counts the number of closed paths of length n in the graph (in a modified sense since L is the Laplacian not the adjacency matrix, so that a path also allows to loop at a point but punished by a penalty factor $-d(x)$ where $d(x)$ is the degree.), one can see it is a **path integral** and $\det(1+xA)$ can be seen as a generating function for the closed paths - or a **zeta function**. In dynamical system theory, where $f : M \to M$ is a map, there is the **Artin-Mazur zeta function** $\exp(\sum_{n=1}^{\infty} x^n|\operatorname{Fix}(f^n)|)/n$. Ruelle combined the two things, the **Fredholm determinant zeta function** and **Artin-Mazur Zeta function** to a more general object

$$\exp(\sum_{n\geq 1} x^n \sum_{p\in \operatorname{Fix}(f^n)} \operatorname{tr}(A^n(p))) ,$$

where $A^n(p) = A(f^{n-1}p) \cdots A(p)$ is the cocycle matrix product of a matrix-valued function $A : M \to M(n, R)$ over the dynamical system $f : M \to M$. This is the **Ruelle zeta function**. If M is the one point space, then this reduces to the Fredholm determinant. If A is the 1×1 matrix 1, then this is the Artin-Mazur zeta function. What is the relation with the classical Riemann zeta function? For a matrix $\log(1 + A)$ with positive eigenvalues $\lambda_1, ..., \lambda_n$, one can look at the zeta function

$$\exp(\zeta_A(s)) = \exp(\sum_k \lambda_k^{-s}) = \exp(\mathrm{tr}((1 + A)^{-s})) = \det((1 + A)^{-s}) .$$

The concept of Zeta function naturally generalizes the concept of determinant and so closed loops, rooted spanning forests, rooted spanning trees.

This prompts the question:

Are there relations between the Tutte polynomial and the zeta function?

If $A = -id/dx$ which has eigenvalues $\lambda_k = k$, then this naturally reduces to the Riemann zeta function (Since the spectrum is symmetric, it is natural only to sum over the positive energies). Anyway, also if A is the Laplacian of a graph, and $s = -k$ is an integer and $\zeta(s) = (\sum_k \lambda_k^{-s})$ is the **zeta function of the graph**, then $\exp(\zeta(s)) = (\det(1 + A))^{-s}$ is the number of spanning forests of k graphs concatendated at single vertices. In some sense, the zeta function of a graph is an analytic continuation of this "number of spanning forests". Since one can see the Riemann zeta function as a limit of the graph zeta functions of circular graphs, one can see the Riemann zeta function "counting rooted forests in concatenated manifolds" related to count "closed paths" with penalty at loops in the simplest case of the circle. And the Zeta function of a manifold M has so an interpretation as counting abstract rooted forests (at least for somebody thinking in terms of nonstandard analysis, where such notions make sense). Now the Minakshisundaraman-Pleijel zeta function for manifolds is difficult to deal with but the analogue for graphs can be studied much easier as it is an analytic function in s. I started to study it for circular graphs and showed that the roots converge to a line.]

This excursion should show how important the Tutte polynomial is: it connects different notions in mathematics: from coloring, over spanning trees and forests and counting closed path in paths, and finally to zeta functions. Back to Tutte: his paper gives also good pictures and intuition: for example if a graph has $e(G)$ loops and $i(G)$ bridges (which Tutte calls **isthmus**, the Greek expression for "neck") and no other edges, then $T_G(x, y) = x^{e(G)} y^{i(G)}$. Tutte also shows that

If two connected subgraphs H, K of G have one vertex x in common, then $T_G(x, y) = T_H(x, y) T_K(x, y)$.

For $(x, y) = (1, 1)$ this gives that the number of spanning trees (Tutte mentions that they have been called "König's Gerüst" once) satisfies this multiplicative property. And for $(x, y) = (2, 1)$ the number of spanning forests (not rooted spanning forests). Why? Because if a tree has k edges, then each is a bridge and there are 2^k possibilities to break bridges and get a forest.

We learned from an article of Welsh (1999) that:

- $T(1, 2)$ counts the number of connected subgraphs if G is connected.
- $T(2, 0)$ counts the number of acyclic orientations of G.

- if G is planar and $xy = 1$ then T becomes the Jones polynomial of the alternating link or knot associated with G, a relation found by Thistlethwaite.
- And for $(x-1)(y-1) = q$, the polynomial $T(x,y)$ becomes the partition function of the Potts model in statistical physics. And more generally, for noninteger q to the partition function of a random cluster model.

For our project relevant is

The polynomial $-(-1)^n x T_G(1-x, 0)$ is the chromatic polynomial of G.

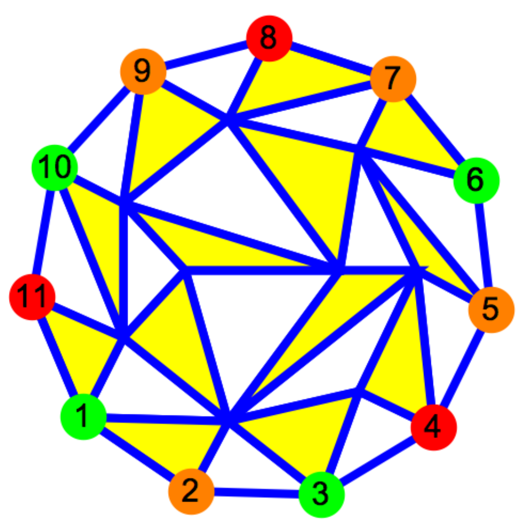
14: Nonorientable graphs of chromatic number 3

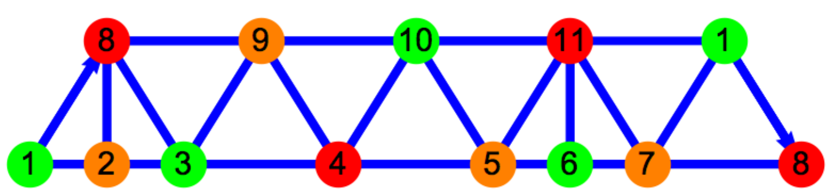
(July 25, 2014)

We have seen projective planes with chromatic number 4 and 5. In the last two weeks we tried to prove that minimal colorability is equivalent to a combination of bipartite structure of the dual graph, monodromy and orientability conditions.
The following example constructed by Jenny destroys the hope to show that orientability, monodromy condition and bipartite dual graph are sufficient too. It is a graph with chromatic number 3 but for which the dual graph is not bipartite. The failure of the bipartite condition and orientability (both considered obstacles for 3 colorability) have somehow canceled in this example.

So, what is equivalent to minimal colorability? In the orientable case, $c(G) = 3$ implies bipartite and the monodromy condition for the fundamental group. And this is also sufficient as long as we stay in the class of orientable graphs. For simply connected graphs, one knows classically (Heawood) that 3 colorability is equivalent to the dual graph being bipartite or (again only for simply connected graphs) to the Eulerian property.

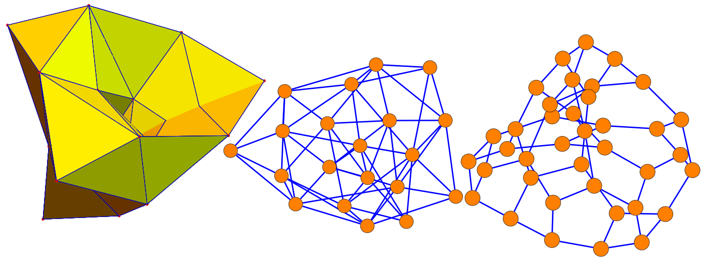
For non-orientable graphs, we still don't know what is necessary and sufficient. Certainly the orientable double cover is 3 colorable and has the properties we have for orientable ones but are these conditions for the cover enough? Its rather unlikely because there could be colorings of the cover which do not project down.





The lower part of the picture shows the Möbius strip which is glued to the boundary of the disc to get the projective plane. It is remarkable that this Moebius strip is 3 colorable! In general, we need 4 or even 5 colors. Here is the graph in 3D, drawn as a network in the plane and the dual graph.

Projective plane: $c(O) = 3$.



The coloring defines a nonzero vector field $\vec{F} = \nabla f$ which has zero curl. As in the continuum, the fundamental group if Z_2. The shortest homotopically nontrivial closed curve has length 5. We would have expected at first that for a surface which has a minimal coloring, the minimal geodesics have lengths which are a multiple of 3. The example shows that unlike in the orientable case, the coloring does not have to come from a bipartite structure.

Here is the chromatic polynomial of this graph: $f_G(x) = (x-2)(x-1)x(x^{17} - 54x^{16} + 1394x^{15} - 22876x^{14} + 267628x^{13} - 2371913x^{12} + 16511370x^{11} - 92317580x^{10} + 420098550x^9 - 1565766049\, x^8 + 4781020655x^7 - 11890082302x^6 + 23774069966x^5 - 37377918871x^4 + 44560023518x^3 - 37890133620x^2 + 20477602060x - 5285032184)$. It of course satisfies $f_G(3) = 6$ and the Euler characteristic is $\chi(G) = 1$ as it has to be. The Laplacian has the eigenvalues $9.73856, 8.65169, 8.28805, 7.74399, 7.68327, 7.65842, 7.56$ $6.94562, 6.53032, 5.76197, 4.84553, 4.6473, \ 3.43845, 3.38737, 2.53026, 2.39938, 2.04911, 0$. For the projective plane with chromatic number 5, we had the eigenvalues $8.44949, 8.37305, 8.37305, 7.43646, 7.43646$ $6.57463, 4.06847, 4.06847, 3.55051, 2.87151, 2.87151, 0$. Is there some significance that we have an integer eigenvalue?

15: A chapter in Krantz and Parks

(July 31, 2014)

Just read the new book of Krantz and Parks, "A mathematical Odyssee" which contains an entire chapter on the 4 color theorem.

Having read a couple of books which Krantz wrote coauthored, I bought this book of Krantz and Parks book "blind", not knowing at all what was inside not even bothering to look at the table of content. My spontaneous buy was triggered by the book cover, which features the Mandelbulb set, the probably most beautiful mathematical object currently known and which is constructed like the Mandelbrot set but by replacing polar coordinates with spherical coordinates in space. Having shown off this YouTube star in my own classes and lamented that despite its beauty, nothing mathematical is known yet, I was curious. Maybe a first breakthrough on establishing some topological? The book does not provide that as the bulb is only mentioned on page 85. But the book provides plenty of stimulation and shows off some modern mathematics. For me it is a great background library book. It is accessible, still not dumbed down. Countless many math books for a more general audience have been written recently. This is a valuable addition and it contains some unique gems and excellent references. I like the style that the individual chapters can be read independently. So, the book can actually be read in a nonlinear way.

(this has been a short review for Amazon. Here are more details:):

The book starts with a chapter on the four color problem. I have already purchased any possible (and impossible) book on that subject and. The chapter is only 19 pages long but lovely. It is a mostly historical account but explains a major difficult of Kempe chains well. The color figures are nice (this helped already Robin Wilson's revised edition on his book). I learned in that chapter more about the proof of Appel and Haken like that it contains quite a few errors and work since on having more reasonable. For me missing was an account on Heinrich Heesch (even so he is mentioned on page 13), but Heesch's case is quite tragic, since he had pioneered the use computer assisted proofs in that field and lost the race, in particular due to funding problems. I like that the chapter addresses also some fundamental issues with computer assisted proofs, the quest of knowledge and certainty and what is required to make a proof a proof.

The second chapter on the mathematics of finance starts with the first steps in accounting. After tracing the historically interesting earliest steps in the development of mathematical accounting language which we see engraved in Babylonian Clay tablets, it goes on with the compound interest formula, exponential growth, the invavoidable Fibonacci sequence, the development of the concepts of stocks and bonds and the description of their prizes using random walk models. It explains terms like derivatives (forward contracts or call options) and especially the Black-Scholes Option pricing, which is a partial differential equation which is in modern times often cited as the reason for recent market anomalies It is a model which becamse a self fulfilling prophecy until the discrepancies became too big. Black-Scholes tries to find the right prize for a call option under the assumption that the log prizes follow a Brownian motion stochastic process, that some random variables have normal distribution and that variance is constant. As recent events have shown volatility can be large, which is an other way to say that variance can fluctuate. Typical for the book, there is an annectote that a Stanford article honoring Scholes for his Nobel prize had messed up the name **Myron Scholes** and wrote **"Moron Schools"** instead.

The third chapter explains some Ramsey theory as pioneered by polymath Frank Ramsey who died only 27 years old in 1930. The chapter starts with the pigeonhole principle, goes to the "happy ending problem" featuring the mathematicians Esther Klein and George Szekeres who married later (hence "the happy ending", a typical Erdoes terminology). A proof of an observation of Klein is given which says that for any 5 points in the plane in general position, four of them must form a convex quadrilateral. An other Ramsey question discussed is to find for given k, the smallest number n of people to make sure that at least k are mutually acquainted or at least k are mutually not acquainted. For k=3, the answer is 6, for k=4, it is 18 but for k=5, one only knows that it is between 43 and 49. While a finite problem, but one can not just check through all the 2^{43} possible cases for $k = 6$, the answer is known to be between 102 and 165. Erdoes is quoted: "if an evil spirit would ask: tell me n(5) or I will exterminate the human race, then it would be best to get all the computing power in the world to try to solve the problem. If the spirit would ask for $n(6)$, then the best strategy for humans would be to try to kill the spirit. And if we could answer the question theoretically, we would be so clever that would not have to be afraid of the spirit." An other story told in the chapter: Erdoes once had been asked by US immigration officials for his opinion on Marx to which he replied that he is not an expert but that there is no doubt that Marx was a great philosopher, an answer which prevented Erdoes to enter the US until the 1960ies.

The fourth chapter deals with dynamical systems. It starts with the Mandelbrot set, which is still after 30 years a beautiful object generated by the quadratic map $f(z) = z^2 + c$ in the complex plane. The set M is the set of parameters c such that if we iterate f with the initial condition z=0, then the orbit stays bounded. For example, for c=0, we have f(0)=0,f(f(0))=0 etc so that c=0 is in the Mandelbrot set. But for c=1, we have f(0)=1, f(f(0))=2, f(f(f(0)))=5 etc so that 1 is not in the Mandelbrot set. For c=i, we have f(0)=i,f(f(i))=-1+i,f(f(f(i)))=-I, f(f(f(f(i))))=-1+i leading to a cycle so that c=i is in the Mandelbrot set. The chapter goes on to differential equations like the harmonic oscillator and the Lorentz system. Then explains fractals, the Cantor set, the Sierpinski triangle and Koch curve, the story of Mandelbrot who popularized the topic in the eighties. Having myself worked in the field of dynamical systems I learned not much new from that. It tells briefly the story of Poincare who submitted a paper trying to prove the stability of the solar system, in which he realized after the article was in press in that there was a crucial mistake. The revision turned out to be a breakthrough paper changing the way on how we look at the n-body problem and dynamical systems in general. There is a section also on Lorenz and the unavoidable "strange attractor".

The fifth chapter is on calculus of variations, extrema problems in infinite dimensions like the Plateau problem, surfaces with minimal area. After some historical account on Euler and Lagrange, the concept of curvature is discussed and Plateau's observation that the mean curvature is constant for minimal surfaces. The topic of minimal surface has relations with differential geometry through curvature to topology (like the study of minimal surfaces which are not of disc type) and complex analysis (Enneper-Weierstrass formula). It is also a nice topic to explain how mathematical concepts can needs expansion: geometric measure theory allows to describe objects with less regularity. In the last century mathematicians have learned to work with generalized functions like the Dirac delta function which is the "derivative" of the signum function sign(x). While the signum function does not have a derivative at $x = 0$ one can look at generalized versions of that by looking at linear functionals of smooth functions and then use the pairing to push over calculus onto the dual side. When done with differential forms, one obtains currents. The book rightly does explain these concepts in an informal way as a full account would require quite a bit of more background. I would have liked to learn more in this chapter about Jesse Douglas, who was one of the first winners of the fields medal and who solved the Plateau problem in 1931. The chapter hints at interesting stories like Garnier or Rados solutions which did not get the same recognition.

The chapter on Euclidean and non-Euclidean geometries revives a classical story. It brings in a bit of planimetry at first, then spherical and hyperbolic geometry. A funny mistakes happens on page 161, where it reads "Shing-Tung Yau (1911-2004)" but the birth dates obviously apply to Shing-Shen Chern, the advisor of Yau. This chapter is probably the weakest in the book, since the story of the creation of non Euclidean geometry has been told so many times already, especially the drama around Bolyai, Lobatchevsky and Gauss, especially the letter of Gauss telling Bolyai that he has discovered this geometry already a long time ago.

The next chapter on special relativity is a bit too informal. It could have been written with a bit more mathematical content like the picture that Lorentz transformations are just hyperbolic rotations. For some reason, the book talks only informally about linear transformations. It would have been an opportunity to introduce them and write $(x, y)- > (cosh(a)x + sinh(a)y, sinh(a)x + cosh(a)y)$ which is the hyperbolic analogue of the usual rotation $(x, y)- > (cos(a)x + sin(a)y, -sin(a)x + cos(a)y)$ Both preserve the area but while rotation leaves circles invariant, the Lorentz transformation preserves hyperbola. But the book chapter contains some interesting historical tidbits like that Poincaré has discovered special relativity before Einstein. (Not mentioned is that Poincare must have been quite upset about this as he never mentioned Einstein even later). There are many stories about why there is no Nobel prize for mathematics.

Chapter 8 explains the idea of wavelets, first telling the story about Fourier theory. It emphasis how important the subject is for applications like sound and movie compression. It tells some success stories in applications like using to enhance older recordings, to compress finger print files, for helping with computer graphics scenes, to do image compressing or DVD or blue-ray videos possible.

Chapter 9 on RSA encryption is also a story which has been told several times already. it is a good place to introduce elementary number theory like modular arithmetic, Fermat's little theorem. There is a nice section on zero knowledge proofs: how to convince somebody that one has the proof without actually revealing the proof.

Chapter 10 addresses the most important problem in computer science, the P-NP problem. The book even claims that the question is considered by some as the most important problem in the mathematical sciences overall. There is an introduction into complexity theory using the concept of Turing machines and how the computations done by these machines can be bootstrapped to formal languages. Its a story not easy to tell (I have gone as a student through an entire course about this).

Chapter 11 deals with prime numbers, primality testing using Fermats little theorem, Rabins test (which is good enough for most applications) as well as the relatively new AKS primality test which tells that if a is relatively prime to n, then $(x - a)^n$ is congruent to $x^n - a$ modulo n if and only if n is prime. This algorithm showed that test for primality is a polynomial task.

Chapter 12 deals with the concept of proof, logic. It first discusses logical statements, quantifiers and then goes on to the Goedel incompleteness theorem. Also this story has been popularized a lot already, first probably by Hofstadter in Goedel-Escher-Bach.

Chapter 13 deals with Fermats last theorem. It starts with a historical account on Diophantus, Pythagorean triples until Kummer's criterion. Then it makes an excursion to algebra, discussing fields, turns to elliptic curves, the modular group and then states the Taniyama-Shimura-Weil conjecture and the story about Wiles proof. Also this chapter has been told a lot already but its nice to have a short version.

The final chapter 14 deals with the Poincaré conjecture and its solution. There is a longer discussion of Thurston's geometrization program. The Ricci flow introduced by Richard Hamilton is described in a geometric manner. The story of Perelman which is now also covered in a couple of books already. There are some footnotes like alternative approaches, and the process of check the work of Perelman.