ON THE CHROMATIC NUMBER OF GEOMETRIC GRAPHS

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Abstract. Two dimensional geometric graphs are finite simple graphs for which all unit spheres are circular graphs with 4 or more vertices. We look at the chromatic number of geometric orientable 2-dimensional graphs and especially at examples with minimal chromatic number 3. We show that this minimal coloring occurs if and only if the dual graph is bipartite and if a monodromy condition holds for the fundamental group of the graph. For any genus g and any $c \in \{3, 4\}$ we give examples of two-dimensional orientable surfaces of genus g. For any $\chi \leq 1$ and $c \in \{3, 4, 5\}$ we prove the existence of two-dimensional non-orientable surfaces of chromatic number c and Euler characteristic χ. This is Oliver's preliminary report of the research last edited August 7, 2014.

1. Introduction

We study the chromatic number for finite simple graphs which are geometric in the sense that the unit spheres are discrete spheres. For two-dimensional graphs, the condition is that every unit sphere $S(x)$ is a circular graph of length larger than 3. We limit us here to two-dimensional graphs but the question extends in an obvious way to higher dimension. In the case of three dimensional graphs for example we deal with graphs for which unit spheres are triangulations of the classical two sphere. In two dimensions, this a different problem than the classical 4 color problem [8, 1, 6, 9, 3, 11] as only geometric graphs with Euler characteristic 2 are planar. Even in the planar case, we deal with geometric graphs which have a discrete Euclidean structure at every point. We can more generally look at "geometric graphs with boundary". Boundary points are vertices for which the unit sphere is a $d-1$ dimensional discrete disc, which means an interval graph in the surface case. The wheel graph W_n for example has a circular graph C_n as its boundary.

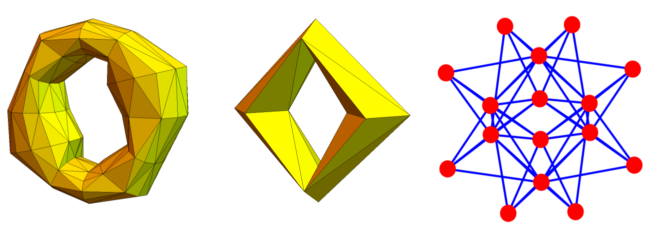

Figure 1. Two geometric graphs with the topology of a torus. For the second torus, we see the abstract graph.

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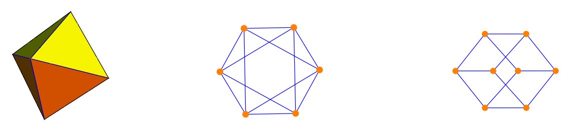
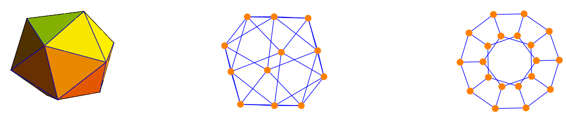
We have reasons to believe that the chromatic number of two-dimensional geometric graphs only takes values in the set $\{3, 4, 5\ \}$. For planar graphs, the chromatic number is either 3 or 4 by the 4 color theorem. We hoped that a more geometric situation allows for new approaches.
Sphere examples with minimal chromatic number $c = 3$ are the octahedron. The icosahedron has chromatic number $c(G) = 4$. We know of implementations of the projective plane with chromatic number $3, 4$ or 5.

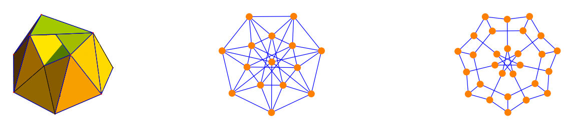
Octahedron: $c(O) = 3$.



Icosahedron: $c(I) = 4$.



Projective: $c(P) = 5$.



It is necessary to point out that the geometric coloring problem considered here differs from the usual questions in the context of topological graph theory. Traditionally, one does not care about the geometric structure of the spheres $S(x)$ which are the subgraphs generated by the vertices directly connected to x. The traditional problem looks at classes of graphs embeddable on surfaces. The most prominent example is the case of "planar graphs", graph which are embeddable on the sphere. The 4 color theorem assures that in this case the chromatic number is 3 or 4 only but the 4 color theorem does not apply for the torus anymore. Since one can place the graph K_7 onto the 2-torus, the chromatic numbers are much larger than in our geometric setup, where we believe the chromatic number is either 3 or 4 for the torus. If we look at a graph G with the topology of the torus, we look at a triangularization of the torus, where every discrete unit sphere is a circular graph. The geometric story is already present in one dimension: the chromatic number of a 1-dimensional graph is 2 or 3 if the set of singularities (vertices where the unit sphere has 3 or more vertices) are isolated. Graphs with this property are the discrete analogue of 1-dimensional varieties and an example is the figure 8 graph. If one allows singularities to be more dense, then Groetzsch has given examples of graphs without triangles, for which the chromatic number is 4. And a construction of Mycielski allows to make it arbitrarily large. Tutte and Cykov first proved independently that for every n, there exist triangle-free graphs with chromatic number n [5].

Since unit balls in the graph naturally can be filled with Euclidean space by placing the vertex with all its neighborhoods into R^d, a d-dimensional graph defines

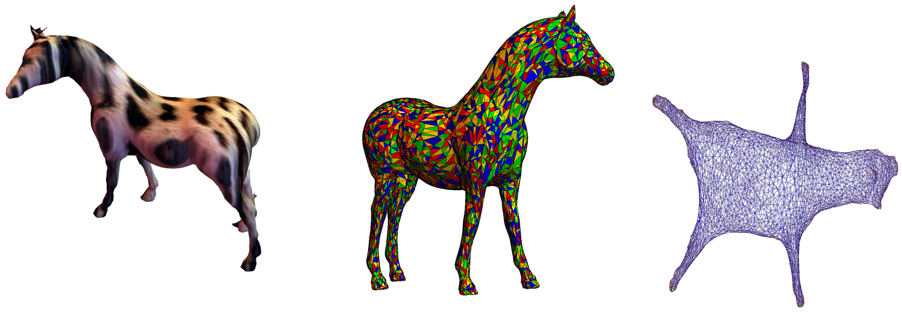


FIGURE 2. A built in geometric structure in Mathematica. It is a planar geometric graph.

an atlas for a topological d-manifold M for which G is a triangularization. Any geometric graph of dimension d defines therefore a d-dimensional topological manifold. We take the point of view that all topological notions for geometric graphs should match the topological notions of the manifold but that we do not want to refer to the manifold. Indeed, cohomology, Euler characteristic, homotopy type, fundamental group all coincide for the discrete and continuous versions. For us, the continuum is just a picture. We refer to the continuum only to motivate things. All notions are discrete.

What upper bounds are known for the chromatic number of geometric graphs? Brooks theorem gives as an upper bound for $c(G)$ the maximal degree plus one. In [2] it was observed that $c(G) \leq 2a(G)$, where $a(G)$ is the arboricity of the graph, the minimal number of forests needed to cover all edges of the graph. By the **Nash-Williams formula** [7, 4] the arboricity is the maximum of $[m_H/(n_H - 1)]$, where n_H is the number of vertices and m_H the number of edges of a subgraph H of G and where $[r]$ is the ceiling function giving the minimum of all integers larger or equal than r. For a complete graph, the arboricity is $[n/2]$. For geometric graphs of Euler characteristic χ we have $v - e + f = \chi, 3f = 2e$ so that $v = \chi + e/3$ and Nash-Williams gives a lower bound $e/(\chi + e/3 - 1)$ showing that the arboricity is at least 2. For the torus, where $\chi(G) = 0$, we have $4 > 3e/(e-3) > 3$ as geometric tori have more than 12 edges. This means that the 2 torus has arboricity 4 giving 8 as an upper bound for the chromatic number of a graph on the torus. We believe that the torus always has $c(G) = 3$ or $c(G) = 4$.

To summarize this summer project:
1) We have a necessary and sufficient condition for $c(G) = 3$ in two dimensions.
2) We can give examples of surfaces with minimal chromatic number 3 of any genus and orientation.
3) For any type of non-orientable surface there are examples with chromatic number 5.

Here are our basic questions which are left at the end of the allocated time for this project. We refer to "surface" as a **discrete two dimensional geometric graph**.

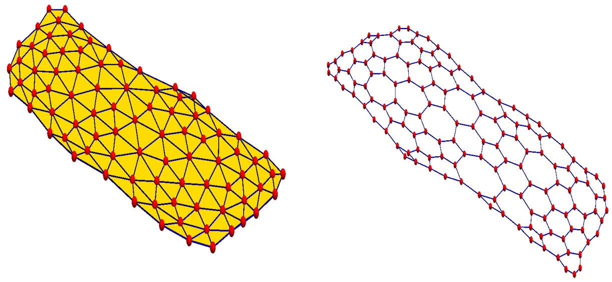


FIGURE 3. A random geometric graph G and its dual graph $\hat{G}$, which has as vertices the triangles of G.

1) Do all orientable surfaces G satisfy $c(G) \leq 4$?
2) Do all surfaces G satisfy $c(G) \leq 5$?
3) Characterize $c(G) = 3$ in the non-orientable case.
4) Characterize $c(G) = 4$ for $d = 3$ dimensional graphs.

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2. Minimal colorings

We address first the problem to characterize geometric graphs with minimal chromatic number 3. One of the conditions involves the **dual graph** $\hat{G}$ of G. This graph has as vertices the 2-dimensional simplices of G. Two such vertices connected if the corresponding triangles share a common edge. The dual graph does not have triangles as this would imply that the original graph contains a K_4 graph. We look also at Z_3-valued 1-forms, alternate functions on the edges. To describe 1-forms in coordinates, we have to fix an orientation of the edges of the graph. A 1-form F attaches a number $F((a,b)) \in Z_3$ to every edge (a,b). It is anti-symmetric in the sense $F((a,b)) = -F((b,a))$. The **curl** of F is a function on oriented triangles $dF((a,b,c)) = F(b,c) - F(a,c) + F(a,b)$, which is the line integral along the boundary of the triangle. We say F is **closed** or **irrotational** if $dF = 0$. We say F is **exact** if there exists a function f on vertices such that $F(a,b) = df((a,b)) = f(b) - f(a)$. We say F has a **stationary point** e, if $F(e) = 0$. We say that a field F satisfies the **monodromy condition** if for every γ in the fundamental group of G, the line integral $\int_\gamma F\, ds$ is zero. The fundamental group in this geometric setup is defined similarly as in the continuum: we just have to say when two closed paths on the graph are homotopic $\gamma_1 \sim \gamma_2$ if one can deform γ_1 to γ_2 by a finite set of deformation steps, where each step consists of S, S^{-1}, T, T^{-1}, where S shortens the leg $(ab)(bc)$ in a triangle with the third edge (ac) or T removes a tail $(ab)(ba)$ to nothing. If G is a triangularization of a manifold M, then two curves γ_i are homotopic with respect to this definition then they are classically homotopic.

Remark. If a d-dimensional connected graph G has chromatic number $c(G) = d+1$ and $p(x)$ is the chromatic polynomial, then $p(d+1) = (d+1)!$. The reason is that we can permute colors so that $(d+1)!$ divides $p(d+1)$ and that if a coloring is given on a simplex, then it is globally determined. The chromatic polynomial of such a graph therefore has the form $p(x) = x(x-1)(x-2)\dots(x-c+1)q(x)$ and we see $p(c) = c!q(c)$. In the minimal geometric situation, we just see that $q(d+1) = 1$. Examples: the chromatic polynomial of the octahedron is $f(x) = (x-2)(x-1)x(-32+29x-9x^2+x^3)$ and indeed $f(3) = 6$. The chromatic polynomial of the icosahedron is $f(x) = (x-3)(x-2)(x-1)x(20170-40240x+36408x^2-19698x^3+6999x^4-1670x^5+260x^6-24x^7+x^8)$ with $f(4) = 240$ but this is not a minimal coloring. For the 4x4 flat torus, we have $f(x) = (x-3)(x-2)(x-1)x(31759570-75683870x+84755347x^2-59183794x^3+28790735x^4-10305586x^5+2788491x^6-575286x^7+89829x^8-10346x^9+833x^{10}-42x^{11}+x^{12})$. Curiously, there are also $f(4) = 240$ colorings here. For the 4×5 torus, we have $f(x) = (x-3)(x-2)(x-1)x*(6919818874-21728763980x+32744308368x^2-31510965969x^3+21720697274x^4-11391220131x^5+4707341165x^6-1564566211x^7+422705253x^8-93098105x^9+16643697x^{10}-2386761x^{11}+268727x^{12}-22925x^{13}+1395x^{14}-54x^{15}+x^{16})$ and again $f(4) = 240$.

Theorem 1. *A geometric orientable 2-dimensional graph has minimal chromatic number 3 if and only if a) the dual graph $\hat{G}$ is bipartite and b) any Z_3 vector field without stationary points satisfies the monodromy condition.*

Proof. The bipartite condition together with orientability defines an irrotational field F without stationary points. The monodromy condition assures that F is a gradient $F = df$. The function f on vertices is the coloring so that $c(G) = 3$.
On the other hand, assume $c(G) = 3$ and f is the coloring, then we have a) a bipartite structure of the dual graph as the coloring defines a permutation which f defines on the vertices of every simplex. A bipartite structure defines an orientation which alternates for adjacent simplices. To show b), we note that every irrotational field F with no stationary point locally defines a coloring g of a simplex which after some permutation agrees with the coloring f on that simplex and which forces $g = \pi(f)$ globally. Therefore, the field F is a gradient field $F = dg$ and so satisfies the monodromy condition. □

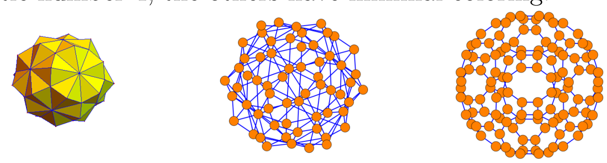
Remarks:
1) For non-orientable graphs, the bipartite structure can fail. There are 3 colorable projective planes which are not bipartite. For non-orientable graphs, a coloring f defines still a vector field ∇f which is nowhere zero and which has zero curl and for which every line integral is zero. But the function does not define an orientation.
2) Fields taking values $H \setminus \{0\}$ where H is a finite Abelian group are called **H-flows** [10] or **nowhere zero H-flows** if "Kirchhoff law is satisfied" which means $\mathrm{div}(F) = d^*F = 0$.
3) In the planar case, the two conditions a),b) collapses to the property Eulerian [9] Theorem 2.3. The result is attributed to Heawood in that case. By Euler-Hierholzer, Eulerian is equivalent that every vertex degree is even.
4) There are examples of bipartite graphs satisfying the monodromy condition which do not define a coloring. These examples are not orientable. An example is a projective plane with chromatic number 5. On the other hand there are non-orientable graphs with minimal chromatic number 3 which have not a bipartite dual.
4) For $d = 1$, the minimal chromatic number 2 happens if and only if G is bipartite.

Here G and the dual graph are equivalent. Because $2 = d+1$, the bipartite condition covers the monodromy condition.
5) For $d = 2$ and orientable graphs, we can assure $c(G) = 3$ by checking that the dual graph has closed loops of even length only. If G is simply connected, then we only have to check that G is Eulerian. In general, Eulerian is weaker. Now we have positively or negative orientations on the d-dimensional simplices which allows us to find the vector field F and so f. Actually, we can write down a possible solution f immediately: start coloring the vertices of a first d dimensional simplex with $d+1$ colors, then take a neighboring simplex and reflect the color of the vertex not on the common face to the other simplex. Continue like that until the entire graph is colored.
6) For $d = 3$, the bipartite condition means that every edge has an even number of tetrahedra hanging on. The monodromy condition for a flat discrete torus T^3 of dimension a, b, c means that all a, b, c are divisible by 4. As we will see below, we already in three dimensions also have to look at "torsion". 7) We were initially tempted to postulate $H^1(G, Z_{d+1}) = 0$ to enforce the monodromy condition. But the cohomology assumption is too strong: there are graphs with $c(G) = d+1$ even so $H^1(G, Z_{d+1})$ is nontrivial. But these cohomology classes F are fields which have stationary points.
8) Since the bipartite property is a spectral property of the adjacency matrix L of the graph, it is natural to ask whether minimal chromatic number $d+1$ can be read off from the spectrum of L together with the adjacency matrix of the dual graph.

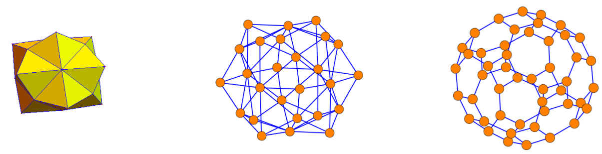
3. Examples

From the platonic solids, only the octahedron and icosahedron are geometric. Here are the 4 Catalan solids which are geometric among all 13. They are all geometric graphs of the topology of the sphere. There are no Archimedian geometric graphs We see the graph with colored faces, the planar graph and the dual graph. The PentakisDodecahedron has chromatic number 4, the others have minimal coloring.

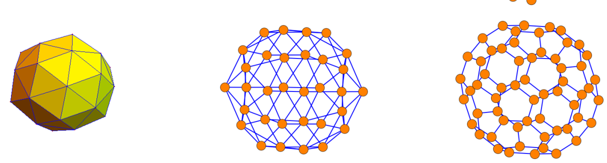
DisdyakisTriacontahedron: $c(O) = 3$.



DisdyakisDodecahedron,: $c(O) = 3$.



PentakisDodecahedron: $c(O) = 4$.



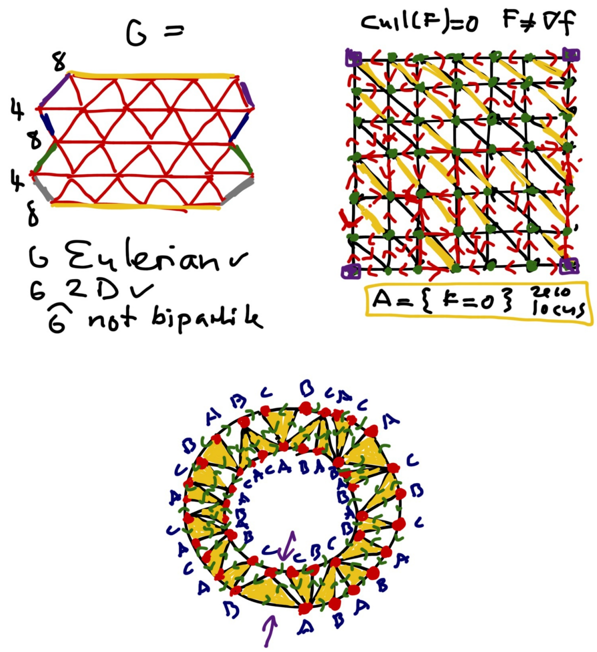
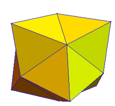
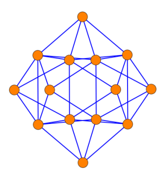
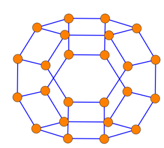
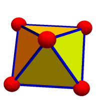
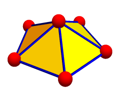
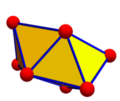
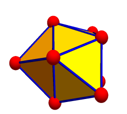
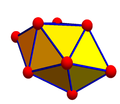
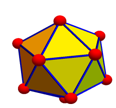


FIGURE 4. Examples: en Eulerian geometric graph for which the dual graph is not bipartite, an example of a cohomology class which is non-trivial even so the graph has a minimal coloring. And finally, an example of a coloring of a ring which is bipartite but for which the monodromy condition fails.

TetrakisHexahedron: $c(O) = 3$.   

From the 6 two dimensional geometric graphs with positive curvature, only the octahedron is 3-colorable. The others have a degree 5 vertex and so $c(G) = 4$. Evaluating the chromatic polynomials at $x = 4$ gives the values $96, 120, 72, 48, 192, 240$.

By construction, for any $g \geq 0$ and $c \in \{3, 4\}$, there is an orientable surface of genus g and chromatic number c. We have seen that for $c \in \{3, 4, 5\}$, there is a projective plane with chromatic number c. We do not know yet whether we can realize nonorientable graphs of any genus with chromatic number 5.
An other example of a non-orientable graph is the Klein bottle which is a connected two projective planes. It can also be described by giving the fundamental domain

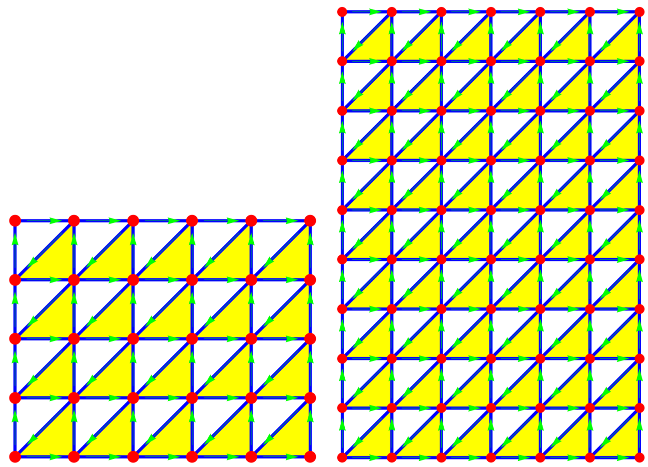


FIGURE 5. Two graphs for which the dual graph is bipartite. The first one has chromatic number 4 as it does not satisfy the monodromy condition of the theorem. The second one has chromatic number 3 and satisfies the monodromy condition.

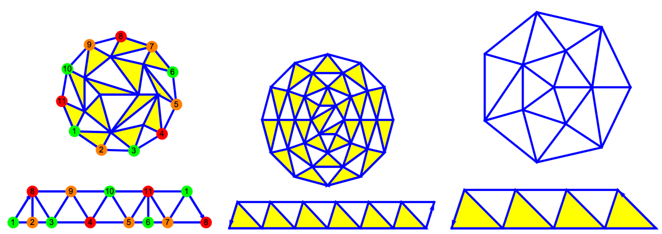


FIGURE 6. Discrete projective planes with chromatic number $3, 4, 5$.

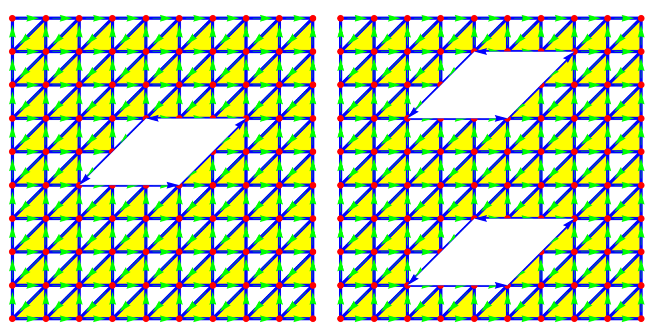


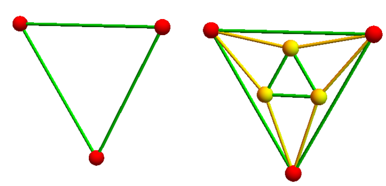
FIGURE 7. To get non-orientable surfaces, take a flat part of a surface and glue in a Moebius strip. To increase the genus, we can cut out two holes and identify the boundaries. The geometry of the holes allows to help tune the chromatic number.

which is a triangulated $n \times m$ rectangle where two boundaries are identified and the other two are anti identified. This also leads to Eulerian surfaces for which the dual graph is not bipartite. Also the monodromy condition is satisfied if n, m are both divisible by 3. We have examples with $c(G) = 3, 4$.

The classification of two-dimensional manifolds tells that any is either a sphere, a connected sum of k tori of genus g or a connected sum of k projective planes. Two surfaces A, B are connected by removing a disc in both and gluing them along the boundary. The Euler characteristic of the glued surface is $(\chi(A) - 1) + (\chi(B) - 1)$. The sphere has Euler characteristic 2, the orientable genus g surfaces have Euler characteristic $2 - 2g$. Since the projective plane has $\chi(G) = 1$, non-orientable surfaces with k glued projective planes have Euler characteristic $2 - k$.

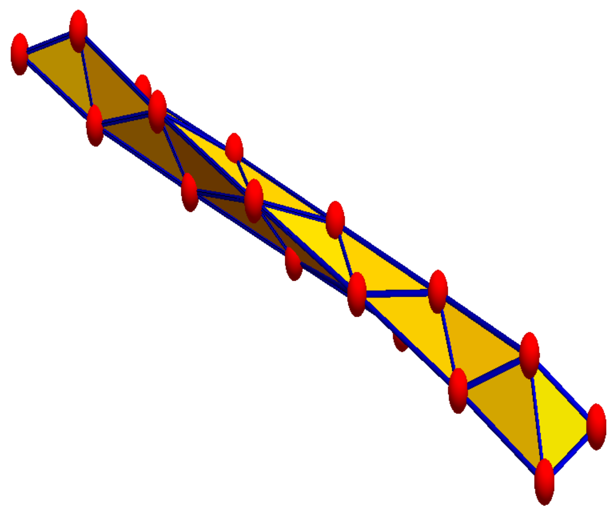
Theorem 2. *a) For $c = 3$ or $c = 4$ there are examples two dimensional orientable graphs of any topological type with chromatic number c.*
b) For $c = 3, 4$ or 5 there are examples of two dimensional nonorientable graphs of any topological type.

Proof. a) The octahedron and icosahedron covers the sphere. We have seen flat tori $T(6n, 6m)$ with $c = 3$ and others lik4 $T(5, 5)$ with $c = 4$. Taking two tori with $c = 3$ and gluing them along 3 colorable subdiscs leads to larger genus surfaces. This covers all orientable surfaces. To get chromatic number 4, we just need to take a 3 colorable example and make a diagonal flip in a flat part of the surface.
b) We have seen a projective plane of chromatic number 3. In general, given a geometric two dimensional graph with $c(3) = 3$ we can refine it and keep the same chromatic number (see figure). By taking two projective planes and makeing sufficiently fine refinements, we can get enough space to glue them along a circle,

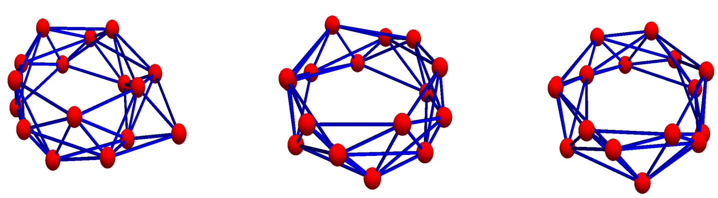


building non-orientable graphs. Again, doing a simple diagonal flip somewhere, distroying the Eulerian condition renders the chromatic number 4. To get higher genus nonorientable surfaces of chromatic number 5, note that if we make refinements of triangles, then we enlarge the graph which can not make $c(G)$ smaller as a coloring for the larger graph would also produce a coloring of the original graph. Note also that these refinements can be done within one triangle and using 3 colors only. We can get enough space so that we can glue two surfaces along circles. The gluing can not decrease the chromatic number as we glue along parts where we use already minimal coloring. $\square$

In higher dimensions, the situation is even more interesting in the non-simply connected case as we can build chains of tetrahedra and identify in different ways introducing a "torsion" which prevents the coloring. Lets note first that the simply connected case is easy: the condition of having a bipartite dual graph is necessary and sufficient. This is the analogue of the Eulerian condition in two dimensions and still works. Lest build a chain of tetrahedra:



And now we identity faces. Even in the orientable case, we can do that in 3 different ways, only one of which leads to a minimal coloring. In the nonorientable case, none leads to a minimal coloring.



But this does not mean that there are no nonorientable three dimensional graph with minimal coloring 4. They might be just harder to find.

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