SELFSIMILARITY AND GROWTH IN BIRKHOFF SUMS FOR THE GOLDEN ROTATION

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ABSTRACT. We study Birkhoff sums $S_k(\alpha) = \sum_{j=1}^{k} X_j(\alpha)$ with

$$X_j(\alpha) = g(j\alpha) = \log|2 - 2\cos(2\pi j\alpha)|$$

at the golden mean rotation number $\alpha = (\sqrt{5} - 1)/2$ with continued fraction p_n/q_n. The summation of such quantities with logarithmic singularity is motivated by critical KAM phenomena [18] and because $g + i\tilde{g} = 2\log(1 - e^{2\pi ix})$ shows that g is the harmonic conjugate to the piecewise linear case $\tilde{g}(x) = 2\pi(x - [x]) - \pi$ studied by Hecke. We relate the boundedness of log averaged Birkhoff sums $S_k/\log(k)$ and the convergence of $S_{q_n}(\alpha)$ with the existence of an experimentally established limit function $f(x) = \lim_{n \to \infty} S_{[xq_n]}(p_{n+1}/q_{n+1}) - S_{[xq_n]}(p_n/q_n)$ on $[0, 1]$ which satisfies a functional equation $f(\alpha) + \alpha^2 f = \beta$ with a monotone function β. The limit $\lim_{n \to \infty} S_{q_n}(\alpha)$ can be expressed in terms of f.

1. INTRODUCTION

Let g be a periodic function of period 1 and average $\int_0^1 g(x)\, dx = 0$. For irrational α, we consider the Birkhoff sums $S_k(\alpha) = \sum_{j=1}^{k} g(j\alpha)$. Let p_n/q_n be the sequence of continued fraction approximants to α. It is known that if g is smooth and if α is sufficiently Diophantine, then S_k stays bounded and S_{q_n} converges [11]. Denjoy-Koksma theory [5, 13, 18] which roots in work of Ostrowski, Hecke and Hardy-Littlewood in the twenties and is now part of ergodic theory [7, 21, 17] assures that if g is of bounded variation, and α is of constant type, then $S_k(\alpha) \leq C\log(k)$ for some constant C which only depends on f. If g is continuous, then the boundedness of the sequence S_k is by Gottschalk-Hedlund [15] equivalent to the existence of a function h such that $g(x) = h(x + \alpha) - h(x)$ [10, 15].

Critical KAM phenomena [18, 27] lead us to the study of Birkhoff sums for the function

$$(1.1) \qquad g(x) = \log(2 - 2\cos(2\pi x)) = 2\log|2\sin(\pi x)| = 2\log|1 - e^{2\pi ix}|$$

We assume throughout this paper that α is the golden mean $\alpha = (\sqrt{5} - 1)/2$. The dynamical system $T(x) = x + \alpha \mod 1$ with this rotation number is the best understood and simplest aperiodic dynamical system.

In this paper we will only consider the Birkhoff sums $S_k(\alpha)$ at $x = 0$. One can sum along orbits of a more general deterministic random walk $S_k(x, \alpha) = \sum_{j=1}^{k} X_j(x, \alpha)$, where $X_j(x, \alpha) = g(x + j\alpha)$ are L^1 random variables over the probability space $(T^1 = [0, 1), dx)$. In our case, the random variables X_j have zero

mean but nonzero quasi-periodic correlation. By Birkhoff's ergodic theorem for uniquely ergodic maps [7] we expect S_k/k to converge to zero for every initial point not hitting the singularity 0. While we know $S_k/k \to 0$, we can look at the behavior of log averages $S_k/\log(k)$. The choice of $\log(k)$ is due to the fact that for this particular Kronecker system and for functions g of bounded variation we have $S_k = O(\log(k))$. It is natural to ask, to which extent limit theorems like the central limit theorem apply, which are so well known when the random variables X_k are independent (see e.g [23, 26]). While classical probability theory has dealt primarily with independent or uncorrelated random variables, the theory of dynamical systems and number theory has motivated to investigate the subject also in situations with correlations.   In the case of irrational rotations, the central limit theorem and the law of iterated logarithm typically use the same scaling function $\log(k)$ if g is of bounded variation. In our particular situation, where g is given in (1.1), we found $\limsup_k S_k/\log(k) = 2$ and $\liminf_k S_k/\log(k) = 0$. By computing Fourier coefficients numerically, we measured that the distribution of $S_k/\log(k)$ to converge weakly to a stable distribution of compact support. One has to compare this with independent random variables of zero mean, where a scaling factor $\sqrt{k}$ is needed to get a limiting distribution and a rescaling by $\sqrt{2k \log\log(k)}$ to have the sequence in a bounded interval almost surely. We will assume that our initial iteration point $x_0 = 0$ of the orbit agrees with the logarithmic singularity. The positive orbit x_n never hits the singularity x_0.

Birkhoff sums of functions with singularities over irrational rotation have been studied by Hardy and Littlewood already [8]. They looked at $g(x) = \sin(x)^{-1}$ and showed that the averaged partial Birkhoff sums S_k/k stay uniformly bounded. In [29] an other non-integrable case $g(x) = (1 - e^{ix})^{-1}$ is considered and a limiting distribution for Birkhoff averages S_k/k is established. Note that our case $g(x) = 2\log|1 - e^{ix}|$ is integrable: $g \in L^1(\mathbb{R})$ and we aim for a limiting distribution of log Birkhoff averages $S_k/\log(k)$ for a particular orbit where α is fixed of constant type and $x_0 = 0$.

Since g has a logarithmic singularity at $x = 0$, the function g is not of bounded variation. When α is Diophantine of constant type, a modified Denjoy-Koksma argument shows that $|S_{q_n-1}(\alpha)|$ grows at most logarithmically so that $|S_k| \leq C\log(k)^2$ (see [18], where the idea there was to truncate the function and to use standard Denjoy-Koksma for a truncated function). The idea is that the orbit $j\alpha(\bmod 1)$, $j = 1, \ldots, q_n - 1$ avoids the singularity 0 and returns at the time q_n just close enough so that S_{q_n-1} grows logarithmically like $\log(q_n)$ while S_{q_n} converges.

A sequence of real numbers x_k has a limiting distribution μ if for every interval A $\mu(A) = (1/n)\lim_{n\to\infty} \sum_{k=1}^n 1_A(x_k) = \mu(A)$. If x_k is a sequence of random variables with the same distribution, then the limiting distribution is the law of the random variable x_k and $\int_{-\infty}^x \rho(x) = F(x)$ is the cumulative distribution function as defined in probability theory. In our case, we look at sequences x_k however which are sums and so not necessarily given by a stationary process. A sequence x_k has a limiting distribution μ if and only if the finite orbit measures $\mu_n = \frac{1}{n}\sum_{k=1}^n \delta_{x_k}$ converge weakly to μ. As the theory characteristic functions in probability theory
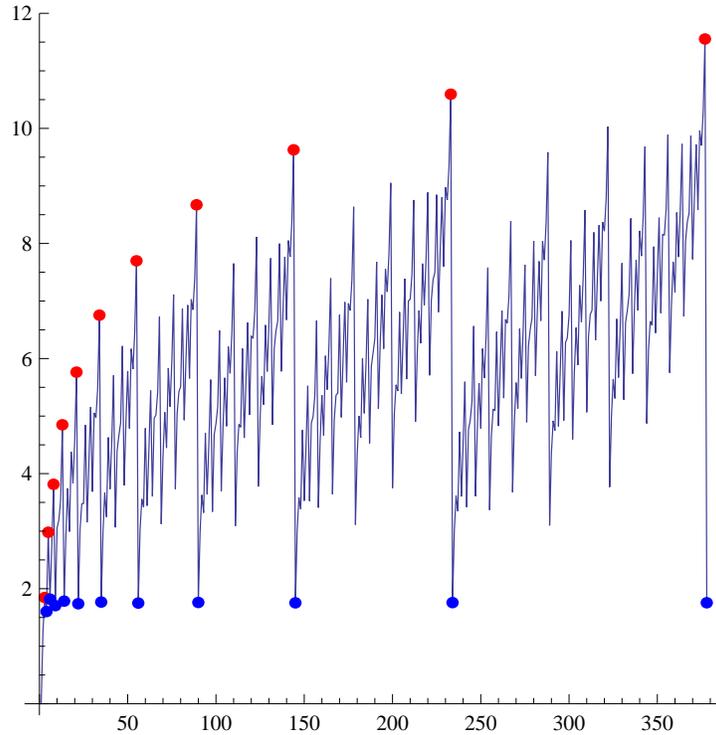
FIGURE 1. The picture shows the sequence $S_k(\alpha)$ between $k = 1$ and $k = q_{14}$.

shows, it is enough to verify that for every test function $\phi_m(x) = e^{imx}$ the sequence $\hat{\mu}_n(m) = \frac{1}{n} \sum_{k=1}^{n} \phi_m(x_k)$ converges to $\int \phi_m(x) \, d\mu(x)$. In other words, we only need to show that the Fourier coefficients $\hat{\mu}_n(m) = \int e^{imx} \, d\mu_n(x)$ of the measure μ_n converge. The limits $\hat{\mu}(m)$ are then the Fourier transform of the limiting measure μ. By Wiener's theorem [16], the distribution measure μ is continuous if and only if $\lim_{l \to \infty} \sum_{k=1}^{l} |\hat{\mu}(k)|^2 = 0$. In the current context of Birkhoff sums this is known as the Wiener-Schoenberg theorem [21]. Limiting distributions have been studied in the past. A close result to the current context is a theorem of Fraczek and Lemancyk [6] which states that for any Birkhoff sum with $f \in C^1(0,1), f'' > 0, \lim_{x \to 0^+} x f'(x) \to c, \hat{f}(n) = O(1/n)$ and irrational α, the Fourier coefficients of an accumulation point μ of limiting distributions satisfy $\mu(\hat{q}_n) \leq \eta < 1$ for large enough n which implies that the measure μ is not purely discrete. The function $g(x) = \log|2 - 2\cos(2\pi x)|$ under consideration here satisfies this logarithmic singularity condition because $g'(x) = 2\pi \cot(\pi x)$, $g'' = -2\pi^2 \cos^2(\pi x)$, $\hat{g}(n) = -1/n$.

We make here the following three experimental observations:

Experimental observation 1.1.

(1) *The sequence $S_{q_n}(\alpha)$ is* **convergent** *and has a finite* **non-zero** *limit.*
(2) *The sequence $S_k(\alpha)/\log(k)$ is* **bounded** *and takes values in the interval $[0,2]$.*
(3) *The sequence $S_k(\alpha)/\log(k)$ has a* **limiting distribution** *in $[0,2]$.*

Figure (1) illustrates the first two observations. This figure shows the sequence $S_k(\alpha)$ between $k = 1$ and $k = q_{14}$. There are two envelopes: a lower one at around $1.75686\ldots$ corresponding to the converging subsequence $S_{q_k}(\alpha), k = 1, \ldots, 14$, and an upper envelope that corresponds to the logarithmically diverging sequence $S_{q_n-1}(\alpha)$. The values $S_{q_n}(\alpha)$ for $n = 1, \ldots, 14$ illustrate the existence of a positive limit. We are able to compute this limit (see (3.3)).

Figure (2) illustrates the limiting distribution for $n = 36$. It appears to have similar features to Bernoulli convolution [12] and is close to symmetric. The right figure shows the Fourier coefficients $\hat{\mu}_{36}(m), m = 1, ..., 500$. It is known [6] that all accumulation points of Fourier coefficients are in absolute value smaller than 1. The numerical experiments suggest that μ has no discrete component and that the Hausdorff dimension of μ is slightly smaller than 1 [25].

These observations go beyond [18], where it was shown that $S_{q_n-1}(\alpha)/\log(q_n)$ is bounded leading to $|S_k(\alpha)| \leq C(\log(k))^2$ for all k. We suggest a renormalization explanation for the first two observations in Section 2.

Birkhoff sums over an irrational rotation are estimated as follows if the function has bounded variation. If p/q is the periodic continued fraction approximation of α, then $|S_q| \leq C$ for all q implies $|S_k| \leq C_1 k^{1-1/r} \log(k)$ with a k independent constant C, if α is of Diophantine type r (see i.e.[13, 5]). In the golden mean case, where $r = 1$, this gives $|S_k| \leq K \log(k)$ for some other constant K.

The convergence 1) implies the boundedness in 2) as indicated in the proof of Denjoy-Koksma. This step of the proof of Denjoy-Koksma's theorem does not need that g is of bounded variation.

Birkhoff sums have been studied in the context of number theory, probability theory and dynamical system theory. Birkhoff's ergodic theorem $S_k/k \to \int_0^1 g(x)\,dx$ relates Birkhoff averages S_k/k with the average of g. If g is continuous, this convergence is independent of the initial point if the dynamical system is an irrational rotation [7]. If the mean $\int g(x)\,dx = 0$, one can ask how fast S_k grows. In the case of rigid rotation, the answer depends on the regularity of g and arithmetic properties of the rotation number. In the case of an irrational rotation we have $S_k = O(1)$ for Diophantine α and smooth g. For functions g of bounded variation, Denjoy-Koksma theory [5] allows estimates $S_k = O(k^r)$ for $0 \leq r < 1$ and of the form $S_k = O(\log(k))$ for α of constant type. There are also lower bounds on the growth rate: Herman [10], using it as a tool for higher dimensional considerations, gave example of Birkhoff sums, where $\limsup S_k \geq \sqrt{k}$. Bièvre and Forni have for any $\epsilon > 0$ examples which achieve $\limsup S_k \geq k^{1-\epsilon}$ [2]. We look here at an L^1-integrable case of unbounded variation. Previous results are by Hecke $g(x) = \lfloor x \rfloor = x - [x]$ [9], by Hardy-Littlewood for $g(x) = \sin(x)^{-1}$ [8], by Bryant-Reznick-Serbinowska
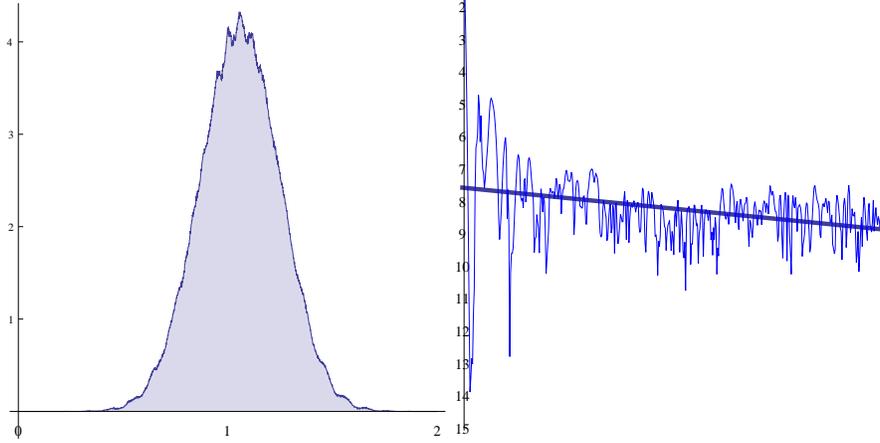
FIGURE 2. The limiting distribution μ of the sequence $S_k(\alpha)/\log(k)$, $k = 1, \ldots, q_{36}$ over the interval $[0, 2]$. The Fourier coefficients $\{\log |\hat{\mu}_{36}(m)|\}_{m=1}^{500}$ of the measure μ_{36} are shown to the right together with a linear fit.

for $g(x) = (-1)^{\lfloor x \rfloor}$ [14], by Sinai-Ulcigrai for $g(x) = 1/(1 - \exp(ix))$ [29] or for functions with logarithmic singularity by Fraczek and Lemancyk [6].

The Hecke case [9] with $\tilde{g}(x) = 2\pi(x - [x]) - \pi$ is particularly relevant to our research. In Hecke's example, which is the conjugate case to our situation because $\tilde{g}$ is the conjugate function with $g + i\tilde{g} = 2\log(1 - e^{2\pi ix})$. The complex case will be discussed at the end. In the Hecke case, $\tilde{h}_n(x) = S_{[xq_n]}(\alpha) - S_{[xq_n]}(p_n/q_n)$ has the property that $\tilde{h}_{2n}(x)$ converges pointwise to the function $\tilde{h}(x) = -cx^2$ with $c = \pi/\sqrt{5}$ and $\tilde{h}_{2n+1}(x)$ converges to cx^2: we use $\alpha - p/q = \pm 1/(\sqrt{5}q^2) + O(1/q^4)$ and the fact that $k\alpha, kp/q$ is always on the same side of the discontinuity for $1 \leq k \leq q - 1$, to get with $q = q_n$

$$\tilde{h}_n(x) = \sum_{k=1}^{[xq]} [\tilde{g}(k\alpha) - \tilde{g}(k\frac{p}{q})] = (-1)^{n+1} \sum_{k=1}^{[xq]} 2\pi \frac{k}{\sqrt{5}q^2} \rightarrow (-1)^{n+1}\pi\frac{x^2}{\sqrt{5}} .$$

An other motivation connects the present study with the field of holomorphic dynamics: consider the nonlinear complex dynamical system $T(z, w) = (cz, w(1 - z))$ in C^2, where $c = \exp(2\pi i\alpha)$ and α is the golden mean. This is one of the simplest quadratic systems which can be written down in $\mathbb{C}^2$. How does the orbit behave on the invariant cylinder $\{|z| = 1\} \times \mathbb{C}$ starting at $(c, 1)$? We have

$$T^n(z, w) = (z_n, w_n) = (c^n z, w(1 - z)(1 - cz)(1 - c^2z) \ldots (1 - c^{n-1}z))$$

and

$$\log|w_n| = \sum_{k=1}^{n} \log|1 - e^{2\pi ik\alpha}| = \frac{S_n}{2}$$

for $(w_0, z_0) = (c, 1)$ because $2\log|1 - e^{ix}| = \log(2 - 2\cos(x))$. The study of the global behavior of the holomorphic map T in $\mathbb{C}^2$ boils down to the Birkhoff sum over the golden circle on a subset because for $r = |z| < 1$, where $g_r(x) = \log|1 - re^{ix}|$

is real analytic and the Birkhoff sum converges by Gottschalk-Hedlund. It follows
that for $r < 1$ the orbits have the graph of a function $A : \{ |z| = r \} \to \mathbb{C}$ as an
attractor. For $r = |z| > 1$, we have $|w_n| \to \infty$. So, all the nontrivial dynamics of
the quadratic map happens on the subset $\{ |z| = 1 \} \times \mathbb{C}$.

We thank the referee for pointing out relations to the ergodic theory of area-
preserving flows on surfaces or interval exchange transformations where Birkhoff
sums with logarithmic singularities matter [3, 4, 20, 22, 28] and especially [6].

2. A RENORMALIZATION PICTURE

In order to shed light on the main observation, we consider only summation in-
tervals of the form $[1, q_n]$ and rescale these to the unit interval $[0, 1]$. More precisely,
let p_n/q_n be the continued fraction of α and let $[t]$ be the largest integer smaller
than t. We consider the piecewise constant function $S_{[xq_n]}(\alpha)$ which has disconti-
nuities at points $x = \dfrac{k}{q_n}$ for which $[xq_n]$ is an integer. Rescaling the sequence this
way, we can talk about functions on the unit interval $[0, 1]$ and can treat all the
sequences in the same space of functions.

By looking at the convergence of Fourier coefficients seen in Figure (2)), we
observed a weak limit $s(x)$ of $s_n(x) = S_{[xq_n]}(\alpha)/\log([xq_n])$ in the sense that

$$\int s_n(x)\phi(x)\,dx \to \int s(x)\phi(x)\,dx$$

for every continuous function ϕ. In order to study the limit better and see pointwise
convergence we look at the sequences

$$S_k(\frac{p_{n+1}}{q_{n+1}}) - S_k(\frac{p_n}{q_n})$$

with $k \in \{1, \ldots, q_n\}$. In the rescaled picture, we consider the functions

$$f_n(x) = S_{[xq_n]}(\frac{p_{n+1}}{q_{n+1}}) - S_{[xq_n]}(\frac{p_n}{q_n})$$

on the interval $[0, 1]$. These functions f_n appear to converge (see Figure (3)).

While some self similarity is present for the original functions $s_n(x)$, where they
only manifest weakly, the new functions f_n have more regularity, appear to con-
verge pointwise almost everywhere and appear to have a self-similarity property:

Unproven hypothesis 2.1.
*The functions f_n converge pointwise almost everywhere to a function f satis-
fying $|f(x)| \le x$ and*

$$(2.1) \qquad\qquad f(\alpha x) + \alpha^2 f(x) = \beta(x) ,$$

where $\beta(x)$ a monotone increasing function which is zero for $x = 0$.

The function β appears to be the anti derivative of a positive measure μ so
that $\beta(x) = \int_0^x d\mu(x)$. Discontinuities occur along the forward orbit $x = n\alpha$,
$n = 1, 2, \ldots$. We observe that the graph $\beta(x)$ looks like a "devil stair case" and is
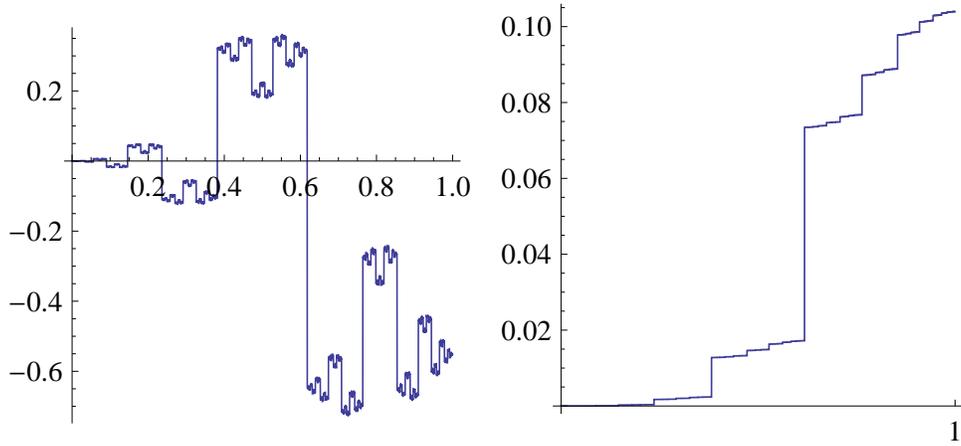close to self similar: the graph of β on $[0, \alpha]$ matches the graph of β on $[0, 1]$ closely:

FIGURE 3.   The "skyline" function $f_{24}(x)$ and the "stair" function $\beta_{24}(x)$. The function β appears monotone with $\beta(1-) = 0.104\ldots$.
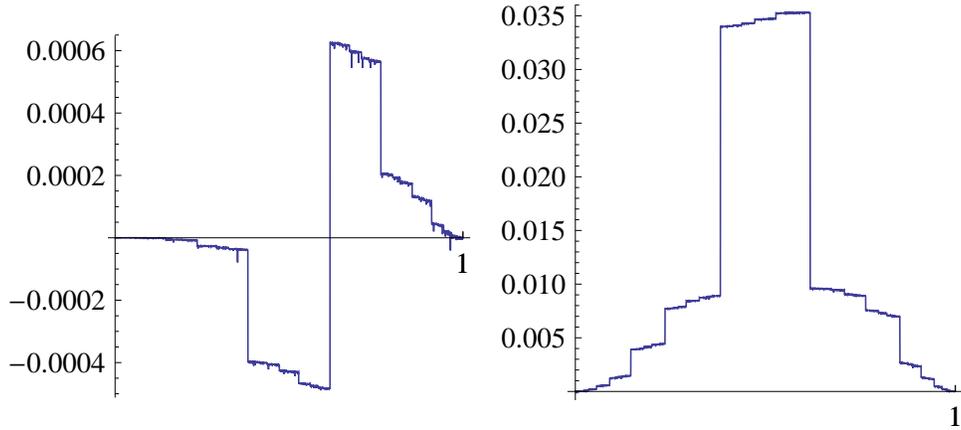

FIGURE 4.   These graphs show a comparison of the entire graph with the lower and upper part of β. To the left we see the function $\beta - \beta_1$, to the right, we see the function $\beta - \beta_2$.

define $a = \beta(\alpha-), b = \beta(\alpha+), c = \beta(1-)$ and the functions $\beta_1(x) = \beta(\alpha x)(c/a)$ and $\beta_2(x) = (\beta(\alpha + (1 - \alpha)x) - b)c/(c - b)$. We observe that they are both close to β. The comparison of the affine scaled graphs β_1 and β_2 to β are seen in Figure (4).

The identity:

$$S_k(\alpha) - S_k(\frac{p_N}{q_N}) = \sum_{n=N}^{\infty} S_k(\frac{p_{n+1}}{q_{n+1}}) - S_k(\frac{p_n}{q_n})$$

suggests the introduction of a sequence of functions:

$$(2.2) \qquad h_n(x) = S_{[xq_n]}(\alpha) - S_{[xq_n]}(\frac{p_n}{q_n})$$

$$(2.3) \qquad f_{n,m}(x) = S_{[xq_n]}(\frac{p_{n+m}}{q_{n+m}}) - S_{[xq_n]}(\frac{p_n}{q_n})$$

By definition, $f_n = f_{n,1}$ and $h_n = f_{n,\infty}$, where we understand $p_\infty / q_\infty = \alpha$.

3. Consequences of the hypothesis

The following statements do follow from the yet unproven hypothesis.

1. The functions h_n converge to a function $h(x)$ determined by f.

Proof.
(i) For every $j > 0$ and a set Y_n of measure $|Y_n|$ converging to 1, we have
$f_{n,j+1}(x) - f_{n,j}(x) = f_{n+j}(\alpha^j x)$. Proof: Take the difference of

$$
\begin{aligned}
f_{n,j+1}(x) &= S_{[xq_n]}(\frac{p_{n+j+1}}{q_{n+j+1}}) - S_{[xq_n]}(\frac{p_n}{q_n}) \\
f_{n,j}(x) &= S_{[xq_n]}(\frac{p_{n+j}}{q_{n+j}}) - S_{[xq_n]}(\frac{p_n}{q_n}) \ .
\end{aligned}
$$

to get

$$ f_{n,j+1}(x) - f_{n,j}(x) = S_{[xq_n]}(\frac{p_{n+j+1}}{q_{n+j+1}}) - S_{[xq_n]}(\frac{p_{n+j}}{q_{n+j}}) \ . $$

Now

$$ f_{n+j}(\alpha^j x) = S_{[x\alpha^j q_{n+j}]}(\frac{p_{n+1}}{q_{n+1}}) - S_{[xq_n]}(\frac{p_n}{q_n}) \ \sim \ S_{[xq_n]}(\frac{p_{n+1}}{q_{n+1}}) - S_{[xq_n]}(\frac{p_n}{q_n}) \ . $$

For fixed j, the sets $Y_n = \{x \mid [xq_n] = [x\alpha^j q_{n+j}] \}$ have measure converging to 1 because $|(q_n - \alpha^j q_{n+j})q_n|$ converges as $n \to \infty$.
(ii) Using $f_n = f_{n,1}$, one obtains that

$$ f_{n,m}(x) - f_n(x) = \sum_{j=0}^{m-1} f_n(\alpha^j x) $$

and $h_n = f_{n,\infty}(x) - f_n(x) = \sum_{j=0}^{\infty} f_{n-1}(\alpha^j x)$. Since by the hypothesis, f_n converges pointwise almost everywhere to a function f, we know that $|f_n(x)| \le |x|$ for large enough n. For fixed n, the sup-norm of $f_n(\alpha^j)$ decays at an exponential rate α^j and h_n is bounded and has a limit

$$(3.1) \qquad h(x) = \sum_{j=0}^{\infty} f(\alpha^j x) \ . $$

The graph of the function h looks similar to the graph of f.

While f was the difference between Birkhoff sums with periodic rotation numbers, the function h shows the difference between Birkhoff sums of irrational and rational rotation numbers in the limit. The function

$$\gamma(x) = h(\alpha x) + \alpha^2 h(x) = \sum_{j=1}^{\infty} \beta(\alpha^j x)$$

is a monotone function too and looks like β.

2. Derivation of the experimental observation: the limit along subsequences are known.

As noted in[18], we have

$$(3.2) \qquad S_{q_n-1}(\frac{p_n}{q_n}) = 2\log(q_n) \ .$$

The cyclotomic formula $\prod_{j=1}^{q-1}(e^{2\pi i j/q} - z) = \dfrac{z^q - 1}{z - 1}$ holds for all positive integers q and complex $z \neq 1$. Applying l'Hospitals rule to the limit $z \to 1$ gives $\prod_{j=1}^{q-1}(e^{2\pi i j/q} - 1) = q$. Equation (3.2) is the logarithm of this identity.

By definition, $h_n(1^-) = S_{q_n-1}(\alpha) - S_{q_n-1}(p_n/q_n)$. Since

$$S_{q_n}(\alpha) - S_{q_n-1}(\alpha) + 2\log(q_n) \to \log(\frac{4\pi^2}{5}) = 2.06632\ldots$$

for $n \to \infty$ (see (3.4) below), we get from equation(3.2)

$$(3.3) \qquad S_{q_n}(\alpha) \to \log(\frac{4\pi^2}{5}) + h(1^-) = 1.75687\ldots$$

where $h(1^-) = \lim_{x \nearrow 1} h(x) = -0.30945\ldots$. We see that the limiting value of $S_{q_n}(\alpha)$ can be read off from the function h which is related to f by (3.1).
(Due to the selfsimilar symmetry, we also know the limit along other subsequences like $S_{q_n+q_{n-1}}(\alpha) \to h(\alpha^-) = 0.24208\ldots$.)

3. Derivation of the experimental observation: The sequence $S_k(\alpha)/\log(k)$ is bounded.

Proof. This part follows from the second part of the proof of Denjoy-Koksma in the cases when α is Diophantine of bounded type. From $S_{q_n}(\alpha) \leq M$ follows $S_k(\alpha) \leq M_2 \log(k)$ for all $k > 0$ with an other constant M_2 independent of k.

4. Derivation of the experimental observation: $S_k(\alpha)/\log(k)$ have accumulation points in $[0,2]$.

The statement

$$\lim_{n \to \infty} S_{q_n}(\alpha)/\log(q_n) = 0$$

follows immediately from the convergence of $S_{q_n}(\alpha)$. The statement

$$\lim_{n \to \infty} S_{q_n-1}(\alpha)/\log(q_n) = 2$$

follows from

$$S_{q_n}(\alpha) - S_{q_n-1}(\alpha) = X_{q_n}(\alpha) = \log|2 - 2\cos(2\pi q_n \alpha)|$$
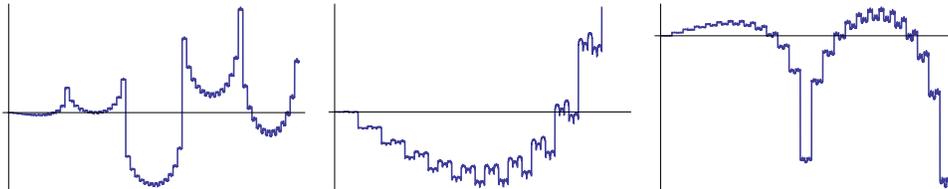
FIGURE 5. For $\alpha = \sqrt{41} - 6$ with continued fraction expansion $[2, 2, 12, 2, 2, 12, \ldots]$ of period $L = 3$, each of the three function sequences $\{f_{3n}\}_{n \in \mathbb{N}}, \{f_{3n+1}\}_{n \in \mathbb{N}}, \{f_{3n+2}\}_{n \in \mathbb{N}}$ seem to converge.

and

$$\lim_{n \to \infty} q_n^2(2 - 2\cos(2\pi q_n \alpha)) = \lim_{n \to \infty} q_n^2(2 - 2\cos(2\pi q_n \alpha - p_n)) = \frac{4\pi^2}{5}$$

which implies with equation(3.2)

$$(3.4) \qquad\qquad X_{q_n}(\alpha) + 2\log(q_n) \to \log(\frac{4\pi^2}{5}) \ .$$

We especially see that the normalized sequence $\{S_k / \log(q_n)\}_{k=1}^{q_n}$ jumps by $-2 + O(1)/\log(q_n)$ when $k = q_n$.

4. Conclusions and Open Questions

The particular Birkhoff sum for the golden mean rotation shows interesting patterns. It is a situation where Denjoy-Koksma falls short of explaining the experimentally measured growth rate. The nature of the limiting distribution ρ of $S_k(\alpha)/\log(k)$ is not yet settled. Taking test functions $\phi(x)_m = e^{imx}$ gives the Fourier coefficients

$$\hat{\mu}(m) = \lim_{n \to \infty} \frac{1}{q_n} \sum_{k=1}^{q_n} \phi_m(\frac{S_k(\alpha)}{\log(q_n)})$$

of this measure. Understanding the still unproven hypothesis could answer the question whether the limiting distribution exists and if yes, how these coefficients decay. In this paper, we have traced the convergence and boundedness questions to the existence of limiting functions given in hypothesis (2.1). Mathematica source code to all the figures can be found in [19].

1. **Other rotation numbers**. For $(\sqrt{2} - 1)/2 = [4, 1, 4, 1, 4, 1, \ldots]$ for example, where the continued fraction expansion has period 2, there appears to be a function f such that f_{2n} converges to f while f_{2n+1} converge to $-f$. For $\sqrt{41} - 6 = [2, 2, 12, 2, 2, 12, \ldots]$, where the continued fraction expansion has period 3, the sequences $f_{3n}, f_{3n+1}, f_{3n+2}$ seem to converge (see Figure 5). That picture shows these three graphs in the case $n = 3$. If α has a continued fraction expansion of period L, we see that f_{Ln} and $S_{q_{Ln}}$ do converge. Similar statements as in hypothesis (2.1) seem to hold: for $L = 2$ with examples like $\alpha = (\sqrt{3}-1)/2 = [2, 1, 2, 1, \ldots]$, we see that f_{2n} converges to a function f and f_{2n+1} converges to a function $\tilde{f}$ and that $\beta(x) = f(\alpha x) + 2\alpha^2 \tilde{f}(x)$ is monotone. For $\alpha = [p, q, p, q, \ldots]$ we observe a monotone function $\beta(x) = qf(\alpha x) + p\alpha^2 \tilde{f}(x)$. This suggests that for quadratic

irrationals, there is presumably a similar story depending on the structure of the continued fraction expansion. The golden mean $(\sqrt{5} - 1)/2 = [1, 1, 1, 1, \ldots]$ is the simplest. The silver ratio $\sqrt{2} - 1 = [2, 2, 2, \ldots]$ is similar to the golden ratio and shows a limit for f_n. For other Diophantine rotation numbers that satisfy a Diophantine or Brjuno condition, the growth is expected to match the estimates given by Denjoy-Koksma in the case of bounded variation.

2. **Different starting point**. Starting at a different point x_0 changes the story. This is not surprising since it is no more the q_n which lead to the close encounters with the logarithmic singularity but by a theorem of Chebychev [17], there are integers $k < q_n, l$ with $|x_0 + k\alpha - l| < 3/q_n$. For the self-similarity structure we have seen here, it looks as if it is important to have the initial point x_0 at the logarithmic singularity.

3. **Other functions**. If $g(x) = \log(2 - 2\cos(x))$ is replaced with an other function $\log(r(x))$ with one logarithmic singularity at 0, the story is just distorted: Assume $r(x)$ is a trigonometric polynomial with a single root at 0 having the property that $r''(0) = 2$. Then $g(x) = \log(r(x))$ shows similar growth rates. The reason is that $g(r(x)) - \log(2 - 2\cos(x)) = \log(r(x)/(2 - 2\cos(x)))$ is now smooth and has bounded variation. Adding a function of bounded variation to g does not change the behavior because of Denjoy-Koksma theory. It is important however that we start the orbit at the critical point $x_0 = 0$ so that we do not hit it in the future.

4. **Selfsimilarity in Hecke's example**. Birkhoff sums for bounded g have been studied first by number theorists like Hardy,Littlewood or Hecke. In the case of a golden rotation, there is a similar structure also but the story is different in that the limiting functions f and h are smooth. Actually, the first and second statements hold for this function. Figure (4) shows the Birkhoff sum in the situation studied by Hecke [9] where $\tilde{g}(x) = 2\pi(x - [x]) - \pi$ is piecewise smooth. It is historically the first example studied for irrational rotation numbers. The Hecke function is the harmonic conjugate to the function g we study here. By looking at the complex case, we can treat both situations together. As mentioned in the introduction, the functions $f = \lim_n f_{2n}$ and $h = \lim_n h_{2n}$ are explicitly known quadratic functions for the Hecke case if the rotation number α is the golden mean.

5. **Replacing the irrational rotation by a smooth circle map**. A similar deformation happens if one replaces the irrational golden rotation with a smooth circle map T with the same rotation as long the conjugated orbit starts at the critical point of g. The reason is that any smooth interval map T with Diophantine rotation number is conjugated to an irrational rotation: $T^n(x) = S^{-1}(S(x) + n\alpha)$ and so $g(T^n x) = g(S^{-1}(S(x) + n\alpha))$. Therefore, if the starting point is $x = S^{-1}(0)$, then $g(T^n x) = h(n\alpha)$ with $h(x) = g(S^{-1}(x))$ so that h and g have the same logarithmic singularity and $g(x) = h(S(x))$. We see that changing to a smooth circle map has the same effect as replacing g with an other function with the same logarithmic singularity and changing the initial point. If the initial point is the same, we see the same behavior but distorted by a Denjoy-Koksma correction for a continuous function g.
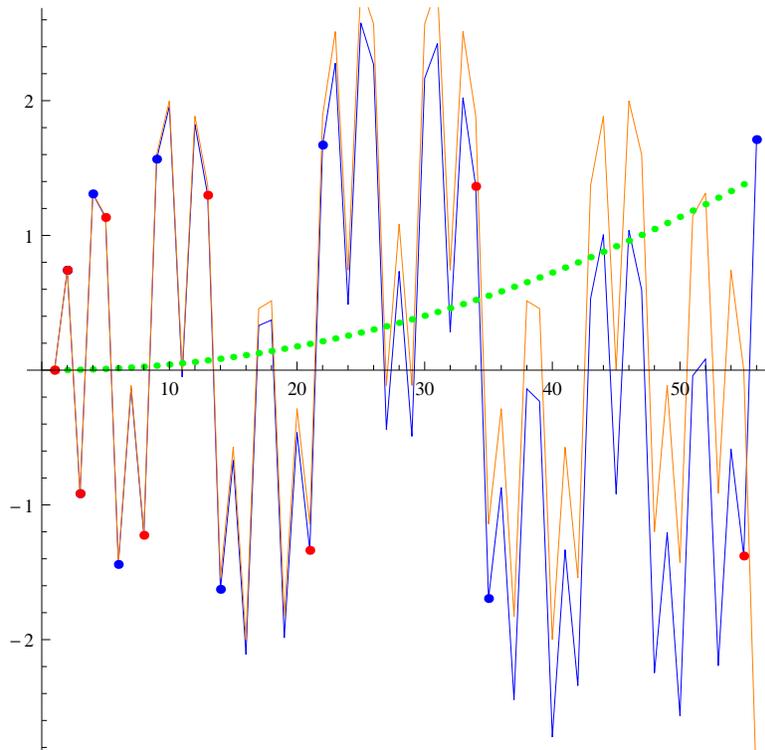
FIGURE 6. We see the Birkhoff sum $\{S_k(\alpha),\ 1 \leq k \leq q_{12}\ \}$ in comparison with $\{S_k(p_{12}/q_{12}),\ 1 \leq k \leq q_{12}\ \}$ in Hecke's case [9], where $\tilde{g}(x) = 2\pi(x - [x]) - \pi$. Here, the functions $\tilde{f}, \tilde{h}$ are explicitly known quadratic functions. The difference function $\tilde{h}(x)$ is shown.

6. **Replacing the irrational rotation by a toral map**. Consider $T(x, y) = (x + \alpha, y + \beta)$ and $g(x, y) = g(2\pi xy)$ and $\gamma = \alpha\beta$, we get Birkhoff sums

$$S_k = \sum_{n=1}^{k} g(n^2\gamma)\ .$$

Illustrations of these Birkhoff sums are called **curlicues** see [1, 24, 4] where $g(x) = \exp(2\pi ix)$ leads to Weyl sums $S_m = \sum_{k=1}^{m} \exp(2\pi ip(n))$ where p is a polynomial. A normalization $S_m/\sqrt{m}$ appears to lead to a limiting distribution [24].

7. **Complex case**. Because of Equation (1.1) the Birkhoff sums originate in products

$$P_n(\alpha) = \prod_{k=1}^{n}(1 - e^{2\pi ik\alpha})$$
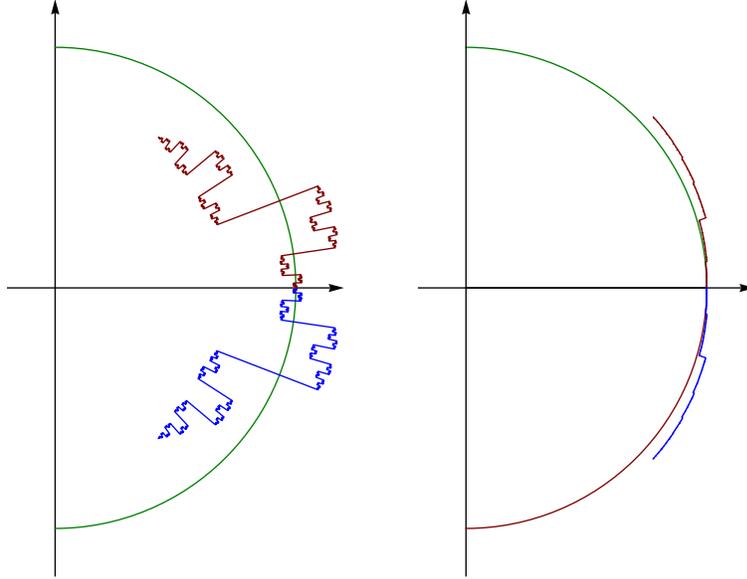
which satisfy

$$\log|P_n(\alpha)| = \frac{S_n}{2}\ .$$

FIGURE 7. The figure to the left shows simultaneously the images
of the rational functions F_{15} and F_{16} from $[0, 1]$ into the complex
plane. It appears that the images of F_{2n} and F_{2n+1} converge point-
wise to complex conjugates. We have $\log |F_n| = f_n$. To the right
we see the image of the function B_{15} and B_{16} in the complex plane.
It appears as if the images of B_{2n} and B_{2n+1} converge pointwise
and that the limits are complex conjugate. We have $\log |B_n| = \beta_n$.

The functions

$$F_n(x) = \prod_{k=1}^{[xq_n]} \frac{1 - e^{2\pi i k \frac{p_{n+1}}{q_{n+1}}}}{1 - e^{2\pi i k \frac{p_n}{q_n}}},$$

on $[0, 1]$ satisfy $f_n = \log |F_n|$. Pointwise convergence of F_n appears to occur modulo
complex conjugation. For odd n, the image of F_n is in the upper half plane, for
even n it is in the lower half plane. The functions

$$B_n(x) = F_n(\alpha x) \cdot F_n(x)^{\alpha^2}$$

which satisfies $\log |B(x)| = \beta(x)$ appear to converge modulo complex conjugation.
Establishing the convergence hypothesis in the complex would of course establish
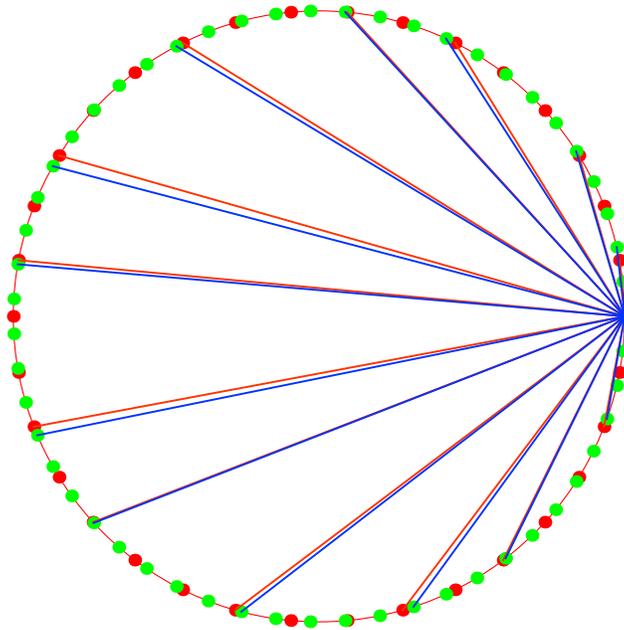the hypothesis in the real case. See Figure (7).

FIGURE 8.   Geometric interpretation of the absolute value of the
complex function F_n as a product of length quotients of regular
polygons with Fibonacci order. The picture illustrates the value
$|F_9(0.4)|$.

To establish the first experimental observation (and so the second), we would
only have to show the existence of the limit

$$\lim_{n\to\infty} \prod_{k=1}^{q_n-1} \frac{1 - e^{2\pi i k\alpha}}{1 - e^{2\pi i k\frac{p_n}{q_n}}} \ ,$$

where p_n/q_n are the Fibonacci quotients converging to the golden mean α.

8. **Geometric interpretation**. Our observations have an elementary geometric
interpretation: take the Fibonacci sequence $\{q_n\}_{n=1}^\infty = (1, 1, 2, 3, 5, 8, 13, 21, \ldots)$
and consider the regular q_{n+1}-gon and q_n-gons in the unit circle. The number

$$|P_{q_n-1}(\frac{q_{n-1}}{q_n})|$$

is the product of the lengths of all the diagonals in the q_n-gon which start from one
point. This product is q_n [18].

The value of $|F_n(x)|$ is a product of ratios between lengths of a subset of all
diagonals in the q_{n+1}-gon and the q_n-gon. See Figure (8).

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