$$\frac{dP}{d\tau} = -\frac{\partial k}{\partial Q}, \qquad \frac{dR}{d\tau} = -\frac{\partial k}{\partial T}, \qquad \frac{dQ}{d\tau} = \frac{\partial k}{\partial P}, \qquad \frac{dT}{d\tau} = 1$$

with Hamiltonian

$$R + k(P, Q, T) = R + \mu k_0(P) + \mu^2 \dots \qquad (2.2.10)$$

and angular coordinates Q, T contain the canonical equations with Hamiltonian (2.2.7). The function (2.2.10) has the form (1.8.1).

An inequality analogous to (1.9.5) and sufficient for the validity of the results of Ch. I, §8 follows from (2.2.6).

Either of these methods can be used to prove the following proposition.

II. *Let* (2.2.7) *be analytic for* $|\mu| < \overline{\mu}$ *in the domain* $|\operatorname{Im} Q, T| \leqslant \rho$, $|P - P_0| \leqslant r$, *where* $|k_0| \leqslant M$, $|k_1 + (\mu) \dots| \leqslant M$, $\left|\frac{d^2 k_0}{dP_0^2}\right| \geqslant \theta > 0$. *Then for every* $\kappa > 0$ *it is possible to find* $\mu_0(\kappa, \overline{\mu}, r, \rho, M, \theta) > 0$ *such that if* $|\mu| < \mu_0$, *then the real torus layer* F, *where* $|P - P_0| \leqslant r$, *is filled with invariant tori with an accuracy up to a residual of measure less than* κ *mes* F *and the distance of each of these invariant tori from a certain torus* $P = $ const *is less than* κ.

Proposition I, and with it also Theorem 1 of 3., easily follows from Proposition II., by virtue of 4.

§3. Adiabatic invariants of conservative systems

We prove in this section the perpetual adiabatic invariance of variables of action in conservative systems with two degrees of freedom.

1. **Adiabatic approximation.** Let us consider a conservative dynamical system with two degrees of freedom X, Y. We shall assume that a change in one of these coordinates, for example X, has little influence on the state of the system. It can then be supposed approximately that there exists a system with one degree of freedom Y depending on the slowly varying parameter μX. The variable of action I_Y (see §2, 1.) corresponds to this system. The magnitude of I_Y is shown to be adiabatically invariant in the following sense.

Let us fix a function of four variables $H(\dots)$. Let X, Y, P_X, P_Y be canonically conjugate variables. We consider a dynamical system defined by the Hamiltonian $H(\mu X, Y; P_X, P_Y)$. If μ is small, then the state of the system varies little if X varies by a quantity of order 1. We fix the values of X and P_X. Then H is transformed into the function $H_Y(Y, P_Y)$. We denote by $I_Y(Y, P_Y)$ the variable of action in the system with Hamiltonian $H_Y(Y, P_Y)$. The magnitude of I_Y depends, however, on the parameters $\mu X, P_X$. For the original system with Hamiltonian $H(\mu X, Y; P_X, P_Y)$ the action I_Y will be an adiabatic invariant in the sense that the variation of

$$I_Y[\mu X(t), Y(t); P_X(t), P_Y(t)] \quad \text{during time } t \sim \frac{1}{\mu} \text{ is small together with } \mu.$$

$H(\mu X, Y; P_X, P_Y)$ can be written in the form $H = H_X(\mu X, P_X; I_Y)$. In order to determine approximately the variation of X, P_X with time it is sufficient to form the canonical equations with Hamiltonian $H_X(\mu X, P_X)$

(depending on the constant parameter I_Y). If in this system with one degree of freedom the motion is periodic, then a variable of action $I_X \sim \frac{1}{\mu}$ can be introduced. In the adiabatic approximation the motion is composed of rapid oscillations of Y with frequency ω_Y and slow oscillations of X with frequency ω_X of order μ, I_Y and μI_X being conserved.

This approximation will be proved below. *We shall prove the perpetual adiabatic invariance of I_Y and μI_X on on the assumption that the mean value*

Fig. 10.

$\overline{\frac{\omega_Y}{\omega_X}}$ *of the ratio* $\frac{\omega_Y}{\omega_X}$ *depends on I_Y for a fixed total energy h.*
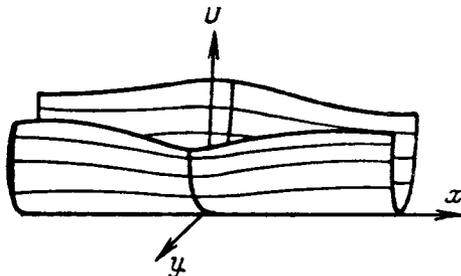
2. **Example.** Let us consider motion in a "potential ditch" pulled out along the x axis (Fig. 10):

$$H = \frac{P_X^2 + P_Y^2 + U(x, Y)}{2}, \text{ where } U = \omega^2 Y^2, \quad \omega = 1 + x^2, \quad x = \mu X, \quad \mu \ll 1.$$

The quantities introduced in 1. take the form

$$H_Y = \frac{P_Y^2 + U}{2} + \frac{P_X^2}{2}, \qquad I_Y = \frac{P_Y^2 + U}{2\omega_Y}, \quad \omega_Y = \omega = 1 + x^2, \quad \overline{\omega}_Y = \frac{2h}{3I_Y} + \frac{1}{3},$$

$$H_X = \frac{P_X^2 + 2I_Y x^2}{2} + I_Y, \quad \mu I_X = \frac{P_X^2 + 2I_Y x^2}{2\sqrt{2I_Y}}, \qquad \omega_X = \mu\sqrt{2I_Y}.$$

Our assumption concerning the dependence of $\overline{\frac{\omega_Y}{\omega_X}}$ on I_Y is fulfilled and therefore I_Y, μI_X are perpetual adiabatic invariants.

At the very bottom of the ditch it is possible to roll away to infinity ($Y = P_Y = I_Y = 0$, $P_X = v$, $X = X_0 + vt$). But if $I_Y \neq 0$, then *the motion takes place in a bounded domain at least for sufficiently small μ* (Fig. 11).
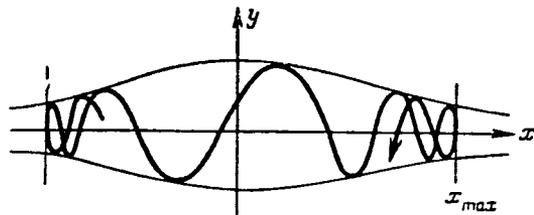

Fig. 11.

For let us fix the values of h and $I_Y \neq 0$ corresponding to the initial conditions we require and then allow μ to tend to zero. For sufficiently small μ we have

$$|I_Y(t) - I_Y(0)| < O(\mu)$$

for all

$$-\infty < t < +\infty.$$

But since

$$h - I_Y(1 + x^2) = \frac{P_X^2}{2} \geqslant 0,$$

motion in the x direction is limited by

$$|x_{max}| = \sqrt{\frac{h}{I_Y} - 1} + O(\mu).$$

3. **Preliminary canonical transformation.** We shall reduce the Hamiltonian of the system to the form (2.2.9). The perpetual adiabatic

invariance of I_Y and μI_X is easily deduced from the existence of the two-dimensional invariant tori of the latter system since these tori divide the three-dimensional level of energy $H = h$.

LEMMA. *Suppose that the Hamiltonian $H(x, Y; P_X, P_Y)$ $(x = \mu X)$ is analytic and for fixed x, P_X defines an oscillating system with action-angle variables $I_Y(x, P_X; h)$, $w_Y(x, Y, P_X, P_Y)$. Then there exists an analytic substitution expressing x, Y, P_X, P_Y in terms of new variables x', w', P', I' such that:*

1) *The functions x, Y, P_X, P_Y are of period 2π with respect to w. As $\mu \to 0$ the variables x', w', P', I' turn into x, w_Y, P_X, I_Y.*

2) *Along the integral curves of the canonical equations*

$$\frac{dP_X}{dt} = -\frac{\partial H}{\partial X}, \quad \frac{dP_Y}{dt} = -\frac{\partial H}{\partial Y}, \quad \frac{dX}{dt} = \frac{\partial H}{\partial P_X}, \quad \frac{dY}{dt} = \frac{\partial H}{\partial P_Y}$$

the canonical equations

$$\frac{dP'}{dw'} = -\frac{\partial K}{\partial x'}, \quad \frac{dx'}{dw'} = \frac{\partial K}{\partial P'} \tag{2.3.1}$$

with Hamiltonian $K(P', x'; w'; h)$ depending on the parameter h are satisfied.

3) *K is of the form $K = -\mu I'$, where*

$$I' = I_0(P', x'; h) + \mu I_1(P', x'; w'; h) + \dots \tag{2.3.2}$$

is an analytic function of period 2π with respect to w' and $I_0(P', x'; h) = I_Y(x', P'; h)$.

PROOF. From the relationship $H(x, Y; P_X, P_Y) = h$ we can express P_Y in the form $P_Y(x, P_X, Y; h)$ and we put $2\pi I_Y(x, P_X; h) = \oint P_Y \, dY$. This relationship defines the function $h(x, P_X; I_Y)$. The generating function $P'X + S(x, P', Y, I')$, where

$$S = \int^Y P_Y [x, P', Y; h(x, P'; I')] \, dY.$$

determines the canonical transformation $X, Y, P_X, P_Y \to X', w', P', I'$ with the help of the formulae

$$X' = X + \frac{\partial S}{\partial P'}, \qquad w' = \frac{\partial S}{\partial I'},$$

$$P_X = P' + \mu \frac{\partial S}{\partial x}, \qquad P_Y = \frac{\partial S}{\partial Y}.$$

We also put $x' = \mu X'$. Since

$$H\{x, Y; P', P_Y[x, P', Y; h(x, P'; I')]\} = h(x, P'; I'),$$

we have

$$H\{x, Y; P_X, P_Y[x, P', Y; h(x, P'; I')]\} = h(x, P'; I') + \mu \frac{\partial S}{\partial x}\frac{\partial H}{\partial P_X} + \dots$$

Therefore, expressing X, Y in terms of the new variables, we obtain

$$H(x, Y; P_X, P_Y) = h(x', P'; I') + \mu \frac{\partial S}{\partial x}\frac{\partial H}{\partial P_X} - \mu \frac{\partial S}{\partial P'}\frac{\partial H}{\partial x} + \dots \tag{2.3.3}$$

(2.3.3) can be written in the form $(h_0 \equiv h)$

$$H(x, Y; P_X, P_Y) = H'(x', P', I', w') = h_0(x', P'; I') + \mu h_1(x', P', I', w') + \ldots,$$

(2.3.4)

where $\mu h_1 + \ldots$ is an analytic function of period 2π with respect to w'.

We shall measure the time by the phase w'. For this purpose in place of P', X'; I', w'; t as independent variables in the expression $P'dX' + I'dw' - Hdt$ we take, respectively, P', X'; $-H$, t; w' (see [4]). We shall consider h as a parameter and w' as the time. The role of the Hamiltonian is played by $-I'(X', P'; w'; h)$, where I' is determined from the equation

$$H'(\mu X', P', I', w') = h.$$

(2.3.5)

The coordinate t is cyclic; discarding the variables $-H$, t we obtain a non-autonomous system with one degree of freedom. We multiply the coordinate X' and the Hamiltonian $-I'$ by the constant μ: $\mu X' = x'$, $-\mu I' = K$. The derivatives of x' and P' with respect to w' are determined by the canonical equations (2.3.1) with Hamiltonian K.

In view of (2.3.5) and (2.3.4) the function I' is of the form (2.3.2) and this proves the lemma.

4. Proof of the perpetual adiabatic invariance of action. In accordance with (2.3.2) the function K is of the form (2.2.9). (2.2.6) follows from the condition formulated at the end of 1. Therefore the reasoning of §2 is applicable. It gives invariant tori and the proof of the perpetual adiabatic invariance of I_Y and μI_X.

§4. Magnetic traps

In this section we consider the motion of a charged particle in a magnetic field. It is assumed that the instantaneous radius of the spiral along which the particle moves is small in comparison with the distances at which the field changes significantly. This condition is fulfilled if the field is large or if it is almost constant, or if the velocity of the particle is small. We shall consider the last case (which does not result in any loss of generality).

We show that in axially-symmetric magnetic traps the adiabatic invariant $\dfrac{W_\perp}{B}$ is perpetually conserved. It therefore follows that such traps are capable of retaining charged particles perpetually.

1. Equations of motion. We assume that the magnetic field **B** is determined by a vector potential **A** which, in polar coordinates r, φ, z has only one component $A_\varphi = A(r, z)$. Then the components of the field strength **B** are

$$B_r = -\frac{\partial A}{\partial z}, \qquad B_z = \frac{1}{r}\frac{\partial rA}{\partial r}.$$

Therefore the lines of force are determined by the equations

$$rA = \text{const}, \qquad \varphi = \text{const}.$$

The Lagrangian of a unit charge of mass 1 with a suitable choice of units is

$$L = \frac{1}{2} (\dot{r}^2 + \dot{z}^2 + r^2 \dot{\varphi}^2) + rA\dot{\varphi}.$$

From this we find the "impulses":

$$p_r = \dot{r}, \quad p_z = \dot{z}, \quad p_\varphi = r^2\dot{\varphi} + rA,$$

and the Hamiltonian:

$$H = \frac{1}{2} \left[p_r^2 + p_z^2 + \frac{(p_\varphi - rA)^2}{r^2} \right].$$

Since φ is cyclic, $p_\varphi = M$ is conserved and it remains to investigate plane motion in the field with potential

$$U(r, z) = \frac{(M - rA)^2}{2r^2}, \tag{2.4.1}$$

where M is a fixed constant.

The function (2.4.1) defines a "potential ditch" with the zero bottom along the line of force $rA = M$. In the neighbourhood of this line we have

$$U(r, z) = \frac{1}{2} B^2 y^2 + \dots, \tag{2.4.2}$$

where y is the distance from the line of force and B is the magnitude of the magnetic field strength on the line of force.

2. Change of variables. In order to apply the results of §3, we introduce curvilinear coordinates x, y into the r, z-plane. We denote by x the arc length along the line of force $rA = M$ from the fixed point 0 to the base of the perpendicular from the point r, z onto this line of force. As in (2.4.2), we shall denote by y the length of this perpendicular (Fig. 12).

Then we have

$$dr^2 + dz^2 = [1 + yk(x)]^2 dx^2 + dy^2,$$

where $k(x)$ is the curvature of the line of force at the point $x, 0$. Therefore

$$p_x = [1 + yk(x)]^2 \dot{x}, \quad p_y = \dot{y},$$

and hence

$$H = \frac{1}{2} \left(\frac{p_x^2}{[1 + yk(x)]^2} + p_y^2 \right) + U, \quad \text{where } U(x, y) = \frac{1}{2} B(x) y^2 + \dots$$
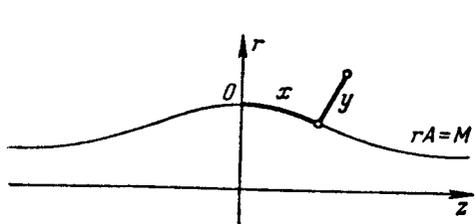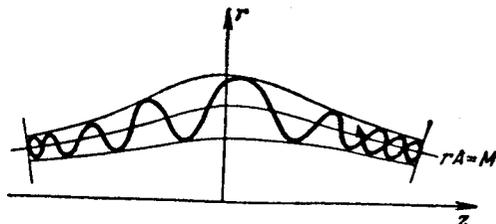

Fig. 12.   Fig. 13.

The case of interest to us is that in which the radius of the Larmor spiral described by the charge around the line of force is very small in comparison with the characteristic dimensions of the field. In the notation introduced this means that we shall consider values of the constant M and the total energy h such that the inequality

$$U \leqslant h$$

defines a strip of width $\sim \mu$ around the line of force $rA = M$ (Fig. 13). It is therefore convenient to introduce new variables X, Y, P_X, P_Y by means of the relationships

$$x = \mu X, \quad y = \mu Y, \quad p_x = \mu P_X, \quad p_y = \mu P_Y.$$

The way in which these variables change with time is described by the canonical equations with Hamiltonian $H' = \mu^{-2}H$:

$$H' = \frac{1}{2}\left(\frac{P_X^2}{[1 + \mu Y k(\mu X)]^2} + P_Y^2\right) + U', \quad U'(\mu X, Y) = \frac{1}{2}B^2(\mu X)Y^2 + \dots,$$

which can be written in the form

$$H' = \frac{P_X^2 + P_Y^2 + B^2(x^2)Y^2}{2} + \mu H_1(x, Y, P_X, P_Y) + \dots,$$

i.e. this function has the form $H'(\mu X, Y; P_X, P_Y; \mu)$ similar to that considered in §3. (It is easy to see that the additional dependence on μ is inessential for the applicability of the arguments of §3.)

3. Perpetual adiabatic conservation of the magnetic moment. In considering the formulation of the result to be obtained let us define more exactly what asymptotic properties we shall be concerned with. We fix the magnetic field **B** and also the self-conserving quantity M. Then we fix the initial value[1] of x and finally choose initial values of y, $\dot{x}$, $\dot{y}$ such that $H \sim \mu^2$, $\mu \ll 1$. For this purpose it is necessary to take y, $\dot{x}$, $\dot{y}$ of order μ. We fix $Y, \dot{X}, \dot{Y}$, take $y = \mu Y$, $\dot{x} = \mu \dot{X}$, $\dot{y} = \mu \dot{Y}$ and then let μ tend to 0.

The method of §3 gives us the following result:

I. *If the magnetic field is analytic and* $B(x) > 0$, $B(x) \to \infty$ *as* $|x| \to \infty$, *then for any* $\kappa > 0$ *it is possible to find* $\mu_0 > 0$ *such that if* $|\mu| < \mu_0$, *then* $|I_Y(t) - I_Y(0)| < \kappa$ *for all* $-\infty < t < +\infty$, *where*

$$I_Y = \frac{P_Y^2 + B^2Y^2}{2B} \ .$$

From this it can be seen at once that a particle for which $I_Y \neq 0$ is *locked in a bounded domain* provided the field **B** increases infinitely as $x \to \infty$ (a trap with stoppers).

The physical meaning of I_Y will become clear if we consider the moment when $Y = 0$. At this moment $\dot{\phi} = 0$ and $\frac{\dot{y}^2}{2} = W_\perp$ (see §1, 2., III). Therefore $\mu^2 I_Y = \frac{W_\perp}{B}$. Thus Proposition I can be formulated as follows:

II. *In an axially-symmetric magnetic trap the magnitude of the magnetic moment* $\frac{W_\perp}{B}$ *is a perpetual adiabatic invariant.*

---

[1]   It is sufficient to fix a bounded domain of initial values of x. The same applies to the fixing of Y, $\dot{X}$, $\dot{Y}$ and M.