THE STABILITY OF THE VAN ALLEN RADIATION BELT*

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Abstract. In this paper we derive two classes of invariant tori which lie on energy surfaces which are large enough to accommodate the charged particles in the Van Allen radiation belt.

Introduction. One of the most fertile fields of mathematical research has been the Störmer problem, which is the study of a charged particle moving in the earth's magnetic field. This problem was originally motivated by phenomena connected with the polar aurora [13]. In the middle 1950's, though, this problem received renewed interest with the discovery of the Van Allen radiation belt. This is a region of space in which charged particles are seemingly "trapped" by the earth's magnetic field for very long periods of time.

In 1968 Braun [1] succeeded in finding two regions where particles were trapped *forever* by the earth's magnetic field. In one of these regions, all particles had a very small latitude (i.e., they stayed near the equator), while in the other region, particles could penetrate to high latitudes. This result is extremely important, because it indicates the possibility of "entrapment", for infinite time, by the earth's magnetic field. However, these regions are not physically relevant, since the energy of the particles in these regions is way too small to be of any physical significance. In this paper we will establish two additional regions, one near the equator, and one with large latitudes, in which particles are trapped forever by the earth's magnetic field, and in which the energy is large enough to be of physical significance.

Remarks. The mathematical treatment in this paper (and in [1]) is an idealized one in that we completely ignore all interactions between particles. A full treatment of this problem requires consideration of how new particles are introduced into the Van Allen belt, and of how old particles are lost through collisions, or collective plasma processes. Indeed, we even make an idealized assumption concerning the nature of the earth's magnetic field. Nevertheless, our results still have significant physical relevance. This is because the special (quasi-periodic) orbits we obtain persist under small perturbations, and most of the effects we have omitted can be considered as a small perturbation of the idealized problem.

The existence of invariant tori near the equator. We assume that the earth's magnetic field is equivalent to that of a magnetic dipole situated at the center of the earth. The motion of a particle in a dipole field can be described, after suitable normalization [5] by the Hamiltonian

$$(1) \qquad H = \frac{1}{2}\left[p_\rho^2 + p_z^2 + \left(\frac{1}{\rho} - \frac{\rho}{r^3}\right)^2 \right],$$

where $r^2 = \rho^2 + z^2$, and

$$\dot{\rho} = p_\rho, \qquad \dot{p}_\rho = \ddot{\rho} = -\frac{\partial H}{\partial \rho},$$

$$\dot{z} = p_z, \qquad \dot{p}_z = \ddot{z} = -\frac{\partial H}{\partial z}.$$

Since H is a function of z^2, it follows immediately that a particle starting in the equatorial plane $z = \dot{z} = 0$ must remain there forever. The motion of the particle in the equatorial plane is then governed by the one dimensional Hamiltonian

$$(2) \qquad H = \frac{1}{2}\left[p_\rho^2 + \left(\frac{1}{\rho} - \frac{1}{\rho^2}\right)^2 \right]$$

and the qualitative properties of all equatorial orbits are now determined by the level curves

$$(3) \qquad E = \frac{1}{2}\left[p_\rho^2 + \left(\frac{1}{\rho} - \frac{1}{\rho^2}\right)^2 \right]$$

in the ρ-p_ρ plane. The level curves (3) have three different forms (see Fig. 1) depending on whether $E > \frac{1}{32}$, $= \frac{1}{32}$ or $< \frac{1}{32}$.
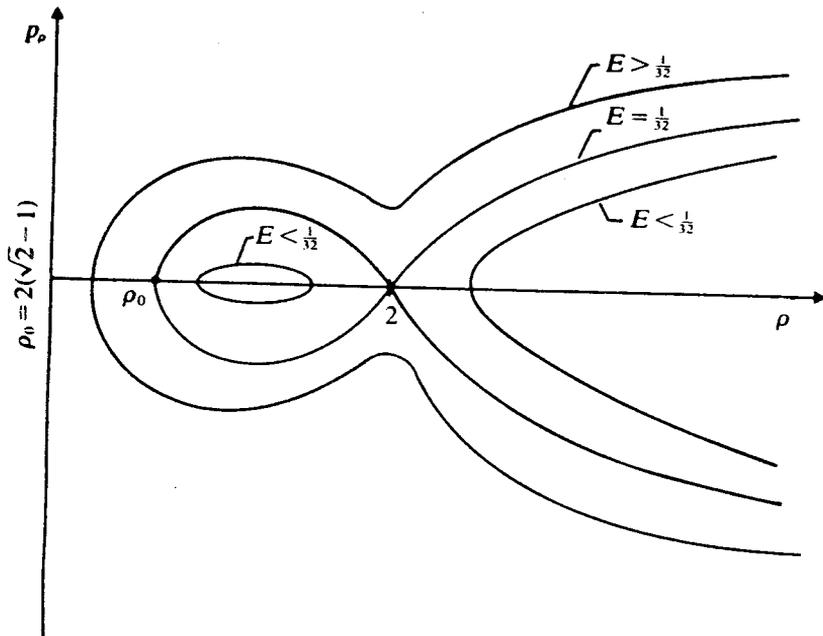

FIG. 1. *The region $H = E$.*

The region of closed curves in Fig. 1 correspond to periodic solutions, an confine our interest to this region.

It is natural to try and establish the stability of these orbits. This was at DeVogelaere in the mid 1950's. His approach was via Hill's equation [10]. differential equations of motion for the Störmer problem for small excursions *out of the* equatorial plane can be written in the form

$$\ddot{z} + \frac{\rho - 1}{\rho^6} z = 0 + O(z^2),$$

$$(4)$$

$$\ddot{\rho} + \frac{1}{\rho^5}(\rho - 1)(\rho - 2) = 0 + O(z^2).$$

Neglecting terms of order z^2 and above, we see that the solutions of the *second equation* of (4) are the equatorial orbits, and are therefore known periodic *functions of time. The*

first equation of (4) is then Hill's equation, if we neglect the $O(z^2)$ term. Its solution can be written in the form

$$z(t) = C e^{\Omega t} \psi(t) + D e^{-\Omega t} \psi(-t),$$

where C and D are arbitrary constants and $\psi(t)$ is periodic in time with the same period as $\rho(t)$. The constant Ω, which is known as the characteristic Poincaré exponent, determines the stability of the orbit. It can be only real, or purely imaginary. If Ω is real and $\neq 0$, then the motion in the z direction grows (within the approximation made) without bound. If Ω is purely imaginary, then the motion is bounded (for time intervals in which the neglected terms have negligible effect) and is therefore stable.

The behavior of Ω as a function of the energy E has been studied by De Vogelaere [4], who found that all orbits are stable (within the approximation made) for

$$E < \frac{1}{8(1.3137)^2}.$$

Of course, De Vogelaere's results are not rigorous, since he has omitted some terms. However, as we shall see shortly, De Vogelaere's results are crucial to a rigorous proof.

A rigorous proof is obtained in the following manner. First, introduce action angle variables [9] R, θ in place of ρ, p_ρ, where R is the area of a closed curve in (3). Then, the Hamiltonian H can be written in the form

(5) $H = H_0(R) + \frac{1}{2}[p_z^2 + H_1(R, \theta)z^2] + H_2(R, \theta)z^4 + O(z^6).$

Let us now fix a closed curve in (3) with area R_0, and make the substitution

$$R = R_0 + \varepsilon^2 R', \qquad z = \varepsilon z',$$

$$\theta = \theta', \qquad p_z = \varepsilon p_z'.$$

Neglecting the primes, we see that our system is still Hamiltonian, with Hamiltonian

(6)
$$H = \frac{H}{\varepsilon^2} = \frac{H_0(R_0)}{\varepsilon^2} + \omega_1 R + \frac{1}{2}[p_z^2 + H_1(R_0, \theta)z^2] + \varepsilon^2 \omega_2 R^2$$

$$+ \frac{\varepsilon^2}{2} \omega_3 z^2 + \varepsilon^2 H_2(R_0, \theta)z^4 + O(\varepsilon^4),$$

where

$$\omega_1 = \frac{\partial H_0(R_0)}{\partial R}, \qquad \omega_2 = \frac{\partial^2 H_0(R_0)}{\partial R^2}, \qquad \omega_3 = \frac{\partial H_1(R_0)}{\partial R}.$$

The constant term H_0/ε^2 can be omitted, since it doesn't effect the equations of motion.

Our next step is to solve (6) for $R = \Phi(\theta, z, p_z, \varepsilon, E)$ on each constant energy surface $H = E$. Introducing θ as the time scale along orbits, we find that

(7) $\dot{z} = -\Phi_{p_z}, \qquad \dot{p}_z = \Phi_z.$

This system is again Hamiltonian, with one degree of freedom, but nonautonomous. To eliminate the dependence of Φ on θ, we follow all solutions from $\theta = 0$ to their next intersection with $\theta = 2\pi$. This gives rise to an area preserving mapping M which can be written in the form

$$M: \begin{pmatrix} z \\ p_z \end{pmatrix}_1 = A_0 \begin{pmatrix} z \\ p_z \end{pmatrix} + \varepsilon^2 A_1(z, p_z) + O(\varepsilon^4),$$

where A_0 is a constant 2×2 matrix.

Now, our crucial result is that the eigenvalues of A_0 are pure imaginary, with absolute value 1. This is exactly De Vogelaere's result. Thus, we can find new coordinates ζ, $\bar{\zeta}$, such that M can be written in the form

$$
\begin{aligned}
\zeta_1 &= e^{i\alpha_0}\zeta + \varepsilon^2 f_1(\zeta, \bar{\zeta}) + O(\varepsilon^4), \\
\bar{\zeta}_1 &= e^{-i\alpha_0}\bar{\zeta} + \varepsilon^2 \bar{f}_1(\bar{\zeta}, \zeta) + O(\varepsilon^4),
\end{aligned}
$$

(8)

for some real number α_0 (which of course depends on R_0). Following Birkhoff (see [12]) we make f_1 a function of $\zeta\bar{\zeta}$. Indeed, we can find new coordinates, which we again call ζ, $\bar{\zeta}$ such that

$$
\zeta_1 = e^{i[\alpha_0 + \varepsilon^2\alpha_1(\zeta\bar{\zeta})]}\zeta + O(\varepsilon^4).
$$

Moreover, the number α_1 can be shown to be nonzero. (This also follows from De Vogelaere's result.) Finally, introducing polar coordinates r, ϕ, we see that M can be written in the form

$$
\begin{aligned}
r_1 &= r + O(\varepsilon^4), \\
\phi_1 &= \phi + \alpha_0 + \varepsilon^2\alpha_1 r^2 + O(\varepsilon^4).
\end{aligned}
$$

(9)

The area-preserving mapping (9) is a twist mapping. Hence, by Moser's theorem [11], it possesses infinitely many closed invariant curves, of nonzero measure. These curves give rise to invariant tori if we construct all solutions which start on these invariant curves. Moreover, any orbit starting between two such invariant tori must remain trapped between these tori forever.

Remark. The tori constructed above all have a physically relevant energy. And even more important, they are stable under perturbation.

Invariant tori which penetrate to high latitudes. One of the key steps in finding invariant tori is to first find stable periodic solutions. Störmer [13], as early as 1907 did extensive work in constructing periodic solutions for the Störmer problem. Later, De Vogelaere [2] established the existence of a whole family of periodic solutions. Just recently, Dragt [7] succeeded in actually locating these periodic solutions, and by a clever topological argument, found additional periodic solutions. Indeed, by using high order Störmer–Crowell integration schemes [6], he was able to accurately compute orbits up to 43 crossings of the equatorial plane.

Once we have found a periodic solution $\zeta(t)$, we proceed in the following manner. First, we can start the solution off in the equatorial plane, with $\dot{z}(0) > 0$, since any periodic orbit with $\dot{z}(0) > 0$ must cross this surface of section infinitely often [3]. Next we expand nearby solutions as power series in the initial conditions. That is to say, if $\rho_0 = a$, $\dot{\rho}_0 = b$, $\dot{z}_0 = c$ and $z_0 = 0$ are the initial values of our periodic solution, then we can write the solution $(\rho(t), \dot{\rho}(t), z(t), \dot{z}(t))$ as power series in $\rho_0 - a$, $\rho_0 - b$ and $\dot{z}_0 - c$. (Here we restrict ourselves to orbits which start in the equatorial plane.) Moreover, the coefficients in these power series, which are functions of time, can be easily computed [10] as the solutions of linear periodic systems.

Our next step is to restrict ourselves to a constant energy surface $H = E$. This will eliminate $\dot{z}(t)$ as an independent variable. Finally, we solve for the time t at which the solution first returns to the equatorial plane, and for which $\dot{z}(t) > 0$. This time t is also a power series in the initial conditions. Putting this all together, and translating coordinates so $\zeta(0) = 0$, we see that we have a mapping M of the equatorial plane into itself

which can be written in the form

$$M: \begin{pmatrix} \rho \\ \dot{\rho} \end{pmatrix}_1 = A \begin{pmatrix} \rho \\ \dot{\rho} \end{pmatrix} + f_2(\rho, \dot{\rho}) + f_3(\rho, \dot{\rho}) + \cdots,$$

where A is a constant 2×2 matrix. Now, the mapping M is area-preserving [8]. This forces the eigenvalues of A to be reciprocals of each other. Thus, if $\zeta(t)$ is stable, then the eigenvalues of A must lie on the unit circle. In this case, we can find new complex coordinates $\xi, \bar{\xi}$ such that M assumes the form

(10)
$$\xi_1 = e^{i\alpha_0}\xi + g_2(\xi, \bar{\xi}) + g_3(\xi, \bar{\xi}) + \cdots.$$

If $\alpha_0 \neq 2\pi(p/q)$, $q \leqq 2n + 1$, then we can put (10) into the normal form [12]

$$\eta_1 = e^{i[\alpha_0 + \alpha_1 \eta\bar{\eta} + \cdots + \alpha_n(\eta\bar{\eta})^n]}\eta + h(\eta, \bar{\eta}),$$

where $h = O(|\eta|^{2n+2})$, and $\alpha_1, \cdots, \alpha_n$ are real. Setting $\eta = \varepsilon r e^{i\theta}$, we see that

(11)
$$r_1 = r + O(\varepsilon^{2n+1}),$$
$$\theta_1 = \theta + \alpha_0 + \alpha_1 \varepsilon^2 r^2 + \cdots + \alpha_n \varepsilon^{2n} r^{2n} + O(\varepsilon^{2n+1}).$$

If either $\alpha_1, \alpha_2, \cdots$, or $\alpha_n \neq 0$, then (11) is a twist map, and consequently possesses infinitely many invariant closed curves.

Now it is quite a tedious procedure to work this all out. However, thanks to Dragt [7], we are spared this agony, since he has computed M numerically near stable (elliptic) periodic solutions, and the twist is evident in his graphical results. This concludes our proof.

Remark. The invariant tori just obtained are not as physically relevant as our first class of tori, since they occupy a very narrow band in phase-space. However, they are still very important since they indicate the very real properties of trapping particles with physically relevant energy which penetrate to high latitudes.

REFERENCES

[1] M. BRAUN, *Particle motions in a magnetic field*, J. Differential Equations, 8 (1970), pp. 294–332.
[2] R. DE VOGELAERE, *On the structure of symmetric periodic solutions of conservative systems*, Contributions to the Theory of Nonlinear Oscillations, Princeton University Press, Princeton, NJ, 1958, pp. 53–84.
[3] ———, *Surface de section dans le problème de Störmer*, Acad. Roy. Belg. Bull. Cl. Sci., 40 (1954), pp. 673–688.
[4] ———, *L'Équation de Hill et le problème de Störmer*, Canad. J. Math., 2 (1950), pp. 440–456.
[5] A. J. DRAGT, *Trapped orbits in a magnetic dipole field*, Rev. Geophysics, 3 (1965), pp. 255–298.
[6] ———, *Lecture notes on classical mechanics*, unpublished.
[7] A. J. DRAGT AND J. M. FINN, *Stability of charged particle motion in a magnetic dipole field*, preliminary report.
[8] ———, *Instability of trapped particle motion in a magnetic dipole field*, Tech. Report No. 75-098, Univ. of Maryland, College Park, MD, 1975.
[9] H. GOLDSTEIN, *Classical Mechanics*, Addison-Wesley, Reading, MA, 1952.
[10] F. JOHN, *Lecture Notes on Ordinary Differential Equations*, Courant Inst. Math. Sci. Lecture Notes, New York, 1965.
[11] J. K. MOSER, *On invariant curves of area preserving mappings of an annulus*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II, no. 1 (1962), pp. 1–20.
[12] C. L. SIEGEL, *Lectures on Celestial Mechanics*, Springer-Verlag, New York, 1970.
[13] C. STÖRMER, *The Polar Aurora*, Clarendon Press, Oxford, 1955.