

Normal form for mirror machine Hamiltonians^{a)}

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(Received 28 June 1978; revised manuscript received 29 December 1978)

A systematic algorithm is developed for performing canonical transformations on Hamiltonians which govern particle motion in magnetic mirror machines. These transformations are performed in such a way that the new Hamiltonian has a particularly simple *normal form*. From this form it is possible to compute analytic expressions for gyro and bounce frequencies. In addition, it is possible to obtain arbitrarily high order terms in the adiabatic magnetic moment expansion. The algorithm makes use of Lie series, is an extension of Birkhoff's normal form method, and has been explicitly implemented by a digital computer programmed to perform the required algebraic manipulations. Application is made to particle motion in a magnetic dipole field and to a simple mirror system. Bounce frequencies and locations of periodic orbits are obtained and compared with numerical computations. Both mirror systems are shown to be insoluble, i.e., trajectories are not confined to analytic hypersurfaces, there is no analytic third integral of motion, and the adiabatic magnetic moment expansion is divergent. It is expected also that the normal form procedure will prove useful in the study of island structure and separatrices associated with periodic orbits, and should facilitate studies of breakdown of adiabaticity and the onset of "stochastic" behavior.

1. INTRODUCTION AND NOTATION

In the study of a complicated dynamical system, one almost invariably seeks at a minimum to learn the answers to two fundamental questions. First, what areas of phase space are in fact accessible to the system for a given trajectory or class of trajectories? Second, where are the periodic and quasiperiodic orbits, and what are their frequencies? Thus, for example, in the study of magnetic mirror machines one uses the magnetic moment "invariant" to "infer" that certain particles will indeed mirror and will not escape through the ends of the machine. In addition, one develops various expressions or runs numerical codes to determine gyro frequencies, bounce frequencies, and those orbits for which these frequencies are commensurate.

The purpose of this paper is to show how these questions can be studied in detail for mirror machines. Our method makes use of algebraic manipulations performed by a digital computer. We are able to produce analytic expressions for the frequencies and initial conditions associated with periodic and quasiperiodic orbits.¹⁻³ These expressions should prove to be useful in the study of island structure and separatrices associated with periodic orbits.^{4,5} In addition, we are able to obtain arbitrarily high order terms in the complete adiabatic magnetic moment expansion. This latter result has already proved useful in demonstrating the "insolubility" of certain mirror machine problems,⁴ and should

facilitate studies of the breakdown of adiabaticity and the onset of "stochastic" behavior.⁶ In particular, it has been shown for certain mirror machines that trajectories are not confined to analytic hypersurfaces in phase space. As a result, the adiabatic magnetic moment expansion is divergent, and one can make no mathematically rigorous statement about confinement or the long-term behavior of orbits.⁴ Such may in fact be the case for all mirror machines.

More precisely, the purpose of this paper is to show that a certain class of Hamiltonians can be brought *systematically* to a particularly simple "normal form" by a sequence of canonical transformations. The class of Hamiltonians of interest will be called "mirror machine" Hamiltonians since they arise naturally in the study of mirror machines designed for plasma containment. By the word "systematically," we mean there exists an algorithm for analytical computation which can be explicitly implemented by a digital computer programmed to perform certain algebraic manipulations.

The meaning of the term "normal form" will be delineated further after the introduction of suitable mathematical machinery. For the moment, we make the following analogy: In the study of a linear operator or matrix, it is often useful to perform similarity transformations to bring the matrix to diagonal or Jordan canonical form. Once this is done, it is a simple matter to read off the eigenvalues and eigenvectors, to evaluate functions of the matrix such as its exponential and inverse, and to find matrices which will commute with the given matrix. In the study of a classical mechanics problem specified by a certain Hamiltonian, one can try to proceed in a similar spirit. One performs canonical transformations on the Hamiltonian in the hope of bringing it to a

^{a)}Supported in part by the National Science Foundation under Grant No. GP-41822X.

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simpler form. Exactly what simple forms a given Hamiltonian can be brought to is not as yet completely known and is still an area requiring active study. (Canonical transformations are in general nonlinear, and thus the problem is intrinsically far more complicated.) However, we will show that there is a normal form for any mirror machine Hamiltonian from which it is possible to compute the frequencies and initial conditions associated with periodic and quasiperiodic orbits. Thus, with the normal form method, it is possible to compute analytic expressions for bounce frequencies and for closed orbits. In addition, the normal form we will describe makes possible the construction of formal integrals of motion for the Hamiltonian in question. Integrals of motion are functions of phase space variables which do not explicitly involve the time and which remain constant on trajectories. In the case of mirror machines, the integral of motion proves to be the *complete adiabatic magnetic moment expansion*. Consequently, it is possible to obtain arbitrarily high order terms in the complete adiabatic magnetic moment expansion providing one is willing to spend sufficient computer time. Finally, the normal form method for mirror machines is related to similar transformation methods which have recently proven to be very useful in such diverse areas as celestial mechanics and molecular physics for both deep mathematical proofs and practical calculations.⁸ Thus we are at the threshold of a unified treatment of a wide variety of classical mechanics problems.

Since our work requires the execution of a long sequence of canonical transformations and also the inversion of these transformations, the remainder of this section is devoted to the development of notation and a brief review of the method of Lie transformations which we have found to be particularly useful. In Sec. 2 we specify the nature of a mirror machine Hamiltonian and develop the normal form algorithm. Section 3 shows how use of the normal form algorithm leads to the construction of integrals of motion. Section 4 illustrates the application of the normal form method to two examples of charged particle motion in magnetic mirror fields, namely the magnetic dipole field and that of a simple model mirror machine. Comparisons are made for these two problems between numerical and analytical results. In particular, we study the frequencies of periodic orbits, the constancy of the series for the complete adiabatic magnetic moment expansion, and the insolubility of the simple model mirror machine. A final section summarizes the conclusions of this paper.

The general problem of interest will have n degrees of freedom described by the canonical coordinates q_1, q_2, \dots, q_n and p_1, p_2, \dots, p_n . For compactness of notation we have found it convenient to treat the q 's and p 's together by introducing the $2n$ variables z_1, \dots, z_{2n} defined by the relations

$$z_i = q_i; \quad z_{n+i} = p_i, \quad i = 1 \text{ to } n. \quad (1.1)$$

The method of Lie transformations makes essential use of Poisson brackets and the Lie algebraic structure associated with them. We shall briefly review here the tools needed for this paper. A more detailed explication with proofs has been given earlier.⁹

Suppose $f(z)$ is a particular function defined on phase space. We associate with f the *Lie operator* F by the rule that if g is any other function, then F acting on g is defined by

$$Fg = [f, g]. \quad (1.2)$$

Here the bracket $[,]$ denotes the Poisson bracket. Note that F is linear.

Next we define the linear operator $\exp(F)$, called the *Lie transformation* associated with F and f , by the rule

$$\exp(F) = \sum_0^{\infty} F^j / j! \quad (1.3)$$

with the convention $F^0 = I$.

Lie transformations have several remarkable properties. Suppose d and e are any two functions. Then we find

$$\exp(F)(de) = (\exp(F)d)(\exp(F)e) \quad (1.4)$$

and

$$\exp(F)[d, e] = [\exp(F)d, \exp(F)e]. \quad (1.5)$$

Consequently, if we define new variables \bar{z}_i by the rule

$$\bar{z}_i(z) = \exp(F)z_i, \quad (1.6)$$

then we have

$$\begin{aligned} [\bar{z}_i, \bar{z}_j] &= [\exp(F)z_i, \exp(F)z_j] \\ &= \exp(F)[z_i, z_j] = [z_i, z_j]. \end{aligned} \quad (1.7)$$

Here we have used (1.5) and the fact that $[z_i, z_j]$ is a number and hence is unchanged by $\exp(F)$. It follows from (1.7) that the new variables $\bar{z}(z)$ are related to the old variables z by a canonical transformation.

Conversely, if the $\bar{z}(z)$ are new variables related to the old variables z by a *canonical transformation near the identity* of the form

$$\bar{z}_i(z) = z_i + \text{higher-degree terms}, \quad (1.8)$$

then it can be shown that there exists a sequence of homogeneous polynomials f_3, f_4, \dots , of degree 3, 4, etc., such that

$$\bar{z}_i(z) = \dots \exp(F_3) \exp(F_4) \exp(F_3) z_i. \quad (1.9)$$

Similarly, the inverse to the transformation (1.8) or (1.9) can be written as

$$z_i(\bar{z}) = \exp(-F_3) \exp(-F_4) \exp(-F_3) \dots \bar{z}_i. \quad (1.10)$$

In this latter expression the f 's are considered as depending on the variables \bar{z} , i.e., $f_3(\bar{z}), f_4(\bar{z}), \dots$; and all Poisson brackets are taken with respect to the variables \bar{z} .

Finally, suppose $\bar{g}(z)$ is a function defined in terms of some other function $g(z)$ by the rule

$$\bar{g}(z) = g(\bar{z}(z)), \quad (1.11)$$

where z and \bar{z} are related by (1.9). Then it follows from consideration of a series expansion of g and repeated use of (1.4) that

$$\bar{g}(z) = \dots \exp(F_3) \exp(F_4) \exp(F_3) g. \quad (1.12)$$

Note that in making canonical transformations, we take the

active viewpoint: trajectories and functions are transformed; the coordinate system remains fixed.

With this preparatory background, we are now able to state more precisely our purpose. Suppose we wish to study the nature of the trajectories governed by a certain mirror machine Hamiltonian. For clarity, we denote this Hamiltonian by the symbol h^{old} . We assume that h^{old} does not depend explicitly on time. Then our aim is to find a sequence of homogeneous polynomials f_3, f_4, \dots , such that the transformed or "new" Hamiltonian h^{new} , given by

$$h^{\text{new}} = \dots \exp(F_3) \exp(F_4) \exp(F_3) h^{\text{old}}, \quad (1.13)$$

has a particularly simple form. By "simple," we mean that h^{new} should only depend on certain combinations of the variables z in such a way that it is easy to find functions i^{new} , called integrals of h^{new} , which do not depend explicitly on time and which satisfy the relation

$$[i^{\text{new}}, h^{\text{new}}] = 0. \quad (1.14)$$

Whenever such an i^{new} can be found, then it is immediately possible to find an associated integral i^{old} of the original Hamiltonian. In analogy to (1.13) we define i^{old} in terms of i^{new} by the rule

$$i^{\text{old}} = \exp(-F_3) \exp(-F_4) \exp(-F_3) \dots i^{\text{new}}. \quad (1.15)$$

We then find, using the definitions and (1.5), that

$$\begin{aligned} [i^{\text{old}}, h^{\text{old}}] &= [\exp(-F_3) \exp(-F_4) \dots i^{\text{new}}, \exp(-F_3) \\ &\quad \times \exp(-F_4) \dots h^{\text{new}}] \\ &= \exp(-F_3) \exp(-F_4) \dots [i^{\text{new}}, h^{\text{new}}] = 0. \end{aligned} \quad (1.16)$$

Hence, i^{old} is an integral of motion for the Hamiltonian h^{old} . That is,

$$\frac{di^{\text{old}}}{dt} = [i^{\text{old}}, h^{\text{old}}] = 0. \quad (1.17)$$

Of course, even when i^{new} is a simple expression in terms of the p 's and q 's as will prove to be the case in Sec. 3, i^{old} will in general be very complicated because of the Lie transformations indicated in (1.15).

Still more will prove to be possible. It is evident from (1.13), with the aid of (1.11) and (1.12), that h^{old} and h^{new} are related by a canonical transformation. We have

$h^{\text{new}}(z) = h^{\text{old}}(\bar{z}(z))$. Therefore, if because of its simple form one can find the frequencies and initial conditions for the periodic and quasiperiodic orbits generated by h^{new} , then it is easy to deduce the equivalent information for h^{old} . We will see in Sec. 4 that this is indeed the case.

2. NORMAL FORM ALGORITHM

We begin by assuming that the canonical coordinates z have been selected in such a way that the origin in phase space is an equilibrium point. Thus if the Hamiltonian h^{old} is expanded about the origin, we obtain an expression of the form

$$h^{\text{old}} = \sum_2^{\infty} h_i^{\text{old}}(z), \quad (2.1)$$

where each h_i^{old} is a homogeneous polynomial of degree i . Next, we assume that the linearized equations of motion about the equilibrium point have m zero frequencies ($m < n$) and $n - m$ nonzero frequencies. That is, we assume that with a suitable choice of coordinates h_2^{old} has the form

$$\begin{aligned} h_2^{\text{old}} &= \frac{1}{2} (p_1^2 + \dots + p_m^2) + (\frac{1}{2} \alpha_{m+1}) (p_{m+1}^2 + q_{m+1}^2) \\ &\quad + \dots + (\frac{1}{2} \alpha_n) (p_n^2 + q_n^2), \end{aligned} \quad (2.2)$$

where all the α 's are positive.

Hamiltonians of this form arise naturally in the study of mirror machines. How this comes about in detail will become apparent in Sec. 4, where we study two explicit examples. Roughly speaking, one can say that a degree of freedom for which a frequency is zero corresponds to motion along a magnetic field line. For this motion there is no restoring force in lowest approximation. By contrast, the degrees of freedom associated with nonzero frequencies correspond to motion across field lines; and in this case there is a restoring force even in first approximation.

As explained in the introduction, our goal is to find functions f_3, f_4, \dots , such that h^{new} given by (1.13) has a simple form. To study systematically what possibilities exist, it is convenient to introduce the notation

$$h^k = \exp(F_k) \exp(F_{k-1}) \dots \exp(F_3) h^{\text{old}}. \quad (2.3)$$

Then we have, for example, the relations

$$h^3 = \exp(F_3) h^{\text{old}}, \quad (2.4a)$$

$$h^{\infty} = h^{\text{new}}, \quad (2.4b)$$

and the recursion formula

$$h^k = \exp(F_k) h^{k-1}, \quad k \geq 3, \quad \text{and} \quad h^2 = h^{\text{old}}. \quad (2.5)$$

In analogy to the notation of (2.1), let us write

$$h^k = \sum_{j=2}^{\infty} h_j^k \quad (2.6)$$

where each term h_j^k is a homogeneous polynomial of degree j . Then from (2.5) we have the relation

$$\sum_{j=2}^{\infty} h_j^k = \exp(F_k) \sum_{i=2}^{\infty} h_i^{k-1}, \quad k \geq 3. \quad (2.7)$$

Evidently, Eq. (2.7) implies the equality of terms of like degree. Our problem is to identify them. Let \mathcal{P}_k denote the set of homogeneous polynomials of degree k , and suppose f_i and g_j are two homogeneous polynomials of degree i and j , respectively. Then, since the Poisson bracket operation involves multiplication and two differentiations, we have the relation

$$[f_i, g_j] \in \mathcal{P}_{i+j-2}. \quad (2.8)$$

Employing this observation and the definition (1.3), we find from (2.7) the relations

$$h_2^k = h_2^{k-1} = h_2^{\text{old}}, \quad (2.9a)$$

$$h_j^k = h_j^{k-1}, \quad j < k \quad \text{and} \quad k \geq 3, \quad (2.9b)$$

$$h_k^k = h_k^{k-1} + F_k h_2^{\text{old}}. \quad (2.9c)$$

We can now draw several conclusions. First, we see that $h_2^{\text{new}} = h_2^{\text{old}}$ and $h_k^{\text{new}} = h_k^k$. Second, the term h_k^{new} depends in a rather complicated way on F_k, F_{k-1}, \dots, F_3 and $h_k^{\text{old}}, h_{k-1}^{\text{old}}, \dots, h_2^{\text{old}}$. Finally, if our goal is to make h_k^{new} "simple," then by (2.9c), our last chance to do so occurs at the stage at which F_k is determined.

It is apparent that (2.9c) is a key relation. In keeping with the notation of Sec. 1, let H_2 be the Lie operator associated with h_2^{old} . Then we have the relation

$$F_k h_2^{\text{old}} = [f_k, h_2^{\text{old}}] = -H_2 f_k. \quad (2.10)$$

Thus, we can also write (2.9c) in the form

$$h_k^k = h_k^{k-1} - H_2 f_k. \quad (2.11)$$

For further discussion, it is useful to regard all polynomials of degree k as elements of a vector space. Then H_2 may be regarded as a linear operator mapping \mathcal{P}_k on to itself. Evidently, the term $H_2 f_k$ consists of all homogeneous polynomials in \mathcal{P}_k that are in the range of the operator H_2 . (A vector y is in the range of an operator A if there exists a vector x such that $y = Ax$.) Thus, with the aid of (2.11) we are able, by a suitable choice of f_k , to adjust h_k^{new} by any polynomial in \mathcal{P}_k lying within the range of H_2 . This is the fundamental result which we shall use in the rest of this paper.

The exploration of the range of a linear operator is facilitated by the introduction of a scalar product. When a scalar product is defined, the Hermitian adjoint of H_2 , denoted by H_2^\dagger , is also defined. Indicating the scalar product operation by angular brackets, we have the relation

$$\langle a, H_2^\dagger b \rangle = \langle H_2 a, b \rangle. \quad (2.12)$$

The virtue of the introduction of a scalar product is that we can then use the result that each subspace \mathcal{P}_k can be decomposed into a direct sum in the form

$$\mathcal{P}_k = \mathcal{R}_k \oplus \mathcal{N}_k, \quad (2.13)$$

where \mathcal{R}_k denotes the range of H_2 and \mathcal{N}_k denotes the null space of H_2^\dagger .

The correctness of this result is easily verified for any operator A . First, note that the range of an operator A is itself a linear vector space. For suppose that y and y' are contained in the range of A . Then there exist vectors x, x' such that $y = Ax$ and $y' = Ax'$. Let α and α' be any two scalars. We have $\alpha y + \alpha' y' = A(\alpha x + \alpha' x')$, and hence $\alpha y + \alpha' y'$ is also in the range of A . Now let the vectors u_1, u_2, \dots , form a basis for the range of A . Without loss of generality, they can be selected to be orthonormal thanks to the Gram-Schmidt process.¹⁰ Second, let v_1, v_2, \dots be the remaining orthonormal basis vectors needed to span the complete space. By construction, the v 's can be taken to be orthogonal to the u 's, and hence to the range of A . They will then also be in the null space A^\dagger . That is, we will have $A^\dagger v_j = 0$. For let w be any vector. We find $\langle w, A^\dagger v_j \rangle = \langle Aw, v_j \rangle = 0$ because Aw is in the range of A . It follows that $A^\dagger v_j = 0$ since w is an arbitrary vector. Conversely, any vector z in the null space of A^\dagger will be orthogonal to the range of A . For if y is in the range of A , we have

$\langle y, z \rangle = \langle Ax, z \rangle = \langle x, A^\dagger z \rangle = 0$. The verification is now complete, because any vector can be written as a linear combination of the u 's and v 's since together they form a basis for the entire space. The portion of the expansion which involves the u 's will be in the range of A , and the remaining portion involving the v 's will be in the null space of A^\dagger .

We next consider the choice of a suitable scalar product. Its discovery requires a bit of trial and error. We have found the following definition to be convenient. Let $|j; m\rangle$ denote the monomial defined by

$$|j_1 \dots j_n; m_1 \dots m_n\rangle = \prod_{i=1}^n \left[\frac{(2j_i)!}{(j_i - m_i)!(j_i + m_i)!} \right]^{1/2} p_i^{j_i - m_i} q_i^{j_i + m_i}. \quad (2.14)$$

In this expression each j_i is positive or zero, each j_i satisfies $-j_i \leq m_i \leq j_i$, and each j_i and m_i is integral or half integral. For $|j; m\rangle$ to belong to \mathcal{P}_k , we require $2(j_1 + \dots + j_n) = k$. The monomials $|j; m\rangle$ are linearly independent and clearly form a basis. Our scalar product will be defined by the requirement that they form an orthonormal basis,

$$\langle j'; m' | j; m \rangle = \delta_{j'j} \delta_{m'm}. \quad (2.15)$$

Here, the quantity δ_{jj} equals +1 if all the indices denoted by j' and j are respectively equal, and it is zero otherwise.

The computation of H_2^\dagger is a simple task. From (2.2) we see that h_2^{old} consist of the squares p_i^2 and q_i^2 . For these functions we use the notation $\text{ad}(p_i^2)$ and $\text{ad}(q_i^2)$ to denote the associated Lie operators since in this case the capital letter convention is not convenient.¹¹ Then using (1.2) and (2.14), we find upon computing the required Poisson bracket that

$$\begin{aligned} \text{ad}(q_i^2) |j_1 \dots j_n; m_1 \dots m_n\rangle &= 2(j_i + m_i + 1)^{1/2} (j_i - m_i)^{1/2} \\ &\times |j_1 \dots j_n; m_1 \dots, m_i + 1, \dots m_n\rangle, \end{aligned} \quad (2.16a)$$

$$\begin{aligned} \text{ad}(p_i^2) |j_1 \dots j_n; m_1 \dots m_n\rangle &= -2(j_i - m_i + 1)^{1/2} (j_i + m_i)^{1/2} \\ &\times |j_1 \dots j_n; m_1 \dots, m_i - 1, \dots m_n\rangle. \end{aligned} \quad (2.16b)$$

We observe that, in analogy to quantum mechanics, $\text{ad}(q_i^2)$ behaves like twice an angular momentum raising operator. Similarly, $\text{ad}(p_i^2)$ behaves like twice the negative of an angular momentum lowering operator.¹² It follows immediately or by direct computation that

$$\text{ad}(q_i^2)^\dagger = -\text{ad}(p_i^2), \quad (2.17a)$$

$$\text{ad}(p_i^2)^\dagger = -\text{ad}(q_i^2). \quad (2.17b)$$

We are ready to specify our choice of f_k in relation (2.11). Using (2.13), we can uniquely write

$$h_k^{k-1} = r_k + n_k, \quad (2.18)$$

where r_k is in the range of H_2 and n_k is in the null space of H_2^\dagger . Next, we require that f_k satisfy the equation

$$H_2 f_k = r_k. \quad (2.19)$$

This equation always has a solution because r_k is in the range

of H_2 by construction.¹³ With this choice for f_k , we find from (2.11) and (2.18) the result

$$h_k^{\text{new}} = h_k^k = n_k, \quad k \geq 3. \quad (2.20)$$

That is, it is always possible to choose f_k in such a way that h_k^{new} for $k \geq 3$ is in the null space of H_2^\dagger .

At this point we should make clear to the reader that we do not maintain that the imposition of (2.19) is always the optimal procedure. Indeed, we are studying other strategies, which will be the subject of another paper.¹⁴ However, the above procedure is clearly a mathematically attractive option worth exploring. We shall see in the next section that it has interesting physical consequences because it leads directly to the formal construction of integrals of motion.

3. INTEGRALS OF MOTION

In the last section, we made a partial exploration of how the choice of the f_k affected the form of h^{new} . We found that by imposing (2.19), it was possible to arrange that each term h_k^{new} in h^{new} (save for h_2^{new}) would be in the null space of H_2^\dagger . In this section we will show that this choice leads to the determination of at least one and perhaps several integrals of motion for h^{new} .

Let us express the function h_2^{old} given by (2.2) in the form

$$h_2^{\text{old}} = c + d, \quad (3.1)$$

where c and d are given by

$$c = \frac{1}{2} \alpha_{m+1} (p_{m+1}^2 + q_{m+1}^2) + \dots + \frac{1}{2} \alpha_n (p_n^2 + q_n^2), \quad (3.2a)$$

$$d = \frac{1}{2} (p_1^2 + \dots + p_m^2). \quad (3.2b)$$

We shall show that c is an integral of motion for h^{new} . That is, c satisfies the equation $[c, h^{\text{new}}] = 0$.

The proof requires a series of steps. First, suppose that $|n_s\rangle$ is a homogeneous polynomial of degree s . Imagine that $|n_s\rangle$ is expanded as a linear combination of the basis vectors $|j; m\rangle$ given by (2.14). Evidently we must have $2(j_1 + \dots + j_n) = s$ for every term in the expansion and hence $2j_i \leq s$ for every factor in each $|j; m\rangle$. It follows that

$$[\text{ad}(p_i^\dagger)]^{s+1} |n_s\rangle = 0 \quad i = 1, 2, \dots, m, \quad (3.3)$$

since $\text{ad}(p_i^\dagger)$ is proportional to a raising operator by (2.16a) and (2.17b).

Let C and D denote the Lie operators associated with c and d , respectively, and consider the quantity $[D^\dagger]^{m(s+1)} |n_s\rangle$. Since all the $\text{ad}(p_i^\dagger)$ commute with each other, we can expand this quantity to obtain an expression of the form

$$[D^\dagger]^{m(s+1)} |n_s\rangle = \sum_{\rho} \beta(\rho) [\text{ad}(p_1^\dagger)]^{\rho_1} \dots [\text{ad}(p_m^\dagger)]^{\rho_m} |n_s\rangle, \quad (3.4)$$

where the $\beta(\rho)$ are certain coefficients. We note that in each term the exponents are non-negative and must satisfy $\rho_1 + \dots + \rho_m = m(s+1)$. It follows that in each term there must be at least one exponent ρ_i such that $\rho_i \geq (s+1)$. This

implies by (3.3) that

$$[\text{ad}(p_i^\dagger)]^{\rho_i} |n_s\rangle = 0. \quad (3.5)$$

Therefore, we must have

$$[D^\dagger]^{m(s+1)} |n_s\rangle = 0. \quad (3.6)$$

From (3.1) we have

$$H_2^\dagger = C^\dagger + D^\dagger, \quad (3.7)$$

Also, from (2.17) and (3.2a), C is antihermitian, that is,

$$C^\dagger = -C. \quad (3.8)$$

Solving (3.7) for D^\dagger and inserting the result into (3.6) gives

$$[C + H_2^\dagger]^{m(s+1)} |n_s\rangle = 0. \quad (3.9)$$

Since C and H_2^\dagger commute, the left-hand side of (3.9) can be expanded to give

$$[C^{m(s+1)} + m(s+1)C^{m(s+1)-1}H_2^\dagger + \dots + (H_2^\dagger)^{m(s+1)}] |n_s\rangle = 0. \quad (3.10)$$

Let us now add the further hypothesis, as our notation may already have suggested, that $|n_s\rangle$ is in the null space of H_2^\dagger ,

$$H_2^\dagger |n_s\rangle = 0. \quad (3.11)$$

Then, all the terms on the left-hand side, except for the first, automatically annihilate $|n_s\rangle$, and we conclude

$$C^{m(s+1)} |n_s\rangle = 0. \quad (3.12)$$

We are almost done. Since C is antihermitian and maps \mathcal{P}_s into itself, we know that its eigenvectors in \mathcal{P}_s must form a complete set in \mathcal{P}_s . Thus we can write an expansion of the form

$$|n_s\rangle = \sum_{\gamma} \beta_{\gamma} |\gamma\rangle, \quad (3.13)$$

where the β_{γ} are certain coefficients and the polynomials $|\gamma\rangle$ are linearly independent and satisfy eigenvector relations of the form

$$C |\gamma\rangle = \nu_{\gamma} |\gamma\rangle. \quad (3.14)$$

Insertion of the expansion (3.13) into (3.12) and use of (3.14) gives the result

$$\sum_{\gamma} \beta_{\gamma} \nu_{\gamma}^{m(s+1)} |\gamma\rangle = 0. \quad (3.15)$$

But, since the polynomials are linearly independent, we must then have $\beta_{\gamma} \nu_{\gamma}^{m(s+1)} = 0$ for every γ , which in turn implies $\beta_{\gamma} \nu_{\gamma} = 0$ for every γ . From this we conclude that

$$C |n_s\rangle = \sum_{\gamma} \beta_{\gamma} \nu_{\gamma} |\gamma\rangle = 0. \quad (3.16)$$

We have shown that if $|n_s\rangle$ is in the null space of H_2^\dagger it must also be in the null space of C .

The result we have been working to prove now follows immediately. Thanks to our normal form algorithm, we have arranged that each h_k^{new} for $k \geq 3$ is in the null space of H_2^\dagger , and hence

$$Ch_k^{\text{new}} = 0 \quad \text{for } k \geq 3. \quad (3.17)$$

Moreover, it is easily checked that

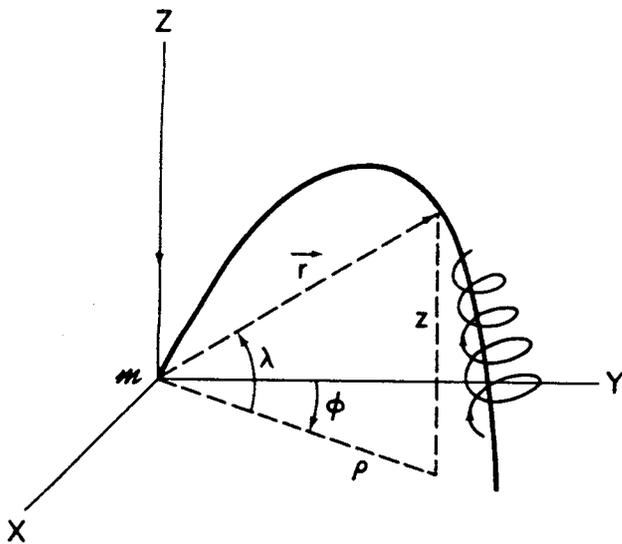


FIG. 1. Motion of a trapped particle in a magnetic dipole field.

$$Ch_2^{\text{new}} = Ch_2^{\text{old}} = [c, h_2^{\text{old}}] = 0. \quad (3.18)$$

Consequently, we have

$$Ch^{\text{new}} = [c, h^{\text{new}}] = 0, \quad (3.19)$$

and c is an integral of motion of h^{new} as advertised.

We have seen that the normal form algorithm of Sec. 2 leads to the determination of an integral of motion i^{new} for h^{new} , namely $i^{\text{new}} = c$. In some cases, for example when the frequencies α_i are irrational in a way as to be incommensurate, it is possible to exhibit additional integrals. This is shown in the Appendix.

It is worth remarking at this point that the normal form algorithm required for mirror machine Hamiltonians is considerably more complicated than that used in celestial mechanics. In the latter case it can be shown that $H_2^+ = -H_2^-$, and then the analysis is far simpler than the preceding has been.

4. EXAMPLES AND APPLICATIONS

In this section we study two Hamiltonian systems which describe the motion of charged particles in magnetic mirror geometries. The first problem considered is that of the motion of a charged particle in a magnetic dipole field, the so-called Störmer problem. This problem is an idealized description of the Van Allen radiation. The second system considered is a simple model mirror machine characterized by the magnetic field given in cylindrical coordinates (ρ, ϕ, z) by

$$\mathbf{B} = (B_0/a^2) [-\rho z \hat{e}_\rho + (a^2 + z^2) \hat{e}_z]. \quad (4.1)$$

The variable a is a typical length scale for the mirror.

Our major tool for dealing with these systems is the use of the normal form algorithm. The algebra involved in carrying out the procedure is very lengthy, but completely routine. Therefore, we have programmed a digital computer to carry out the necessary steps. In brief, we have written routines using the language FORMAL to carry out the decompo-

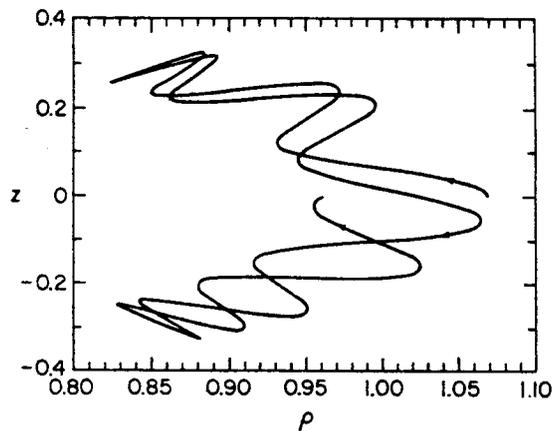


FIG. 2. A trapped orbit as seen in ρ, z coordinates. The initial conditions are $z = 0, \rho = 1.07, \dot{\rho} = 0, \dot{z} = 0.0355$.

sition (2.18), solve (2.19) for f_k , and perform (2.5) to move from h^{k-1} to h^k . The calculations were performed through sixth order for the Störmer problem [i.e., $k = 6$ in Eq. (2.3)] and through ninth order for the model mirror machine. Typical calculations required 45 min of Univac 1108 time. We expect that specially written routines for the same purpose which are currently under development will require considerably less computer time.

Figure 1 shows the motion of a typical particle trapped by a magnetic dipole field. When the equations of motion are written in cylindrical coordinates, the axial symmetry associated with a dipole field and scaling of space and time can be used to reduce the problem to the determination of the orbits governed by the reduced Hamiltonian

$$h(p_\rho, p_z, \rho, z) = \frac{1}{2}(p_\rho^2 + p_z^2) + \frac{1}{2}(1/\rho - \rho/r^2)^2. \quad (4.2)$$

That is, due to axial symmetry, the problem is reduced to one having two degrees of freedom. Once the motion in the ρ, z plane is determined so that $\rho(t)$ and $z(t)$ are known, $\phi(t)$ can be found by a quadrature. Details are given in Refs. 2 and 4. Figure 2 shows a typical orbit as it appears in the ρ, z plane.

The Hamiltonian (4.2) is not in the form of a power series, and consequently we cannot apply the normal form algorithm directly. However, we observe from Figs. 1 and 2

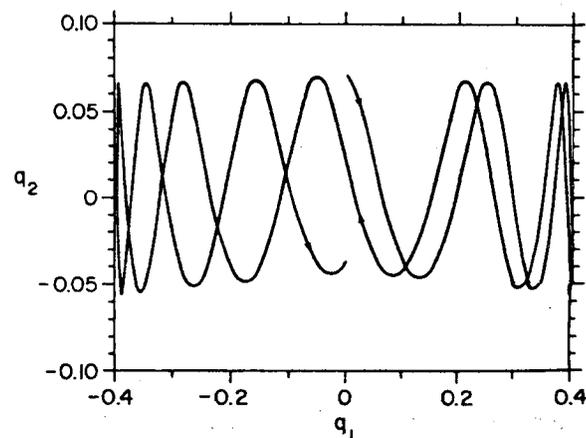


FIG. 3. The orbit of Fig. 2 as it appears in dipolar coordinates.

that the motion consists of gyration about a field line superimposed upon motion along a field line. We therefore introduce orthogonal dipolar coordinates q_1 and q_2 given by

$$q_1 = z/r^3, \quad q_2 = r^3/\rho^2 - 1. \quad (4.3)$$

Roughly speaking, the coordinate q_1 describes motion along the confining field line, i.e., the guiding center motion, and q_2 describes motion perpendicular to the field line. This fact is illustrated in Fig. 3, which displays the orbit of Fig. 2 as it appears in dipolar coordinates. The motion has now been separated, in first approximation, into oscillations about $q_2 = 0$ superimposed upon motion along the q_1 axis.

Now let p_1 and p_2 be momenta canonically conjugate to q_1 and q_2 . Then after calculation,¹⁵ one finds that the Hamiltonian (4.2) when expressed in terms of these new variables has a power series expansion. Explicitly, employing the notation (2.1), one finds for the first four terms

$$\begin{aligned} h_2^{\text{old}} &= \frac{1}{2}p_1^2 + \frac{1}{2}(p_2^2 + q_2^2), \\ h_3^{\text{old}} &= \frac{1}{2}(-4q_2^3 - 6q_2p_1^2), \\ h_4^{\text{old}} &= \frac{1}{2}(6p_2^2q_1^2 + 21q_2^2p_1^2 + 10q_2^4 + 3q_1^2q_2^2 + 9q_1^2p_1^2). \end{aligned} \quad (4.4)$$

Note that h_2^{old} has the form (2.2).

Before continuing, we should make a remark about the transformation (4.3). It can be shown that for $q_2 = 0$, the transformation from $\lambda = \tan^{-1}(\rho/z)$ to q_1 has a singularity for complex values of q_1 and is analytic only for $|q_1| < 3(3)^{1/2}/16$.

As a consequence, neither the Hamiltonian (4.4) nor the results derived from its normal form are expected to have meaning for $|q_1| > 3\sqrt{3}/16$.

The stage is set for the application of the normal form algorithm of Sec. 2. The coefficients of the power series expansion (4.4) are inserted into a properly coded computer program. Sometime later the coefficients for the f_k generating (2.3) and the coefficients for the h_k^{new} emerge.

All the results obtained are too lengthy to record here. We find, for example, that f_3 is given by

$$f_3 = -2p_2q_2^2 - 3p_1^2p_2 - \frac{4}{3}p_2^3. \quad (4.5)$$

The higher f 's rapidly become much longer expressions and are of little direct interest.

The normal form Hamiltonian is of direct interest. Through terms of order 6, and using a notation similar to (3.2a), that is $c_2 = \frac{1}{2}(p_2^2 + q_2^2)$, the normal form Hamiltonian is given by the expression

$$\begin{aligned} h^{\text{new}} &= \frac{1}{2}p_1^2 + c_2 + \left(\frac{9}{2}c_2q_1^2 - \frac{15}{2}c_2^2\right) \\ &+ \left(\frac{39}{8}c_2q_1^4 + \frac{9}{16}c_2^2q_1^2 + \frac{15}{2}c_2^3\right). \end{aligned} \quad (4.6)$$

We see that h^{new} is of the functional form

$$h^{\text{new}} = \frac{1}{2}p_1^2 + g(c_2, q_1). \quad (4.7)$$

That is, the normal form Hamiltonian is "simple" in the sense that it depends on the variables p_2, q_2 only in the combination $c_2 = \frac{1}{2}(p_2^2 + q_2^2)$. This is, of course, what is to be expected because according to (3.19), c_2 is an integral of motion for h^{new} .

It is also of interest to record some terms of the integral of motion i^{old} obtained from (1.15) with $i^{\text{new}} = c_2$. They are also calculated by our computer program. We write $i^{\text{old}} = i_2^{\text{old}} + i_3^{\text{old}} + i_4^{\text{old}} + \dots$, etc. The first few terms are given by

$$\begin{aligned} i_2^{\text{old}} &= \frac{1}{2}(p_2^2 + q_2^2), \\ i_3^{\text{old}} &= -3p_1^2q_2 - 2q_2^3, \\ i_4^{\text{old}} &= \frac{27}{8}p_1^2p_2^2 + \frac{117}{8}p_1^2q_2^2 + \frac{3}{4}p_2^2q_1^2 + \frac{15}{4}p_2^2q_2^2 - \frac{3}{4}q_1^2q_2^2 \\ &\quad - \frac{3}{2}p_1p_2q_1q_2 + \frac{9}{2}p_1^4 + \frac{15}{8}p_2^4 + \frac{55}{8}q_2^4. \end{aligned} \quad (4.8)$$

The industrious reader is invited to verify for himself that the Poisson bracket (1.17) in fact vanishes using the expansions (4.4) and (4.8). Incidentally, that it does so has been verified directly as a check by a computer programmed Poisson bracket routine.

The series i^{old} has great utility in the examination of the nature of motion in a dipole field in fine detail. It can be used, among other things, to show that the Størmer problem is insoluble. What this means is described extensively elsewhere.⁴ We shall give a parallel but much abbreviated treatment of the question of insolubility later on in this section when we discuss the model mirror machine example.

As advertised in the first section of this paper, the normal form Hamiltonian may be sufficiently simple that it is possible to find the frequencies and initial conditions for periodic and quasiperiodic orbits. We shall now see that this is the case for the Størmer problem.

Observe that the Hamiltonian (4.6) is of the form

$$h^{\text{new}} = \beta(c_2) + \frac{1}{2}p_1^2 + \frac{1}{2}\omega^2(c_2)q_1^2 + \eta(c_2)q_1^4 + \dots, \quad (4.9)$$

where

$$\beta(c_2) = c_2 - \frac{15}{2}c_2^2 + \frac{15}{2}c_2^3 + \dots,$$

$$\omega^2(c_2) = 9c_2 + \frac{9}{8}c_2^2 + \dots,$$

$$\eta(c_2) = \frac{39}{8}c_2 + \dots. \quad (4.10)$$

Since c_2 is an integral of motion and therefore constant in time, we conclude that the motion in p_1, q_1 governed by h^{new} is that of an anharmonic oscillator described by the c_2 dependent parameters $\omega^2(c_2), \eta(c_2)$, etc. This circumstance suggests that we should attempt to bring the quadratic part of (4.9) to the form $(\frac{1}{2}\alpha)(p_1^2 + q_1^2)$, which is analogous to (3.2a) as far as the variables p_1, q_1 are concerned, and then we should again apply some normal form algorithm.

Consider the canonical transformation generated by the function g_2 given by

$$g_2 = \frac{1}{2}p_1q_1 \log \omega(c_2). \quad (4.11)$$

Note that g_2 is homogeneous of degree 2 as far as the variables q_1, p_1 are concerned. We find that

$$\begin{aligned} \exp(G_2)c_2 &= c_2, \\ \exp(G_2)p_1 &= p_1\omega^{1/2}, \\ \exp(G_2)q_1 &= q_1/\omega^{1/2}. \end{aligned} \quad (4.12)$$

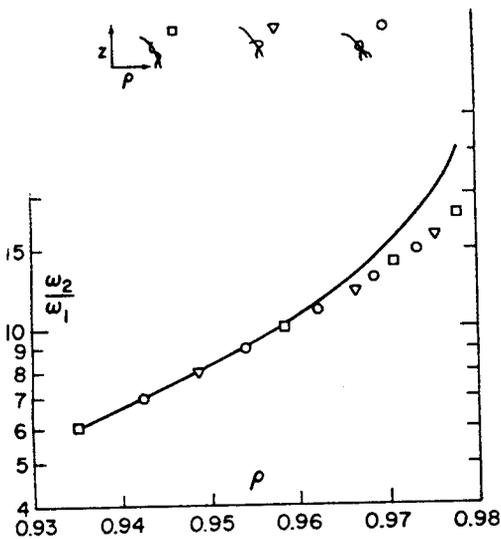


FIG. 4. The frequency ratio ω_2/ω_1 , as computed from Eq. (4.17), plotted as a function of the value of ρ . The other initial conditions are $z = 0, \dot{\rho} = 0$, and $h = 0.002907$. For comparison, actual periodic orbits obtained numerically are plotted as points. Initial conditions for these orbits were supplied by R. DeVogelaere (private communication). The type of orbit is shown at the top of the figure.

Consequently, we find that

$$\exp(G_2)h^{\text{new}} = \beta(c_2) + \frac{1}{2}\omega(c_2)(p_1^2 + q_1^2) + \gamma(c_2)q_1^4 + \dots, \quad (4.13)$$

where

$$\gamma(c_2) = \eta/\omega^2. \quad (4.14)$$

We see that apart from the term $\beta(c_2)$, which plays no role in the determination of the p_1, q_1 motion, the transformed Hamiltonian (4.13) has a quadratic part of the form (3.1) with the term of the form (3.2b) completely absent. Now consider the Lie operator associated with this quadratic part. It will be antihermitian because of the analog of (3.8), and hence the null spaces of the analogs of H_2 and H_2^\dagger will coincide in this case.

Let h^* denote the result of applying the normal form algorithm a second time. We write $h^* = \dots \exp(G_4) \exp(G_3) \exp(G_2) h^{\text{new}}$ where the functions g_3, g_4, \dots which lead to G_3, G_4, \dots and which may involve c_2 as a parameter are still to be determined. It follows from the discussion in the previous paragraph that one can arrange to have h^* lie in the null space of $\text{ad}(p_1^2 + q_1^2)$. If this is done, h^* will only depend on the variables p_1, q_1 in the combination $c_1 = \frac{1}{2}(p_1^2 + q_1^2)$. Of course, h^* may also depend on c_2 . After calculation employing the normal form algorithm, we find the explicit result

$$h^*(c_1, c_2) = \beta(c_2) + \omega(c_2)c_1 + 13c_1^2/(16 + 2c_2) - 25857/62208c_2^{-1/2} [1 + (c_2/8)]^{-5/2} c_1^3 + \dots \quad (4.15)$$

Let us introduce variables ϕ_1 and ϕ_2 which are canonically conjugate to the variables c_1 and c_2 by means of the equations

$$q_i = (2c_i)^{1/2} \sin \phi_i, \quad p_i = (2c_i)^{1/2} \cos \phi_i. \quad (4.16)$$

Evidently the pair c_i, ϕ_i for each i are action-angle variables. Consequently, we have

$$\dot{\phi}_i = \left(\frac{\partial h^*}{\partial c_i} \right) = \omega_i(c_1, c_2). \quad (4.17)$$

Since the c_i are integrals of motion (h^* is independent of the ϕ_i thanks to the normal form algorithm), the frequencies ω_i are independent of the time. Consequently, Eq. (4.17) can be integrated directly to give

$$\phi_i = \omega_i t + \phi_i^0. \quad (4.18)$$

Thus, combining (4.16) through (4.18), we find that the motion of the q_i, p_i is periodic with the frequencies ω_1 and ω_2 . Since h^{old} and h^* are related by a canonical transformation, it follows that the motion described by h^{old} must be quasi-periodic with the two fundamental frequencies ω_1 and ω_2 . In particular, orbits for which the ratio ω_2/ω_1 is a rational number will be closed, and therefore will be completely periodic.

Suppose we consider all the orbits for which the energy has a particular set value and for which the fundamental frequency ratio is rational, $\omega_2/\omega_1 = m/n$. Since h^* and the ω_i depend only on c_1 and c_2 , these two conditions determine the values of the integrals c_1 and c_2 . However, ϕ_1^0 and ϕ_2^0 are undetermined. Consequently, we expect that those periodic orbits having a particular energy and frequency ratio will form a two-dimensional surface in phase space. However, direct numerical integration of orbits for the Störmer problem shows that this is not the case. Instead, one finds that for a fixed energy, there are only a finite number of orbits having a specified rational value for ω_1/ω_2 . Put another way, for a fixed energy, the closed (and therefore periodic) orbits in phase space are isolated curves, and do not form a two-dimensional surface.¹⁶ It follows that the normal form process that we have described in this paper must be divergent for the Störmer problem.

We are currently working on a different normal form procedure in order to overcome this difficulty.¹⁴ However, we wish to point out here that the apparently formal expres-

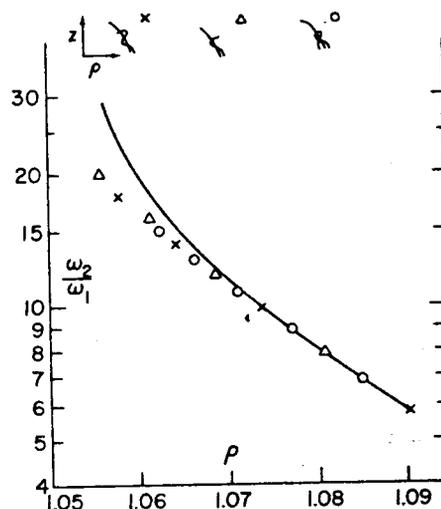


FIG. 5. A continuation of Fig. 4 showing analogous results for those orbits whose initial conditions are such that $\rho > 1$.

TABLE I. A comparison of initial conditions for periodic orbits estimated using the normal form (4.21) in (4.17) and initial conditions obtained exactly by the numerical integration of trajectories. There are no orbits with rotation number less than 10 for this energy ($h = 0.01$).

Rotation number ω_2/ω_1	Estimated q_2	Numerically determined q_2
10	0.1413772	0.1413772
11	0.1349294	0.1349264
12	0.1294129	0.1294038
13	0.124624	0.124608
14	0.120415	0.120393
15	0.116678	0.116650
16	0.113330	0.113298
17	0.110308	0.110272
18	0.107561	0.107523
19	0.105050	0.105009
20	0.102743	0.102700
22	0.098638	0.098592
24	0.095083	0.095035
27	0.090542	0.090493
30	0.086722	0.086672
33	0.083447	0.083397
36	0.080596	0.080547
39	0.078084	0.078036

sions (4.15) and (4.17) for the ω_i still are useful. For when periodic orbits are located numerically and ω_2/ω_1 is then computed from (4.15) and (4.17) using the numerically determined initial conditions, one finds that the ratio is in fact nearly rational. Figures 4 and 5 illustrate how well this works. From these figures we see that the results are quite accurate for nearly equatorial orbits, but are worse for orbits whose mirror points are further down the guiding field line. Moreover, we remark that for initial values of ρ satisfying $0.9383 < \rho < 1.082$, which includes most of the interesting region of the figures, the value of q_1 exceeds $3\sqrt{3}/16$ somewhere on the trajectory, and consequently we expect trouble on that basis alone. Thus, the accuracy of our results is much better than we have any reason to expect.

We devote the remainder of this section to a discussion of orbits in the simple model mirror machine whose magnetic field is given by (4.1). In particular, we shall focus our attention on those orbits for which $p_\phi = 0$. These orbits do not encircle the central axis of the mirror machine, but instead pass continually through it. Hence, one cannot use energy conservation arguments to preclude their extending arbitrarily far down the length of the machine. After suitable scaling, we arrive at the simple Hamiltonian

$$h^{\text{old}} = \frac{1}{2}p_1^2 + \frac{1}{2}(p_2^2 + q_2^2) + \frac{1}{2}q_1^2q_2^2 + \frac{1}{8}q_1^4q_2^2. \quad (4.19)$$

Here q_1 is proportional to z and q_2 is proportional to ρ .

In this case the Hamiltonian is considerably simpler than that for the Størmer problem. Consequently, we were able to carry out the normal form algorithm to higher order within the computer time available. The result through terms of ninth order is

$$h^{\text{new}} = \frac{1}{2}p_1^2 + c_2 + \frac{1}{2}c_2q_1^2 + \frac{1}{16}c_2^2q_1^4 + \left(\frac{1}{16}c_2^3q_1^2 - \frac{5}{48}c_2^2q_1^4\right). \quad (4.20)$$

As before, we may now perform a variant of the normal form algorithm a second time to arrive at a final Hamiltonian which only depends on the variables c_1 and c_2 . The

$$h^*(c_1, c_2) = c_2 + (c_2 + \frac{1}{8}c_2^2 + \frac{1}{8}c_2^3)^{1/2}c_1 - \frac{5}{32}(1 + \frac{1}{8}c_2 + \frac{1}{8}c_2^2)c_1c_2^2 + \dots \quad (4.21)$$

A numerical study of the periodic orbits for a fixed value of the energy shows that they are again isolated, and do not form a two dimensional surface. In particular, various periodic orbits have initial conditions of the following form $q_1 = 0; p_2 = 0; q_2$ variable and taking on various discrete values; and p_1 determined by energy conservation once q_1, q_2 , and p_2 have been specified. Table I shows the initial value of q_2 for various periodic orbits in the case for which the equation $h^{\text{old}} = 0.01$ fixes the energy. Each integer in the column labeled "rotation number" is defined to be the number of oscillations undergone by the variable q_2 during one oscillation of q_1 . It corresponds physically to the number of gyrations a charged particle makes during one complete mirroring cycle, and should ideally equal the ratio ω_2/ω_1 computed by the normal form algorithm. This is very nearly the case. The column labeled "estimated q_2 " is the value of q_2 computed by requiring that the ratio ω_2/ω_1 , computed using (4.21) and (4.17) be exactly equal to the rotation number in question. (We have, of course, also required that $h^{\text{old}} = 0.01, p_2 = 0$, and $q_1 = 0$.)

The agreement between the columns "estimated q_2 " and "numerically determined q_2 " is remarkably good, and illustrates the utility of the normal form approach even though the series employed must ultimately be divergent. We observe that the agreement for the model machine problem is much better than it was for the Størmer problem. This is because in (4.20) the ratio of the coefficient of the terms involving q_1^2 is smaller by a factor of c_2 than the corresponding ratio for (4.6). This circumstance can be traced to the fact that (4.19) contains no terms of order 3 [whereas (4.4) does], and is a special feature of orbits with $p_\phi = 0$.

Because of its simplicity, the model mirror machine problem is an ideal context in which to examine the integral of motion produced by the normal form algorithm. We have computed i^{old} through terms of ninth order using (1.15) with $i^{\text{new}} = c_2$. The entire expression is too lengthy to record, and is best transferred directly from one computer program to another. The first few terms are given by

$$\begin{aligned} i_2^{\text{old}} &= \frac{1}{2}(p_2^2 + q_2^2), \\ i_3^{\text{old}} &= 0, \\ i_4^{\text{old}} &= \frac{1}{8}(p_2^2p_2^2 - p_2^2q_2^2 - 2p_2^2q_1^2 + 2q_1^2q_2^2 + 4p_2p_1q_1q_2). \end{aligned} \quad (4.22)$$

Examination of these terms and those of higher order shows that i^{old} contains within it all the terms in the power series expression for the magnetic moment, E_1/B :

$$\begin{aligned} E_1/B &= \frac{1}{2}p_2^2/(1 + \frac{1}{2}q_1^2) + \frac{1}{2}q_2^2 + \frac{1}{4}q_1^2q_2^2 \\ &= \frac{1}{2}(p_2^2 + q_2^2) + \frac{1}{4}(q_1^2q_2^2 - p_2^2q_1^2) + \dots \end{aligned} \quad (4.23)$$

There are also additional terms in i^{old} beyond these. They represent corrections required to make up the complete adiabatic magnetic moment expansion.

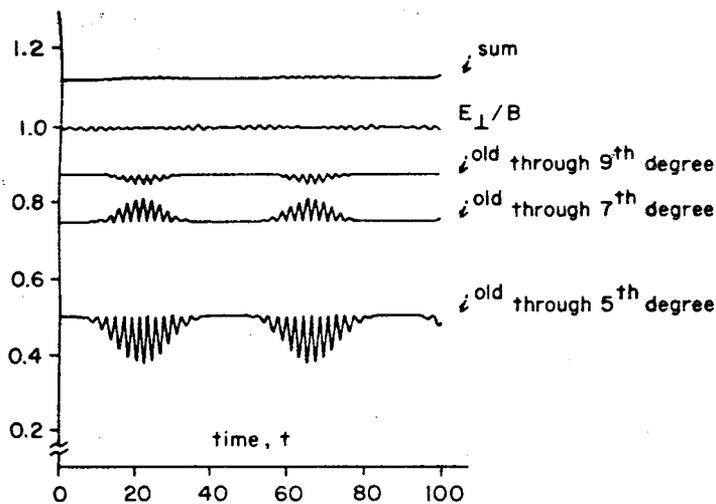


FIG. 6. The quantities i^{old} through various orders, E_{\perp}/B , and i^{sum} as functions of time for the orbit with initial conditions $q_1 = q_2 = 0$ and $\dot{q}_1 = 0.0707$, $\dot{q}_2 = \frac{1}{10}$. The mirror point is at $q_1 \approx 1.0$, and the magnetic field strength at the mirror point is 1.5 times its value at $q_1 = 0$. The vertical scale is arbitrary, and to separate the curves a different constant has been added to each.

Since i^{old} , as a series, contains all the terms in (4.23), it is reasonable to sum these terms explicitly. When this is done, we will obtain a quantity, denoted by i^{sum} , which contains E_{\perp}/B exactly plus all terms of the remainder of the complete adiabatic magnetic moment expansion through the highest degree to which i^{old} is computed. The quantity i^{sum} may be expected to be more nearly constant than i^{old} . More explicitly, through terms of order nine we write

$$i^{\text{sum}} = i^{\text{old}} - \frac{1}{2}p_2^2(1 - \frac{1}{2}q_1^2 + \frac{1}{4}q_1^4 - \frac{1}{8}q_1^6) + \frac{1}{2}p_2^2/(1 + \frac{1}{2}q_1^2). \quad (4.24)$$

Note that the series for the magnetic moment E_{\perp}/B given in (4.23) diverges for $|q_1| > (2)^{1/2}$. We therefore do not expect that the function i^{old} truncated at a finite degree will be a good integral of motion for orbits which mirror beyond $|q_1| = (2)^{1/2}$. But this does not rule out the possibility of i^{sum} being very nearly constant, when truncated at high degree.

To examine the constancy of i^{old} and i^{sum} as integrals of motion, we have integrated numerically the equations of motion. Figure 6 shows graphs of i^{old} , taken through 5th, 7th, and 9th order respectively, as a function of time over an orbit. The orbit starts at $q_1 = 0$ at $t = 0$, mirrors at $t \approx 22$, recrosses the median plane $q_1 = 0$ at $t \approx 44$, mirrors again at $t \approx 66$, etc. It is evident that the constancy of i^{old} improves as more terms in the series are included. (We will see later, however, that the series must ultimately diverge, so that improvement cannot continue indefinitely.) Also shown is the quantity i^{sum} , computed from (4.24) with i^{old} taken through ninth degree. It remains remarkably constant over the entire orbit. Finally, the quantity E_{\perp}/B is plotted to demonstrate that i^{old} is superior to the magnetic moment except at mirror points, and that i^{sum} is superior everywhere. Results of several numerical integration runs for a range of initial conditions show that this behavior holds in general.

We close this section with the presentation of numerical evidence that the model mirror machine problem, like the Störmer problem, is insoluble. What it means for a classical mechanics problem to be "insoluble" and how this can come about has been described in detail elsewhere.⁴ In essence, it means that there are, in fact, no analytic functions of the p 's and q 's which are integrals of motion and therefore satisfy (1.17). Consequently, trajectories in phase space are not confined to lie on analytic hypersurfaces. Instead, they may wander in a very complicated way, and are sufficiently complex so as to preclude their explicit representation. It also follows that the complete adiabatic magnetic moment expansion is divergent. Finally, it can also be shown that any of the standard methods of classical mechanics, such as perturbation series or solution of the Hamiltonian-Jacobi equation, must also fail. In particular, there is no known way of predicting the long-term behavior of trajectories; and in the case of mirror machines, long-term containment cannot be mathematically guaranteed.

One method of testing numerically for the existence of integrals of motion is to plot the values of some independent pair of variables each time a trajectory in phase space crosses the median plane $q_1 = 0$. If the result of such a plot is a collection of points which have the appearance of lying on a smooth curve, then the existence of an integral is suggested, although not proved. By contrast, if no such regularity occurs, the existence of an integral is ruled out. Figure 7 shows variations in i^{old} (through ninth order) plotted against p_2 for successive median plane crossings. It is evident that these points are scattered, and the existence of an integral of motion is precluded. Thus, the model mirror machine, despite its simplicity, is insoluble. More detail may be found in Ref. 3.

5. CONCLUDING SUMMARY

In Secs. 1 and 2 a partial exploration was made of the

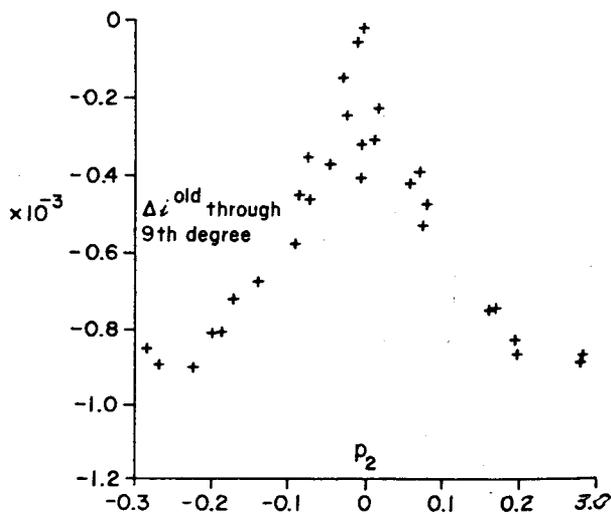


FIG. 7. Changes in the value of i^{old} through ninth degree plotted versus p_2 for successive median plane crossings. The initial conditions for this orbit are $q_2 = 0.299$, $p_2 = 0$, $q_1 = 0$, and p_1 , determined by the energy condition $H^{\text{old}} = \frac{1}{10}$.

effect of canonical transformations on mirror machine Hamiltonians, and a certain normal form was shown to be always possible. In Sec. 3 it was shown that this normal form led to the existence of a formal integral. Section 4 treated two examples of mirror machines, and normal form methods were used to obtain expressions for bounce frequencies and the location of periodic orbits. The lengthy algebraic calculations required were performed by computer. Good agreement was found between analytical and numerical results for periodic orbits. In addition, it was shown that the integral of motion produced by the normal form method is in fact a series for the complete adiabatic magnetic moment expansion. Finally, it was shown that, like the Störmer problem, the simple model mirror machine does not possess an analytic third integral of motion. Therefore its complete adiabatic magnetic moment expansion is divergent, phase space trajectories are not confined to lie on analytic hypersurfaces, and the problem, despite its apparent simplicity, is insoluble. In particular, there is no mathematical guarantee of long-term containment.

ACKNOWLEDGMENT

The first author is indebted to the Institut des Hautes Etudes Scientifiques where this work was begun. He wishes to thank Professor N. Kuiper and Professor L. Michel for their fine hospitality. We are both indebted to the University of Maryland Computer Science Center, which provided the computer time essential to this research, and to the National Science Foundation for their partial support.

APPENDIX

Suppose that the frequencies α_i in (2.2) exhibit some degree of incommensurability.¹⁷ In particular, we assume that there are l and only l linearly independent equations of commensurability between the frequencies $\alpha_{m+1}, \dots, \alpha_n$. We write these relations in the form

$$\sum_{j=m+1}^n \mathcal{M}_{ij} \alpha_j = 0, \quad i = 1, \dots, l. \quad (A1)$$

Equivalently, we write

$$\mathcal{M} \alpha = 0 \quad (A2)$$

where α is a vector with entries $\alpha_{m+1} \dots \alpha_n$, and \mathcal{M} is an $l \times (n - m)$ matrix with integer coefficients, rank l , and indexed so that $m + 1 \leq j \leq n$.

Using a notation similar to (3.2a), we write

$$c = \sum_{j=m+1}^n \alpha_j f_j \quad (A3)$$

where

$$c_j = \frac{1}{2}(p_j^2 + q_j^2). \quad (A4)$$

We also write

$$C = \sum_{j=m+1}^n \alpha_j C_j \quad (A5)$$

It is evident that the various Lie operators C_j mutually commute, since they involve differentiation with respect to dif-

ferent variables. Also, from (2.17) they are all antihermitian. Finally, they each map \mathcal{P}_k on to itself for each value of k . It follows that there exists a basis in which the C_j are all simultaneously diagonal.

We next assert that the eigenvalues of each C_j are the integers multiplied by i . To see this, we introduce monomials in the single pair q_s, p_s defined by

$$|j_s; m_s\rangle = (p_s + iq_s)^{j_s + m_s} (p_s - iq_s)^{j_s - m_s}. \quad (A6)$$

Here, as before, the quantities j_s and m_s are integral or half integral. We find, after simple computation, that

$$C_s |j_s; m_s\rangle = -2im_s |j_s; m_s\rangle. \quad (A7)$$

Since any monomial in the various variables can be built of products of the form (A6), and since these monomials obviously form a basis, our assertion is proved.

Let $|\lambda_{m+1} \dots \lambda_n\rangle$ denote an eigenvector of the various C_j . It is constructed from products of monomials of the form (A6). We have

$$C_j |\lambda_{m+1} \dots \lambda_n\rangle = i\lambda_j |\lambda_{m+1} \dots \lambda_n\rangle, \quad (A8)$$

where the λ 's are integers. It follows from (A5) and (A8) that

$$C |\lambda_{m+1} \dots \lambda_n\rangle = \left(i \sum_{j=m+1}^n \alpha_j \lambda_j \right) |\lambda_{m+1} \dots \lambda_n\rangle. \quad (A9)$$

Since C is antihermitian, we may rewrite (A9) in the form

$$\langle \lambda_{m+1} \dots \lambda_n | C = \left(i \sum_{j=m+1}^n \alpha_j \lambda_j \right) \langle \lambda_{m+1} \dots \lambda_n |. \quad (A10)$$

Now take the scalar product of both sides of (A10) with the vector $|h^{new}\rangle$. We find, using (3.19), the result

$$0 = \langle \lambda_{m+1} \dots \lambda_n | C | h^{new}\rangle = \left(i \sum_{j=m+1}^n \alpha_j \lambda_j \right) \langle \lambda_{m+1} \dots \lambda_n | h^{new}\rangle. \quad (A11)$$

Thus, for $|\lambda_{m+1} \dots \lambda_n\rangle$ to appear in h^{new} , the eigenvalues λ_j must necessarily satisfy the relation

$$\sum_{j=m+1}^n \alpha_j \lambda_j = 0. \quad (A12)$$

By the hypothesis (A1) or (A2), there are only l linearly independent relations of the form (A12) with integer coefficients. It follows that for each set of λ 's obeying (A12) there must be other coefficients $\beta_1^\lambda \dots \beta_l^\lambda$ such that

$$\lambda_j = \sum_{i=1}^l \beta_i^\lambda \mathcal{M}_{ij} \quad (A13)$$

or, using matrix and vector notation,

$$\lambda = \tilde{\mathcal{M}} \beta^\lambda. \quad (A14)$$

We are almost done. Let $\gamma_{m+1} \dots \gamma_n$ be a set of real numbers such that the vector γ with entries γ_j is an eigenvector of \mathcal{M} with eigenvalue zero,

$$\mathcal{M} \gamma = 0. \quad (A15)$$

By the nature of \mathcal{M} , there will be $n - m - l$ such vectors γ which are linearly independent. Using (A14) and (A15), we find that the scalar product between any λ obeying (A12) and any γ obeying (A15) vanishes,

$$(\lambda, \gamma) = (\tilde{\mathcal{M}} \beta^\lambda, \gamma) = (\beta^\lambda, \mathcal{M} \gamma) = 0. \quad (A16)$$

Now define operators C_γ by the rule

$$C_\gamma = \sum_{j=m+1}^n \gamma_j C_j. \quad (\text{A17})$$

There will $n - m - l$ such operators which are linearly independent. They evidently have the property that

$$C_\gamma |\lambda_{m+1} \dots \lambda_n\rangle = i(\lambda, \gamma) |\lambda_{m+1} \dots \lambda_n\rangle = 0 \quad (\text{A18})$$

when (A12) holds. Since the only vectors $|\lambda_{m+1} \dots \lambda_n\rangle$ occurring in the expansion of h^{new} are those for which (A12) holds, we must also have

$$C_\gamma h^{\text{new}} = 0. \quad (\text{A19})$$

It follows that the $n - m - l$ functions i_γ^{new} defined by $i_\gamma^{\text{new}} = c_\gamma$, where

$$c_\gamma = \sum_{j=m+1}^n \gamma_j f_j \quad (\text{A20})$$

are integrals of motion for h^{new} .

We note that, in this notation, c_α is the integral already found in Sec. 3. We also remark that in the extreme case in which $l = 0$, i.e., all the α 's are irrational in a way as to be completely incommensurate, then each c_j is an integral of motion for h^{new} .

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⁷See, for example J. Moser, *Stable and Random Motions in Dynamical Systems* (Princeton U. P., Princeton, NJ, 1973); B.V. Chirikov, "A universal instability of many dimensional oscillator systems," *Phys. Rep.* (in press).

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¹⁰See, for example, P.R. Halmos, *Finite Dimensional Vector Spaces* (Van Nostrand, Princeton, NJ, 1958), p. 127.

¹¹For those familiar with Lie Algebras, we remark that the associated Lie operators constitute the *adjoint* representation of the underlying Poisson bracket Lie Algebra; hence the notation. See, for example, M. Hausner and J.T. Schwartz, *Lie Groups; Lie Algebras* (Gordon & Breach, New York, 1968). Computation by the reader may be simplified by noting that $\text{ad}(q^2) = 2q(\partial/\partial p)$, etc.

¹²See, for example, E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1961), p. 361.

¹³Note that although (2.19) always has a solution, the solution is not unique. Given a solution, one can always produce a new solution simply by adding on a function in the null space of H_2 . The implications of this freedom of choice have not been fully explored. In our calculations to date we have imposed the further requirement that each f_k be orthogonal to the null space of H_2 . When considering the question of uniqueness, it should also be noted that the choice of scalar product, which in turn defines H_2^+ ; by (2.12), is also not unique. This degree of freedom also has not been fully explored.

¹⁴For a study of those orbits which eventually close on themselves, and hence are periodic, a different procedure may be more useful. It is known that such orbits correspond to fixed points of a Poincare surface of section map or its iterates, and that these fixed points are in turn the locale or origin of island or homo-heteroclinic behavior. (See Ref. 4.) A preliminary step toward the construction of integrals in these so called "resonance" cases has been made in Ref. 15.

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