Dedicated to Odon Godart

IV. ON THE STRUCTURE OF SYMMETRIC PERIODIC SOLUTIONS OF CONSERVATIVE SYSTEMS, WITH APPLICATIONS[1]

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INTRODUCTION

If one wants to study a given algebraic curve, one will start to investigate the properties of such a curve near an ordinary point, but this will not furnish any information on the behavior of the curve in the large; the knowledge of the singular points will furnish one of the tools for such a study in the large and the classification of these singular points into double points, triple points, cusps, etc., will be of utmost importance. A similar situation arises in the study of differential equations. Let us take those of the type

$$\frac{d^2x}{dt^2} = \frac{\partial}{\partial x} U(x, y), \qquad \frac{d^2y}{dt^2} = \frac{\partial}{\partial y} U(x, y) \quad ;$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2U(x, y) \quad .$$

ere the properties of existence and uniqueness of a solution or of the behavior near a solution, do not give any information on the set of all solutions in the large; moreover it is difficult to arrive at the complete nowledge of one solution for an infinite time except for periodic solutions, and for asymptotic solutions to periodic solutions. Periodic solutions may be taken as analogous to the singular points of an algebraic urve and their knowledge and classification may furnish a good tool to dvance in the study of differential equations.

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[1] This paper is extracted from a Report bearing the same title prepared nder the sponsorship of the United States Air Force through the Office of cientific Research of the Air Research and Development Command (Report o. 2 (1954), ix + 74 pp., 3 Tables, 1[illegible] Fig.).

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$$T_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - ky \\ kx - y \end{pmatrix}$$

$$T_2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (x + (k^2 - 1) y)^2 \\ y \end{pmatrix}$$

$$T_3 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \Bigg($$

A classification of periodic solutions is considered here and is made dependent on a correspondence of solutions of the differential equation with points in the two dimensional space R^2. Such a correspondence can easily be established in general, if there exists a surface which is crossed by every trajectory. This surface is called surface of section by Poincaré; to successive intersections of the surface of section by the trajectories correspond successive points A and B in R^2. The correspondence of B to A defines a transformation T of R^2 into itself, and the problem of finding periodic solutions of the differential equations is equivalent to the determination of fixed points of the transformation T.

Unfortunately it is not, ih general, easy to obtain a simple surface of section. Hence, it makes sense to consider the similar transformation that can be deduced from the successive intersections of part of a surface satisfying less stringent conditions. This transformation furnishes those periodic solutions which cross this surface and this gives at least a partial answer to the problem.

In a certain number of applications, such as the billiard ball problem and the restricted three bodies problem, the transformation T is the product of two involutions $M_1 M_0$. We shall restrict ourselves in this paper to such a transformation and shall study first the properties of T. Many of these properties are explicit or implicit in Birkhoff's work. It will appear that some periodic points under powers of T are contained in the intersections $\mathcal{M}_{n,p}$ of the sets $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_n, \ldots, \mathcal{M}_p, \ldots$ of the points invariant under M_0, M_1 and sets generalizing M_0 and M_1.

This permits a classification of the so-called symmetric periodic points $\mathcal{M}_{n,p}$.

One typical application is given in the second part of the paper and concerns symmetric periodic solutions of conservative systems of two degrees of freedom having a line of symmetry and reversible, i.e., for which any trajectory may be followed in either sense. It is shown that under some conditions the transformation T in this application is topological and the sets $\mathcal{M}_n$ are continuous curves. This imposes, when the invariant curves $\mathcal{M}_n$ are known for $n < p$, restrictive conditions on $\mathcal{M}_p$, hence the fact that some knowledge of the more complicated periodic solutions can be obtained from the less complicated ones. The conservative property of T is not explicitly mentioned here but is implicit because of the conservative system which is at the origin of T.

When a family of differential equations is considered, the periodic points corresponding to the periodic solutions vary continuously and give rise to the structure of periodic solutions. This then is investigated to a great extent for a typical conservative problem, the problem of

Störmer, bringing together all the known results, throwing light on the role of the essential singularity of the problem and presenting new results which had escaped former analysis.

As a help to the reader, we mention a more elementary application on conservative transformations in the plane which are products of involutions, which we discovered recently and of which the following mapping is an example.

$$X = -y + k(kx-y)^2 + \left[x - ky + (k^2-1)(kx-y)^2\right]^2$$
$$Y = x - (kx-y)^2 - k\left[x - ky + (k^2-1)(kx-y)^2\right]^2 .$$

CHAPTER I. PERIODIC POINTS OF SYMMETRIC TRANSFORMATIONS

1. Definitions. Let us consider a one to one mapping T of a set E onto itself and an involution R of E. The study of both transformations is of interest only if some relation exists between them. We shall here consider that TR is also an involution and call T a symmetric transformation with respect to R. Such mappings, products of two involutions were already considered by Birkhoff [1, I p. 727, II p. 412, 439, 668, 718] [2, p. 186].

For the sake of simplicity we let (n being any integer)

$$P_n = T^n, \quad M_n = T^nR .$$

By hypothesis, $M_0^2 = 1$, $M_1^2 = 1$ and it is easy to prove that

(1) $$T^nR = RT^{-n}$$

(2) $$P_nP_q = P_{n+q}$$

(2') $$P_nM_q = M_{n+q}$$

(2'') $$M_nP_q = M_{n-q}$$

(2''') $$M_nM_q = P_{n-q} ;$$

the last relation gives, when $n = q$, the generalization of the hypothesis

$$M_n^2 = 1 .$$

It follows also, that every product of a finite number of mappings T and R is a mapping P_n or M_n.

2. Invariant Sets. We define the subsets $\mathcal{P}_n$ and $\mathcal{M}_n$ of E as the sets of points left invariant under the mappings P_n and M_n; $\mathcal{P}_n$ for $n > 0$ is evidently the set of all the points of period n or divisor of n and $\mathcal{P}_{-n} = \mathcal{P}_n$.

We shall also consider the sets

(3) $$\mathcal{P}_{n,q} = \mathcal{P}_n \cap \mathcal{P}_q$$

(4) $$\mathcal{M}_{n,q} = \mathcal{M}_n \cap \mathcal{M}_q$$

where $n \neq q$; we will in general assume $n > q$.

We shall now deduce some relations; first

(5) $$\mathcal{M}_{n,q} \subset \mathcal{P}_{n-q} \quad .$$

For, if $A \in \mathcal{M}_{n,q}$,

$$M_n A = A, \quad M_q A = A$$

and

$$M_n M_q A = P_{n-q} A = A \quad .$$

From a similar argument we obtain

(6) $$\left(\mathcal{M}_q \cap \mathcal{P}_{n-q}\right) \subset \mathcal{M}_n \quad .$$

Also, as a corollary of (6),

$$\left(\mathcal{M}_q \cap \mathcal{P}_{n-q}\right) \subset \mathcal{M}_{n,q}$$

and because of (4) and (5),

$$\mathcal{M}_{n,q} \subset \mathcal{M}_q \text{ and } \mathcal{P}_{n-q}$$

and so,

(7) $$\mathcal{M}_q \cap \mathcal{P}_{n-q} = \mathcal{M}_{n,q} \quad .$$

We remind ourselves also that

(8) $$\mathcal{P}_n \subset \mathcal{P}_{kn}$$

(9) $$\mathcal{P}_{n,q} = \mathcal{P}_{(n,q)} .$$

Also, if $k - k' + 1 \neq 0$

(10) $$\mathcal{M}_{n,q} \subset \mathcal{M}_{k(n-q)+n,\ k'(n-q)+q}$$

for, because of (5) and (8), if $A \in \mathcal{M}_{n,q}$,

$$P_{n-q}A = A, \quad P_{k(n-q)}A = A$$

and because of (2') and $M_n A = A$,

$$M_{k(n-q)+n}A = A \quad ;$$

similarly we have

$$M_{k'(n-q)+q}A = A \quad ;$$

the two indices in the second set of (10) must be different, hence the restriction on k and k'.

In the special case $q = 0$, $k = -1$ and $k' = -1$,

$$\mathcal{M}_{n,o} \subset \mathcal{M}_{o,-n}$$

and because of the symmetry of this relation

(11) $$\mathcal{M}_{n,o} = \mathcal{M}_{o,-n} .$$

Finally corresponding to (8) and (9) we have

(12) $$\mathcal{M}_{n,o} \subset \mathcal{M}_{kn,o}$$

(12') $$\mathcal{M}_{n,1} \subset \mathcal{M}_{k(n-1)+1,1}$$

(13) $$\mathcal{M}_{n,o} \cap \mathcal{M}_{q,o} = \mathcal{M}_{(n,q),o}$$

(13') $$\mathcal{M}_{n,1} \cap \mathcal{M}_{q,1} = \mathcal{M}_{(n-1,q-1)+1,\ 1} .$$

For the applications we have in mind, we introduce the following terminology: $\mathcal{M}_n$ will be called a set of symmetric points and $\mathcal{M}_{n,q}$, a set of doubly symmetric points, so that the relations (5) and (7) are a more precise form of the following theorems:

THEOREM 1. Every doubly symmetric point is periodic under T.

THEOREM 2. Every symmetric periodic point is doubly symmetric.

3. Transformations of the Invariant Sets. All the sets of symmetric points may be deduced from $\mathcal{M}_o$ and $\mathcal{M}_1$ by means of powers of the transformation T because

(14) $$\mathcal{M}_{2n} = P_n\mathcal{M}_o$$

(14') $$\mathcal{M}_{2n+1} = P_n\mathcal{M}_1$$

for, in general

(15) $$P_q\mathcal{M}_n = \mathcal{M}_{n+2q} \quad .$$

As corollary,

(16) $$P_r\mathcal{M}_{n,q} = \mathcal{M}_{n+2r,q+2r} \quad .$$

Also because of

$$\mathcal{M}_{-n} = P_{-n}\mathcal{M}_n = M_oM_n\mathcal{M}_n = M_o\mathcal{M}_n \quad ,$$

we have

(17) $$\mathcal{M}_{-n} = R\mathcal{M}_n \quad .$$

Up to a transformation P_n all doubly symmetric sets $\mathcal{M}_{n,q}$ may be reduced to the sets

$\mathcal{M}_{2k,o}$ when n and q are even ,

$\mathcal{M}_{2k-1,o}$ when n is odd and q is even ,

$\mathcal{M}_{2k+1,1}$ when n and q are odd ,

$\mathcal{M}_{2k-2,-1}$ when n is even and q is odd ;

for instance, because of (16),

$$P_{-j}\mathcal{M}_{2k+2j,2j} = \mathcal{M}_{2k,o} \quad .$$

The last set may be reduced to the second one because of (16) and (11);

(18) $$P_{-k+1}\mathcal{M}_{2k-2,-1} = \mathcal{M}_{0,-2k+1} = \mathcal{M}_{2k-1,0} \;;$$

but no further reduction is possible.

4. <u>Classification of Symmetric Periodic Points</u>. Because of the Theorems 1 and 2, symmetric periodic points are doubly symmetric and conversely, so that we have to classify in a unique manner the points of the sets $\mathcal{M}_{n,q}$. But one of the iterates of these points is contained in

($\mathcal{A}$) $$\mathcal{M}_{2k,0}, \quad \mathcal{M}_{2k-1,0}, \quad \mathcal{M}_{2k+1,1} \quad .$$

and conversely, iterates of points of these sets are points in $\mathcal{M}_{n,q}$. It is thus sufficient to classify points of ($\mathcal{A}$).

Points are in different sets ($\mathcal{A}$) for different values of the indices, but because of (12) to (13') each point is contained in only one of the sets

($\mathcal{B}$) $$\mathcal{E}^{o}_{k} \qquad \mathcal{D}^{o}_{k} \qquad \mathcal{C}^{o}_{k} \qquad (k > 0)$$

defined by

$$\mathcal{E}^{o}_{k} = \mathcal{M}_{2k,0} - \bigcup_{0<n<2k} \mathcal{M}_{n,0}$$

$$\mathcal{D}^{o}_{k} = \mathcal{M}_{2k-1,0} - \bigcup_{0<n<2k-1} \mathcal{M}_{n,0}$$

$$\mathcal{C}^{o}_{k} = \mathcal{M}_{2k+1,1} - \bigcup_{1<n<2k+1} \mathcal{M}_{n,1} \quad .$$

The properties of the points of the set ($\mathcal{B}$) are as follows:

i) the period is exactly and respectively

$$2k, \qquad 2k-1, \qquad 2k$$

ii) -a point in a set $\mathcal{E}^{o}_{k}$ is in $\mathcal{M}_{0}$ as well as its k^{th} iterate because if $A \in \mathcal{E}^{o}_{k}$ and $P_k A = B$, then $A \in \mathcal{M}_{0,-2k}$, hence $M_0 A = A$, $M_{-2k} A = A$, $M_0 B = B$; similarly

-a point in the set $\mathcal{C}_k^o$ is in $\mathcal{M}_1$ as well as its k^{th} iterate and

-a point in the set $\mathcal{D}_k^o$ is in $\mathcal{M}_o$ and its $(k-1)^{th}$ iterate is in $\mathcal{M}_{-1}$.

As for the sets $(\mathcal{B})$, we have

iii) The period of the sets $\mathcal{C}_k^o$ and $\mathcal{E}_k^o$ is k and that of $\mathcal{D}_k^o$ is $2k-1$.

The iterates of the sets $(\mathcal{B})$ will be denoted

$$\mathcal{C}_k^1,\ \mathcal{C}_k^2,\ \ldots,\ \mathcal{C}_k^{k-1}\ ;$$

$$\mathcal{D}_k^1,\ \mathcal{D}_k^2,\ \ldots,\ \mathcal{D}^{k-1},\ \mathcal{D}_k^{-k+1},\ \ldots,\ \mathcal{D}_k^{-1}\ ;$$

$$\mathcal{E}_k^1,\ \mathcal{E}_k^2,\ \ldots,\ \mathcal{E}_k^{k-1}\ .$$

And we have

(19) $$\mathcal{C}_k^j = R\,\mathcal{C}_k^{k-j-1}$$

(19') $$\mathcal{D}_k^j = R\,\mathcal{D}_k^{-j}$$

(19'') $$\mathcal{E}_k^j = R\,\mathcal{E}_k^{k-j}\ .$$

5. Other Properties. From a similar theorem on the indices follows:

THEOREM 3. The sets $\mathcal{M}_{n,o}$ are partitioned by the sets $\mathcal{E}_i$, $\mathcal{D}_j$ where $2i$ and $2j-1$ are all the divisors of n. The sets $\mathcal{M}_{2n+1,1}$ are partitioned by the sets $\mathcal{C}_i$, $\mathcal{D}_j$ where $2i$ and $2j-1$ are all the divisors of $2n$.

From this theorem, one may deduce the partition of any set $\mathcal{M}_{n,q}$. Table I gives the partition of $\mathcal{M}_{n,q} = \mathcal{M}_{q,n}$ corresponding to some values of n and q: $|n| \leq 4$, $|q| \leq 4$, $n \neq q$.

If the sets $\mathcal{M}_{n,p}$ are obtained in the order of increasing n and for a fixed n in the order of decreasing p, parenthesis indicate the first time the sets $\mathcal{C}_k^j$, $\mathcal{D}_k^j$, $\mathcal{E}_k^j$ are obtained for a fixed k, whatever be j; brackets indicate the first time the sets are obtained for

TABLE I

The Partitions of the Sets $\mathcal{M}_{n,q}$ are Given Below n, q.
(For the conventions see Section 5.)

-4,-3: $\mathcal{D}_1^0$	-4,-2: $\mathcal{D}_1^0\ \mathcal{E}_1^0$	-4,-1: $\mathcal{D}_1^0\ \mathcal{D}_2^1$	-4,0: $\mathcal{D}_1^0\ \mathcal{E}_1^0\ \mathcal{E}_2^0$
-3,-2: $\mathcal{D}_1^0$	-3,-1: $\mathcal{C}_1^0\ \mathcal{D}_1^0$	-3,0: $\mathcal{D}_1^0\ \mathcal{D}_2^0$	-4,1: $\mathcal{D}_1^0\ (\mathcal{D}_3^{-2})$
-2,-1: $\mathcal{D}_1^0$	-2,0: $\mathcal{D}_1^0\ \mathcal{E}_1^0$	-3,1: $\mathcal{C}_1^0\ (\mathcal{C}_2^0)\ \mathcal{D}_1^0$	-4,2: $\mathcal{D}_1^0\ \mathcal{D}_2^1\ \mathcal{E}_1^0\ (\mathcal{E}_3^1)$
-1,0: $\mathcal{D}_1^0$	-2,1: $\mathcal{D}_1^0\ (\mathcal{D}_2^{-1})$	-3,2: $\mathcal{D}_1^0\ (\mathcal{D}_3^1)$	-4,3: $\mathcal{D}_1^0\ (\mathcal{D}_4^{-2})$
1,-1: $(\mathcal{C}_1^0)\mathcal{D}_1^0$	2,-2: $\mathcal{D}_1^0\ \mathcal{E}_1^0\ (\mathcal{E}_2^1)$	3,-3: $\mathcal{C}_1^0\ (\mathcal{C}_3^1)\ \mathcal{D}_1^0\ \mathcal{D}_2^0$	4,-4: $\mathcal{D}_1^0\ \mathcal{E}_1^0\ \mathcal{E}_2^0\ (\mathcal{E}_4^2)$
1,0: $\longrightarrow (\mathcal{D}_1^0)$	2,-1: $\mathcal{D}_1^0\ (\mathcal{D}_2^1)$	3,-2: $\mathcal{D}_1^0\ (\mathcal{D}_3^{-1})$	4,-3: $\mathcal{D}_1^0\ (\mathcal{D}_4^2)$
2,1: $\longrightarrow \mathcal{D}_1^0$	2,0: $\mathcal{D}_1^0\ (\mathcal{E}_1^0)$	3,-1: $\mathcal{C}_1^0\ (\mathcal{C}_2^1)\ \mathcal{D}_1^0$	4,-2: $\mathcal{D}_1^0\ \mathcal{D}_2^{-1}\ \mathcal{E}_1^0\ (\mathcal{E}_3^2)$
3,2: $\longrightarrow \mathcal{D}_1^0$	3,1: $\mathcal{C}_1^0\ \mathcal{D}_1^0$	3,0: $\mathcal{D}_1^0\ [\mathcal{D}_2^0]$	4,-1: $\mathcal{D}_1^0\ [\mathcal{D}_3^2]$
4,3: $\longrightarrow \mathcal{D}_1^0$	4,2: $\mathcal{D}_1^0\ \mathcal{E}_1^0$	4,1: $\mathcal{D}_1^0\ \mathcal{D}_2^{-1}$	4,0: $\mathcal{D}_1^0\ \mathcal{E}_1^0\ [\mathcal{E}_2^0]$

fixed k with the exception that sets related by (19) to (19'') are considered as equivalent; braces are used when this equivalence is not considered.

THEOREM 4. If all the $\mathcal{M}_i$ are known for $|i| < n$, from the knowledge of $\mathcal{M}_n$ and $\mathcal{M}_{-n} = R\mathcal{M}_n$, we deduce the new sets

$$\mathcal{E}_{n-1}, \quad \mathcal{E}_n, \quad \mathcal{D}_n \text{ when } n \text{ is even and}$$

$$\mathcal{C}_{n-1}, \quad \mathcal{C}_n, \quad \mathcal{D}_n \text{ when } n \text{ is odd.}$$

For instance, when $n = 2p$ the only new doubly symmetric sets are $\mathcal{M}_{2p,-2p+2}$, $\mathcal{M}_{2p,-2p+1}$ and $\mathcal{M}_{2p,-2p}$ because the others

$$\mathcal{M}_{2p,-2p+k} \qquad (3 \leq k < 4p)$$

are equivalent up to a transformation P_k to

$$\mathcal{M}_{2p-2,-2p+k-2} \quad .$$

The new sets are equivalent to

$$\mathcal{M}_{4p-2,0}, \quad \mathcal{M}_{4p-1,0} \quad \text{and} \quad \mathcal{M}_{4p,0} \quad ;$$

when these sets are partitioned with the aid of Theorem 3, only the improper divisors give new sets ($\mathcal{C}$), hence the theorem.

6. Non Existence of Symmetric Periodic Points. Let us consider two sets E_1 and E_2 which form with $\mathcal{M}_0$ a partition of E, we have

THEOREM 5. If the mappings of E_1 and $\mathcal{M}_0$ are contained in E_1, the sets $\mathcal{D}_n$ and $\mathcal{E}_n$ are empty.

For, $T(E_1 \cup \mathcal{M}_0) \subset E_1$ is equivalent to

$$T(\mathcal{C} E_2) \subset E_1$$

from which we deduce

$$E_2 \supset \mathcal{C}(T^{-1} E_1) = T^{-1}(\mathcal{C} E_1)$$

and so

$$\mathcal{M}_{-2} = T^{-1}\mathcal{M}_0 \subset E_2;$$

hence $\mathcal{M}_{0,-2} = \emptyset$; on the other hand by iteration of the hypothesis, $T\mathcal{M}_0 \subset E_1$ and $T(E_1) \subset E_1$,

$$\mathcal{M}_{2p} = T^p \mathcal{M}_o \subset E_1$$

and so

$$\mathcal{M}_{2p,-2} = \emptyset \quad \text{for} \quad p > 0$$

but the sets $\mathcal{D}_n^o$ and $\mathcal{E}_n^o$ are contained in some $T\mathcal{M}_{2p,-2} = \emptyset$ $(p \geq o)$ and so these sets and their iterates are empty.

If E_3, E_4 and $\mathcal{M}_{-1}$ form a partition of E, it is easy to prove

THEOREM 6. If the mapping of E_3 and the mapping of $\mathcal{M}_{-1}$ are contained in E_3, the sets $\mathcal{C}_n$ and $\mathcal{D}_n$ are empty.

Also if E_3 and E_4 form a partition of E we have

THEOREM 6 bis. If $\mathcal{M}_1$ and the mapping of E_3 are contained in E_3 and if $\mathcal{M}_{-1}$ is contained in E_4, the sets $\mathcal{C}_n$ and $\mathcal{D}_n$ are empty.

CHAPTER II. APPLICATION TO A CLASS OF CONSERVATIVE PROBLEMS OF TWO DEGREES OF FREEDOM

The properties of Chapter 1 were suggested by the following application.

Let us consider the canonical form of a conservative dynamical problem of two degrees of freedom

(20) $$x'' = \partial U/\partial x$$

(21) $$y'' = \partial U/\partial y$$

for which the following first integral is easily found

(22) $$\dot{x}^2 + \dot{y}^2 = 2U(x, y, a) + h \quad .$$

Let us think of U as an analytic function in x and y; weaker conditions are studied in Section 13. In the problems which are naturally of two degrees of freedom, the constant a does not appear and h is the integration constant; when the equations are deduced from a problem of three degrees of freedom with an ignorable coordinate, the constant a is the conjugate coordinate and often h is fixed if a suitable system of units is chosen. Hence there is no restriction if we write $h = 0$ or

$$\dot{x}^2 + \dot{y}^2 = 2U(x, y, a) \quad . \tag{22'}$$

Different spaces may be used:

1) The four dimensional phase space,
2) The three dimensional surface (22') if a is fixed,
3) A two dimensional surface of section [20], [I, II p. 70], if a is fixed; but such a surface does not exist in general.
4) The two dimensional subspace $\mathcal{S}$ of the plane x, y : $2U \geq 0$.

7. Introduction of Symmetry. We shall now make the further hypothesis $U(x, -y, a) = U(x, y, a)$, which means that the problem is symmetrical with respect to $y = 0$.

$y = 0$ is a solution of (21) and the solution on the axis of symmetry is reduced to a problem of one degree of freedom (20).

When a is fixed, a solution of the problem which crosses $y = 0$ is determined up to a symmetry by the values of x and $\dot{x}$ at the time of crossing because $\dot{y}$ is determined up to the sign by (22'). The two dimensional subspace $\mathcal{T}$ of the plane $x, \dot{x}$

$$2U(x, o, a) - \dot{x}^2 \geq 0 \quad ,$$

is a generalization of a surface of section: this surface is made up of analytic pieces, but some discussion will have to be made when there are double points on the boundary and for regular boundedness [I, II, p. 70]; we do not ask that all the trajectories cross $y = 0$.

If we are looking for periodic solutions of (20) and (21), symmetry is very important, because then symmetric periodic solutions may exist and these are much more easy to find than non-symmetric periodic orbits as will be indicated in Section 10. We shall consider here the symmetry defined above, but what follows may be generalized to other cases.

The two subspaces $\mathcal{S}$ and $\mathcal{T}$ will mostly be used. Each point A of $\mathcal{T}$ not on the boundary gives the initial conditions of two symmetric curves τ_A in $\mathcal{S}$ crossing the line of symmetry $y = 0$ at $\bar{A}$ and followed in a definite sense and conversely. The points on the boundary of $\mathcal{T}$ correspond to the solution $y = 0$, $\dot{x}^2 = 2U$.

8. Transformations of the Set $\mathcal{T}$. Let us consider a trajectory corresponding to a point A of $\mathcal{T}$; if this trajectory again crosses $y = 0$, then to this point A there corresponds a point B of $\mathcal{T}$ and the differential equations define a transformation $\overline{T}$ of $\mathcal{T}$ into itself, $\overline{T}^{-1}$ is obtained by following the same trajectory in the other sense giving, if the trajectory crosses $y = 0$ at D: $\overline{T}^{-1} A = D$. Let us now define the involution $\overline{R}$ of $\mathcal{T}$ as the reflexion about $\dot{x} = 0$; this corresponds in $\mathcal{S}$ to the trajectory starting at the same point on $y = 0$ in the opposite direction. If $\overline{R}A = C$, we have $\overline{TRC} = B$ and $\overline{TRB} = C$ and so $\overline{TR}$ is also an involution.

If $\overline{T}$ is a one to one mapping of $\mathcal{T}$ onto itself, i.e., if to each A corresponds a B and a D, then the application of Part I is immediate and $E = \mathcal{T}$; if not, E will be the largest subset of $\mathcal{T}$ for which the transformation T deduced from $\overline{T}$ is one to one onto E. An a priori knowledge of E is then, in general, difficult to obtain, but is not necessary.

9. Periodic Orbits and Symmetric Periodic Orbits. Solutions of (20) and (21) will be called here indifferently, trajectories or orbits. It is immediate that the sets $\mathcal{P}_n$ of E correspond to the periodic orbits τ crossing $y = 0$ at least once and conversely. We have now to find the meaning of the sets $\mathcal{M}_n$. $\mathcal{M}_0$ is the set of points invariant under R, i.e., the set of points on $\dot{x} = 0$ in E. These points correspond to trajectories perpendicular to $y = 0$ in $\mathcal{S}$, i.e., to trajectories symmetric with respect to the x axis or trajectories symmetric with respect to the plane $y = 0$, $\dot{x} = 0$ in the phase space. $\mathcal{M}_{2n}$ is the transform of $\mathcal{M}_0$ under T^n (14); these points corresponds to the n^{th} crossing of trajectories symmetric with respect to $y = 0$. If A is a point of $\mathcal{M}_1$, TRA = A; to A corresponds in the phase space a trajectory τ_A with initial conditions $X(x_0, y_0 = 0, \dot{x}_0, \dot{y}_0)$, to RA corresponds the trajectory τ_{RA} with initial conditions $Y(x_0, 0, -\dot{x}_0, -\dot{y}_0)$ and

$$\dot{x}(-t, X) = -\dot{x}(t, Y) .$$

After a time $2t_1$, τ_{RA} crosses $y = 0$ at X, and we have, because of TRA = A

$$\dot{x}(2t_1, Y) = \dot{x}(0, X) ;$$

but because of the uniqueness in the phase space, this was true at any preceding time, for instance $-t_1$, and so

$$\dot{x}(t_1, Y) = \dot{x}(-t_1, X) = -\dot{x}(t_1, Y) = 0 .$$

For the same reason $\dot{y}(t_1, Y) = 0$. The orbit will have one point of zero velocity or in $\mathcal{S}$ a point on the boundary $2U = 0$, and A is the next point of intersection with $y = 0$. Because of (14') $\mathcal{M}_{2n+1}$ is the set in E of the orbits starting from the zero velocity line and crossing $y = 0$ for the n^{th} time. These orbits are symmetric in the phase space about $\dot{x} = 0$, $\dot{y} = 0$. Symmetric points correspond to symmetric orbits in the preceding sense.

The sets $\mathcal{M}_n$ are defined in E, but we extend their definition in $\mathcal{T}$ in the obvious manner; they will be, in general, lines which intersect in doubly symmetric points $\mathcal{M}_{n,q}$, which correspond (Theorem 1) to symmetric periodic orbits; the converse is also true because of Theorem 2.

10. On the Advantage of Symmetry. Actually the sets $\mathcal{M}_n$ in $\mathcal{T}$ will be obtained by integrating the differential equations (20) and (21) with initial conditions $(x, y = 0) \in S$, $\dot{x} = 0$, $\dot{y}$ given by (22') or with the initial conditions (x, y) satisfying $U(x, y) = 0$, $\dot{x} = \dot{y} = 0$. The sets $\mathcal{M}_{n,q}$ do not give all the periodic solutions, namely those which do not cross $y = 0$, and those which cross $y = 0$ and are not symmetric. The last ones may in general only be obtained with much more work, because we have then to integrate all the solutions starting with any value of x, $\dot{x}$ in $\mathcal{T}$ up to the second crossing with $y = 0$ in $\mathcal{S}$; by using the symmetry argument, the integration with the indicated initial conditions up to the first crossing with $y = 0$ is sufficient.

Periodic orbits which do not cross $y = 0$ may be obtained in the case where there exists another plane of symmetry in the phase space, which could be for instance the $x = 0$, $\dot{y} = 0$ plane.

But to deduce from this, that all non-symmetric periodic solutions are obtained, we must have more knowledge on the surfaces of section and prove that every trajectory crosses in the phase space either $x = 0$, $\dot{y} = 0$ or $y = 0$, $\dot{x} = 0$.

11. Classification of Periodic Orbits. Because of the correspondence of the symmetric periodic points and the symmetric periodic orbits, Section 4 gives us a classification of symmetric periodic orbits.

With the notation introduced at the end of Section 7, if

$$A \in \mathcal{E}_k^o \subset \mathcal{M}_{2k,o}$$

we have with $B = T^k A$,

$$M_o A = A, \quad M_o B = B \quad \text{(Section 4, II)}$$

$$\mathcal{E}_k^1 \ni A_i = T^i A = T^i RA = RT^{-i} A = R(T^{k-i} B) \quad i = 1, 2, \ldots, k-1 ;$$

all the points A_i being different because of the definition of $\mathcal{E}_k^0$. The corresponding trajectory τ_A crosses $y = 0$ perpendicularly at $\bar{A}$ and $\bar{B}$ and has exactly $k - 1$ other points of intersection $\bar{A}_i$ with $y = 0$, such a trajectory will be noted ϵ_k.

Similarly to $A \in \mathcal{C}_k^0$, corresponds a trajectory γ_k with exactly k points of intersection with $y = 0$ and with two points of zero velocity. Also to $A \in \mathcal{D}_k^0$, corresponds a trajectory δ_k with exactly k points of intersection with $y = 0$ at one of which ($\bar{A}$) the crossing is perpendicular and with two points of zero velocity symmetric with respect to $y = 0$.

Using Table 1, one sees that from the computation of $\mathcal{M}_1$, δ_1 and γ_1 may be obtained, i.e., the periodic orbits symmetric or not with respect to $y = 0$ and having two points of zero velocity. The additional computation of $\mathcal{M}_2$ gives every orbit ϵ_1, ϵ_2 and δ_2, i.e., the symmetric periodic orbits with respect to $y = 0$ with two points where the crossing is perpendicular and without or with an additional crossing point and the symmetric periodic orbits with respect to $y = 0$ with two symmetric points of zero velocity and two points on $y = 0$, one at which the crossing is perpendicular. And so on.

12. <u>Non Existence of Symmetric Periodic Solutions</u>. If in the Theorem 5, $E_1 = (\mathcal{J} \cap \dot{x} > 0)$ and $E_2 = (\mathcal{J} \cap \dot{x} < 0)$ one has

> THEOREM 7. If all trajectories starting from $y = 0$ with $\dot{y} > 0$ and $\dot{x} \geq 0$ and which cross $y = 0$ are such that at the first crossing point $\dot{x} > 0$, there is no symmetric periodic orbit of type δ or ϵ.

Moreover, if $E_3 = E_1$ and $E_4 = E_2$ we derive from Theorem 6 bis:

> THEOREM 8. If the hypothesis of Theorem 7 are verified and if all trajectories starting with zero velocity and $y > 0$ and which cross $y = 0$ are such that at the first crossing point $\dot{x} > 0$, there is no symmetric periodic orbit.

The additional hypothesis means that $\mathcal{M}_1 \subset E_3$ hence

$$\mathcal{M}_{-1} = R\mathcal{M}_1 \subset RE_3 = E_4 \quad .$$

REMARK. As such the two preceding theorems are not very useful, but will become so when T is a topological transformation.

13. Properties of the Transformation T.

THEOREM 9. Let $\mathcal{G}' \subset \mathcal{G}$ be a closed bounded domain in which $U(x, y, a)$ has continuous bounded first partial derivatives in x and y and in which these derivatives satisfy Lipschitz's condition. Let $I(x_0)$ be a closed set of intervals in x_0 such that

$$\left(I(x_0),\ y_0 = 0\right) \in \mathcal{G}'$$

and $\mathcal{T}'$ such that

$$\mathcal{T}' = \mathcal{T} \cap \left(I(x_0),\ \dot{x}_0\right) \quad ,$$

then to every point $P_0 \in \mathcal{T}'$ corresponds a unique solution of the differential equations (20) and (21); if this solution crosses again $y = 0$ after a finite time t_1, and before leaving $\mathcal{G}'$, the corresponding point P_1 is in a set $\mathcal{T}'_1$; and the original point is in the set $\mathcal{T}'_0$; the correspondence between $\mathcal{T}'_0$ and $\mathcal{T}'_1$ is one to one.

PROOF. The system of first order differential equation (20) and (21) satisfies the condition for existence and uniqueness of a solution with initial conditions at $t = t_0$, $(x_0, y_0) \in \mathcal{G}'$ and $\dot{x}_0$, $\dot{y}_0$ satisfying (22); moreover those solutions are continuous functions of x_0, y_0, $\dot{x}_0$, $\dot{y}_0$ and $(t - t_0)$ in $\mathcal{G}'$ [28, II, 141]; hence the correspondence between some subsets of $\mathcal{T}$ is one to one and it is easy to see that $\mathcal{T}'_0$ and $\mathcal{T}'_1$ are such subsets. It is not so that those solutions are necessarily continuous functions of the initial conditions alone, for, t may increase indefinitely along a solution without these solutions leaving $\mathcal{G}'$, for instance when the solution is asymptotic to a periodic solution.

THEOREM 10. If besides the hypothesis of Theorem 9, U admits in $\mathcal{G}'$ continuous bounded second partial derivatives which satisfy a Lipschitz condition, the correspondence between $\mathcal{T}'_0$ and $\mathcal{T}'_1$ is continuous and thus topological.

We shall restrict ourselves at first to the open domains of $\mathcal{T}_0'$ and $\mathcal{T}_1'$, a solution corresponding to a point in $\mathcal{T}'$ has then continuous first partial derivatives in x_0, y_0, $\dot{x}_0$, $\dot{y}_0$ and $t - t_0$. In particular, if $y_0 = 0$ and $\dot{y}_0$ is determined by (22), y is a continuous function of x_0, $\dot{x}_0$, and $t - t_0$, since $\dot{y}_0$ is by (3) a continuous function of x_0 and $\dot{x}_0$. Now y becomes zero for $t_1 = t - t_0$, is differentiable and its derivative with respect to t does not become zero at P_1 (otherwise the trajectory would be on $y = 0$ and P_1 on the boundary of $\mathcal{T}'$), nor in the vicinity of P_1 (since $\ddot{y}_1$ is bounded by (21)). Hence there exists a unique differentiable function $t_1 = \tau(x_0, \dot{x}_0)$ which satisfies identically $y(x_0, \dot{x}_0, t_1) = 0$ [28, I, 141-142]. If we replace t_1 by φ in the continuous and differentiable functions x and $\dot{x}$, we obtain a result even stronger than the one stated.

We have now to discuss points on the boundary of $\mathcal{T}'$. If such a point is not on the boundary of $\mathcal{T}$, it is immediate that x and $\dot{x}$ are continuous and differentiable for any variation of x_0, $\dot{x}_0$ which points to the interior of $\mathcal{T}'$. If the point is on the boundary of $\mathcal{T}$, we need to consider more closely the solutions of the differential equations near $y = 0$. If $x(t)$, $y = 0$ is a solution, any solution in the vicinity $x + \xi$, η, satisfies the differential equations

(23) $$\ddot{\xi} = \left(\frac{\partial^2 U}{\partial x^2}\right)\xi$$

(24) $$\ddot{\eta} = \left(\frac{\partial^2 U}{\partial y^2}\right)\eta$$

(25) $$\dot{\xi}\dot{x} = \ddot{x}\xi$$

with errors of the type $\epsilon(|\xi| + |\eta|)$, ϵ tending to zero with ξ and η if the time is bounded.

$$\left(\frac{\partial^2 U}{\partial x \partial y}\right) = 0$$

because of the symmetry of U. The first equation is equivalent to the third, and (25) shows that the variation ξ is of the form $A\dot{x}$. For any initial condition $\eta = 0$, $\dot{\eta} \neq 0$, the solution of the second equation at any finite time cannot be zero at the same time as $\dot{\eta}$, and so at P_1, $\eta = 0$ can be solved explicitly in t and continuity and differentiability follow easily for points on the boundary of $\mathcal{T}$, for directions interior to $\mathcal{T}'$. The proof of the theorem is completed if the points on the boundary are considered as limit points of corresponding sequences in the interior of $\mathcal{T}$. This asks for the condition of finite time entering in the definition of $\mathcal{T}'$. The following interpretation must then be given to points L on the boundary of $\mathcal{T}$: if L is the limit of a sequence of points L_1 interior to $\mathcal{T}$, the time t_1 after which the corresponding trajectories cross $y = 0$ tends to a finite time t as L_1 tends to L. Theorems 9 and 10 may serve to prove under certain conditions, continuity of the curves $\mathcal{M}_{2n}$ as iterates of $\mathcal{M}_0$. It is also possible to write similar conditions which will serve to prove continuity for $\mathcal{M}_{2n+1}$ as obtained from a set of initial conditions on $U = 0$.

To investigate other properties of the curves $\mathcal{M}_n$, many hypotheses may be studied. We shall give as illustration the following theorems and considerations.

14. The Case of a Connected Bounded Domain $\mathcal{S}$.

THEOREM 11. If the domain $\mathcal{S}$ is bounded, connected, and containing a segment PQ on $y = 0$; if in this closed domain $U(x, y, a)$ has continuous bounded second partial derivatives in x and y and these derivatives satisfy Lipschitz conditions; if PQ is a line of section, i.e., if every trajectory through any point crosses PQ after a finite positive time, the correspondence T is topological and the invariant curves $\mathcal{M}_n$ are continuous.

This follows quite directly from the two preceding theorems. In this case the set $I(x_0)$ is the closed segment PQ and $\mathcal{T}'_0 \equiv \mathcal{T}'_1 \equiv \mathcal{T}$. The invariant curves $\mathcal{M}_{2n}$ are the iterates of the continuous segment PQ, and the curves $\mathcal{M}_{2n+1}$ are iterates of $\mathcal{M}_1$ which is continuous by a reasoning analogous to the one in Theorems 9 and 10 because of the continuity with respect to the initial conditions corresponding to the curve $U = 0, y \geqq 0$.

The end points of the curves are on the boundary of $\mathcal{T}$, and correspond to trajectories infinitely close to the line of symmetry. Properties of the immediate vicinity of $y = 0$ may hence furnish some properties on symmetric periodic orbits; for instance.

THEOREM 12. Let us consider two trajectories p and q infinitely near the line of symmetry, such that $\dot{y} = 0$ (hence $\dot{x} = 0$) at P and Q respectively and limited to a section corresponding to the segment PQ; if their total number of intersections with the open segment PQ is m, under the conditions of Theorem 11, there exists an odd number of periodic solutions of type

δ if m is different from one and an even number if m equals one. The case where p or q crosses PQ at P or Q requires a special investigation. It is also necessary for the hypothesis of Theorem 11 to be fulfilled that the periodic orbit on $y = 0$ has a non-real characteristic exponent $(k < 1)$.

The solutions near the axis of symmetry are given by Hill's equation (see, for instance, [5])

(26) $$\ddot{\eta} = \left(\frac{\partial^2 U}{\partial y^2} \right) \eta$$

where the partial derivative is computed on the periodic solution. The Sturm-Liouville Theorem applies and the roots of two solutions of (26) separate each other, so that if one of the sections of the trajectories considered has at least two points of intersection with PQ, the other will have at least one. However, the first point of intersection of both sections gives one end point of $\mathcal{M}_1$, hence if $m > 2$ both end points are in regions of $\mathcal{T}$ separated by $y = 0$; if $m = 2$ both sections must have one intersection with PQ and $\mathcal{M}_1$ has also an odd number of intersections with PQ; if $m = 1$ both end points are not separated by PQ and the number of intersections with PQ is even; if $m = 0$, we must use Korteweg's Theorem [29, p. 404] which states that if $\eta_p, \eta_{p+1}, \eta_{p+2}$ are the values of η for $t = pT, (p + 1)T, (p + 2)T$ where T is the period of the orbit on $y = 0$, we have

(27) $$\eta_{p+2} + \eta_p = 2k\, \eta_{p+1}, \qquad k = \text{constant} = \cosh \Omega T,$$

Ω being called the characteristic exponent. Hence the trajectories p, q, when extended, will be such that at P and Q or Q and P their ordinate η is proportional to

(28) $$1,\ \eta_1,\ d_1,\ c_1,\ \eta_1 d_2,\ \ldots,\ c_n,\ \eta_1 d_{n+1},\ \ldots$$

(29) $$1,\ \eta_2,\ d_1,\ c_1,\ \eta_2 d_2,\ \ldots,\ c_n,\ \eta_2 d_{n+1},\ \ldots$$

with $c_n = \cosh n\Omega T$

$$d_n = 2 \cosh \Omega T - d_{n-1} - d_{n-2} \quad ,$$

$$d_{-1} = d_1 = \cosh \frac{\Omega T}{2} \quad \text{or} \quad d_n = \cosh (n + 1/2)\Omega T \quad .$$

If the orbit on $y = 0$ has an odd instability, [19], [5], i.e.,

$k = c_1 \leqq -1$, then the second and third members of both series (28) and (29) alternate in sign, for $\eta_1 d_1$ and $\eta_2 d_2$ are positive $(m = 0)$ and so the end points of η_1 are separated by $\dot{x} = 0$.

If $k > -1$, we can always choose Ω so that d_1 is positive, hence η_1 and η_2 are positive. If now two consecutive terms of the series (28) are of opposite sign, it will be so for the corresponding terms of (29). The first alternation in sign must occur for corresponding couples in the two series and once again the end points of $\mathcal{M}_1$ are separated by $\dot{x} = 0$.

The exceptional case corresponds to a series for which c_n and d_n are all positive, implying $k = \cosh \Omega T \geqq 1$. In the case $k = 1$, there exists at least one periodic solution for η. That p is such a solution is impossible, for if we take the family of orbits with their initial points on $U = 0$ tending to P, the time of the first crossing would become infinite, as indicated by the behavior of p; in all other cases the solution for η is of the form

$$t\phi(t) + \Psi(t)$$

where ϕ and Ψ are periodic in t with the same period as the orbit on $y = 0$. This indicates that this orbit is unstable, and the same is true if $k > 1$. In these cases the time of crossing of p or q may be made as large as we want as the initial η becomes small. This however is excluded by hypothesis. (See end of proof of Theorem 10.) In the case $k = -1$, at least one of the orbits, p or q, crosses PQ at Q or P and the variational equations of first order (26) is not sufficient to derive the behavior of $\mathcal{M}_1$ near its end point.

As a corollary we mention that the number of intersections of the curves $\mathcal{M}_1$ and $\mathcal{M}_2$ is of the same parity because every intersection with PQ of $\mathcal{M}_2$ and not $\mathcal{M}_1$ corresponds to a curve ϵ_1 which has two points on PQ. Other topological properties of the invariant curves may be deduced from Section 1; for instance, $\mathcal{M}_p$ and $\mathcal{M}_q$ $(p, q > 0)$ intersect only at known points or at symmetric points of known points. It is then possible, if all $\mathcal{M}_n$'s are known for $n < p$, to determine a domain, sometimes much smaller than $\mathcal{S}$ which contains $\mathcal{M}_p$, since most of its intersections with $\mathcal{M}_0$ and all of its intersections with $\mathcal{M}_n (n < p)$ are known.

This in turn will furnish an approximation of the new periodic solutions to which the knowledge of $\mathcal{M}_p$ leads.

15. Continuous Variation of the Parameter a. A continuous variation of the parameter a will now be considered. In this case, if U is a continuous function of a, the invariant curves will vary continuously.

this indicates that the periodic points appear and disappear in pairs [20] except if the point of disappearance is on the boundary of $\mathcal{T}$ (and on $\mathcal{M}_0$); in this case, an orbit disappears by flattening on $y = 0$, this may be given the interpretation that two symmetric orbits disappear on $y = 0$, they coincide if they are of the type δ or ϵ. If two points in $\mathcal{E}_k$ appear, they give only one orbit ϵ_k and an interpretation must be given to the theorem of Poincaré referred to.

At first sight one would think that in general the contact of $\mathcal{M}_k$ and $\mathcal{M}_0$ will be of first order but just as often the contact may be of second order. To make this clear let us consider for instance a family of periodic solutions of type δ_1, the invariant $k = \cosh \alpha\overline{T}$ (1) is a continuous function of the parameter and may take values $+1$, or -1, in this case orbits near δ_1 appear or disappear. This has been studied in [9]. The results are as follows: When $k = -1$, either

a) $\mathcal{M}_1$ is orthogonal to $\mathcal{M}_0$ and near δ_1 exist orbits of type γ_1, or

b) $\mathcal{M}_2$ has an inflection point and near δ_1 exist orbits of type ϵ_1. Inflection point in this context means that the order of contact is at least two.

When $k = 1$, either

c) there exist near δ_1 non symmetric periodic orbits which cross δ_1 near the points of zero velocity and have a double point on $y = 0$, or

d) $\mathcal{M}_1$ and $\mathcal{M}_2$ have an inflection point and near δ_1 exist orbits of type δ_1 distinct from the original family.

Conversely, if at a point where $\mathcal{M}_1$ crosses $\mathcal{M}_0$

i) $\mathcal{M}_1$ is orthogonal to $\mathcal{M}_0$, we have $k = -1$, case a) and orbits γ_1,

ii) $\mathcal{M}_1$ has an inflection point with $\mathcal{M}_0$, it will be so for $\mathcal{M}_2$ and $k = +1$, case d) and orbits δ_1,

iii) $\mathcal{M}_2$ has an inflection point with $\mathcal{M}_0$, but not $\mathcal{M}_1$, we have $k = -1$, case b) and orbits ϵ_1,

iv) $\mathcal{M}_2$ is orthogonal to $\mathcal{M}_0$, $k \neq \pm 1$, there exists an orbit in the vicinity of δ_1 which starts orthogonally to $y = 0$, crosses δ_1 after $t = \overline{T}$

(1) $\overline{T}$ is here the half period $T/2$ i.e., the time between two consecutive crossings of $y = 0$ by δ_1, this is the natural definition as follows from [9].

on $y = 0$ and will be again orthogonal to $y = 0$ after $t = 2\overline{T}$, this means that $\cosh \Omega T = -1$ or $\cosh \Omega\overline{T} = 0$ and that the orbit is of type ϵ_2.

v) $\mathcal{M}_2$ is tangent to $\mathcal{M}_{-1}$ but no other preceding property is satisfied; $k = \pm 1$, there exists orbits in the vicinity of δ_1 which starts with zero velocity crosses $y = 0$ the second time orthogonally; in this case after $t = 3\overline{T}$ again the velocity is zero and $\cosh \Omega 3\overline{T} = 1$, hence $\cosh \Omega\overline{T} = -\frac{1}{2}$ and the orbits are of type δ_2. The points of zero velocity being on one side of δ_1, we may expect for values of γ_1 nearby two orbits of type δ_2, one with the points of zero velocity on one side of δ_1, and one with points of zero velocity on the other side. Case c) can not be discovered from the behavior of $\mathcal{M}_1$ and $\mathcal{M}_2$.

The similar results for periodic solutions of type ϵ_1 crossing $y = 0$ at X and Y as follows: When $k = -1$,

a'), b') $\mathcal{M}_2$ is orthogonal to $\mathcal{M}_0$ either at X or at Y = TX and near ϵ_1 exist orbits of type ϵ_2.

When $k = +1$, either

c') there exists near ϵ_1 a non symmetric orbit crossing ϵ_1 at X and T, or

d') $\mathcal{M}_2$ has an inflection point with $\mathcal{M}_0$ and near ϵ_1 exist orbits of the same type ϵ_1 distinct from the original family. Conversely, if at a point where $\mathcal{M}_2$ crosses $\mathcal{M}_0$ but not $\mathcal{M}_1$

vi) $\mathcal{M}_2$ is orthogonal to $\mathcal{M}_0$, we have $k = -1$, case a) or b) and orbits ϵ_2,

vii) $\mathcal{M}_2$ has an inflection point with $\mathcal{M}_0$, we have $k = +1$, case d) and orbits ϵ_1; case c') can not be discovered from the behavior of $\mathcal{M}_2$.

Another interesting property is that if for a certain value of $a = a_0$ one orbit of type ϵ_1 appears, corresponding to I on $\dot{x} = 0$, $\mathcal{M}_2$ must have a contact or order at least two with $\dot{x} = 0$ at I. Let us note the order of the contact at I by k_1 for $\mathcal{M}_1$ and k_2 for $\mathcal{M}_2$, with the convention $k = 0$ when the curve $\mathcal{M}$ crosses $\dot{x} = 0$ at I but is not tangent at $\dot{x} = 0$ and $k = -1$ when $\mathcal{M}$ does not pass through I.

on $y = 0$ and will be again orthogonal to $y = 0$ after $t = 2\overline{T}$, this means that $\cosh \Omega T = -1$ or $\cosh \Omega\overline{T} = 0$ and that the orbit is of type ϵ_2.

v) $\mathcal{M}_2$ is tangent to $\mathcal{M}_{-1}$ but no other preceding property is satisfied; $k = \pm 1$, there exists orbits in the vicinity of δ_1 which starts with zero velocity crosses $y = 0$ the second time orthogonally; in this case after $t = 3\overline{T}$ again the velocity is zero and $\cosh \Omega 3\overline{T} = 1$, hence $\cosh \Omega\overline{T} = -\frac{1}{2}$ and the orbits are of type δ_2. The points of zero velocity being on one side of δ_1, we may expect for values of γ_1 nearby two orbits of type δ_2, one with the points of zero velocity on one side of δ_1, and one with points of zero velocity on the other side. Case c) can not be discovered from the behavior of $\mathcal{M}_1$ and $\mathcal{M}_2$.

The similar results for periodic solutions of type ϵ_1 crossing $y = 0$ at X and Y as follows: When $k = -1$,

a'), b') $\mathcal{M}_2$ is orthogonal to $\mathcal{M}_0$ either at X or at $Y = TX$ and near ϵ_1 exist orbits of type ϵ_2.

When $k = +1$, either

c') there exists near ϵ_1 a non symmetric orbit crossing ϵ_1 at X and T, or

d') $\mathcal{M}_2$ has an inflection point with $\mathcal{M}_0$ and near ϵ_1 exist orbits of the same type ϵ_1 distinct from the original family. Conversely, if at a point where $\mathcal{M}_2$ crosses $\mathcal{M}_0$ but not $\mathcal{M}_1$

vi) $\mathcal{M}_2$ is orthogonal to $\mathcal{M}_0$, we have $k = -1$, case a) or b) and orbits ϵ_2,

vii) $\mathcal{M}_2$ has an inflection point with $\mathcal{M}_0$, we have $k = +1$, case d) and orbits ϵ_1; case c') can not be discovered from the behavior of $\mathcal{M}_2$.

Another interesting property is that if for a certain value of $a = a_0$ one orbit of type ϵ_1 appears, corresponding to I on $\dot{x} = 0$, $\mathcal{M}_2$ must have a contact or order at least two with $\dot{x} = 0$ at I. Let us note the order of the contact at I by k_1 for $\mathcal{M}_1$ and k_2 for $\mathcal{M}_2$, with the convention $k = 0$ when the curve $\mathcal{M}$ crosses $\dot{x} = 0$ at I but is not tangent at $\dot{x} = 0$ and $k = -1$ when $\mathcal{M}$ does not pass through I.

We know that $k_2 \geqq 1$, we must disprove $k_2 = 1$. If k_2 would be one, $\mathcal{M}_2$ would cross $\dot{x} = 0$ for a let us say smaller than a_0 at two points I_1 and I_2, these points are of period two under the transformation T. If $I_2 = T(I_1)$, and a increases to a_0, I_2 and I_1 tend to I and the orbit at I is of type δ_1 and not ϵ_1 unless $k_2 > 1$. If $J_1 = T(I_1)$ and $J_2 = T(I_2)$ when a increases to a_0, I_3 and I_4 tend to a point J iterate of I, distinct from I, this means that for $a = a_0$ two orbits of type ϵ_1 appeared and not one, this is not an ordinary situation.

CHAPTER III. APPLICATION TO STÖRMER'S PROBLEM

15. Generalities. As application of the preceding results, we will choose a typical conservative problem of two degrees of freedom which is of interest in physics, namely the study of the motion of an electrical particle in the magnetic field of an elementary dipole; the Lagrangian of the problem is

$$L = T + e\,\vec{v}\cdot\vec{A}$$

where T is the kinetic energy, e the charge of the particle, $\vec{v}$ its velocity and $\vec{A}$ the potential vector of a magnetic dipole of moment M situated at the origin in the direction of the z axis; this vector in polar coordinates may be taken as tangent to a parallel, positive in the West direction and of length $M \cos\lambda/r^2$; this gives the Hamiltonian

$$H = \frac{1}{2m}\left[p_r^2 + r^{-2}p_\lambda^2 + \left((r^{-2}\cos^{-2}\lambda)p_\varphi - Mer^{-1}\cos\lambda\right)^2\right]$$

hence the Hamiltonian and the momentum p_φ are constant [27]. When p_φ is not positive, Störmer has proven [21, p. 23] that no periodic solution exists; when p_φ is positive, with as unities m, v and Me and with the change of variable

$$p_\varphi r = e^x, \qquad d\sigma = e^{-2x}p_\varphi^3\,dt$$

the equations deduced from the new Hamiltonians are:

$$\ddot{x} = ae^{2x} - e^{-x} + e^{-2x}\cos^2\lambda \equiv X = \partial U/\partial x \tag{30}$$

$$\ddot{\lambda} = \left(e^{-2x}\cos^2\lambda + 1 + \tan^2\lambda\right)\tan\lambda \equiv \Lambda = \partial U/\partial\lambda \tag{31}$$

$$\dot{x}^2 + \dot{\lambda}^2 = ae^{2x} - 1 - \tan^2\lambda + 2e^{-x} - e^{-2x}\cos^2\lambda \equiv 2U \tag{32}$$

here $a = p_\varphi^{-4}$ is used, $\gamma_1 = -\gamma = p_\varphi/2$ are equivalent parameters. To every solution in the meridian plane r, λ correspond an infinity of trajectory in the space obtained by integrating the equation for the longitude:

(33) $$\dot{\varphi} = \cos^{-2}\lambda - e^{-x} .$$

This integration is a trivial problem and the properties of the trajectories in the three dimensional space are deduced easily from those in the meridian plane, for instance to a periodic solution of period Σ in this plane correspond trajectories on the surface of revolution around the z axis having the periodic solution as directrix. If

$$\Phi = \int_0^\Sigma \left(\cos^{-2}\lambda - e^{-x}\right) d\sigma$$

is commensurable with π, all corresponding trajectories are periodic in space, if Φ is incommensurable each trajectory recurs infinitely often to the neighborhood of an initial state on the surface of revolution and may be classified as almost periodic; hence the general problem is easily solved if the problem in the x, λ plane, is; for an example see [11].

Let us now consider some general properties of the equations (30, 31).

16. Properties. The equations are of the type discussed in Part II, λ playing the role of the variable y, the x axis is the axis of symmetry; the portion $\mathcal{G}$ of the x, λ plane, $U \geq 0$, has been described as follows (see for instance [27]), the boundary $U = 0$ is made up of a branch asymptotic for $x = -\infty$ to $\lambda = \pi/2$ cutting the x axis orthogonally at P with $x = g$ and when $0 < a < 1/16$ of two similar branches one asymptotic for $x = -\infty$ to $\lambda = \pi/2$ cutting $\lambda = 0$ at Q with $g_1 > g$, the other asymptotic for $x = +\infty$ to $\lambda = \pi/2$ cutting $\lambda = 0$ at $g_2 > g$; $\mathcal{G}$ is then formed of two disconnected regions; when $a > 1/16$ the two last branches become connected and do not cross $\lambda = 0$, $\mathcal{G}$ is then formed of one connected region. When $a = 1/16$ the boundary has a double point at $x = \log 2$, $\lambda = 0$ and a special discussion is needed. When $a = 0$, $\mathcal{G}$ reduces to a line

(34) $$e^x = \cos^2\lambda$$

called the thalweg, all points of this line are equilibrium points and this case will be excluded in the following.

For every finite point X and Λ are analytic; by a change of variable one finds that the points at infinity are regular with one exception. The point $x = -\infty$, $\lambda = \pi/2$ is an essential singularity and has been studied by Störmer [22, p. 145, 235], [23], [26, 1947, 1949], the proof that there exists a solution through that point has been furnished by Malmquist [18], but no proof of uniqueness is available although this seems highly probable.

The general results on periodic solutions are as follows: every periodic solution is in the domain $x < f = \log (2\gamma_1^{4/3}) < g_2$ [25, p. 61]; this shows that no periodic solution extends to $+\infty$ and exists in the region which extends to $+\infty$ in the case $a < 1/16$. Every periodic solution crosses the x axis [9], [7].

There is only one equilibrium point, which is the double point on the boundary of $\mathcal{S}$ when $a = 1/16$. Also any trajectory asymptotic to the equilibrium point or a periodic solution which does not extend to infinity crosses the x axis after a finite time.

It has also been proven that no periodic solutions exist for a greater than 4 [6] because then $f < g$. A lot of specific results on periodic solutions have been obtained for the value $\gamma_1 = 0.97$ by Störmer [24], [26, 1950] and on a certain number of families of periodic solutions, the principal one [24], [13], [14], the oval family [3], [4], and the horseshoe family [9] and the family on the equator [5]. It will be seen now how the method of Part II leads very naturally to those results.

17. Surface of Section. The concept of surface of section, being especially useful to find periodic solutions, we will not consider the case $\gamma_1 \leq 0$ where no such solutions exist; when $\gamma_1 > 0$ we have to distinguish three cases, we will treat first the most straightforward one.

i) $\gamma_1 > 1$ or $a < 0.0625$. It was proven by Graef that for these values of the parameter every trajectory crosses the line $\lambda = 0$ or is asymptotic to this line [12, p. 30, Th. VII and convention p. 28 Th. V]. Furthermore if a trajectory is asymptotic to $\lambda = 0$, the time between successive crossings is finite [7, Section 7], hence the segment PQ defined in the preceding section is a line of section (see also [7, Section 8]) for every trajectory of the region $\mathcal{S}_1$ of $\mathcal{S}$ not extending to $+\infty$. To $\mathcal{S}_1$ corresponds a connected bounded domain $\mathcal{T}$ inside the strip $g \leq x \leq g_1$ with boundary

$$\dot{x}^2 = 2U(x, 0, a) \quad . \tag{35}$$

Unfortunately the Theorem 11 can not be used, because $\mathcal{S}_1$ is not bounded and because the point at infinity is an essential singularity of U. We have to restrict $\mathcal{S}_1$, let us therefore use the coordinates of Lemaître-Bossy [16], obtained as follows: we draw from the point (x, λ) a perpendicular to the thalweg (34), the distance to the thalweg is u, the latitude of the point on the perpendicular and the thalweg is y; the Jacobian of the transformation is positive and when $\lambda = 0$ we have $y = 0$, $x = u$ and $\dot{x} = \dot{u}$. Let us follow the trajectories in the space $u, \dot{u}, y$; the trajectories which are extremum ($\dot{y} = 0$) for a given value of $y = y_0$ are represented in the plane $y = y_0$ by points on a closed curve c_0 approximated by [16]:

$$\dot{u}^2 + (\cos y \cos v)^{-2} u^2 = a \cos^4 y$$

with $\tan v = 2 \tan y$. A point in the interior of c_0 corresponds to a trajectory having an extremum in y_0 greater than y_0 or passing through the origin. These trajectories will be represented on another plane $y = y_1 < y_0$ by curve c_1' interior to c_1 because every trajectory has only one extremum in y between each crossing of $y = 0$ [7]. For instance when $y = 0$, there exists a set of closed curves c_i' each one corresponds to trajectories having their extremum at $y = y_i > 0$ and c_i' is interior to c_j' if $y_i > y_j$. Let us now consider

$$\mathcal{S}_i' = \mathcal{S}_i \cap (|y| \leq y_i) \quad .$$

$\mathcal{S}_1'$ is a closed bounded domain in which U is analytic, if we exclude from $\mathcal{T}$ the domain inside c_i' and call this $\mathcal{T}_1'$, and define $\mathcal{T}_0' = R\mathcal{T}_1'$, to every point $p_0 \in \mathcal{T}_0'$ corresponds a unique solution of the differential equations (30) to (32) and this solution crosses again $\lambda = 0$ after a finite time at a corresponding point $P_1 \in \mathcal{T}_1'$ hence by Theorem 10 the correspondance is topological. Moreover, the part of the invariant curves $\mathcal{M}_1$ and $\mathcal{M}_2$ contained in $\mathcal{T}_1'$ is continuous. If it is true, as conjectured by Störmer [22, p. 145], that there is only one trajectory through the origin(1) and if for this trajectory the return is by convention the same path backwards, this will mean that the curves c_i' converge to the point 0 corresponding to the trajectory through the origin, when y_i tends to $\frac{\pi}{2}$. The transformation T should then be topological in $\mathcal{T}'$ and the $\mathcal{M}_n$ continuous. We will suppose this to be true in the following; if the conjecture is disproved, the results here stated will need an interpretation.

As no trajectory through the origin crosses the first time $\lambda = 0$ orthogonally, $\mathcal{M}_2$ does not pass through 0 and is analytic, $\mathcal{M}_1$ is made up of two branches corresponding to the two branches of the boundary of $\mathcal{S}$ which converge towards 0.

ii) $\gamma_1 = 1$ or $a = 0.0625$. This case is similar to the preceding one except for the fact that the boundary of $\mathcal{S}$ and $\mathcal{T}$ has a

(1) In the following trajectory through the origin is by convention a trajectory through the origin in the r, λ coordinates, i.e., a trajectory passing through the essential singularity at infinity in the negative direction of the x axis.

double point, this does not alter any conclusion of i.

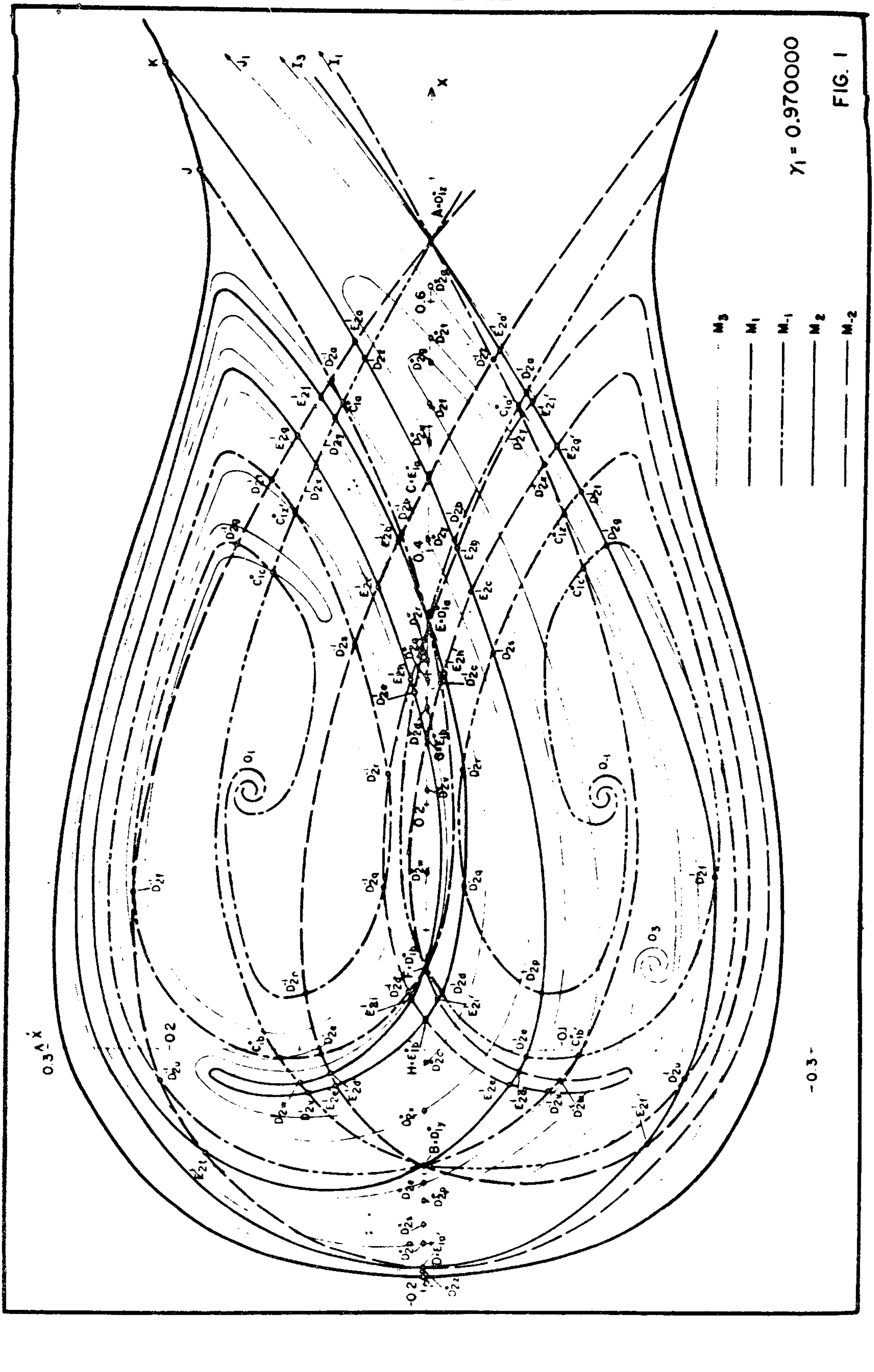
iii) $0 < \gamma_1 < 1$ or $a > 0.0625$. Here the situation is not as nice as in the preceding cases; in this case a trajectory starting from $\lambda = 0$ either crosses again $\lambda = 0$ or is unbounded, hence there does not exist a nice closed section of the $x, \dot{x}$ for which the transformation T transforms this section into itself. But the ideas of the second part of this paper may still be used because every periodic trajectory is bounded by the line $x = f$. Let us bound the domain inside of (35) by the line $x = f$ and call this $\mathcal{T}$; if $A \in \mathcal{T}$, TA is not necessarily in $\mathcal{T}$ or may even not exist if the trajectory goes to infinity before crossing $\lambda = 0$, but if A is periodic all the iterates of A are in $\mathcal{T}$; the invariant lines $\mathcal{M}_n$ are not entirely in $\mathcal{T}$ but their intersections $\mathcal{M}_{n,p}$ are necessarily in $\mathcal{T}$, hence to find the periodic solutions it is not necessary to know which part of $\mathcal{T}$ is transformed into itself but only to find the part of the iterates of $\mathcal{M}_0$ and $\mathcal{M}_1$ which is inside $\mathcal{T}$. This can be done in succession and we know that the iterate of a part of $\mathcal{M}_p$ not in $\mathcal{T}$ will not be in $\mathcal{T}$.

We will from here on summarize the content of the Report bearing the same title as this paper.

18. Data on the Computations. A sufficient number of points on the curves $\mathcal{M}_1$ and $\mathcal{M}_2$ have been determined for a sufficient number of values of the parameter a or γ_1. The choice has been made with great care and was dependent on preceding computation and on extensive computations done with a desk computer. The computations were done on a high-speed computer (the SEAC of the National Bureau of Standards). In the Report we have indicated clearly how the initial conditions were determined, which method of integration was used and which method of checking was involved. A total of 1288 trajectories were integrated for some 21 values of the parameter. The trajectories near $y = 0$ have also been integrated with another method to give the boundary points of $\mathcal{M}_1$ and $\mathcal{M}_2$. It may be interesting to note that the time for the integration of one step was 0.35 sec. for the high-speed computer versus 25 min. for the desk computer. For a precision of five to six decimal places, from 40 to 70 steps were necessary, in the mean.

19. Presentation of the Results. The results have been presented in the form of tables which are available on microfilm and in the form of diagrams, one for each value of the parameter γ_1; for $\gamma_1 = 0.97$, a detailed study has been made which leads to two diagrams, one of which is enclosed (Figure 1). Some data about Figure 1 will now be given.

The subspace $\mathcal{T}$ is bounded by the curve KJP and its symmetric.



$\gamma_1 = 0.970000$

FIG. 1

The curves $\mathcal{M}_1$ and $\mathcal{M}_2$ have been determined by computation; each point corresponds to a trajectory with special initial conditions (Section 9). $\mathcal{M}_{-1}$ and $\mathcal{M}_{-2}$ have been deduced by symmetry. The two branches of $\mathcal{M}_1$ spiral towards a point O_1, which corresponds to a trajectory σ tending to the singularity $x = -\infty$, $\lambda = \frac{\pi}{2}$. The curves $\mathcal{M}_1$ have been deformed in the neighborhood of O_1 to simplify the drawing (of $\mathcal{M}_3$). The intersections of $\mathcal{M}_{-2}$, $\mathcal{M}_{-1}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ give periodic points labeled according to their symmetry (Section 4); the letter subscript distinguishes different periodic solutions having the same symmetry. A to H correspond to periodic solutions which are particularly important.

The curve $\mathcal{M}_3$ has been obtained using only topological properties and its known intersections with the curves $\mathcal{M}_1$ and $\mathcal{M}_2$ deduced from Table I. Its drawing is inexact, the curve may be more complicated but not less complicated than what appears in Figure 1. Its intersections with the other curves and its symmetric (not drawn) give new symmetric periodic solutions. $\mathcal{M}_4$ has been constructed on another diagram. $\mathcal{M}_3$ has two branches which spiral towards a point O_3 which correspond to the second intersection of the trajectory σ with $\lambda = 0$.

20. Interpretation of the Results.

From the diagrams it is possible to obtain results on the appearance of and disappearance of the symmetric periodic solutions. In particular, the orbits of type δ_1 and ϵ_1 have been studied (points A to H ...) and the results on their existence and stability, which had been obtained by special methods [3], [4], [14], [17] are deduced easily. New results include the study of the disappearance of orbits by passage to a limit position on $\lambda = 0$ related to [5], the proof of existence of many new periodic solutions which could hardly be obtained by the classical methods.

Analytic computations of the curves $\mathcal{M}_1$ and $\mathcal{M}_2$ which correspond to orbits close to the singular trajectory σ have been made using results of Lemaitre and Bossy [16] and the analytic computations of the same curves when γ_1 is large have also been done.

In one word the above method permits the unification of all the known results on Störmer's problem; at the same time it provides new results and new insight on the difficulties which one should expect for similar problems.

21. Remarks on the Classification.

Before I conclude, I should like to make some remarks on the classification here given as applied to conservative problems. The reader will have no difficulty extending these remarks to any system of differential equations for which this classification can be used.

When a continuous variation of the parameter "a" was investigated, we saw that an orbit kept its classification except when the orbit disappeared; hence for a continuous variation of "a", the number of double points of the orbit on $y = 0$ remains the same. This is a special case of a more general theorem of Birkhoff [1, II, p. 60].

From this theorem we see that we can refine the classification by considering the double points of the orbit outside $y = 0$. It would be of interest to consider in more detail this subclassification which has, no doubt, a certain importance.

22. Conclusion. In the Introduction we have indicated the aims of this paper. We should like as conclusion, to bring out the relation of these investigations with connected research.

First of all, nothing has been gained here on problems such as transitivity or stability of systems of differential equations, but results are obtained which are much more detailed than those given by the general fixed point theorems, because not only is the existence of more than one or two periodic solutions proven, but the relations of the different periodic solutions with each other is obtained, i.e., the structure of the periodic solution is established.

There is a possibility that these investigations could lead to the proof, under certain hypotheses, of the density of the periodic solutions; in that case, every solution can be approximated during a finite time by a periodic solution.

The method here used has no essential limitations and enables dealing with problems with large non linearities as opposed to the perturbation methods which deal with small non linearities, but which generally fails otherwise (see, for instance, the application of perturbation method to Störmer's problem by G. Lemaitre [15]).

It is proper to mention that some methods (Van der Pol method and others) may succeed in special cases even with large non linearities.

Finally, we should like to mention that the results here obtained are not only of interest in themselves but because of their application to the region problems.

In these problems one considers two domains R_1, R_2 in the phase space and asks, for instance, the proportion of trajectories with initial conditions in R_1 that enter R_2. Solution to these problems depends on the knowledge of unstable periodic trajectories and trajectories asymptotic to these. Examples of this application are given for the cosmic rays' problem by Lemaitre, Vallarta, Bouckaert, Albagli Hunter, Yong-Li, and DeVogelaere (see references of [27]) and for a problem of chemical physics (on transitions rates) by Boudart and DeVogelaere [10].

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