Math 1a Exams, Harvard 2011-2014

This is a one-document version of exams given Spring 2011, Spring 2012, Spring 2013 and 2014. The first year 2011 was primed with two practice exams which were not given before.

You find the exams in the first part and the solutions in the second.

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3/4/2014: First Midterm Exam

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The function $f(x) = \exp(-x^2) - 1$ has the root $x = 0$.

2) T F If f is continuous function and odd then 0 is a root of f.

3) T F $\log(\log(e)) = 0$, if log is the natural log.

4) T F The chain rule assures that $\frac{d}{dx}\sin^2(x) = 2\cos(x)$.

5) T F The function $f(x) = x^2/(1-x^2)$ is continuous everywhere on the real axes.

6) T F The function $\arctan(x)$ is the inverse function of the function $\tan(x)$.

7) T F The Newton method is $T(x) = x - f'(x)/f''(x)$.

8) T F $\cos(3\pi/2) = 0$.

9) T F If a function f is continuous on $[-1, 1]$ and $f(1) = 1, f(-1) = -1$, then there is $-1 < x < 1$, where $f(x) = 0$.

10) T F The chain rule assures that $\frac{d}{dx}g(1/x) = -g'(1/x)/x^2$.

11) T F We have $\lim_{x\to\infty}(2x + 1)/(3x - 1) = 2/3$.

12) T F If 1 is a root of f, then $f'(x)$ changes sign at 1.

13) T F If $f''(0) < 0$ and $f''(1) > 0$ then there is a point in $(0, 1)$, where f has an inflection point.

14) T F The intermediate value theorem assures that the equation $f(x) = x^2 - \cos(x) = 0$ has a root.

15) T F The function $f(x) = x/\sin(x)$ is continuous everywhere if $f(0)$ is suitably defined.

16) T F $f'(x) = 0$ and $f'''(0) < 0$ at $x = 0$ assures that f has a maximum at $x = 0$.

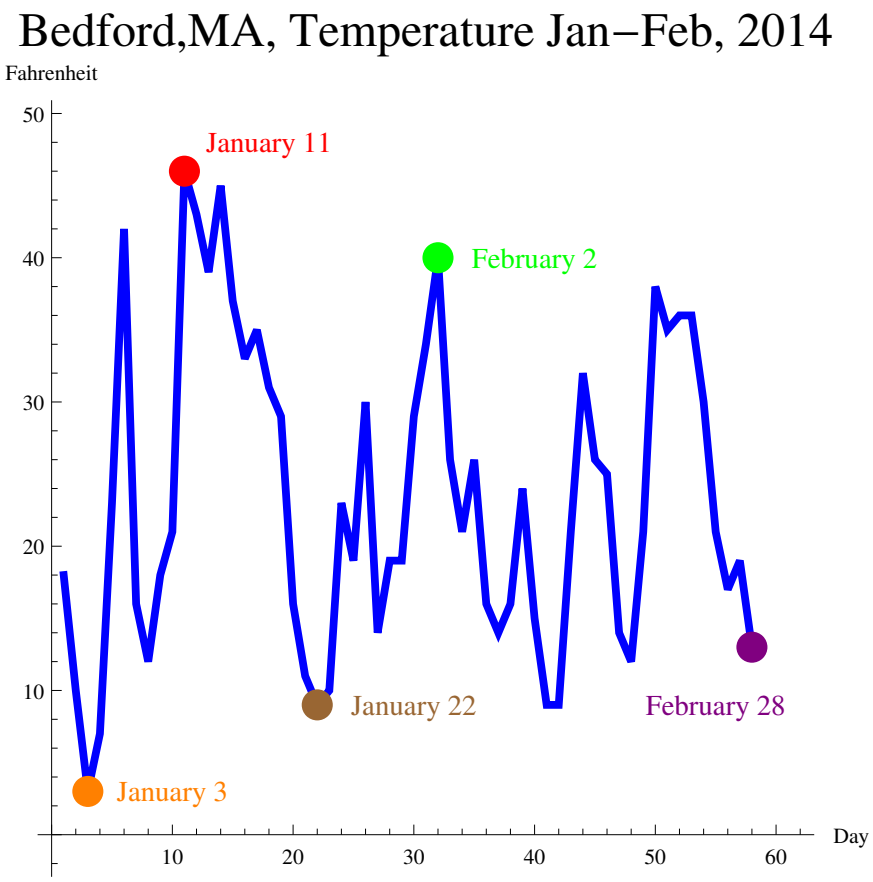
17) T F If f is constant, then $f(x + h) - f(x)/h = 0$ for all $h > 0$.

18) T F The quotient rule is $\frac{d}{dx}(f/g) = (f'(x)g'(x) - f(x)g(x))/(g'(x))^2$.

19) T F $\sin(2\pi) + \tan(2\pi) = 0$.

20) T F It is true that $e^{x\log(5)} = x^5$.

Problem 2) Matching problem (10 points) No justifications are needed.



In this winter, the **polar vortex** ruled the weather in Boston. The above graph shows the temperatures of the first two months of 2014 measured at the **Hanscom field** in Bedford, MA. While temperatures are measured hourly, you can assume that temperature is a continuous function of time. Remember that "global maximum" includes being local too so that only one entry in each line of the table below needs to be checked.

a) (5 points) Check what applies, by checking one entry in each of the 5 dates.

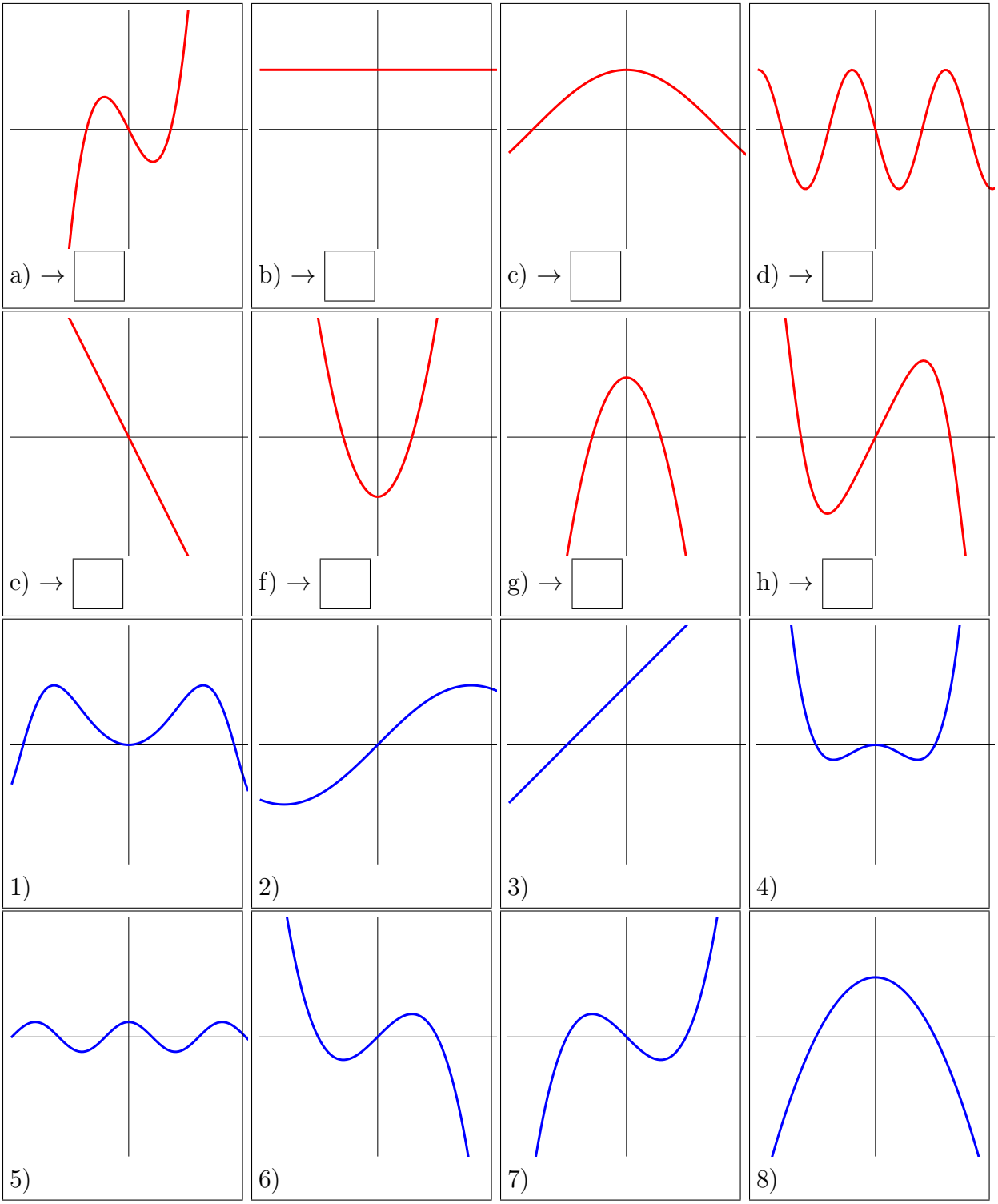
Date	local maximum	local minimum	global maximum	global minimum
January 3				
January 11				
January 22				
February 2				
February 28				

b) (2 points) Which theorem assures that on the closed interval $[0, 59]$ of 59 days, there is a global maximal temperature?

c) (3 points) Argue by citing a theorem why there is a time at which the temperature at Bedford was exactly 25 degree Fahrenheit.

Problem 3) Matching problem (10 points) No justifications are needed.

In the first pictures, we see the first derivatives f'. Match them with the functions f in 1-8. Note that the functions above are the derivative and the functions below are the functions.



Problem 4) Continuity (10 points)

Each of the following functions has a point x_0, where the function is not defined. Find the limit $\lim_{x\to x_0} f(x)$ or state that the limit does not exist.

a) (2 points) $f(x) = \frac{1-2x^3}{1-x}$, at $x_0 = 1$.

b) (2 points) $f(x) = \sin(\sin(5x))/\sin(7x)$, at $x_0 = 0$.

c) (2 points) $f(x) = \frac{\exp(-3x)-1}{\exp(2x)-1}$, at $x_0 = 0$.

d) (2 points) $f(x) = \frac{2x}{\log(x)}$, at $x_0 = 0$.

e) (2 points) $f(x) = \frac{(x-1)^{10}}{(x+1)^{10}}$, at $x_0 = -1$.

Problem 5) Derivatives (10 points)

Find the derivatives of the following functions. If you use a differentiation rule, note which one you use.

a) (2 points) $f(x) = \sqrt{\log(x+1)}$.

b) (3 points) $f(x) = 7\sin(x^3) + \frac{\log(5x)}{x}$.

c) (3 points) $f(x) = \log(\sqrt{x}) + \arctan(x^3)$.

d) (2 points) $f(x) = e^{5\sqrt{x}} + \tan(x)$.

Problem 6) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions:

a) (2 points) $f(x) = \frac{\exp(3x)-\exp(-3x)}{\exp(5x)-\exp(-5x)}$.

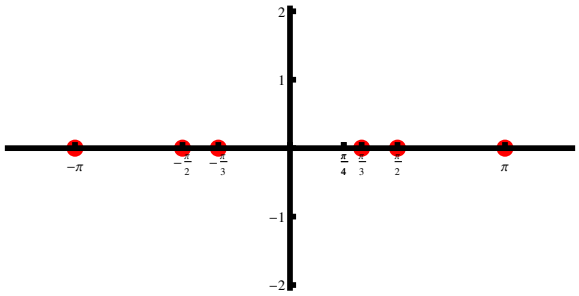
b) (3 points) $f(x) = \frac{\cos(3x)-1}{\sin^2(x)}$.

c) (3 points) $f(x) = [\arctan(x) - \arctan(0)]/x$.

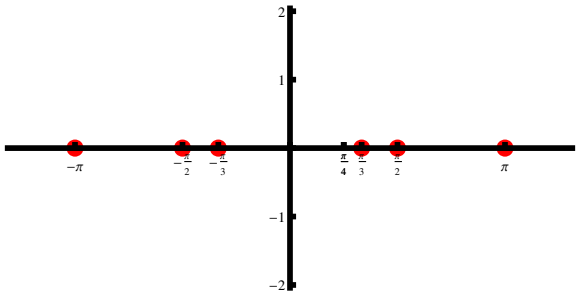
d) (2 points) $f(x) = \frac{\log(7x)}{\log(11x)}$.

Problem 7) Trig functions (10 points)

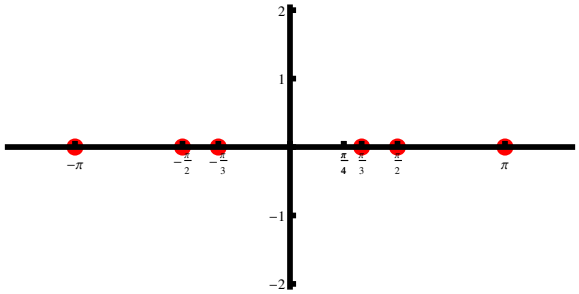
a) Draw the sin function and mark the values of $\sin(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



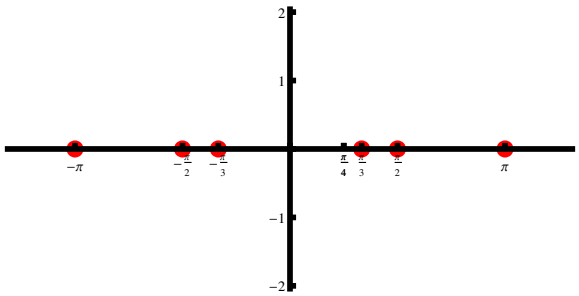
b) Draw the cos function and mark the values of $\cos(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



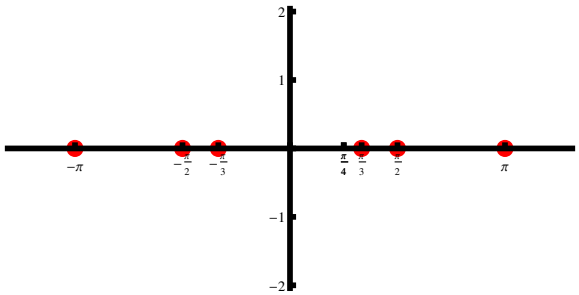
c) Draw the tan function and mark the values of $\tan(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



d) Draw the cot function and mark the values of $\cot(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



e) Draw the sinc function $f(x) = \sin(x)/x$ and mark the points $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2,$.



Problem 8) Extrema (10 points)

Last week, Oliver got a new batch of strong **Neodym magnets**. They are ring shaped. Assume the inner radius is x, the outer radius y is 1 and the height is $h = x$, we want to maximize the surface area $A = 2\pi(y - x)h + 2\pi(y^2 - x^2)$. This amount of maximizing

$$f(x) = 2\pi(1 - x)x + 2\pi(1 - x^2)$$

a) (2 points) Using that $f(x)$ is a surface area, on what interval $[a, b]$ needs f to be considered?

b) (3 points) Find the local maxima of f inside the interval.

c) (3 points) Use the second derivative test to verify it is a maximum.

d) (2 points) Find the global maximum on the interval.

Problem 9) Trig and Exponential functions (10 points)

Simplify the following terms. log denotes the natural log and $\log_{10}$ the log to the base 10. Each result in a)-c) is an integer or a fraction

a) (2 points) $\exp(\log(2)) + e^{3\log(2)}$

b) (2 points) $\log(1/e) + \exp(\log(2)) + \log(\exp(3))$.

c) (2 points) $\log_{10}(1/100) + \log_{10}(10000)$

d) (4 points) Produce the formula for $\arccos'(x)$ by taking the derivative of the identity

$$\cos(\arccos(x)) = x \ .$$

Your answer should be simplified as we did when deriving the derivatives of $\arcsin, \arctan$ in class or when you derived the derivative of arccot and $\operatorname{arcsinh}, \operatorname{arccosh}$ in the homework.

3/4/2014: First hourly Practice A

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Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) T F If f is concave up on $[0,1]$ and concave down on $[1,2]$ then 1 is an inflection points.

2) T F The function $f(x) = \exp(x)$ has the root $x = 1$.

3) T F $\log(\exp(1)) = 1$, if log is the natural log and $\exp(x) = e^x$ is the exponential function.

4) T F The chain rule assures that $d/dx f(f(x)) = f'(f(x))f'(x)$.

5) T F The function $x^2/(1+x^2)$ is continuous everywhere on the real axes.

6) T F The function $\cot(x)$ is the inverse of the function $\tan(x)$.

7) T F The Newton method is $T(x) = x + f(x)/f'(x)$.

8) T F $\cos(\pi/2) = 1/2$.

9) T F If a function f is differentiable on $[-1,1]$, then there is a point x in that interval where $f'(x) = 0$.

10) T F The chain rule assures that $d/dx(g(x^2)) = 2xg'(x^2)$.

11) T F We have $\lim_{x\to\infty}((x^2+1)/x^2) = 1$

12) T F An inflection point is a point, where the function $f'(x)$ changes sign.

13) T F If $f''(-2) > 0$ then f is concave up at $x = -2$.

14) T F The intermediate value theorem assures that the continuous function $x + \sin(x) = 0$ has a root.

15) T F We can find a value b and define $f(0) = b$ such that the function $f(x) = (x^{28}-1)/(x^2-1)$ is continuous everywhere.

16) T F If the third derivative $f'''(x)$ is negative and $f''(x) = 0$ then f has a local maximum at x.

17) T F If $f(x) = x^2$ then $Df(x) = f(x+1) - f(x)$ has a graph which is a line.

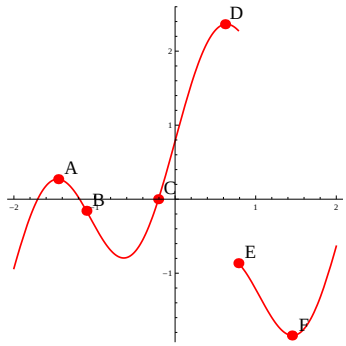
18) T F The quotient rule is $d/dx(f/g) = f'(x)/g'(x)$.

19) T F With $Df(x) = f(x+1) - f(x)$, we have $D(1+a)^x = a(1+a)^x$.

20) T F It is true that $\log(5)e^x = e^{x\log(5)}$ if $\log(x)$ is the natural log.

Problem 2) Matching problem (10 points) No justifications are needed.

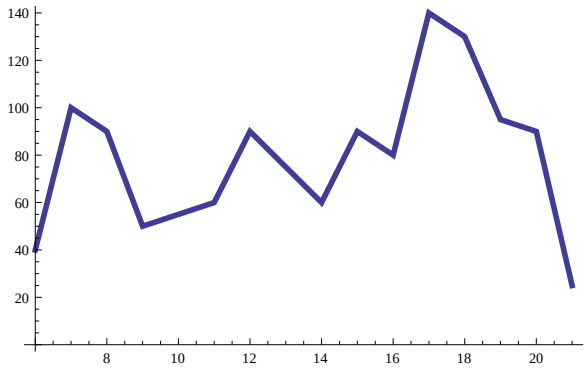
a) (6 points) You see the graph of a function $f(x)$ defined on $[-2, 3]$. Various points $(x, f(x))$ are marked. Match them:



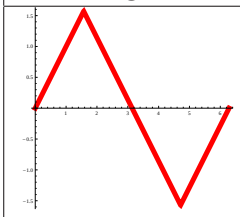
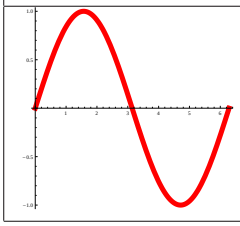
Point x is	Fill in A-F
Local maximum	
Root	
Inflection point	
Discontinuity	
Global maximum	
Local minimum	

b) (2 points) Last week, the **Harvard recreation** published a graph of a function $f(x)$ which shows the number of people at the **Mac** gym as a function of time. At 5 o'clock, there are in average 120 visitors, at 9 in the morning, there are 60 people working out. By the intermediate value theorem, there must be a moment at which exactly $\pi^4 = 97.5...$ visitors are present. This is obviously nonsense. Where is the flaw?

Reason	Check one
No differentiability	
Statistical glitch	
No Continuity	
Inaccurate data	

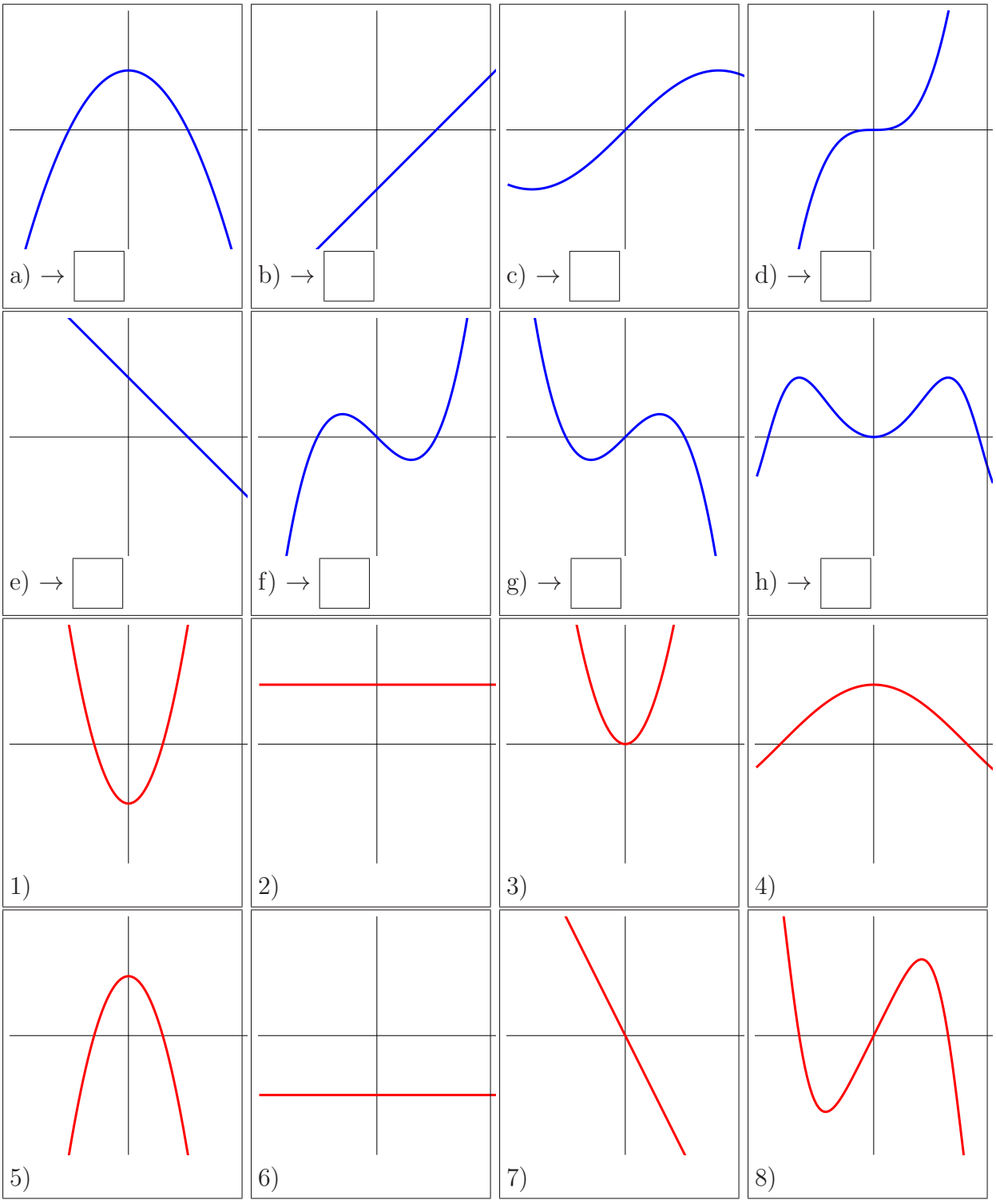


c) (2 points) In front of the **"Class of 1959 Chapel"** at the Harvard business school is an amazing clock: a marble tower contains a steel pole and a large bronze ball which moves up and down the pole indicating the time of the day. As the ball moves up and down the pole, lines with equal distance on the tower indicate the time. At noon, the sphere is at the highest point. At midnight it is at the bottom. It moves the same distance in each hour. If we plot the hight of the sphere as a function of time, which graph do we see?

The height function	Check which applies
	
	

Problem 3) Matching problem (10 points) No justifications are needed.

Match the graph of the functions f in $a) - h)$ with the derivatives f' in 1)-8).



Problem 4) Continuity (10 points)

Each of the following functions has a point x_0, where the function is not defined. Find the limit $\lim_{x \to x_0} f(x)$ or state that the limit does not exist.

a) (2 points) $f(x) = \frac{x^3-8}{x-2}$, at $x_0 = 2$

b) (2 points) $f(x) = \sin(\sin(\frac{1}{x})) - \tan(x)$, at $x_0 = 0$

c) (2 points) $f(x) = \frac{\cos(x)-1}{x^2}$, at $x_0 = 0$

d) (2 points) $f(x) = \frac{\exp(x)-1}{\exp(5x)-1}$, at $x_0 = 0$

e) (2 points) $f(x) = \frac{x-1}{x}$, at $x_0 = 0$

Problem 5) Derivatives (10 points)

Find the derivatives of the following functions. If you use a differentiation rule, note which one you use.

a) (2 points) $f(x) = \sin(7x) + (1 + x^2)$.

b) (2 points) $f(x) = \frac{\sin(7x)}{(1+x^2)}$.

c) (2 points) $f(x) = \sin(7 + x^2)$.

d) (2 points) $f(x) = \sin(7x)(1 + x^2)$.

e) (2 points) $f(x) = \sin(7x)^{(1+x^2)}$

Problem 6) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions f at $x = 0$:

a) (2 points) $f(x) = \frac{(1-\exp(11x))}{(1-\exp(3x))}$

b) (2 points) $f(x) = \frac{\sin(\sin(5x))}{\sin(7x)}$

c) (2 points) $f(x) = \frac{\log(x)}{\log(5x)}$

d) (2 points) $f(x) = \frac{x^2\cos(x)}{\sin^2(x)}$

e) (2 points) $f(x) = \frac{(1+1/x^2)}{(1-1/x^2)}$

Problem 7) Trig functions (10 points)

A triangle with side lengths $3, 4, 5$ has a right angle. Let $\alpha < \beta < \gamma$ denote the angles ordered by

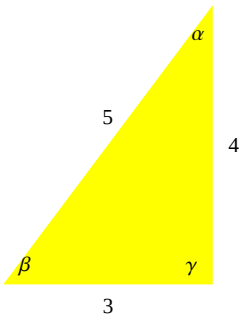
size.

a) (4 points) What are the numerical values of $\cos(\alpha), \cos(\beta), \cos(\gamma), \sin(\gamma)$?

b) (2 points) Find the numerical value of $\tan(\alpha)$ and $\cot(\alpha)$.

The next problem is independent of the previous two.

c) (4 points) Find the derivative of the inverse function of $\arcsin(x)$ by starting with the identity $x = \sin(\arcsin(x))$. Your derivation of $\arcsin'(x)$ should convince somebody who does not know the identity already.



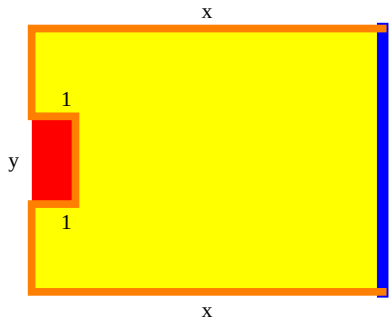
Problem 8) Extrema (10 points)

A tennis field of width x and length y contains a fenced referee area of length 2 and width 1 within the field and an already built wall. The circumference a fence satisfies $2x + y + 2 = 10$, (an expression which still can be simplified). We want to maximize the area $xy - 2$.

a) (2 points) On which interval $[a, b]$ does the variable x make sense? Find a function $f(x)$ which needs to be maximized.

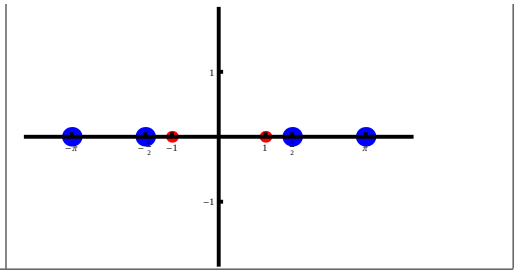
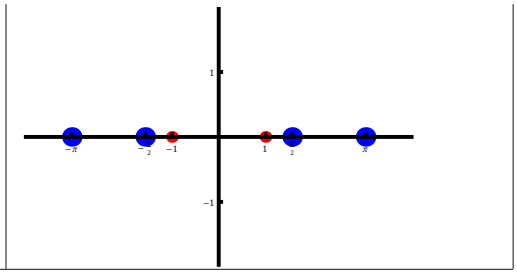
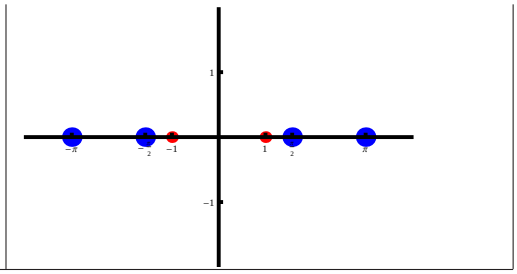
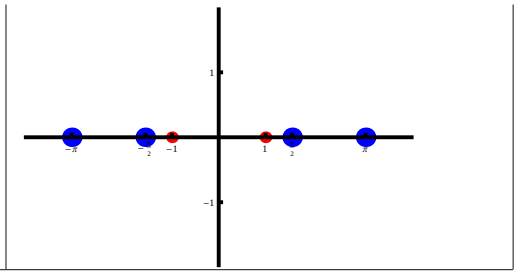
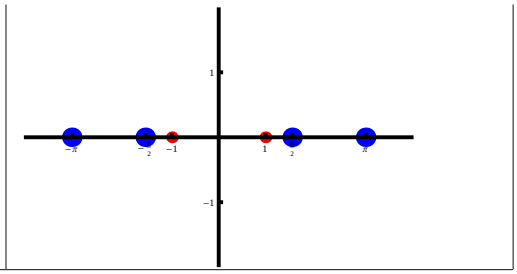
b) (6 points) Find the local maximum of x and check it with the second derivative test.

c) (2 points) What is the global maximum of f on $[a, b]$?



Problem 9) Trig functions (10 points)

In the following five problems, find the numerical value and then draw the graph of the function.

Problem	Answer	Graph
a) (2 points) What is $\sin(\pi/3)$? Plot $\sin(x)$.		
b) (2 points) What is $\cos(5\pi/2)$? Plot $\cos(x)$.		
c) (2 points) Find $\arctan(1)$ Plot $\arctan(x)$.		
d) (2 points) What is $\log(1)$ Plot $\log\|x\|$.		
e) (2 points) What is $\arcsin(\sqrt{3}/2)$. Plot $\arcsin(x)$		

Problem 10) Exponential functions (10 points)

Simplify the following terms. log denotes the natural log and $\log_{10}$ denotes the log to the base 10. All answers are integers.

a) (2 points) $\exp(\log(2))$
b) (2 points) $e^{\log(2)3}$
c) (2 points) $\log(\log(e))$
d) (2 points) $\exp(\log(2) + \log(3))$
e) (2 points) $\log_{10}(10000)$

3/4/2014: First hourly Practice B

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- Please write neatly and except for multiple choice problems, give computations. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.
- All unspecified functions are assumed to be smooth: one can differentiate arbitrarily.
- The actual exam has a similar format: TF questions, multiple choice and then problems where work needs to be shown.

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5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points) No justifications are needed.

1) T F The function $\cot(x)$ is the inverse of the function $\tan(x)$.

2) T F We have $\cos(x)/\sin(x) = \cot(x)$

3) T F $\sin(3\pi/2) = -1$.

4) T F The function $f(x) = \sin(x)/x$ has a limit at $x = 0$.

5) T F For the function $f(x) = \sin(\sin(\exp(x)))$ the limit $\lim_{h\to 0}[f(x+h)-f(x)]/h$ exists.

6) T F If a differentiable function $f(x)$ satisfies $f'(3) = 3$ and is f' is odd then it has a critical point.

7) T F The l'Hopital rule assures that the derivative satisfies $(f/g)' = f'/g'$.

8) T F The intermediate value theorem assures that a continuous function has a derivative.

9) T F After healing, the function $f(x) = (x+1)/(x^2-1)$ is continuous everywhere.

10) T F If f is concave up on $[1, 2]$ and concave down on $[2, 3]$ then 2 is an inflection point.

11) T F There is a function f which has the property that its second derivative f'' is equal to its negative f.

12) T F The function $f(x) = [x]^4 = x(x-h)(x-2h)(x-3h)$ has the property that $Df(x) = 4[x]^3 = 4x(x-h)(x-2h)$, where $Df(x) = [f(x+h) - f(x)]/h$.

13) T F The quotient rule is $d/dx(f/g) = (f'g - fg')/g^2$ and holds whenever $g(x) \neq 0$.

14) T F The chain rule assures that $d/dxf(g(x)) = f'(g(x)) + f(g'(x))$.

15) T F If f and g are differentiable, then $(3f + g)' = 3f' + g'$.

16) T F For any function f, the Newton step $T(x)$ is continuous.

17) T F One can rotate a four legged table on an arbitrary surface such that all four legs are on the ground.

18) T F The fundamental theorem of calculus relates integration S with differentiation D. The result is $DSf(x) = f(x), SDf(x) = f(x) - f(0)$.

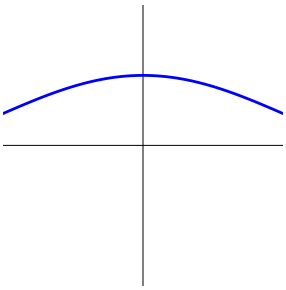
19) T F The product rule implies $d/dx(f(x)g(x)h(x)) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

20) T F Euler and Gauss are the founders of infinitesimal calculus.

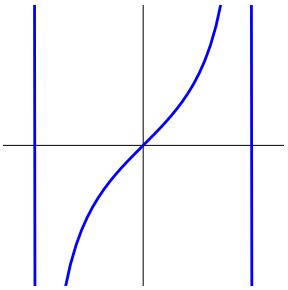
Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their graphs.

Function	Fill in 1-8
$x^2 - x$	
$\exp(-x)$	
$\sin(3x)$	
$\log(\lvert x\rvert)$	
$\tan(x)$	
$1/(2+\cos(x))$	
$x - \cos(6x)$	
$\sin(3x)/x$	



1)



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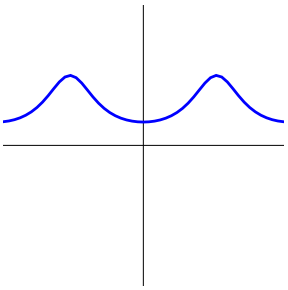

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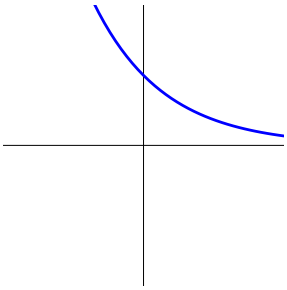

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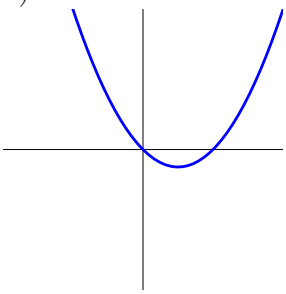

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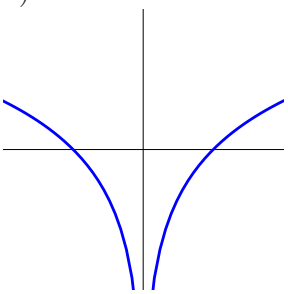

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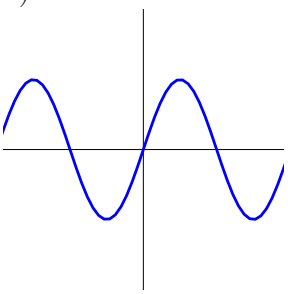

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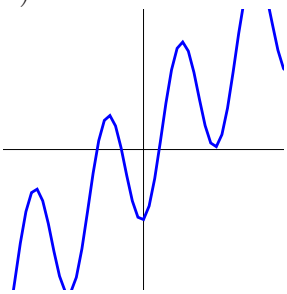
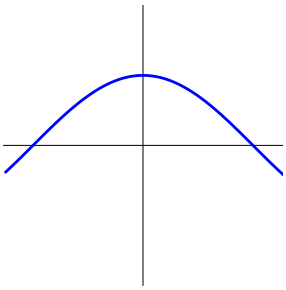


8)

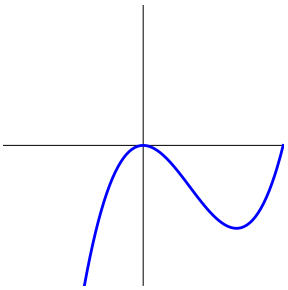
Problem 3) Matching problem (10 points) No justifications are needed.

Match the following functions with their derivatives.

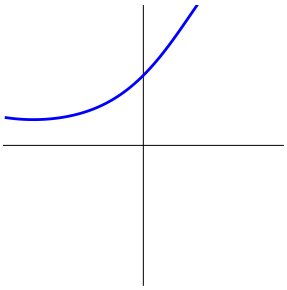
Function	Fill in the numbers 1-8
graph a)	
graph b)	
graph c)	
graph d)	
graph e)	
graph f)	
graph g)	
graph h)	



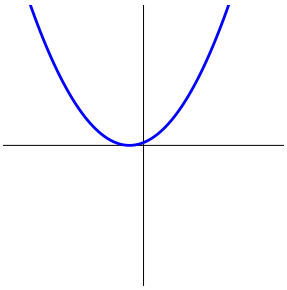
a)



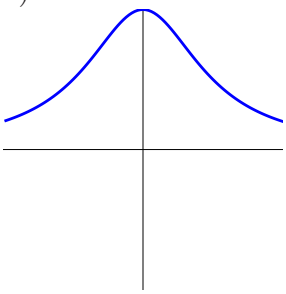
b)



c)



d)



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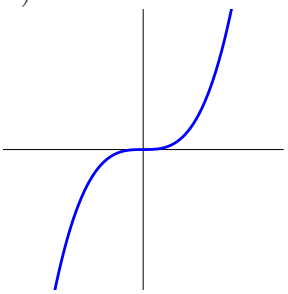

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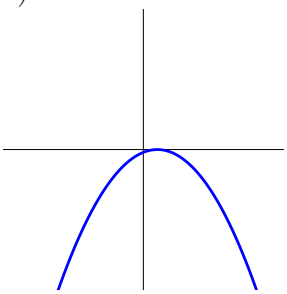

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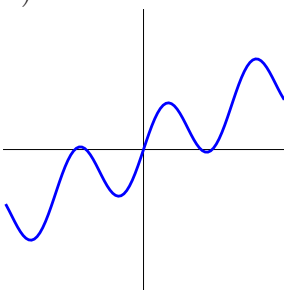
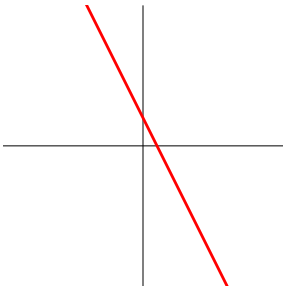
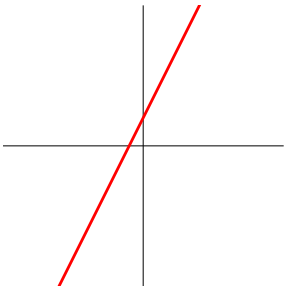


h)



1)



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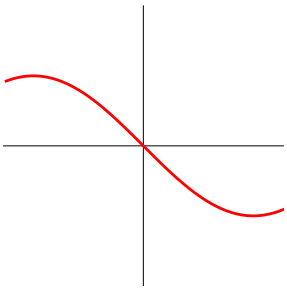

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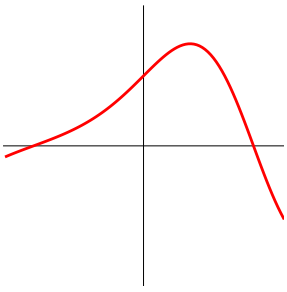

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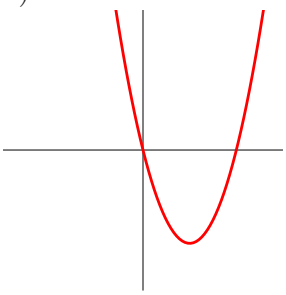

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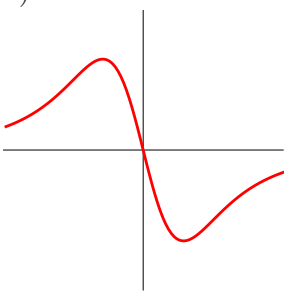

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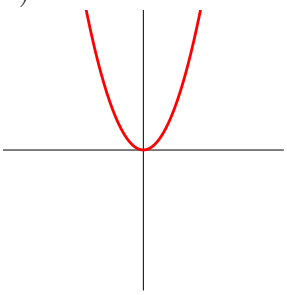

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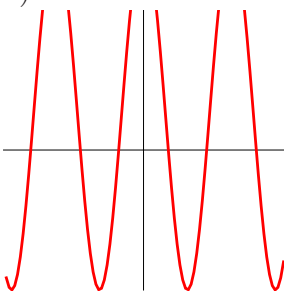


8)

Problem 4) Functions (10 points) No justifications are needed

Match the following functions with simplified versions. In each of the rows, exactly one of the choices A-C is true.

Function	Choice A	Choice B	Choice C	Enter A-C
$\frac{x^4-1}{x-1}$	$1+x+x^2+x^3$	$1+x+x^2$	$1+x+x^2+x^3+x^4$	
2^x	$e^{2\log(x)}$	$e^{x\log(2)}$	$2^{e\log(x)}$	
$\sin(2x)$	$2\sin(x)\cos(x)$	$\cos^2(x)-\sin^2(x)$	$2\sin(x)$	
$(1/x)+(1/(2x))$	$1/(x+2x)$	$3/(2x)$	$1/(x+2x)$	
e^{x+2}	$e^x e^2$	$2e^x$	$(e^x)^2$	
$\log(4x)$	$4\log(x)$	$\log(4)\log(x)$	$\log(x)+\log(4)$	
$\sqrt{x}^3$	$x^{3/2}$	$x^{2/3}$	$3\sqrt{x}$	

Problem 5) Roots (10 points)

Find the roots of the following functions

a) (2 points) $7\sin(3\pi x)$

b) (2 points) $x^5 - x$.

c) (2 points) $\log|ex|$.

d) (2 points) $e^{5x} - 1$

e) (2 points) $8x/(x^2+4) - x$.

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \cos(3x)/\cos(10x)$

b) (2 points) $f(x) = \sin^2(x)\log(1+x^2)$

c) (2 points) $f(x) = 5x^4 - 1/(x^2+1)$

d) (2 points) $f(x) = \tan(x) + 2^x$

e) (2 points) $f(x) = \arccos(x)$

Problem 7) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ of the following functions:

a) (2 points) $f(x) = (x^6 - 3x^2 + 2x)/(1 + x^2 - \cos(x))$.

b) (2 points) $f(x) = (\cos(3x) - 1)/(\cos(7x) - 1)$.

c) (2 points) $f(x) = \tan^3(x)/x^3$.

d) (2 points) $f(x) = \sin(x)\log(x^6)$

e) (2 points) $f(x) = 4x(1 - x)/(\cos(x) - 1)$.

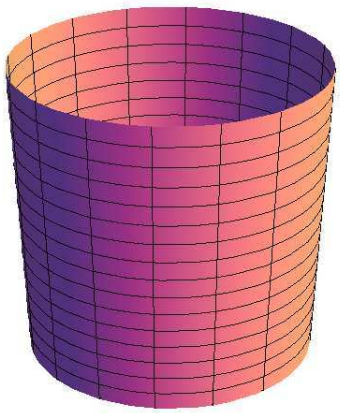
Problem 8) Extrema (10 points)

a) (5 points) Find all local extrema of the function $f(x) = 30x^2 - 5x^3 - 15x^4 + 3x^5$ on the real line.

b) (5 points) Find the global maximum and global minimum of the function $f(x) = \exp(x) - \exp(2x)$ on the interval $[-2, 2]$.

Problem 9) Extrema (10 points)

A cup of height h and radius r has the volume $V = \pi r^2 h$. Its surface area is $\pi r^2 + \pi r h$. Among all cups with volume $V = \pi$ find the one which has minimal surface area. Find the global minimum.



Problem 10) Newton method (10 points)

a) (3 points) Produce the first Newton step for the function $f(x) = e^x - x$ at the point $x = 1$.

b) (4 points) Produce a second Newton step.

c) (3 points) Find the Newton step map $T(x)$ if the function $f(x)$ is replaced by the function $3f(x)$.

3/4/2014: First hourly Practice C

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The function arcsin(x) is defined as $1/\sin(x)$.

2) T F The function $f(x) = \sin(1/x^2)$ can be defined at 0 so that it becomes a continuous everywhere on the real line.

3) T F The function $x/\sin(x)$ can be defined at $x = 0$ so that it becomes a continuous function on the real line.

4) T F The function $f(x) = \sin^2(x)/x^2$ has the limit 1 at $x = 0$.

5) T F The function $f(x) = 1/\log|x|$ has the limit 1 at $x = 0$.

6) T F The function $f(x) = (1 + h)^{x/h}$ has the property that $Df(x) = [f(x + h) - f(x)]/h = f(x)$.

7) T F $\cos(3\pi/2) = 1$.

8) T F If a function f is continuous on the interval $[3, 10]$, then it has a global maximum on this interval.

9) T F The reciprocal rule assures that $d/dx(1/g(x)) = 1/g(x)^2$.

10) T F If $f(0) = g(0) = f'(0) = g'(0) = 0$ and $g''(0) = f''(0) = 1$, then $\lim_{x\to 0}(f(x)/g(x)) = 1$

11) T F An inflection point is a point where the function $f''(x)$ changes sign.

12) T F If $f''(x) > 0$ then f is concave up at x.

13) T F The chain rule assures that $d/dxf(g(x)) = f'(x)g'(x)$.

14) T F The function $f(x) = 1/x + \log(x)$ is continuous on the interval $[1, 2]$.

15) T F If we perform the Newton step for the function $\exp(x)$, we get the map $T(x) = x - 1$.

16) T F The graph of the function $f(x) = x/(1 + x^2)$ has slope 1 at 0.

17) T F There is a differentiable function for which $f'(0) = 0$ but for which 0 is not a local extremum.

18) T F The second derivative test assures that $x = p$ is a local minimum if $f'(p) = 0$ and $f''(p) < 0$.

19) T F The identity $(x^7 - 1)/(x - 1) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ holds for all $x \neq 1$.

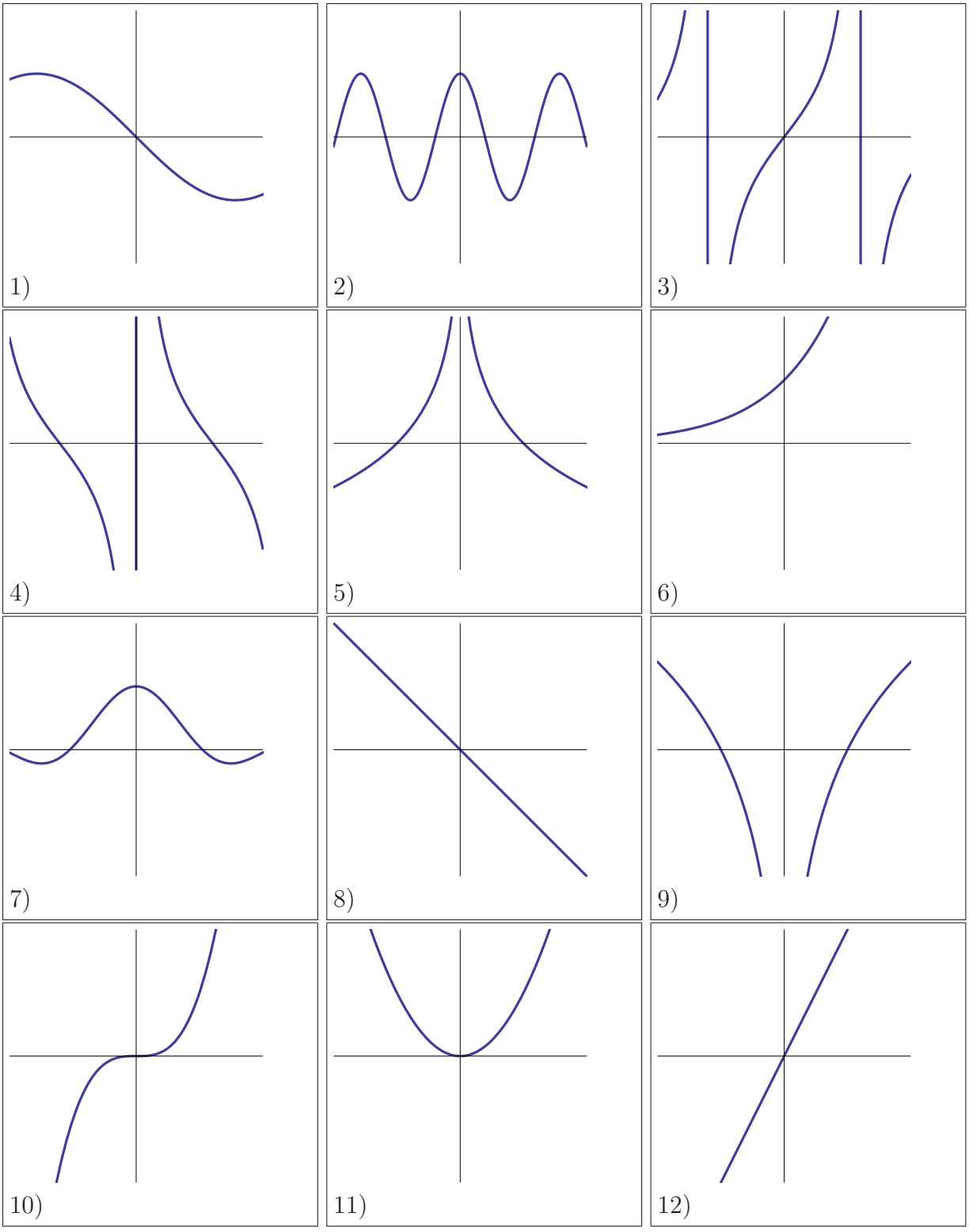
20) T F The slope of the tangent at a point $(x, f(x))$ of the graph of a differentiable function f is equal to $1/f'(x)$.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the functions with the graphs. Naturally, only 10 of the 12 graphs will appear.

Function	Enter 1-12
$\cot(x)$	
$\cos(2x)$	
$2x$	
$\tan(x)$	
$\log(1/\lvert x\rvert)$	

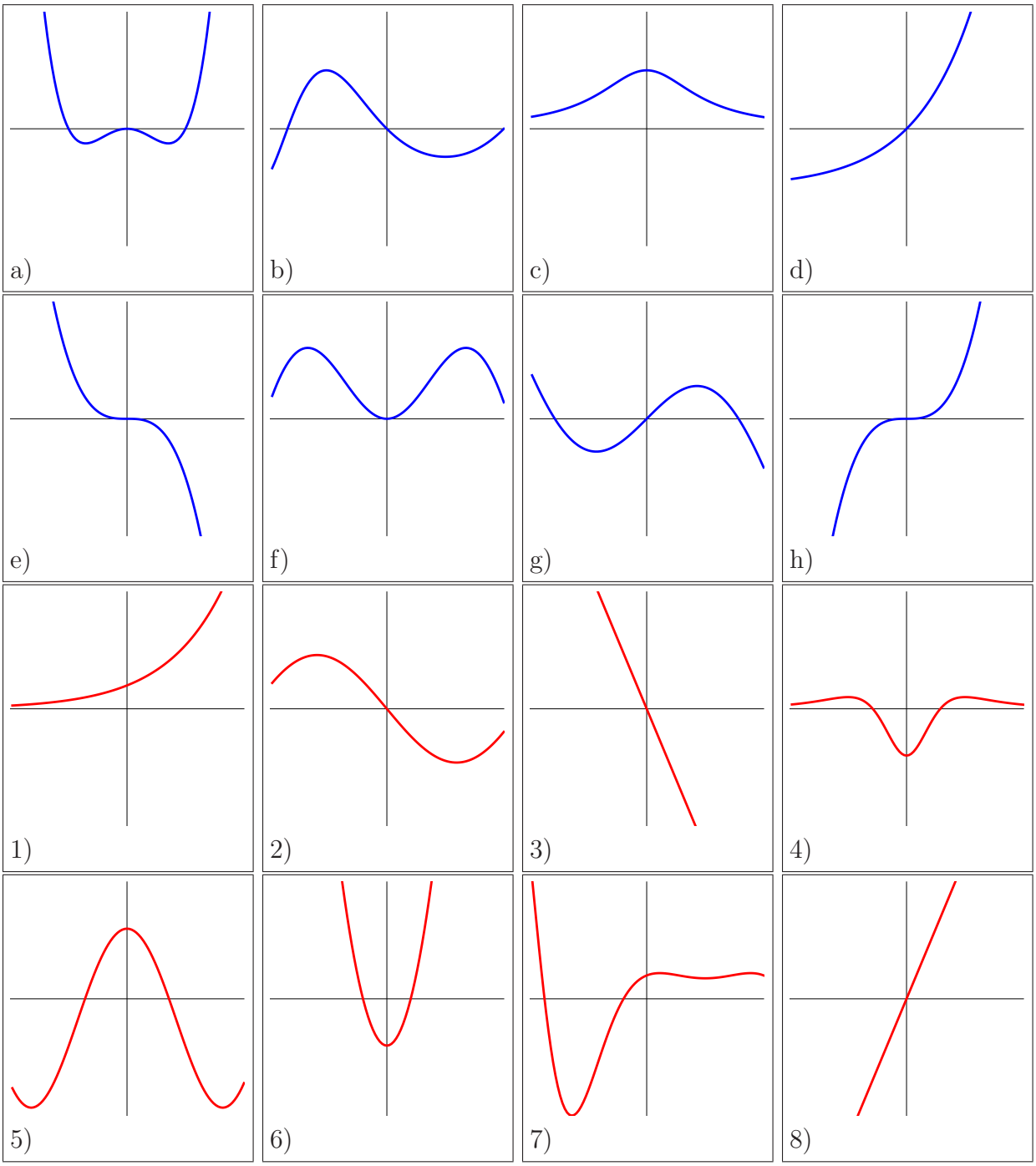
Function	Enter 1-12
x^2	
$\exp(x)$	
$-\sin(x)$	
x^3	
$\operatorname{sinc}(x)$	



Problem 3) Matching problem (10 points) No justifications are needed.

Match the graph of the functions f in $a) - h)$ with the second derivatives f'' in 1)-8).

Function	Second derivative (Enter 1- 8 here)
a)	
b)	
c)	
d)	
e)	
f)	
g)	
h)	



Problem 4) Continuity (10 points)

Some of the following functions might a priori not be defined yet at the point a. In each case, decide whether f can be made a continuous function by assigning a value $f(a)$ at the point a. If no such value exist, state that the function is not continuous.

a) (2 points) $f(x) = \frac{(x^3-1)}{(x-1)}$, at $x = 1$

b) (2 points) $f(x) = \sin(\frac{1}{x}) + \cos(x)$, at $x = 0$

c) (2 points) $f(x) = \sin(\frac{1}{\log(|x|)})$, at $x = 0$

d) (2 points) $f(x) = \log(|\sin(x)|)$, at $x = 0$

e) (2 points) $f(x) = \frac{(x-1)}{x}$, at $x = 0$

Problem 5) Chain rule (10 points)

a) (2 points) Write $1 + \cot^2(x)$ as an expression which only involves the function $\sin(x)$.

b) (3 points) Find the derivative of the function $\operatorname{arccot}(x)$ by using the chain rule for

$$\cot(\operatorname{arccot}(x)) = x \ .$$

c) (2 points) Write $1 + \tan^2(x)$ as an expression which only involves the function $\cos(x)$.

d) (3 points) Find the derivative of the function $\arctan(x)$ by using the chain rule for

$$\tan(\arctan(x)) = x \ .$$

Remark: even if you should know the derivatives of arccot or arctan, we want to see the derivations in b) and d).

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \frac{\cos(3x)}{\cos(x)}$

b) (2 points) $f(x) = \sin^2(x)\log(1 + x^2)$

c) (2 points) $f(x) = 5x^4 - \frac{1}{x^2+1}$

d) (2 points) $f(x) = \tan(x) + \exp(-\sin(x^2))$

e) (2 points) $f(x) = \frac{x^3}{(1+x^2)}$

Problem 7) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions f at $x = 0$ or state (providing reasoning as usual) that the limit does not exist.

a) (2 points) $f(x) = \frac{\sin(3x)}{\sin(x)}$

b) (2 points) $f(x) = \frac{\sin^2(x)}{x^2}$

c) (2 points) $f(x) = \sin(\log(|x|))$

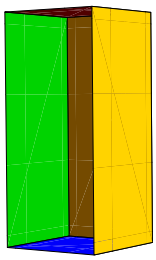
d) (2 points) $f(x) = \tan(x)\log(x)$

e) (2 points) $f(x) = \frac{(5x^4-1)}{(x^2+1)}$

Problem 8) Extrema (10 points)

A rectangular shoe-box of width x, length x and height y is of volume 2 so that $x^2y = 2$. The surface area adds up three rectangular parts of size $(x \times y)$ and 2 square parts of size $(x \times x)$ and leads to

$$f = 2x^2 + 3xy \ .$$



a) (2 points) Write down the function $f(x)$ of the single variable x you want to minimize.

b) (6 points) Find the value of x for which the surface area is minimal.

c) (2 points) Check with the second derivative test, whether the point you found is a local minimum.

Problem 9) Global extrema (10 points)

In this problem we study the function $f(x) = 3x^5 - 5x^3$ on the interval $[-2, 2]$.

a) (2 points) Find all roots of f.

b) (3 points) Find all local extrema of the function.

c) (3 points) Use the second derivative test to analyze the critical points, where applicable.

d) (2 points) Find the **global** maximum and minimum of f on the interval $[-2, 2]$.

Problem 10) Newton (10 points)

Perform one Newton step for the function $f(x) = x^5 - x$ starting at the point $x = 3$.

3/4/2014: First hourly Practice D

Your Name:

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- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
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- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] 1 is the only root of the log function on the interval $(0, \infty)$.

2) [T] [F] $\exp(\log(5)) = 5$, if log is the natural log and $\exp(x) = e^x$ is the exponential function.

3) [T] [F] The function $\cos(x)+\sin(x)+x^2$ is continuous everywhere on the real axes.

4) [T] [F] The function $\sec(x) = 1/\cos(x)$ is the inverse of the function $\cos(x)$.

5) [T] [F] The Newton method allows to find the roots of any continuous function.

6) [T] [F] $\sin(3\pi/2) = -1$.

7) [T] [F] If a function f is continuous on $[0, \infty)$, then it has a global maximum on this interval.

8) [T] [F] The reciprocal rule assures that $d/dx(1/g(x)) = -1/g(x)^2$.

9) [T] [F] If $f(0) = g(0) = f'(0) = g'(0) = 0$ and $g''(0) = f''(0) = 1$, then $\lim_{x\to 0}(f(x)/g(x)) = 1$

10) [T] [F] An inflection point is a point, where the function $f''(x)$ changes sign.

11) [T] [F] If $f''(3) > 0$ then f is concave up at $x = 3$.

12) [T] [F] The intermediate value theorem assures that a continuous function has a maximum on a finite interval.

13) [T] [F] We can find a value b and define $f(1) = b$ such that the function $f(x) = (x^6 - 1)/(x^3 - 1)$ is continuous everywhere.

14) [T] [F] Single roots of the second derivative function f'' are inflection points.

15) [T] [F] If the second derivative $f''(x)$ is negative and $f'(x) = 0$ then f has a local maximum at x.

16) [T] [F] The function $f(x) = [x]^3 = x(x + h)(x + 2h)$ satisfies $Df(x) = 3[x]^2 = 4x(x + h)$, where $Df(x) = [f(x + h) - f(x)]/h$.

17) [T] [F] The quotient rule is $d/dx(f/g) = (fg' - f'g)/g^2$.

18) [T] [F] The chain rule assures that $d/dxf(g(x)) = f'(g(x))f'(x)$.

19) [T] [F] With $Df(x) = f(x + 1) - f(x)$, we have $D2^x = 2^x$.

20) [T] [F] Hôpital's rule applied to the function $f(x) = \text{sinc}(x) = \sin(x)/x$ gives us the fundamental theorem of trigonometry.

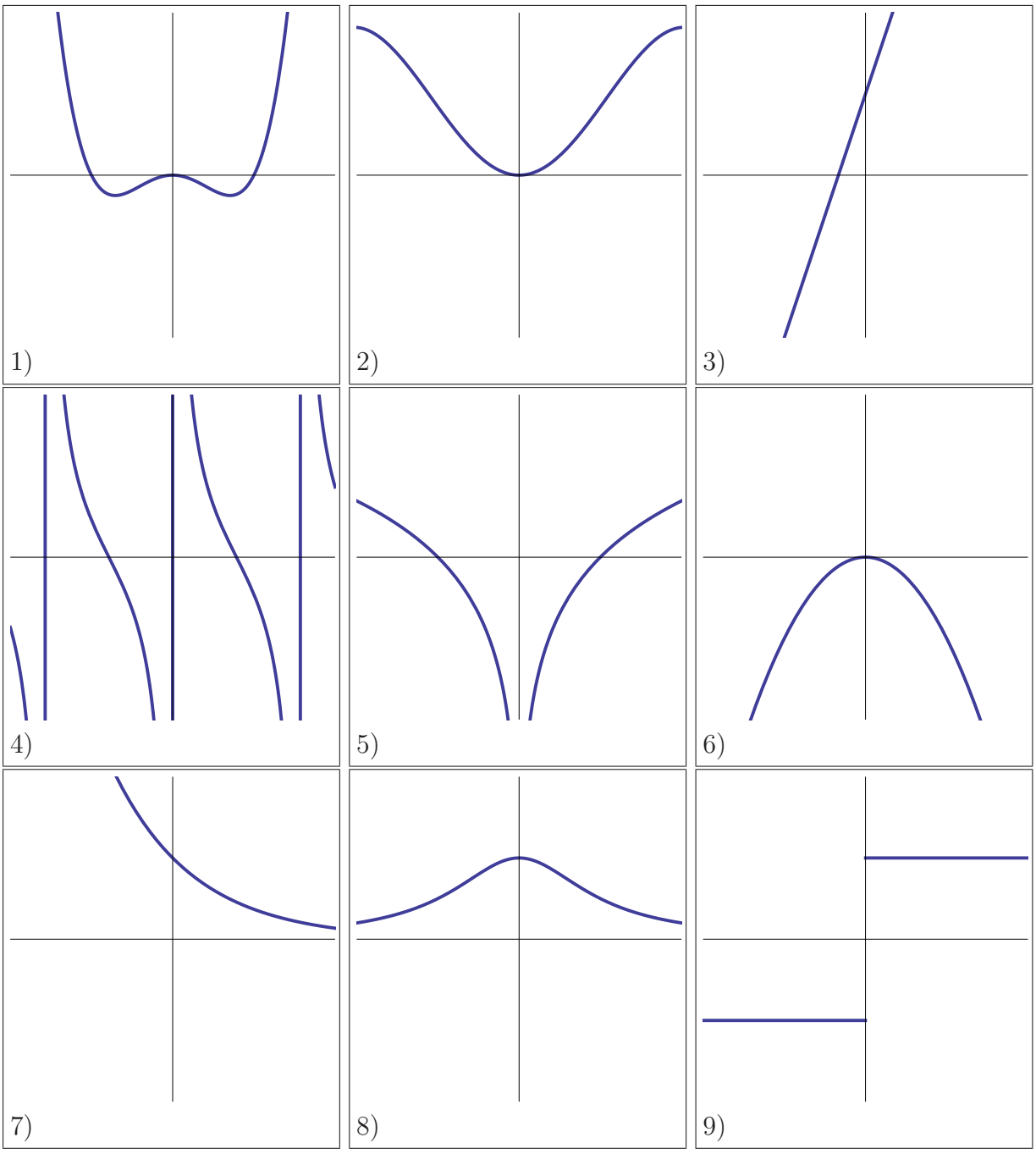
Problem 2) Matching problem (10 points) No justifications are needed.

Match the functions with the graphs.

Function	Enter 1-9
$1/(1+x^2)$	
$\cot(2x)$	
$3x+1$	

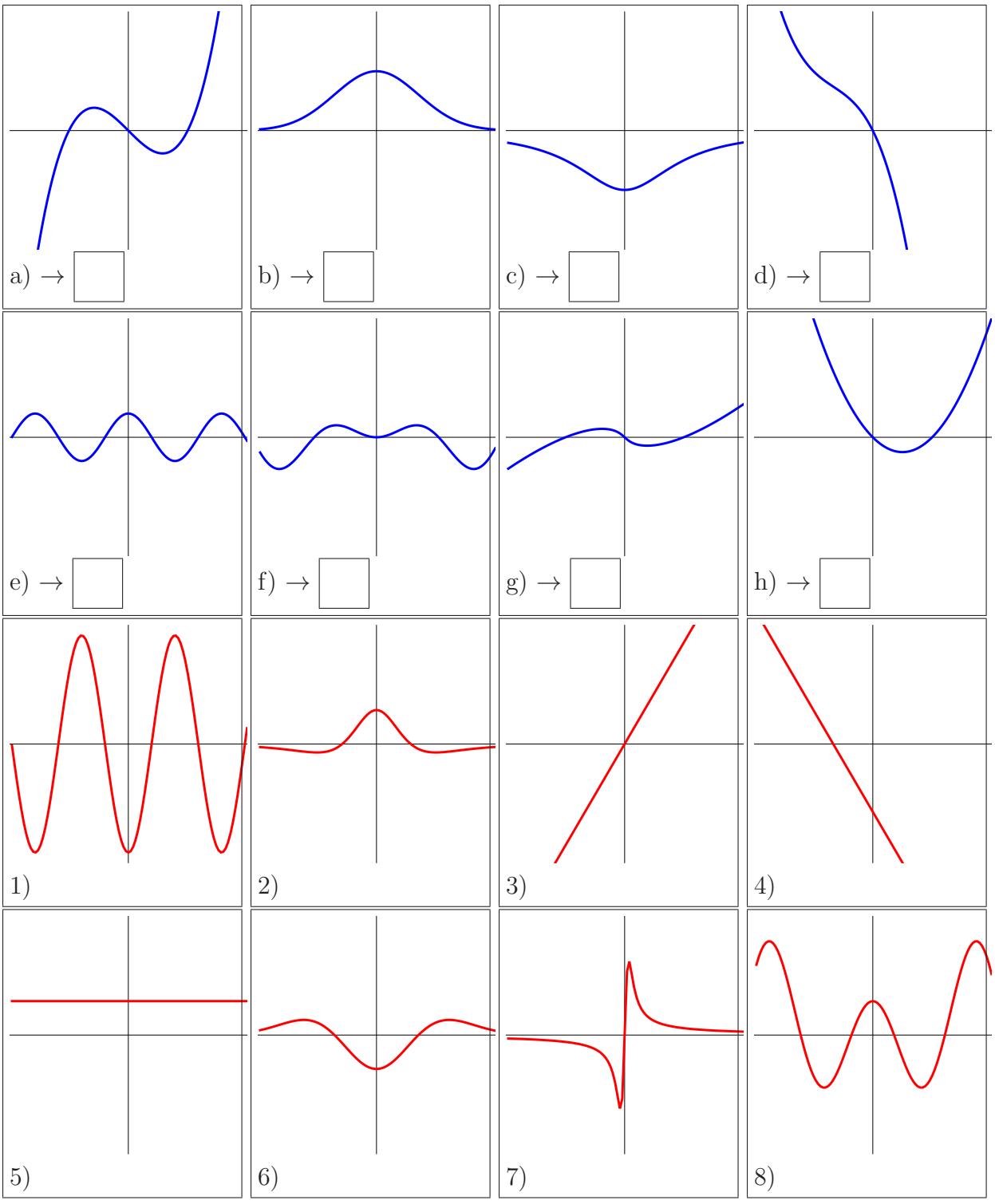
Function	Enter 1-9
$x\sin(x)$	
$\exp(-x)$	
$\log(\lvert x\rvert)$	

Function	Enter 1-9
$\text{sign}(x)$	
x^4-x^2	
$-x^2$	



Problem 3) Matching problem (10 points) No justifications are needed.

Match the graph of the functions f in $a)-h)$ with the second derivatives f'' in 1)-8).



Problem 4) Continuity (10 points)

Decide whether the function can be healed at the given point in order to be continuous everywhere on the real line. If the function can be extended to a continuous function, give the value at the point.

a) (2 points) $f(x) = \frac{(x^3-8)}{(x-2)}$, at $x = 2$

b) (2 points) $f(x) = \sin(\sin(1/x)) - \tan(x)$, at $x = 0$

c) (2 points) $f(x) = \frac{\cos(x)-1}{x^2}$, at $x = 0$

d) (2 points) $f(x) = (\exp(x) - 1)/(\exp(5x) - 1)$, at $x = 0$

e) (2 points) $f(x) = \frac{(x-1)}{x}$, at $x = 0$

Problem 5) Chain rule (10 points)

In the following cases, we pretend not to know the formula for the derivative of log or arctan and again recover it using the chain rule.

b) (2 points) Rederive the derivative of the square root function $\text{sqrt}(x) = \sqrt{x}$ by differentiating

$$(\text{sqrt}(x))^2 = x$$

and solving for $\text{sqrt}'(x)$.

b) (4 points) Rederive the derivative of the logarithm function $\log(x)$ by differentiating

$$\exp(\log(x)) = x$$

and solving for $\log'(x)$.

c) (4 points) Rederive the formula for the derivative of the arctan function $\arctan(x)$ by differentiating the identity

$$\tan(\arctan(x)) = x$$

and using $1 + \tan^2(x) = 1/\cos^2(x)$ to solve for $\arctan'(x)$.

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \frac{5\sin(x^6)}{x}$ for $x > 0$

b) (2 points) $f(x) = \tan(x^2) + \cot(x^2)$ for $x > 0$

c) (2 points) $f(x) = \frac{1}{x} + \log(x^2)$ for $x > 0$

d) (2 points) $f(x) = x^6 + \sin(x^4)\log(x)$ for $x > 0$

e) (2 points) $f(x) = \log(\log(x))$ for $x > 1$

Problem 7) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions f at $x = 0$ or state that the limit does not exist. State the tools you are using.

a) (2 points) $f(x) = x^2 + x + \sin(1 - \cos(x))$

b) (2 points) $f(x) = \frac{x^3}{\sin(x^3)}$

c) (2 points) $f(x) = x^3/\sin(x)^2$

d) (2 points) $f(x) = x^3 + \text{sign}(x)$

e) (2 points) $f(x) = \cos(x^4) + \cos(\frac{1}{x})$

Problem 8) Extrema (10 points)

In the following problem you can ignore the story if you like and proceed straight go to the question:

Story: a cone shaped lamp designed in 1995 by **Verner Panton** needs to have volume $\pi r^2 h = \pi$ to be safe. To minimize the surface area $A = \pi r\sqrt{h^2 + r^2}$, we minimize the square A^2 and so $\pi^2 r^2(h^2 + r^2)$. From the volume assumption, we get $r^2 = 1/h$ so that we have to minimize $(\pi^2/h)(h^2 + 1/h) = \pi^2 f(h)$.

Which height h minimizes the function

$$f(h) = h + \frac{1}{h^2} \ ?$$

Use the second derivative test to check that you have a minimum.

Problem 9) Global extrema (10 points)

An investment problem leads to the profit function

$$f(x) = x - 2x^2 + x^3 \ ,$$

where $x \in [0, 2]$. Find the local and global maxima and minima of f on this interval and use the second derivative test.

4/8/2014: Second midterm exam

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1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F For any continuous function f we have $\int_0^1 3f(t)\,dt = 3\int_0^1 f(t)\,dt$.

2) T F For any continuous function $\int_0^3 f(t)\,dt = 3\int_0^1 f(t)\,dt$.

3) T F For any continuous function $\int_0^1 1 - f(t)\,dt = 1 - (\int_0^1 f(t)\,dt)$.

4) T F The anti-derivative of $\tan(x)$ is $-\log(\cos(x)) + C$.

5) T F The fundamental theorem of calculus implies that $\int_1^3 f'(x)\,dx = f(3) - f(1)$.

6) T F The integral $\pi\int_0^1 x^2\,dx$ gives the volume of a cone of height 1.

7) T F The anti-derivative of $1/\cos^2(x)$ is $\tan(x)$.

8) T F The function $F(x) = \int_0^x \tan(t^2)\,dt$ has the derivative $\tan(x^2)$.

9) T F If the area $A(r(t))$ of a disk changes in a constant rate, then the radius $r(t)$ changes in a constant rate.

10) T F The identity $\frac{d}{dx}\int_1^2 \log(x)\,dx = \log(2) - \log(1)$ holds.

11) T F If $xy = 3$ and $x'(t) = 1$ at $(3, 1)$ then $y' = 1$.

12) T F If $f < 1$, then $\int_0^2 f(x)\,dx$ can be bigger than 1.

13) T F An improper integral is an improperly defined definite indefinite integral.

14) T F The anti derivative $F(x)$ of $f(x)$ satisfies $F'(x) = f(x)$.

15) T F A parameter value c for which the number of minima are different for parameters smaller or larger than c is called a catastrophe.

16) T F If f is unbounded at 0, then $\int_0^1 f(x)\,dx$ is infinite.

17) T F If $f(-1) = 0$ and $f(1) = 1$ then $f' = 2$ somewhere on $(-1, 1)$.

18) T F The anti-derivative of $\log(x)$ is $x\log(x) - x + C$, where log is the natural log.

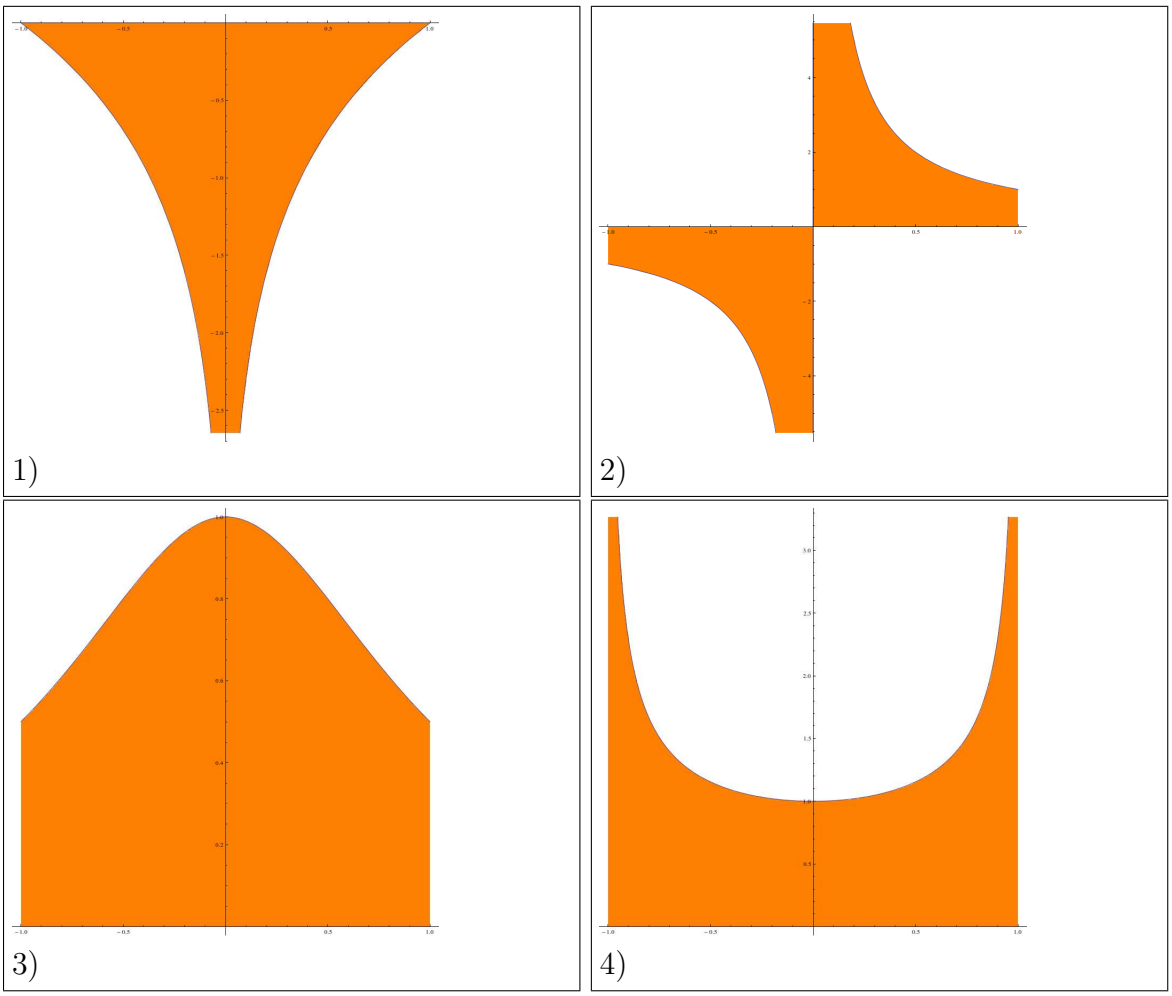
19) T F The sum $\frac{1}{n}[(\frac{0}{n})^2 + (\frac{1}{n})^2 + ... + (\frac{n-1}{n})^2]$ converges to $1/3$ in the limit $n \to \infty$.

20) T F The **improper integral** $\int_1^\infty \frac{1}{x^2}\,dx$ represents a finite area.

Problem 2) Matching problem (10 points) No justifications are needed.

a) (4 points) Match the following integrals with the regions and indicate whether the integral represents a finite area.

Integral	Fill in 1-4	Finite?
$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx$		
$\int_{-1}^{1} \frac{1}{x}\, dx$		
$\int_{-1}^{1} \frac{1}{1+x^2}\, dx$		
$\int_{-1}^{1} \log\lvert x\rvert\, dx$		



b) (6 points) Which of the following properties are always true. This means which are true for all choices of continuous functions and all choices of a, b, c.

Identity	Check if true
$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$	
$\int_a^b f(x)\, dx + \int_a^b g(x)\, dx = \int_a^b f(x) + g(x)\, dx$	
$\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$	
$\int_a^b f(x)^2\, dx = (\int_a^b f(x)\, dx)^2$	
$\int_a^a f(x)\, dx = 0$	
$\int_a^b f(x)\, dx = \int_b^a f(x)\, dx$	

Problem 3) (10 points)

Fill in the missing part into the empty box to make a true statement:

a) (2 points)

$\frac{d}{dx} \int_0^x f(t)\, dt =$ [ ] by the **fundamental theorem of calculus**.

b) (2 points)

$\int_0^x f(t)\, dt =$ [ ] by the **fundamental theorem of calculus**.

c) (2 points)

The **mean value theorem** tells there is exists $a < x < b$ with $\frac{f(b)-f(a)}{b-a} =$ [ ]

d) (2 points)

A **probability distribution** satisfies $\int_{-\infty}^{\infty} f(x)\, dx = 1$ and [ ] for all x.

e) (2 points)

For an improper integral $\int_a^b f(x)\, dx$, either $a = \infty$ or $b = \infty$ or f is [ ] on $[a, b]$.

Problem 4) Area computation (10 points)

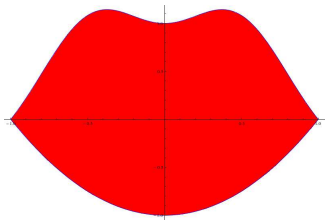
The region enclosed by the graphs of

$$f(x) = x^2 - 1$$

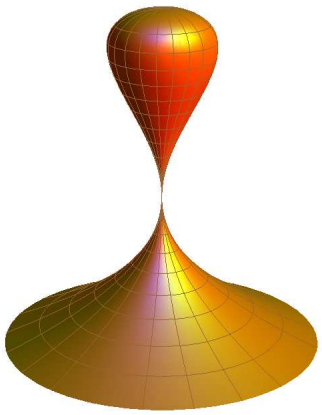
and

$$g(x) = 1 - x^2 + (1 - \cos(2\pi x))/6$$

models of the lips of **Rihanna**. Find the area.



Problem 5) Volume computation (10 points)



The **kiss** is a solid of revolution for which the radius at height z is

$$z^2\sqrt{1-z}$$

and where $-1 \leq z \leq 1$. What is the volume of this solid? The name "kiss" is the official name for this quartic surface. Indeed, the top part has the shape of a **Hershey Kiss**. P.S. Creative **"exam product placement"** like this has been invented and patented by Oliver himself ...

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. Each of the problems produces a numerical answer.

a) (2 points) $\int_0^1 \sqrt{1+x}\, dx$.

b) (2 points) $\int_0^1 \frac{1}{1+x^2}\, dx$

c) (2 points) $\int_2^e \frac{5}{3+x}\, dx$

d) (2 points) $\int_0^1 \frac{1}{\sqrt{1-x}}\, dx$.

e) (2 points) $\int_0^1 (x+1)^{10}\, dx$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives:

a) (2 points) $\int e^{23x} - x^{23}\, dx$

b) (2 points) $\int \frac{2}{x+3} + x^{1/23}\, dx$

c) (2 points) $\int \frac{23}{1+x} + 23\tan(x)\, dx$

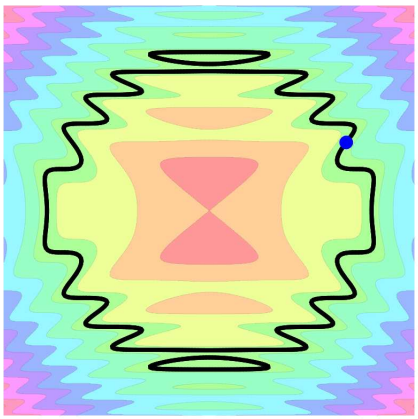
d) (2 points) $\int \frac{1}{\sin^2(x)} + \frac{1}{x^3}\, dx$

e) (2 points) $\int \cos^2(3x)\, dx$

Jim Carrey in the movie "The number 23"

Problem 8) Implicit differentiation and related rates (10 points)

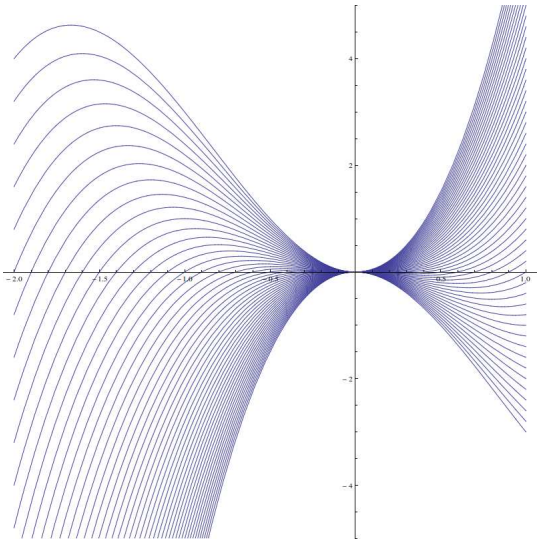
a) (5 points) Find the slope y' of the curve $x^2 + y^2 + \sin(x^2 - 4y^2) = 5$ at $x = 2, y = 1$.



b) (5 points) We deal with functions $x(t), y(t)$ where t is time. If $x^4 + 2y^4 = 3$ and $x'(t) = t$, find $y'(t)$ at the point $(x, y) = (1, 1)$.

Problem 9) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = 2x^3 + cx^2$, where c is a parameter.



a) (2 points) Find the critical points of $f_3(x)$.

b) (2 points) Find the critical points of $f_{-3}(x)$.

c) (2 points) Check that 0 is always a critical point.

d) (2 points) For which c is 0 a minimum?

e) (2 points) For which c does the catastrophe occur?

4/8/2014: Second midterm practice A

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1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) T F If f is a continuous function then $\int_0^x f(t)\ dt$ is an area and therefore positive.

2) T F The anti-derivative of $\operatorname{arccot}(x)$ is $-\log(\sin(x)) + C$.

3) T F The fundamental theorem of calculus implies that $\int_0^3 f''(x)\ dx = f'(3) - f(0)$.

4) T F The volume of a cylinder of height 3 and radius 5 is given by the integral $\int_0^3 \pi 5^2\ dx$.

5) T F The antiderivative of $\tan(x)$ is $1/\cos^2(x)$.

6) T F The mean value theorem implies that the derivative of $\sin(x)$ in the interval $[0, \pi/2]$ is $2/\pi$ somewhere.

7) T F The function $F(x) = \int_0^x \sin(t^2)\ dt$ has the derivative $\sin(x^2)$.

8) T F The level of wine in a parabolic glass changes with a constant rate if the volume decreases in a constant rate.

9) T F The identity $\frac{d}{dx}\int_0^1 \sin(x)\ dx = \sin(1)$ holds.

10) T F If a solid is scaled by a factor 2 in all directions then its volume increases by a factor 8.

11) T F If $x^2 - y^2 = 3$ and $x'(t) = 1$ at $(2, 1)$ then $y' = 1$.

12) T F If $f(x)$ is smaller than $g(x)$ for all x, then $\int_0^1 f(x) - g(x)\ dx$ is negative.

13) T F Every improper integral defines an infinite area.

14) T F The anti derivative of $f'(x)$ is equal to $f(x) + c$.

15) T F Catastrophes can explain why minima can change discontinuously.

16) T F If f is discontinuous at 0, then $\int_{-1}^1 f(x)\ dx$ is infinite.

17) T F If $f(-\infty) = 0$ and $f(\infty) = 1$ then $f' = 1$ somewhere on $(-\infty, \infty)$.

18) T F The anti-derivative of $1/x$ is $\log(x) + C$, where log is the natural log.

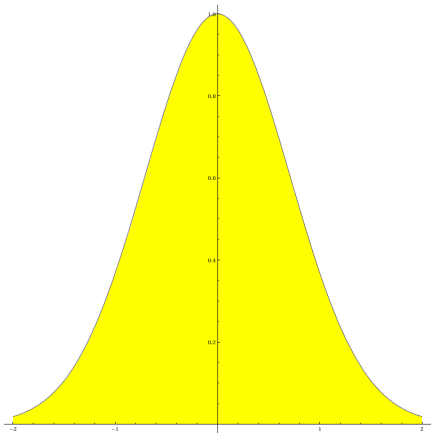
19) T F A catastrophe is defined as a critical point of f which is a minimum.

20) T F The integral $\int_0^\infty 1/x^2\ dx$ represents a finite area.

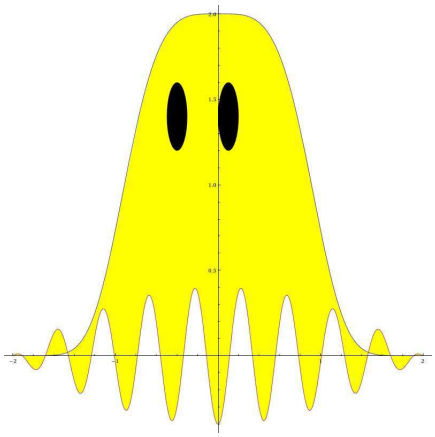
Problem 2) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the following integrals with the regions. Graphs 1) and 2) are inspired by a cartoon by Matthew Freeman (J Epidemiol. Community Health. 2006 January; 60(1): 6)

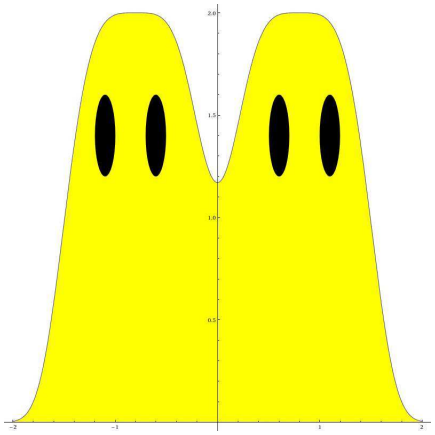
Integral	Fill in 1-4
$\int_{-2}^{2}(4-x^2)\cos^2(14x)/10-(4-x^2)\cos(14x)/15dx$	
$\int_{-2}^{2} 2\exp(-3(x+0.8)^4)+2\exp(-3(x-0.8)^4\ dx$	
$\int_{-2}^{2}\exp(-x^2)\ dx$	
$\int_{-2}^{2} 2\exp(-x^4)-(x^2-4)\cos(14x)/10\ dx$	



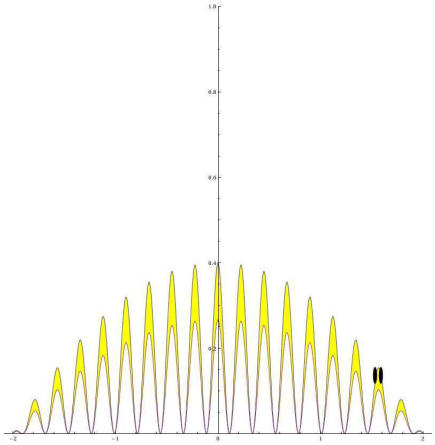
1) Normal distribution



2) Paranormal distribution



3) Abnormal distribution



4) Wormal distribution

b) (4 points) Which of the following statements follows from Rolle's theorem? Check only one.

Result	Check
If $f(0)=-1$ and $f(1)=1$ then there is x with $0\le x\le 1$ with $f'(x)=2$	
If $f(0)=1$ and $f(1)=1$ then there is a critical point x of f in $(0,1)$	
If $f(0)=1$ and $f(1)=1$ then there is point where $f(x)=2$ in $(0,1)$	
If $f(0)=1$ and $f(1)=1$ then there is point where $f''(p)=0$ in $(0,1)$	

Problem 3) (10 points)

a) (4 points) Having seen some applications of integration and differentiation, complete the table:

Function f	Antiderivative F
Probability density function	
	Total cost
	Mass
Area	
	Velocity
Power	
Velocity	

b) (2 points) We have seen two methods to find roots $f(x) = 0$ of equations. Both methods need some assumptions on the functions: Choose from the following: "differentiability", "continuity", "positivity".

Method	Assumption which f has to satisfy
Disection method	
Newton method	

c) (2 points) Which is more general? In each row, check one box.

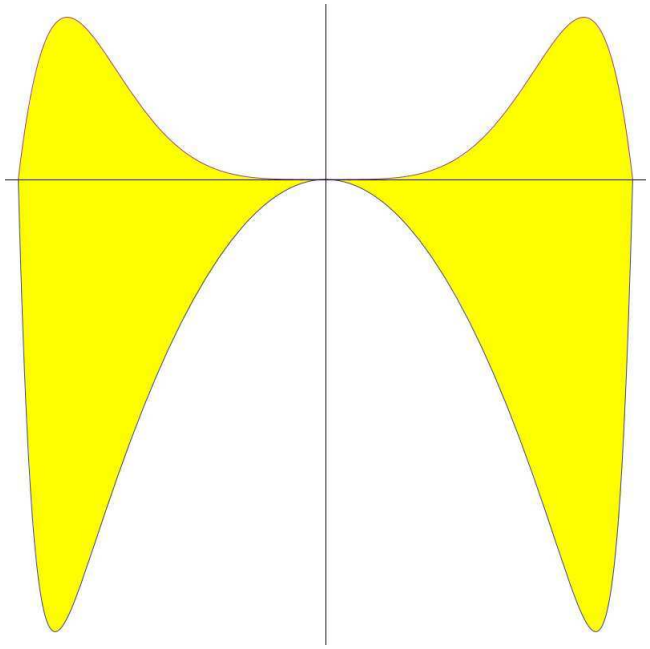
	Related rates	Implicit differentiation	
	Rolles theorem	Intermediate value theorem	

d) (2 points) Which integral is finite? Chose one!

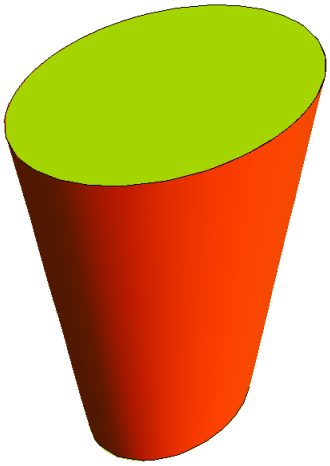
Integral	finite	infinite	
$\int_1^\infty 1/\sqrt{x}\, dx$			
$\int_1^\infty 1/x^2\, dx$			

Problem 4) Area computation (10 points)

The region enclosed by the graphs of $f(x) = x^{20} - x^2$ and $g(x) = x^4 - x^8$ is a cross section for a catamaran sailing boat. Find the area.

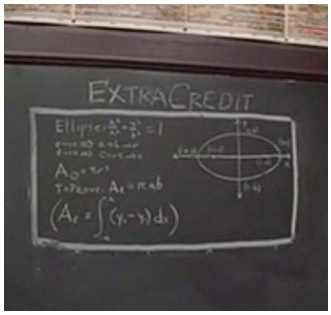


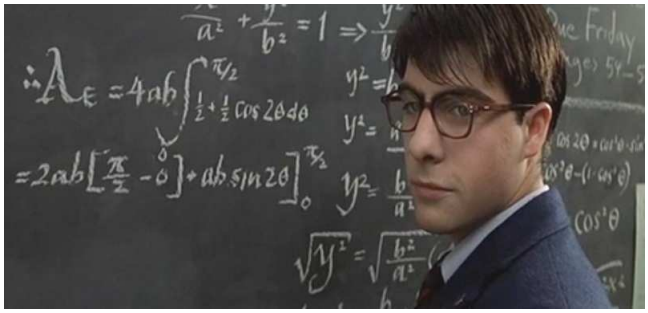
Problem 5) Volume computation (10 points)



An ellipse with diameters $2b$ and $2a$ has area πab. Find the volume of part of a cone whose height is between $z = 3$ and $z = 5$ for which the cross section at height z is an ellipse with parameters $a = 2z$ and $b = 3z$.

Remark. We will see later the area formula. In the movie "Rushmore", the teacher tells about the problem: "I put that up as a joke. It's probably the hardest geometry equation in the world".





Screen shots from the movie Rushmore shows a blackboard where the formula for the ellipse is computed using trig substitution. You might spot a double angle formula. We will come to that.

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. Each of the problems produces a numerical answer.

a) (2 points) $\int_0^1 (x-1)^4 \, dx$

b) (2 points) $\int_0^1 x^{1/3} \, dx$.

c) (2 points) $\int_0^{\sqrt{3}} \frac{6}{1+x^2} \, dx$

d) (2 points) $\int_{-2}^{e-3} \frac{5}{3+x} \, dx$

e) (2 points) $\int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx$.

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (2 points) $\int e^{7x} - \sqrt{x} \, dx$

b) (2 points) $\int \frac{5}{x+1} + 7\cos^2(x) \, dx$

c) (2 points) $\int \frac{11}{1+x^2} + 9\tan(x) \, dx$

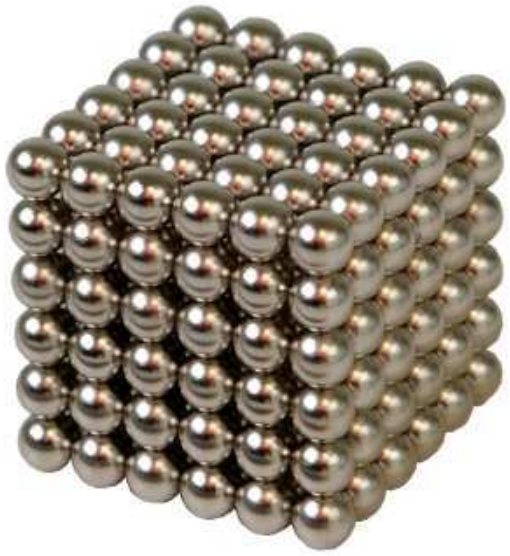
d) (2 points) $\int \frac{4}{\cos^2(x)} + \frac{2}{\sin^2(x)} \, dx$

e) (2 points) $\int 2x\cos(x^2) \, dx$

Problem 8) Implicit differentiation and related rates (10 points)

a) (5 points) Find the slope y' of the curve $x^2 y = \sin(xy) + (y-1)$ at $x = \pi/2, y = 1$.

b) (5 points) A magnetic Neodym metal cube of length x is heated and changes the volume in time at a rate $V' = 1$. At which rate does the length $x(t)$ of the cube change, when the volume is $V = 27$?



Neodym magnets. Soon outlawed since kids can swallow them, leading to a change of topology of their intestines. Dangerous stuff! Gun bullets can be obtained more easily, naturally because they can not be swallowed ...

Problem 9) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = x^6 - cx^4 - cx^2$, where c is a parameter.

a) (4 points) Verify that f has a critical point 0 for all c.

b) (3 points) Determine whether 0 is a minimum or maximum depending on c.

c) (3 points) For which c does a catastrophe occur?

Problem 10) Basic integrals (10 points)

Find the anti derivatives. You have to solve in 10 seconds each. For every second over that limit, one point of the entire exam will be taken off. So, for example: if you use 62 seconds for the following 5 problems, you have used 12 seconds too much and 12 points are taken off from your exam. Don't worry, we do not assign negative points; your final score will always remain a number between 0 and 110 points. To fill a loophole in that setup: if you choose not do the problems, 50 points are taken off.

a) (2 points) e^{-2x}.
b) (2 points) $\cos(15x)$.
c) (2 points) 2^x.
d) (2 points) $1/(1-x)$
e) (2 points) $1/(1+x^2)$

4/8/2014: Second midterm practice B

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] The anti-derivative of $\tan(x)$ is $-\log(\cos(x)) + C$.

2) [T] [F] The fundamental theorem of calculus implies that $\int_0^1 f'(x)\ dx = f(1) - f(0)$.

3) [T] [F] The volume of truncated pyramid with a base square length 2 and top square length 3 is given by the integral $\int_2^3 x^2\ dx$.

4) [T] [F] The derivative of $\arctan(x)$ is $1/\cos^2(x)$.

5) [T] [F] The mean value theorem implies $\int_a^b f'(x)\ dx = f'(c)(b-a)$ for some c in the interval (a, b).

6) [T] [F] If $F(x) = \int_0^x f(t)\ dt$ has an critical point at $x = 1$ then f has a root at $x = 1$.

7) [T] [F] The anti-derivative of the derivative of f is equal to $f + C$ where C is a constant.

8) [T] [F] If we blow up a balloon so that the volume V changes with constant rate, then the radius $r(t)$ changes with constant rate.

9) [T] [F] The identity $\frac{d}{dx}\int_5^9 f(x)\ dx = f(9) - f(5)$ holds for all continuous functions f.

10) [T] [F] Two surfaces of revolution which have the same cross section area $A(x)$ also have the same volume.

11) [T] [F] If $x^2 + y^2 = 2$ and $x(t), y(t)$ depend on time and $x' = 1$ at $x = 1$ then $y' = -1$ is possible.

12) [T] [F] The identity $\int_2^9 7f(x)\ dx = 7\int_2^9 f(x)\ dx$ is true for all continuous functions f.

13) [T] [F] The improper integral $\int_1^\infty 1/x\ dx$ in the sense that $\int_1^R 1/x\ dx$ converges for $R \to \infty$ to a finite value.

14) [T] [F] If $f_c(x)$ has a local minimum at $x = 2$ for $c < 1$ and no local minimum anywhere for $c > 1$, then $c = 1$ is a catastrophe.

15) [T] [F] An improper integral is an indefinite integral which does not converge.

16) [T] [F] If $f(-5) = 0$ and $f(5) = 10$ then $f' = 1$ somewhere on the interval $[-5, 5]$.

17) [T] [F] The sum $\frac{1}{n}\sum_{k=0}^{n-1}\frac{k}{n} = \frac{1}{n}[\frac{0}{n} + \frac{1}{n} + \cdots + \frac{n-1}{n}]$ is a Riemann sum to the integral $\int_0^1 x\ dx$.

18) [T] [F] The anti-derivative of $\mathrm{sinc}(x) = \sin(x)/x$ is equal to $\sin(\log(x)) + C$.

19) [T] [F] The anti-derivative of $\log(x)$ is $1/x + C$.

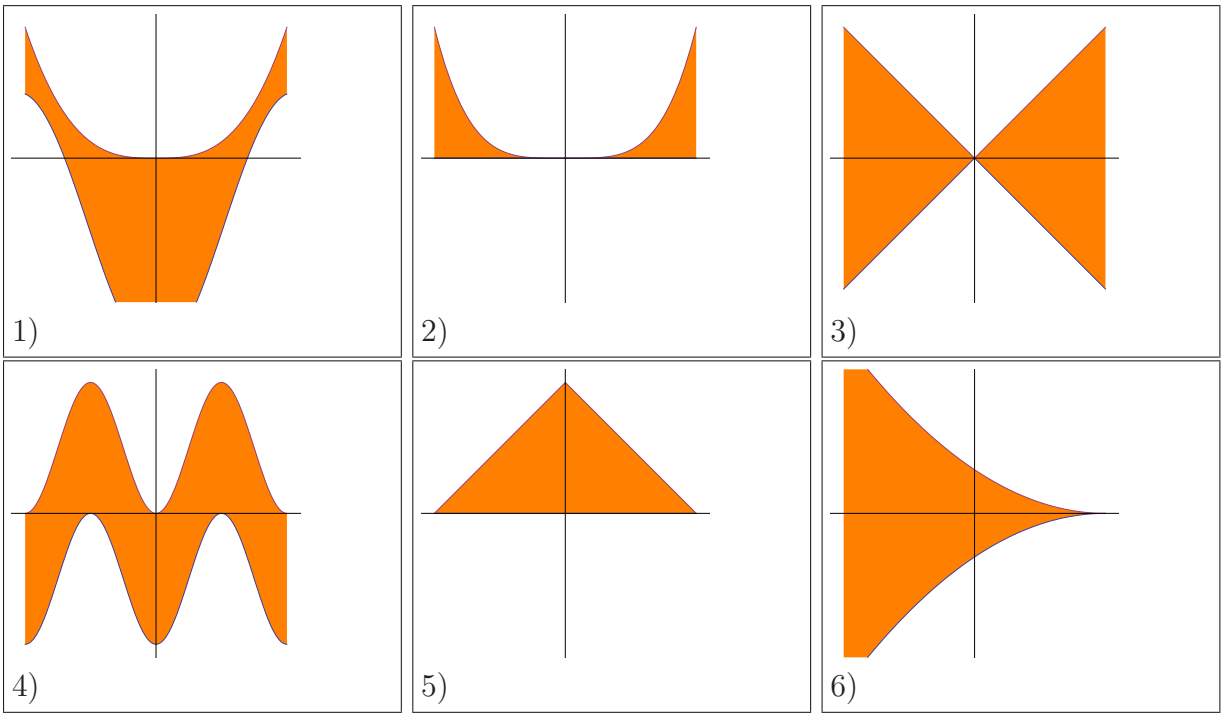
20) [T] [F] We have $\int_0^x tf(t)\ dt = x\int_0^x f(t)\ dt$ for all functions f.

Problem 2) Matching problem (10 points) No justifications are needed.

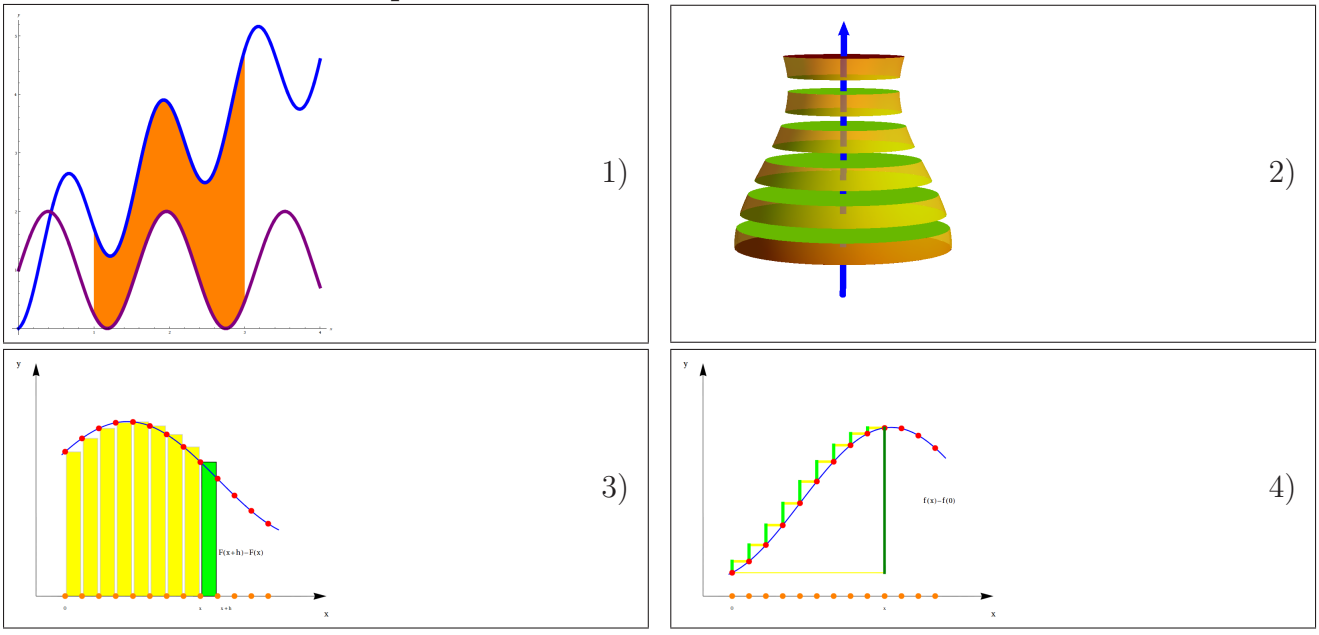
a) (6 points) Match the integrals with the pictures.

Integral	Enter 1-6
$\int_{-1}^{1}(1-x)^2\,dx$	
$\int_{-1}^{1}\lvert x\rvert\,dx$	
$\int_{-1}^{1}x^4\,dx$	

Integral	Enter 1-6
$\int_{-1}^{1}\lvert x\rvert^3-\cos(3x)\,dx$	
$\int_{-1}^{1}[\sin^2(\pi x)-\cos^2(\pi x)]\,dx$	
$\int_{-1}^{1}1-\lvert x\rvert\,dx$	



b) (4 points) Match the concepts: each of the 4 figures illustrates one of the formulas which are the centers of the **mind map** we have drawn for this exam:



Formula	Enter 1-4
$\int_a^b A(z)\,dz$	
$\int_a^b g(x)-f(x)\,dx$	

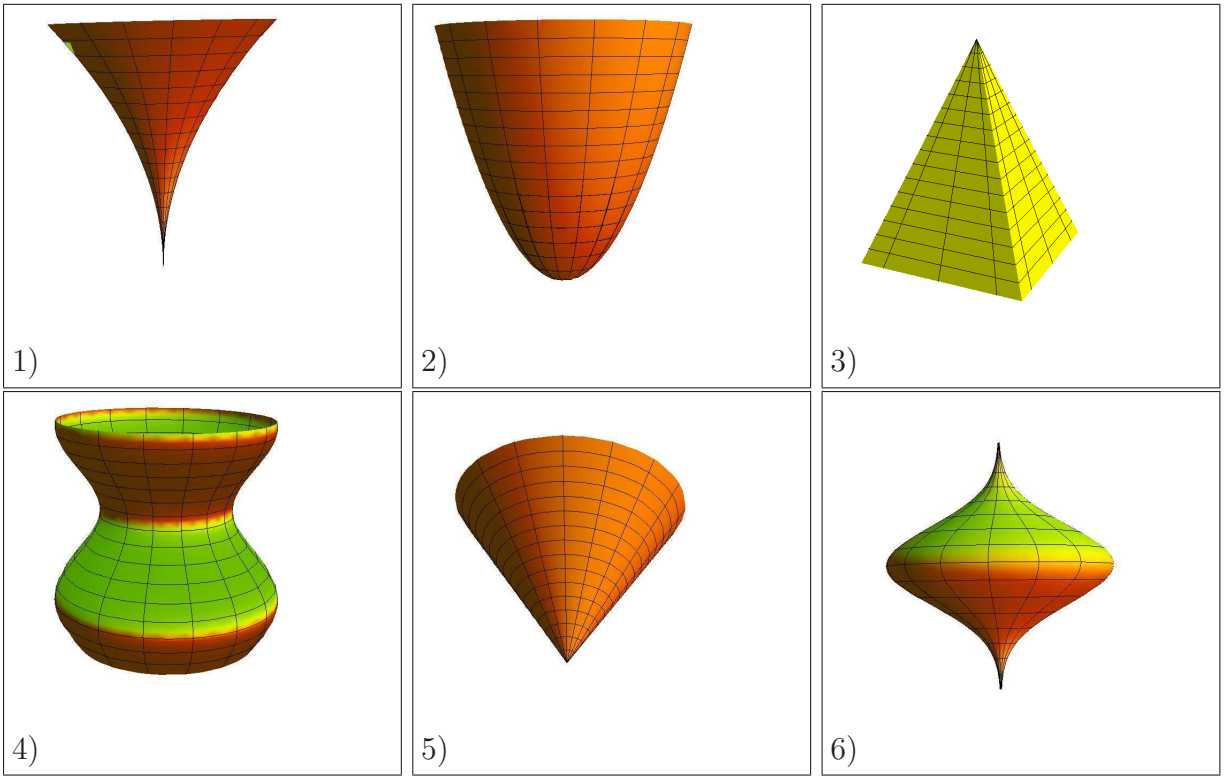
Formula	Enter 1-4
$\frac{d}{dx}\int_0^x f(t)\,dt=f(x)$	
$\int_0^x f'(t)\,dt=f(x)-f(0)$	

Problem 3) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the volumes of solids.

Integral	Enter 1-6
$\int_0^1 \pi z^4 \, dz$	
$\int_0^1 \pi z \, dz$	
$\int_0^1 \pi(4+\sin(4z)) \, dz$	

Integral	Enter 1-6
$\int_{-1}^1 \pi e^{-4z^2} \, dz$	
$\int_0^1 \pi z^2 \, dz$	
$\int_0^1 (1-z)^2 \, dz$	



b) (4 points) Fill in the missing word which links **applications** of integration.

The probability density function is the		of the cumulative distribution function.
The total cost is the		of the marginal cost.
The volume of a solid is the		of the cross section area function.
The velocity of a ball is the		of the acceleration of the ball.

Problem 4) Area computation (10 points)

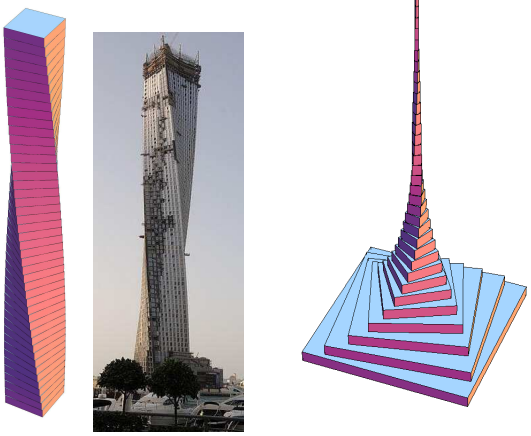
Find the area of the region enclosed the graphs of $y = x^4 - 12$ and $y = 8 - x^2$.

Problem 5) Volume computation (10 points)

The **infinity tower** in Dubai of height 330 meters has floors which can rotate. After much delay, it is expected to be completed this year. Inspired by the name "infinity", we build a new but twisted science center for which the side length of the square floor is

$$l(z) = \frac{1}{1+z} .$$

Find the volume of this new **Harvard needle building** which extends from 0 to ∞. We are the best!



Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. You should get a definite real number in each case.

a) (2 points) $\int_0^\infty e^{-x}\, dx$

b) (3 points) $\int_0^1 x^{1/5} + x^3\, dx$.

c) (3 points) $\int_{-1}^1 \frac{1}{1+x^2}\, dx$

d) (2 points) $\int_0^{e-1} \frac{2}{1+x}\, dx$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (2 points) $\int \frac{3}{\sqrt{1+3x}} + \cos(x)\, dx$

b) (3 points) $\int e^{x/5} - 7x^6 + \frac{4}{x^2+1}\, dx$

c) (2 points) $\int \frac{4}{e^{4x+5}} + 3\sin(x)\, dx$

d) (3 points) $\int \frac{1}{\sin^2(x)} + \frac{4}{x}\, dx$

Problem 8) Implicit differentiation and related rates (10 points)

a) (5 points) The implicit equation

$$x^3 + y^4 = y + 1$$

defines a function $y = y(x)$ near $(x, y) = (-1, -1)$. Find the slope $y'(x)$ at $x = -1$.

b) (5 points) An ice cube of side length x melts and changes volume V with a rate $V' = -16$. What is the rate of change of the length x at $x = 4$?

Problem 9) Catastrophes (10 points)

Verify first for each of the following functions that $x = 0$ is a critical point. Then give a criterium for stability of $x = 0$. The answer will depend on c.

a) (3 points) $f(x) = x^5 + 2x^2 - cx^2$.

b) (3 points) $f(x) = x^4 + cx^2 - x^2$.

Determine now in both examples for which parameter c the catastrophe occurs

c) (2 points) in the case $f(x) = x^5 + 2x^2 - cx^2$.

d) (2 points) in the case $f(x) = x^4 + cx^2 - x^2$.

4/8/2014: Second midterm practice C

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
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- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The formula $\int_0^x f''(x)\,dx = f'(x) - f'(0)$ holds.

2) T F The area of the upper half disc is the integral $\int_{-1}^1 \sqrt{1-x^2}\,dx$

3) T F If the graph of the function $f(x) = x^2$ is rotated around the interval $[0,1]$ in the x axes we obtain a solid with volume $\int_0^1 \pi x^4\,dx$.

4) T F The function $f(x) = e^x$ is the only anti derivative of e^x.

5) T F If f has a critical point at 1, then $F(x) = \int_0^x f(t)\,dt$ has an inflection point at 1.

6) T F Catastrophes are parameter values c for a family of functions $f_c(x)$, for which a local minimum of f_c disappears.

7) T F The volume of a cylinder of height and radius 1 minus the volume of a cone of height and radius 1 is half the volume of a sphere of radius 1.

8) T F Rolle's theorem tells that if $0 < c < 1$ is a critical point of f and $f(0) = f(1)$, then the critical point is in the interval $[0,1]$.

9) T F Rolle also introduced the notation $|x|^{1/3}$ for roots.

10) T F Integrals are linear: $\int_0^x f(t) + g(t)\,dt = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt$.

11) T F The function $\mathrm{Li}(x) = \int_2^x dt/\log(t)$ has an anti-derivative which is a finite construct of trig functions.

12) T F There is a region enclosed by the graphs of x^5 and x^6 which is finite and positive.

13) T F The integral $\int_{-1}^1 1/x^4\,dx = -1/(5x^5)|_{-1}^1 = -1/5 - 1/5 = -2/5$ is defined and negative.

14) T F Gabriel's trumpet has finite volume but infinite surface area.

15) T F A function $f(x)$ is a probability density function, if $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

16) T F The mean of a function on an interval $[a,b]$ is $\int_a^b f(x)\,dx$.

17) T F The cumulative probability density function is an antideriative of the probability density function.

18) T F The integral $\int_{-\infty}^{\infty}(x^2-1)\,dx$ is finite.

19) T F The total prize is the derivative of the marginal prize.

20) T F The acceleration is the anti-derivative of the velocity.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their anti derivatives. Of course only 6 of the 30 functions will appear.

Function	Antiderivative Enter 1-30
$\cos(3x)$	
$\sin(3x)$	
$3x$	

Function	Antiderivative Enter 1-30
$1/(3x)$	
$\tan(3x)$	
$1/(1+9x^2)$	

1) $\sin(3x)$
2) $-\sin(3x)/3$
3) $\sin(3x)/3$
4) $-3\sin(3x)$
5) $3\sin(3x)$
6) $\cos(3x)$
7) $-\cos(3x)/3$
8) $\cos(3x)/3$
9) $-3\cos(3x)$
10) $3\cos(3x)$
11) $\log(x)/3$
12) $1/(3-x)$
13) $1/(3x)$
14) $\log(x/3)$
15) $-1/(3x^2)$
16) $3x^2$
17) $x^2/2$
18) $3x^2/2$
19) 3
20) x^2
21) $\arctan(3x)/3$
22) $3\arctan(3x)$
23) $1/(1+9x^2)$
24) $3/(1+9x^2)$
25) $-3/(1+x^2)$
26) $1/\cos^2(3x)$
27) $\log(\cos(3x))$
28) $-\log(\cos(3x))/3$
29) $\log(\cos(3x))/3$
30) $3/\cos^3(3x)$

Problem 3) Matching problem (10 points) No justifications are needed.

Which of the following formulations is a Riemann sum approximating the integral $\int_0^3 f(x)\,dx$ of $f(x) = x^2$ over the interval $0, 3$.

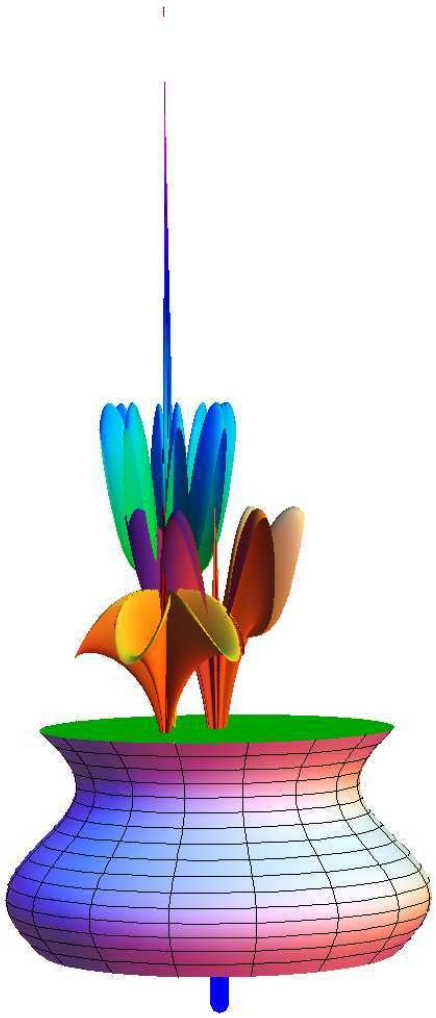
Sum	Check if this is the Riemann sum.
$n\sum_{k=0}^{n-1}(3k/n)^2$	
$\frac{1}{n}\sum_{k=0}^{n-1}(3k/n)^2$	
$n\sum_{k=0}^{3n-1}(k/n)^2$	
$\frac{1}{n}\sum_{k=0}^{3n-1}(k/n)^2$	

Problem 4) Area computation (10 points)

Find the area of the region enclosed by the three curves $y = 6 - x^2, y = -x$ and $y = x$ which is above the x axes.

Problem 5) Volume computation (10 points)

Emma R. grows magical plants in a pot which is a rotationally symmetric solid for which the radius at position x is $5 + \sin(x)$ and $0 \leq x \leq 2\pi$. Find the volume of the pot.



Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (5 points) $\int_1^2 x^{1/5} + x^4 + 1/x \; dx$.

b) (5 points) $\int_1^3 2x + \sin(x-1) + \cos(x+2) \; dx$

Problem 7) Anti-derivatives (10 points)

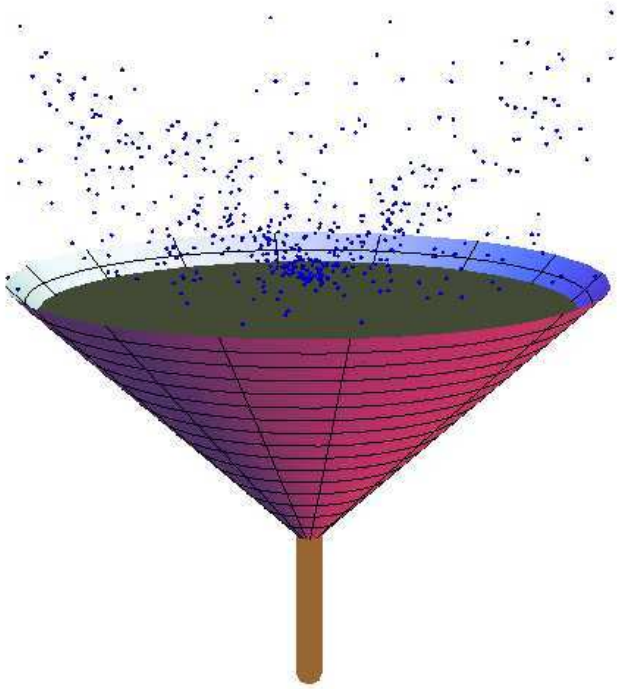
Find the following anti-derivatives

a) (5 points) $\int \frac{3}{\sqrt{1-x^2}} + x^4 + \frac{1}{1+x^2} \; dx$

b) (5 points) $\int \frac{1}{x-2} + \frac{1}{x+4} + \frac{2}{x-1}\, dx$

Problem 8) Related rates (10 points)

A coffee machine has a filter which is a cone of radius z at height z. Coffee drips out at a rate of 1 cubic centimeter per second. How fast does the water level sink at height $z = 10$?



Problem 9) Implicit differentiation (10 points)

Find the derivatives $y' = dy/dx$ of the following implicitly defined functions:

a) (5 points) $x^5 + 3x + y = e^y$.

b) (5 points) $\sin(x^2 - 2y) = y - x$.

Problem 10) Improper integrals (10 points)

Evaluate the following improper integrals or state that they do not exist

a) (3 points) $\int_1^\infty 1/\sqrt{x}\ dx$.

b) (2 points) $\int_0^1 \sqrt{x}\ dx$.

c) (3 points) $\int_0^\infty 2xe^{-x^2}\ dx$.

d) (2 points) $\int_0^\infty \frac{1}{x}\ dx$

4/8/2014: Second midterm practice D

Your Name:

- Start by writing your name in the above box.
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1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The formula $\int_0^x f''(x)\,dx = f'(x) - f'(0)$ holds.

2) T F The area of the lower half disc is the integral $\int_{-1}^1 -\sqrt{1-x^2}\,dx$

3) T F If the graph of the function $f(x) = x^2$ is rotated around the interval $[0,1]$ we obtain a solid with volume $\int_0^1 \pi x^4\,dx$.

4) T F The identity $d/dx \int_0^x f''(t)\,dt = f'(x)$ holds.

5) T F There is a point in $[0,1]$, where $f'(x) = 0$ if $f(x) = x^3 - x^2 + 1$.

6) T F The fundamental theorem of calculus assures that $\int_a^b f'(x)\,dx = f(b) - f(a)$.

7) T F If f is differentiable on $[a,b]$, then $\int_a^b f(x)\,dx$ exists.

8) T F The integral $\int_0^{\pi/2} \sin(\sin(x))\,dx$ is positive.

9) T F The anti-derivative of an anti-derivative of f is equal to the derivative of f.

10) T F If a function is positive everywhere, then $\int_a^b f(x)\,dx$ is positive too.

11) T F If a differentiable function is odd, then $\int_{-1}^1 f(x)\,dx = 0$.

12) T F If $f_c(x)$ is a function with a local minimum at 0 for all $c < 0$ and no local minimum in $[-1,1]$ for $c > 0$, then $c = 0$ is called a catastrophe.

13) T F The term "improper integral" is a synonym for "indefinite integral".

14) T F The function $F(x) = x\sin(x)$ is an antiderivative of $\sin(x)$.

15) T F The mean value theorem holds for every continuous function.

16) T F Newton and Leibniz were best buddies all their life. Leibniz even gave once the following famous speech: "You guys might not know this, but I consider myself a bit of a loner. I tend to think of myself as a one-man wolf pack. But when my sister brought Isaac home, I knew he was one of my own. And my wolf pack ... it grew by one.

17) T F Any function $f(x)$ satisfying $f(x) > 0$ is a probability density function.

18) T F The moment of inertia integral I can be used to compute energy with the relation $E = \omega^2 I/2$ where ω is the angular velocity.

19) T F If $0 \le f(x) \le g(x)$ then $0 \le \int_0^1 f(x)\,dx \le \int_0^1 g(x)\,dx$.

20) T F The improper integral $\int_0^\infty 1/(x^4+1)\,dx$ is finite.

Problem 2) Matching problem (10 points) No justifications are needed.

From the following functions there are two for which no elementary integral is found. Find them. You can find them by spotting the complement set of functions which you can integrate.

Function	Antiderivative is not elementary	Function	Antiderivative is not elementary
e^{-x^2}		$1/\log(x)$	
$\sin(3x)$		$\tan(3x)$	
$1/x$		$\arctan(3x)$	

Problem 3) Matching problem (10 points) No justifications are needed.

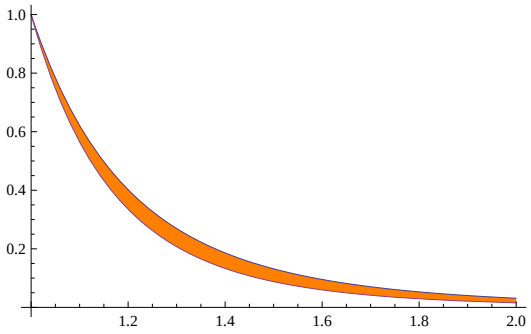
Which of the following problems are related rates problems? Several answers can apply.

Problem	Related rates?
Find the volume of a sphere in relation to the radius.	
Relate the area under a curve with value of the curve.	
If $x^3 + y^3 = 5$ and $x' = 3$ at $x = 1$, find y'.	
Find the rate of change of the function $f(x) = \sin(x)$ at $x = 1$	
Find r' for a sphere of volume V satisfying $d/dtV(r(t)) = 15$.	
Find the inflection points of $f(x) = x^3 + 3x + 4$.	
Find the global maxima of $f(x) = x^4 + x^3 - x$.	

Problem 4) Area computation (10 points)

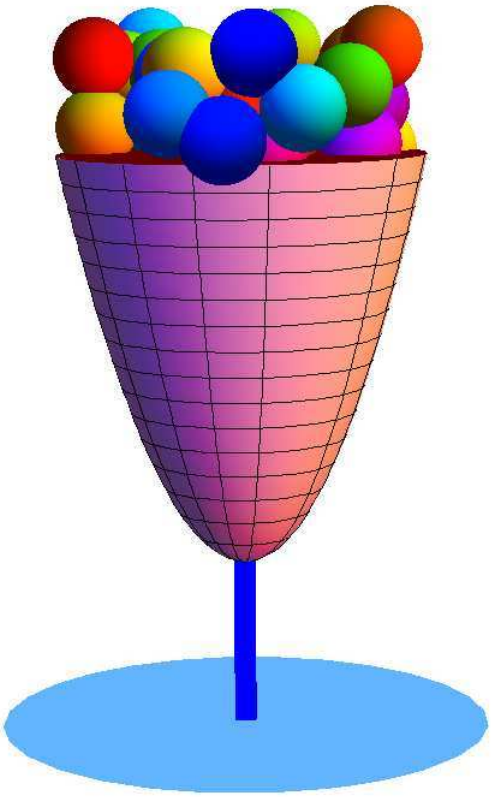
a) (5 points) Find the area of the region enclosed by the curves $3 - x^4$ and $3x^2 - 1$.

b) (5 points) Find the area of the region between $1/x^6$ and $1/x^5$ from $x = 1$ to $x = \infty$.



Problem 5) Volume computation (10 points)

Cody eats some magic "Bertie Botts Every Flavor Beans" from a cup which is a rotationally symmetric solid, for which the radius at position x is $\sqrt{x}$ and $0 \leq x \leq 4$. Find the volume of Cody's candy cup.



Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (5 points) $\int_1^2 x + \tan(x) + \sin(x) + \cos(x) + \log(x)\ dx$.

b) (5 points) $\int_1^3 (x+1)^3\ dx$

Problem 7) Anti derivatives (10 points)

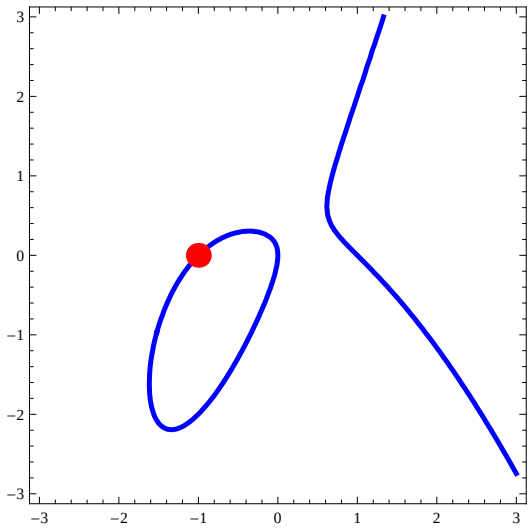
Find the following anti-derivatives

a) (5 points) $\int \sqrt{x^3}\ dx$

b) (5 points) $\int 4/\sqrt{x^5}\ dx$

Problem 8) Implicit differentiation (10 points)

The curve $y^2 = x^3 + 2xy - x$ is an example of an **elliptic curve**. Find dy/dx at the point $(-1, 0)$ without solving for y first.



Problem 9) Applications (10 points)

The probability density of the exponential distribution is given by $f(x) = (1/2)e^{-x/2}$.
The probability to wait for for time x (hours) to get an idea for a good calculus exam problem is $\int_0^x f(x)\, dx$. What is the probability to get a good idea if we wait for $T = 10$ (hours)?

Problem 10) Applications (10 points)

What is the **average value** of the function

$$f(x) = 4 + 1/(1 + x^2)$$

on the interval $[-1, 1]$?

5/17/2014: Final Exam

Your Name:

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- All functions f if not specified otherwise can be assumed to be smooth so that arbitrary many derivatives can be taken.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) T F $\cos(17\pi/4) = \sqrt{2}/2$.

2) T F The tangent function is monotonically increasing on the open interval $(-\pi/2, \pi/2)$.

3) T F The arccot function is monotonically increasing from $\pi/4$ to $3\pi/4$.

4) T F If f is a probability density function, then $\int_{-\infty}^{\infty} f(x)\, dx = 0$

5) T F $\frac{d}{dx} e^{\log(x)} = 1$.

6) T F If $f''(0) = -1$ then f has a local maximum at $x = 0$.

7) T F The improper integral $\int_{-1}^{1} 1/|x|\, dx$ is finite.

8) T F The function $-\cos(x) - x$ has a root in the interval $(-100, 100)$.

9) T F If a function f has a local maximum in $(0, 1)$ then it also has a local minimum in $(0, 1)$.

10) T F The anti derivative of $1/(1 - x^2)$ is equal to $\arctan(x)$.

11) T F The function $f(x) = (e^x - e^{2x})/(x - x^2)$ has the limit 1 as x goes to zero.

12) T F If you listen to the sound $e^{-x} \sin(10000x)$, then it gets louder and louder as time goes on.

13) T F The function $f(x) = e^{x^2}$ has a local minimum at $x = 0$

14) T F The function $f(x) = (x^{55} - 1)/(x - 1)$ has the limit 1 for $x \to 1$.

15) T F If the total cost $F(x)$ of an entity is extremal at x, then we have a break even point $f(x) = g(x)$.

16) T F The value $\int_{-\infty}^{\infty} xf(x)\, dx$ is called the expectation of the PDF f.

17) T F The trapezoid rule is an integration method in which the left and right Riemann sum are averaged.

18) T F $\tan(\pi/3) = \sqrt{3}$.

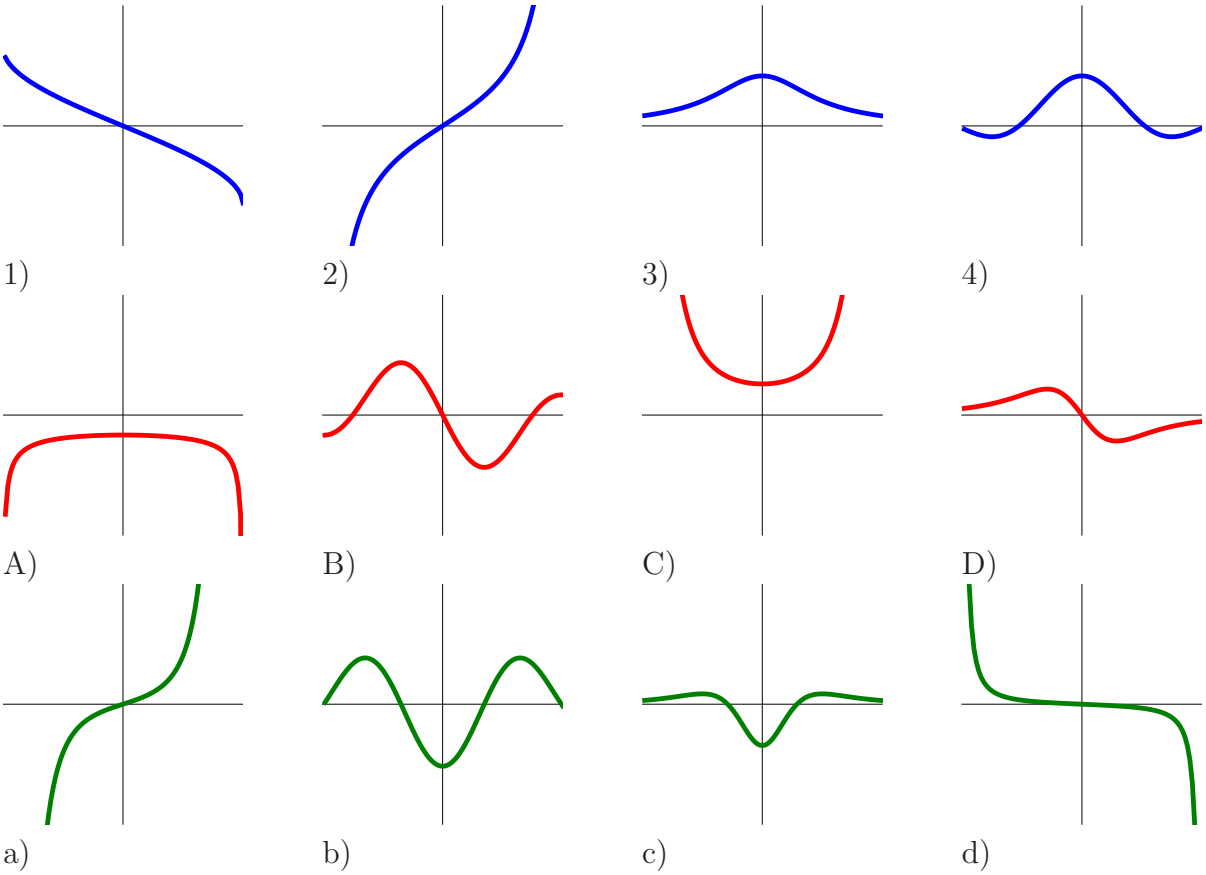
19) T F A Newton step for the function f is $T(x) = x + \frac{f(x)}{f'(x)}$.

20) T F $\sin(\arctan(1)) = \sqrt{3}$.

Problem 2) Matching problem (10 points) No justifications needed

(5 points) Match the functions names with their graphs (1-4) their derivatives (A-D) (middle row) and second derivatives (a-d) (last row).

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$\sin(x)/x$			
$\tan(x)$			
$\arcsin(x)$			
$1/(1+x^2)$			

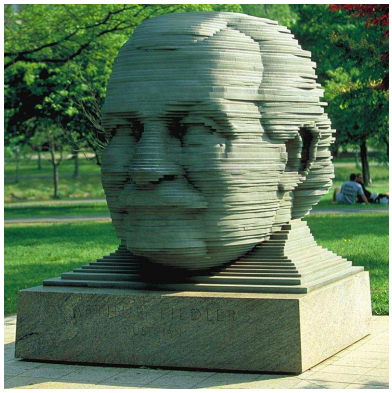


(5 points) Which of the following limits exists in the limit $x \to 0$.

Function	exists	does not exist
$\sin^4(x)/x^4$		
$1/\log\|x\|$		
$\arctan(x)/x$		
$\log\|x\|/(x-1)$		
$\cos(x)/(x-1)$		
$(x^{10}-1)/(x-1)$		

Problem 3) Matching or short answer problem (10 points). No justifications are needed.

a) (4 points) On the Boston Esplanade is a sculpture of **Arthur Fiedler** (1894-1979) a long-time conductor of the Boston Pops Orchestra. His head is sliced into thin slices. Assume that the thickness of each level is $h = 1.5$ inch and the area of each of the 100 slices k is $A(k)$. Which formula gives the volume of the head? (One applies.)



Formula	Check if true
$1.5[A(1) + \cdots + A(100)]$	
$\frac{1}{1.5}[A(1) + \cdots + A(100)]$	

Formula	Check if true
$1.5[\frac{1}{A(1)} + \cdots + \frac{1}{A(100)}]$	
$\frac{1.5}{100}[A(1) + \cdots + A(100)]$	

b) (4 points) The summer has arrived on May 12 for a day before it cooled down again. Harvard students enjoy the **Lampoon pool** that day in front of the **Lampoon castle**. Assume the water volume at height z is $V(z) = 1 + 5z - \cos(z)$. Assume water evaporates at a rate of $V'(z) = -1$ gallon per day. How fast does the water level drop at $z = \pi/2$ meters? Check the right answer: (one applies)

Rate	Check if true
-6	
$-1/6$	

Rate	Check if true
-4	
$-1/4$	

c) (2 points) Speaking of weather: the temperature on May 13 in Cambridge was 52 degrees Fahrenheit. The day before, on May 12, the temperature had been 85 degrees at some point and had us all dream about **beach time**. Which of the following theorems assures that there was a moment during the night of May 12 to May 13 that the temperature was exactly 70 degrees? (One applies.)

Theorem	check if true
Mean value theorem	
Rolle theorem	

Theorem	check if true
Intermediate value theorem	
Bolzano theorem	

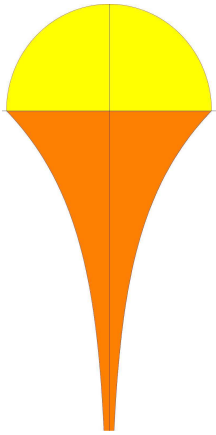
Problem 4) Area computation (10 points)

Find the area enclosed by the graphs of the functions

$$f(x) = \log|x|$$

and

$$g(x) = \sqrt{1-x^2}\ .$$



Problem 5) Volume computation (10 points)

The lamps near the front entrance of the **Harvard Malkin Athletic Center** (MAC) have octagonal cross sections, where at height z, the area is

$$A(z) = 2(1+\sqrt{2})(1+z)^2$$

with $0 \leq z \leq 3$. What is the volume of the lamp?

Problem 6) Improper integrals (10 points)

Which of the following limits $R \to \infty$ exist? If the limit exist, compute it.

a) (2 points) $\int_1^R \sin(2\pi x)\ dx$

b) (2 points) $\int_1^R \frac{1}{x^2}\ dx$

c) (2 points) $\int_1^R \frac{1}{\sqrt{x}}\ dx$

d) (2 points) $\int_1^R \frac{1}{1+x^2}\ dx$

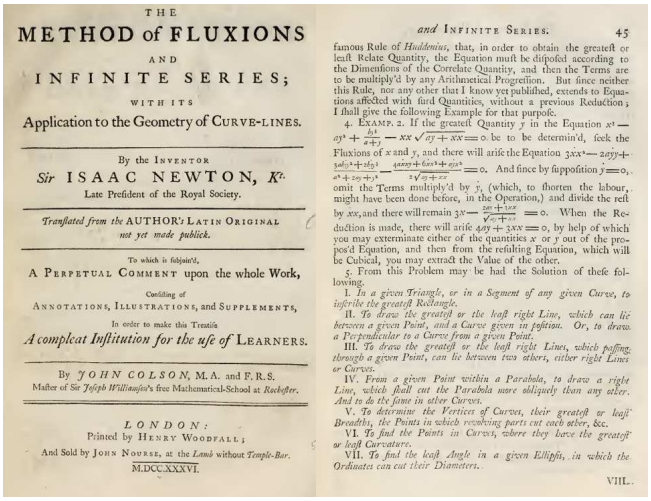
e) (2 points) $\int_1^R x\ dx$

Problem 7) Extrema (10 points)

In Newton's masterpiece **"The Method of Fluxions"** on the bottom of page 45, Newton asks: "In a given triangle or in a segment of any given curve, to inscribe the greatest rectangle." Lets be more specific and find rectangle with largest area

$$A = xy$$

in the triangle given by the x-axes, y-axes and line $y = 2 - 2x$. Use the second derivative test to make sure you have found the maximum.



THE
METHOD of FLUXIONS
AND
INFINITE SERIES;
WITH ITS
Application to the Geometry of CURVE-LINES.

By the INVENTOR
Sir ISAAC NEWTON, Kt.
Late Prefident of the Royal Society.

Tranſlated from the AUTHOR'S LATIN ORIGINAL
not yet made publick.

To which is ſubjoin'd,
A PERPETUAL COMMENT upon the whole Work,
Conſiſting of
ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,
In order to make this Treatiſe
A compleat Inſtitution for the uſe of LEARNERS.

By *JOHN COLSON*, M.A. and F.R.S.
Maſter of Sir *Joſeph Williamſon*'s free Mathematical-School at *Rocheſter*.

LONDON:
Printed by HENRY WOODFALL;
And Sold by JOHN NOURSE, at the *Lamb* without *Temple-Bar*.
M.DCC.XXXVI.

and INFINITE SERIES. 45

famous Rule of *Huddenius*, that, in order to obtain the greateſt or leaſt Relate Quantity, the Equation muſt be diſpoſed according to the Dimenſions of the Correlate Quantity, and then the Terms are to be multiply'd by any Arithmetical Progreſſion. But ſince neither this Rule, nor any other that I know yet publiſhed, extends to Equations affected with ſurd Quantities, without a previous Reduction; I ſhall give the following Example for that purpoſe.

4. EXAMP. 2. If the greateſt Quantity y in the Equation $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay + xx} = 0$ be to be determin'd, ſeek the Fluxions of x and y, and there will ariſe the Equation $3\dot{x}x^2 - 2a\dot{y}y + \frac{3a\dot{x}y^2 + 2b\dot{y}y^3}{a^2 + 2ay + y^2} - \frac{4\dot{x}xxy + 6\dot{x}xy^2 + 2\dot{y}x^3}{2\sqrt{ay + xx}} = 0$. And ſince by ſuppoſition $\dot{y} = 0$, omit the Terms multiply'd by $\dot{y}$, (which, to ſhorten the labour, might have been done before, in the Operation,) and divide the reſt by $\dot{x}x$, and there will remain $3x - \frac{2ay + 3xx}{\sqrt{ay + xx}} = 0$. When the Reduction is made, there will ariſe $4ay + 3xx = 0$, by help of which you may exterminate either of the quantities x or y out of the propoſ'd Equation, and then from the reſulting Equation, which will be Cubical, you may extract the Value of the other.

5. From this Problem may be had the Solution of theſe following.

I. *In a given Triangle, or in a Segment of any given Curve, to inſcribe the greateſt Rectangle.*

II. *To draw the greateſt or the leaſt right Line, which can lie between a given Point, and a Curve given in poſition. Or, to draw a Perpendicular to a Curve from a given Point.*

III. *To draw the greateſt or the leaſt right Lines, which paſſing through a given Point, can lie between two others, either right Lines or Curves.*

IV. *From a given Point within a Parabola, to draw a right Line, which ſhall cut the Parabola more obliquely than any other. And to do the ſame in other Curves.*

V. *To determine the Vertices of Curves, their greateſt or leaſt Breadths, the Points in which revolving parts cut each other,* &c.

VI. *To find the Points in Curves, where they have the greateſt or leaſt Curvature.*

VII. *To find the leaſt Angle in a given Ellipſis, in which the Ordinates can cut their Diameters.*

VIII.

Problem 8) Integration by parts (10 points)

a) (5 points) Find

$$\int (1 + x + x^2 + x^3 + x^4)(\sin(x) + e^x)\, dx\ .$$

b) (5 points) Find

$$\int \log(x)\frac{1}{x^2}\, dx\ .$$

Problem 9) Substitution (10 points)

a) (5 points) **"One,Two,Three,Four Five, once I caught a fish alive!"**

$$\int \frac{(1 + 2x + 3x^2 + 4x^3 + 5x^4)}{(1 + x + x^2 + x^3 + x^4 + x^5)}\, dx\ .$$

b) (5 points) A **"Trig Trick-or-Treat"** problem:

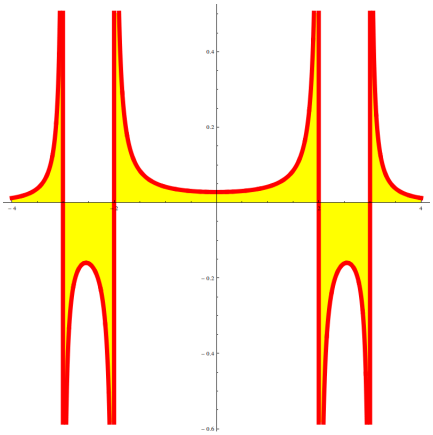
$$\int (1 - x^2)^{-3/2} + (1 - x^2)^{-1/2} + (1 - x^2)^{1/2}\, dx\ .$$

Problem 10) Partial fractions (10 points)

Integrate

$$\int_{-1}^{1} \frac{1}{(x+3)(x+2)(x-2)(x-3)}\, dx\ .$$

The graph of the function is shown to the right.



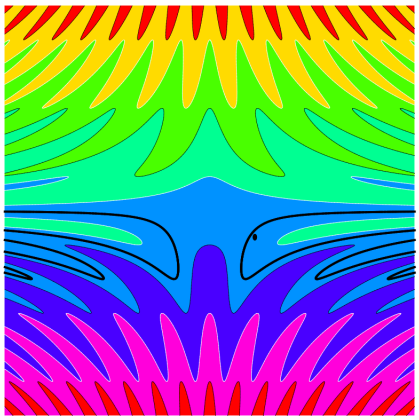
Lets call it the **friendship graph**.

Problem 11) Related rates or implicit differentiation. (10 points)

Assume $x(t)$ and $y(t)$ are related by

$$(\cos(xy) - y) = 1\ .$$

We know that $x' = 2$ at $(x, y) = (\pi/2, -1)$. Find y' at this point.



P.S. The figure shows other level curves of a **monster function**. The traced out curve is the curve under consideration.

Problem 12) Various integration problems (10 points)

a) (2 points) $\int_0^{2\pi} 2\cos^2(x) - \sin(x)\ dx$

b) (2 points) $\int x^2 e^{3x}\ dx$

c) (2 points) $\int_1^{\infty} \frac{1}{(x+2)^2}\ dx$

d) (2 points) $\int \sqrt{x}\log(x)\ dx$

e) (2 points) $\int_1^e \log(x)^2\ dx$

Problem 13) Applications (10 points)

a) (2 points) [**Agnesi density**]
The CDF of the PDF $f(x) = \pi^{-1}/(1+x^2)$ is

b) (2 points) [**Piano man**]

The upper hull of $f(x) = x^2 \sin(1000x)$ is the function [ ]

c) (2 points) [**Rower's wisdom**]

If f is power, F is work and $g = F/x$ then $f = g$ if and only if $g'(x) =$ [ ]

d) (2 points) [**Catastrophes**]

For $f(x) = c(x-1)^2$ there is a catastrophe at $c =$ [ ]

e) (2 points) [**Randomness**]

We can use chance to compute integrals. It is called the [ ] method.

5/17/2014: Final Practice A

Your Name:

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1		20
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4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) [T] [F] $\frac{d}{dx}e^{e^x} = e^x$.

2) [T] [F] A function f which is concave down at 0 satisfies $f''(0) \leq 0$.

3) [T] [F] The integral $\int_{1/2}^{1} \log(x)\ dx$ is positive. Here $\log(x) = \ln(x)$ is the natural log.

4) [T] [F] The function $x + \sin(\cos(\sin(x)))$ has a root in the interval $(-10, 10)$.

5) [T] [F] The function $x(1-x) + \sin(\sin(x(1-x)))$ has a maximum or minimum inside the interval $(0, 1)$.

6) [T] [F] The derivative of $1/(1+x^2)$ is equal to $\arctan(x)$.

7) [T] [F] The limit of $\sin^{100}(x)/x^{100}$ for $x \to 0$ exists and is equal to 100.

8) [T] [F] The function $f(x) = (1-e^x)/\sin(x)$ has the limit 1 as x goes to zero.

9) [T] [F] The frequency of the sound $\sin(10000x)$ is higher than the frequency of $\sin(\sin(3000x))$.

10) [T] [F] The function $f(x) = \sin(x^2)$ has a local minimum at $x = 0$

11) [T] [F] The function $f(x) = (x^5 - 1)/(x - 1)$ has a limit for $x \to 5$.

12) [T] [F] The average cost $g(x) = F(x)/x$ of an entity is extremal at x for which $f(x) = g(x)$. Here, $f(x)$ denotes the marginal cost and $F(x)$ the total cost.

13) [T] [F] The mean of a probability density function is defined as $\int f(x)\ dx$.

14) [T] [F] The differentiation rule $(f(x)^{g(x)})' = (f'(x))^{g(x)}g'(x)$ holds for all differentiable functions f, g.

15) [T] [F] $\sin(5\pi/6) = 1/2$.

16) [T] [F] Hôpital's rule assures that $\sin(10x)/\tan(10x)$ has a limit as $x \to 0$.

17) [T] [F] A Newton step for the function f is $T(x) = x - \frac{f'(x)}{f(x)}$.

18) [T] [F] A minimum x of a function f is called a catastrophe if $f'''(x) < 0$.

19) [T] [F] The fundamental theorem of calculus implies $\int_{-1}^{1} g'(x)\ dx = g(1) - g(-1)$ for all differentiable functions g.

20) [T] [F] If f is a differentiable function for which $f'(x) = 0$ everywhere, then f is constant.

Problem 2) Matching problem (10 points) No justifications needed

a) (2 points) One of three statements A)-C) is not the part of the fundamental theorem of calculus. Which one?

A)	$\int_0^x f'(t)\,dt = f(x) - f(0)$
B)	$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$
C)	$\int_a^b f(x)\,dx = f(b) - f(a)$

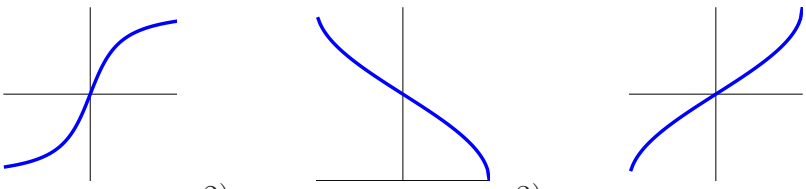
b) (3 points) Biorythms can be fascinating for small kids, giving them a first exposure to trig functions and basic arithmetic. The "theory" tells that there are three functions $p(x) = \sin(2\pi x/23)$ (Physical) $e(x) = \sin(2\pi x/28)$ (Emotional) and $i(x) = \sin(2\pi x/33)$ (Intellectual), where x is the number of days since your birth. Assume **Tuck**, the pig you know from the practice exams, is born on October 10, 2005. Today, on May 11, 2014, it is 2670 days old. Its biorythm is $E = 0.7818, P = -0.299, I = -0.5406$. It is a happy fellow, tired, but feeling a bit out of spirit, like the proctor of this exam feels right now. Which of the following statements are true?

Check if true	
	i) One day old Tuck had positive emotion, intellect and physical strength.
	ii) Among all cycles, the physical cycle takes the longest to repeat.
	iii) Comparing with all cycles, the physical increases fastest at birth.

c) (4 points) Name the statements:

$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$ is called the	.
Rule $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$ is called	.
$\int_0^x f'(t)\,dt = f(x) - f(0)$ is called	.
The PDF $f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ is called the	

d) (1 point) Which of the following graphs belongs to the function $f(x) = \arctan(x)$?

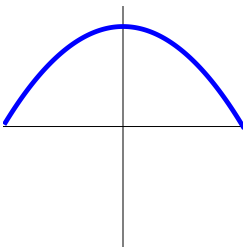


1) 2) 3)

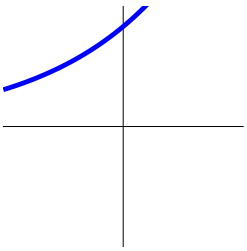
Problem 3) Matching or short answer problem (10 points). No justifications are needed.

a) (4 points) Match the functions (a-d) (top row) with their derivatives (1-4) (middle row) and second derivatives (A-D) (last row).

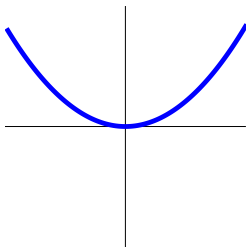
Function a)-d)	Fill in 1)-4)	Fill in A)-D)
graph a)		
graph b)		
graph c)		
graph d)		



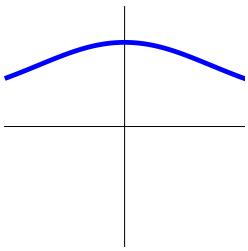
a)



b)



c)



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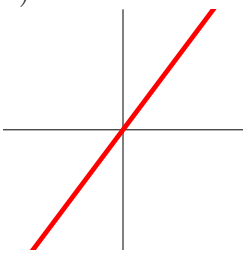
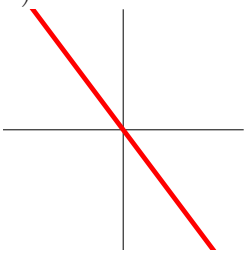


1)



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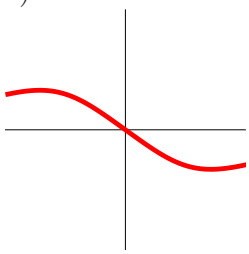

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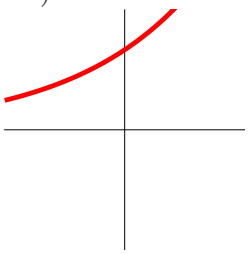

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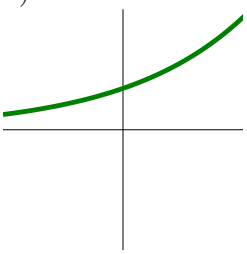
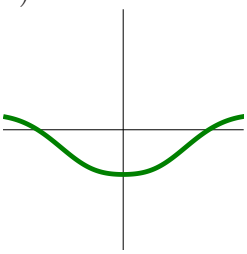


A)



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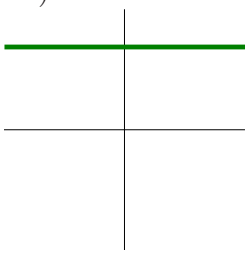

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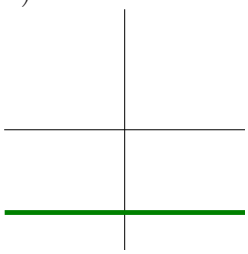
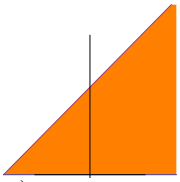


D)

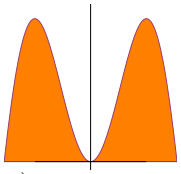
b) (4 points) Match the following integrals with the areas in the figures:

Integral	Enter 1-4
$\int_{-\pi}^{\pi} x \sin(x)\, dx.$	
$\int_{-\pi}^{\pi} \exp(-x^2)\, dx.$	

Integral	Enter 1-4
$\int_{-\pi}^{\pi} \pi + x\, dx.$	
$\int_{-\pi}^{\pi} 1 - \sin(x^3/\pi^3)\, dx.$	



1)



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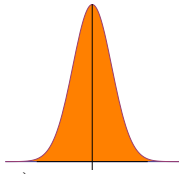

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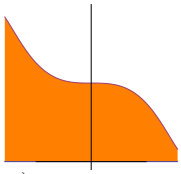


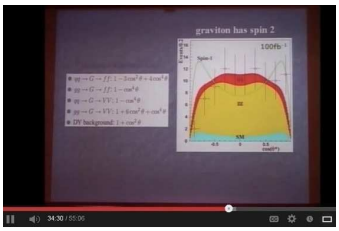
4)

c) (2 points) Name two different numerical integration methods. We have seen at least four.

Your first method	
Your second method	

Problem 4) Area computation (10 points)

A slide in a lecture of Harvard physicist **Lisa Randall** shows the area between two functions. Lisa is known for her theory of "branes" which can explain why gravity is so much weaker than electromagnetism. Assist Lisa and write down the formula for the area between the graphs of $1 - \cos^2(x)$ and $1 - \cos^4(x)$, where $0 \leq x \leq \pi$. Find the area.



Lisa Randall - Extra Dimensional Particle Resonances at the LHC (1/3)

Hint. Lisa already knows the identity

$$\cos^2(x) - \cos^4(x) = \cos^2(x)(1 - \cos^2(x)) = \cos^2(x)\sin^2(x) \ .$$

Problem 5) Volume computation (10 points)

Find the volume of the solid of revolution for which the radius at height z is

$$r(z) = \sqrt{z \log(z)}$$

and for which z is between 1 and 2. Here, log is the natural log. Naturalmente!

Problem 6) Improper integrals (10 points)

a) (5 points) Find the integral or state that it does not exist

$$\int_1^\infty \frac{7}{x^{3/4}}\, dx \ .$$

b) (5 points) Find the integral or state that it does not exist

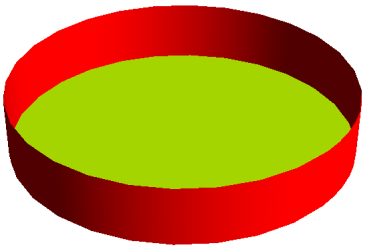
$$\int_1^\infty \frac{13}{x^{5/4}}\, dx \ .$$

Problem 7) Extrema (10 points)

A **candle holder** of height y and radius x is made of aluminum. Its total surface area is $2\pi xy + \pi x^2 = \pi$ (implying $y = 1/(2x) - x/2$). Find x for which the volume

$$f(x) = x^2 y(x)$$

is maximal.



Problem 8) Integration by parts (10 points)

a) (5 points) Find

$$\int (x+5)^3 \sin(x-4)\, dx\ .$$

b) (5 point) Find the indefinite integral

$$\int e^x \cos(2x)\, dx\ .$$

Don't get dizzy when riding this one.

Problem 9) Substitution (10 points)

a) (3 points) Solve the integral $\int \log(x^3)x^2\, dx$.

b) (4 points) Solve the integral $\int x\cos(x^2)\exp(\sin(x^2))\, dx$.

c) (3 points) Find the integral $\int \sin(\exp(x))\exp(x)\, dx$.

Problem 10) Partial fractions (10 points)

a) (5 points) Find the definite integral

$$\int_1^5 \frac{1}{(x-2)(x-3)(x-4)}\, dx\ .$$

(Evaluate the absolute values $\log|\cdot|$ in your answer. The improper integrals exist as a Cauchy principal value).

b) (5 points) Find the indefinite integral

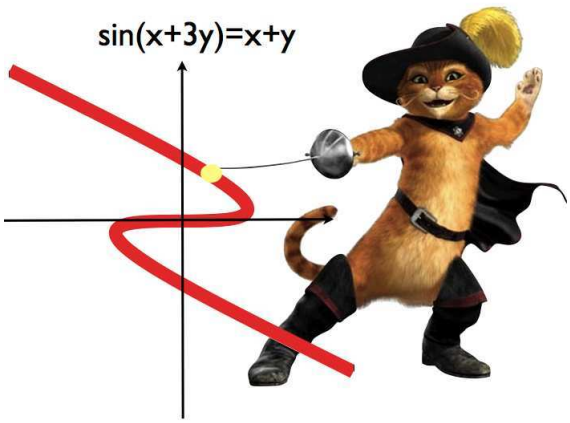
$$\int \frac{1}{x(x-1)(x+1)(x-2)}\, dx\ .$$

Problem 11) Related rates or implicit differentiation. (10 points)

a) (5 points) Assume

$$x^4(t) + 3y^4(t) = 4y(t)$$

and $x'(t) = 5$ at $(1, 1)$. What is y' at $(1, 1)$?



b) (5 points) What is the derivative $y'(x)$ at $(0, 0)$ if

$$\sin(x + 3y) = x + y\ .$$

Problem 12) Various integration problems (10 points)

Find the anti-derivatives of the following functions:

a) (2 points) $f(x) = x\log(x) + \frac{1}{1+x^2}$.

b) (3 points) $f(x) = \frac{2x}{x^2+1} + \frac{1}{x^2-4}$.

c) (2 points) $f(x) = \sqrt{16 - x^2} + \frac{1}{\sqrt{1-x^2}}$.

d) (3 points) $f(x) = \log(x) + \frac{1}{x\log(x)}$.

Problem 13) Applications (10 points)

a) (3 points) Find the CDF $\int_0^x f(t)\, dt$ for the PDF which is $f(x) = \exp(-x/3)/3$ for $x \geq 0$ and 0 for $x < 0$.

b) (2 points) Perform a single Newton step for the function $f(x) = \sin(x)$ starting at $x = \pi/3$.

c) (3 points) Check whether the function $f(x) = 1/(2x^2)$ on the real line $(-\infty, \infty)$ is a probability

density function.

d) (2 points) A rower produces the power $P(t)$ is $\sin^2(10t)$. Find the energy when rowing starting at time $t = 0$ and ending at $t = 2\pi$.

5/17/2014: Final Practice B

Your Name:

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- All functions f if not specified otherwise can be assumed to be smooth so that arbitrary many derivatives can be taken.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) [T] [F] The definite integral $\int_0^{2\pi} \sin^2(5x)\, dx$ is zero.

2) [T] [F] The intermediate value theorem assures that the function $\exp(\sin(x))$ has a root in the interval $(0, 2\pi)$.

3) [T] [F] $\frac{d}{dx}\cos(4x) = -4\sin(4x)$.

4) [T] [F] If $f''(1) < 0$ then 1 is a local maximum of f.

5) [T] [F] The derivative of $1/x$ is $\log(x)$ for all $x > 0$.

6) [T] [F] The limit of $\sin(3x)/(5x)$ for $x \to 0$ exists and is equal to $3/5$.

7) [T] [F] The function $(e^t - 1)/t$ has the limit 1 as t goes to zero.

8) [T] [F] The derivative of $f(f(x))$ is $f'(f'(x))$ for any differentiable function f.

9) [T] [F] A monotonically increasing function f has no point x, where $f'(x) < 0$.

10) [T] [F] The function $f(x) = \exp(-x^2)$ has an inflection point x somewhere on the real line.

11) [T] [F] The function $f(x) = (1 - x^3)/(1 + x)$ has a limit for $x \to -1$.

12) [T] [F] If we know the marginal cost for all quantities x as well as the total cost for $x = 1$ we know the total cost for all x.

13) [T] [F] The function f which satisfies $f(x) = 0$ for $x < 0$ and $f(x) = e^{-x}$ for $x \geq 0$ is a probability density function.

14) [T] [F] The differentiation rule $(f \cdot g)' = f'(g(x)) \cdot g'(x)$ holds for all differentiable functions f, g.

15) [T] [F] Hôpital's rule assures that $\cos(x)/\sin(x)$ has a limit as $x \to 0$.

16) [T] [F] A Newton step for the function f is $T(x) = x - \frac{f(x)}{f'(x)}$.

17) [T] [F] The family of functions $f_c(x) = cx^2$ where c is a parameter has a catastrophe at $x = 0$.

18) [T] [F] The fundamental theorem of calculus implies $\int_{-x}^{x} f'(t)\, dt = f(x) - f(-x)$ for all differentiable functions f.

19) [T] [F] If f is a smooth function for which $f''(x) = 0$ everywhere, then f is constant.

20) [T] [F] The function $f(x) = \sin(x)/(1 - \cos(x))$ can be assigned a value $f(0)$ such that $f(x)$ is continuous at 0.

Problem 2) Matching problem (10 points) Only short answers are needed.

We name some important concepts in this course. To do so, please complete the sentences with one or two words. Each question is one point.

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ is called the		of f.
$f'(x) = 0, f''(x) > 0$ implies that x is a		of f.
The sum $\frac{1}{n}[f(0) + f(1/n) + f(2/n) + \ldots + f((n-1)/n) + f(1)]$ is called a		sum.
If $f(0) = -3$ and $f(4) = 8$, then f has a root on the interval $(0, 4)$ by the		theorem.
There is a point $x \in (0, 1)$ where $f'(x) = f(1) - f(0)$ by the		theorem.
The expansion rate $r'(t)$ can be obtained from $d/dtV(r(t)) = -5$ by the method of		rates.
The anti derivative $\int_{-\infty}^{x} f(t)\, dt$ of a probability density function f is called the		function.
A point x for which $f(x) = 0$ is called a		of f.
A point x for which $f''(x) = 0$ is called an		of f.
At a point x for which $f''(x) > 0$, the function is called		up.

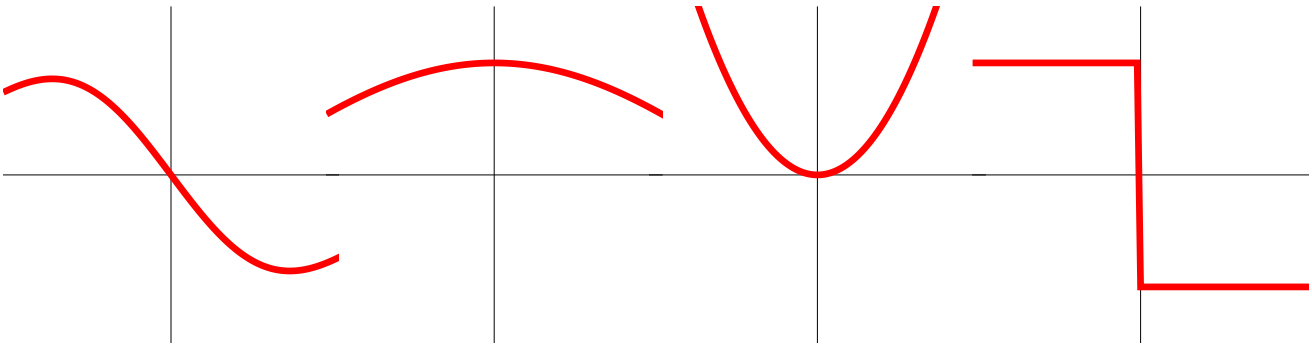
Problem 3) Matching or short answer problem (10 points). No justifications are needed.

a) (4 points) Find the relation between the following functions:

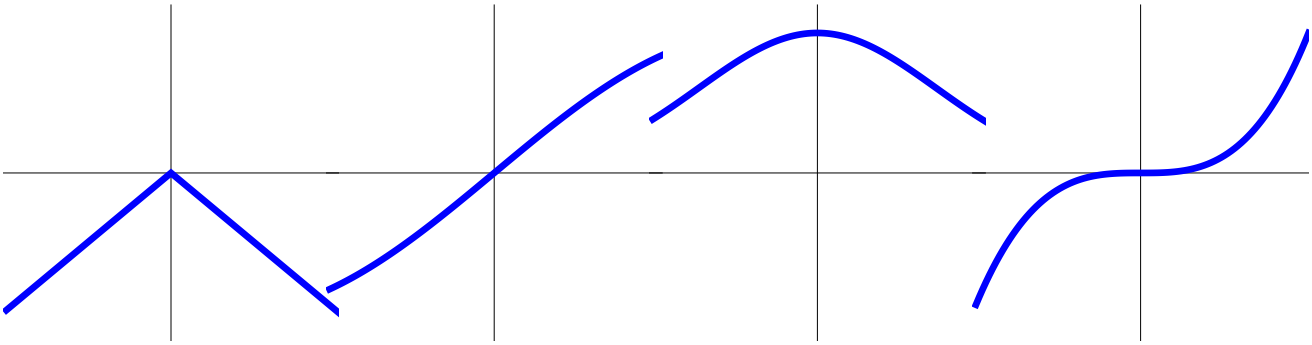
function f	function g	$f = g'$	$g = f'$	none
$\log\lvert\sin(x)\rvert$	$\cot(x)$			
$1/\cos^2(x)$	$\tan(x))$			
x^5	$5x^4$			
$1/x^2$	$-1/x$			
$\sin(\log(x))$	$\cos(\log(x))/x$			

b) (3 points) Match the following functions (a-d) with a choice of **anti-derivatives** (1-4).

Function a)-d)	Fill in 1)-4)
graph a)	
graph b)	
graph c)	
graph d)	



a) b) c) d)



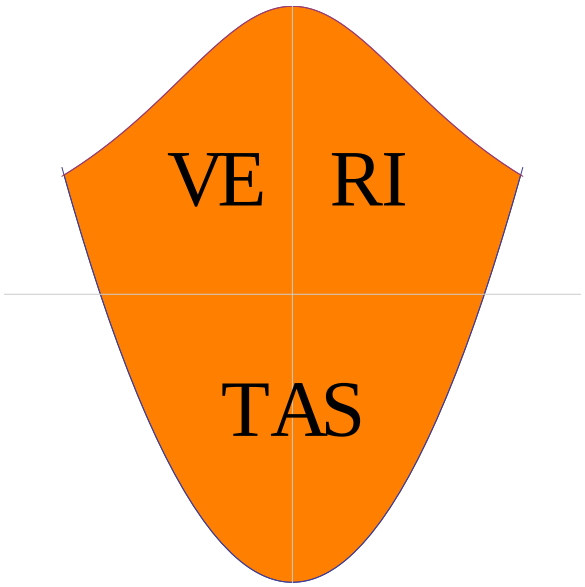
1) 2) 3) 4)

c) (3 points) Find the limits for $x \to 0$

Function f	$\lim_{x\to 0} f(x)$
$x/(e^{2x}-1)$	
$(e^{2x}-1)/(e^{3x}-1)$	
$\sin(3x)/\sin(5x)$	

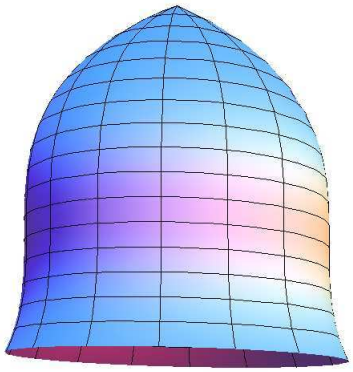
Problem 4) Area computation (10 points)

Find the area of the shield shaped region bound by the two curves $1/(1+x^2)$ and x^2-1.



Problem 5) Volume computation (10 points)

Did you know that there is a scaled copy of the **liberty bell** on the campus of the Harvard business school? Here we compute its volume. Find the volume of the rotationally symmetric solid if the radius $r(z)$ at height z is $r(z) = 8-(z-1)^3$ and the height z of the bell is between 0 and 3.



Problem 6) Improper integrals (10 points)

a) (5 points) Find the integral or state that it does not exist

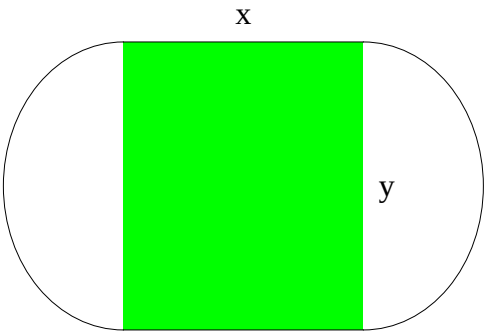
$$\int_1^\infty \frac{1}{x^4}\, dx\ .$$

b) (5 points) Find the integral or state that it does not exist

$$\int_1^\infty \frac{1}{x^{3/2}}\, dx\ .$$

Problem 7) Extrema (10 points)

The Harvard stadium has a track which encloses a rectangular field of dimensions x, y. The circumference of the track is $400 = 2\pi y + 2x$ and is fixed. We want to maximize the area xy for a play field. Which x achieves this?



Problem 8) Integration by parts (10 points)

Find the antiderivative:

$$\int (x-1)^4 \exp(x+1)\, dx\ .$$

Problem 9) Substitution (10 points)

a) (3 points) Solve the integral $\int e^{x^2} 2x\, dx$.

b) (3 points) Solve the integral $\int 2x \log(x^2)\, dx$.

c) (4 points) Find the integral $\int e^{-2e^x} e^x\, dx$.

Problem 10) Partial fractions (10 points)

a) (5 points) Find the definite integral

$$\int_1^5 \frac{1}{(x-4)(x-2)}\,dx\ .$$

b) (5 points) Find the indefinite integral

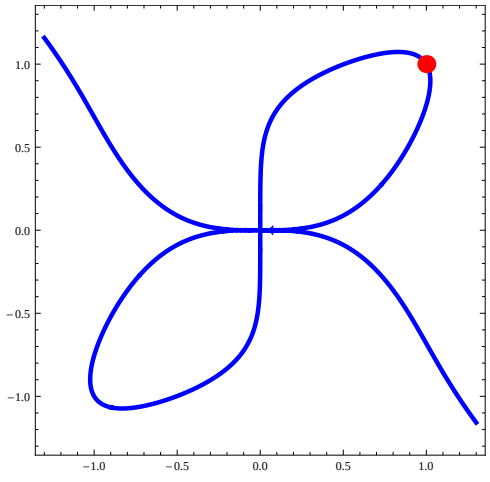
$$\int \frac{1}{(x-1)(x-3)(x-5)}\,dx\ .$$

Problem 11) Related rates (10 points)

The coordinates of a car on a freeway intersection are $x(t)$ and $y(t)$. They are related by

$$x^7 + y^7 = 2xy^2\ .$$

We know $x' = 3$ at $x = 1, y = 1$. Find the derivative y'.



Problem 12) Various integration problems (10 points)

Find the anti-derivatives of the following functions:

a) (2 points) $f(x) = \sin^5(x)\cos(x)$.

b) (3 points) $f(x) = \frac{1}{x^2+1} + \frac{1}{x^2-1}$.

c) (2 points) $f(x) = \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}}$.

d) (3 points) $f(x) = \log(x) + \frac{1}{\log(x)}$.

Problem 13) Applications (10 points)

a) (5 points) We know the total cost $F(x) = -x^3 + 2x^2 + 4x + 1$ for the quantity x. In order to find the positive **break-even point** x satisfying $f(x) = g(x)$, where $g(x) = F(x)/x$ is the total cost and $f(x) = F'(x)$ is the marginal cost, we do - how sweet it is - find the maximum of the average cost $g(x) = F(x)/x$. Find the maximum!

b) (5 points) We know the **"velocity"**, **"acceleration"** and **"jerk"** as the first second and third derivative of position. The fourth, fifth and sixth derivatives of position as a function of time are called **"snap"**, **"crackle"** and **"pop"** according to characters used in a cereal add. Assume we know the snap $x''''(t) = t$. Find $x(t)$ satisfying $x(0) = x'(0) = x''(0) = 0, x'''(0) = 0$.

5/17/2014: Final Exam Practice C

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
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4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The quantum exponential function $\exp_h(x) = (1+h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$ for $h > 0$.

2) T F The function $\mathrm{sinc}(x) = \sin(x)/x$ has a critical point at $x = 0$.

3) T F The limit of $1/\log(1/|x|)$ for $x \to 0$ exists.

4) T F The strawberry theorem tells that for any $f(x)$, its anti-derivative $F(x)$ and $g(x) = F(x)/x$ the points $f = g$ are the points where $g' = 0$.

5) T F The function $f(x) = \tan(x)$ has a vertical asymptote at $x = \pi/2$.

6) T F The function $x/(1+x)$ converges to 1 for $x \to \infty$ and has therefore a horizontal asymptote.

7) T F The function $f(x) = \tan(x)$ is odd: it satisfies $f(x) = -f(-x)$.

8) T F The function $\sin^3(x)/x^2$ is continuous on the real line.

9) T F With $Df(x) = f(x+1)-f(x)$ we have $D(fg)(x) = Dfg(x+1)+f(x)Dg(x)$.

10) T F If f has a critical point 0 then f has a minimum or maximum at 0.

11) T F The limit of $[\frac{1}{3+h} - \frac{1}{3}]/h$ for $h \to 0$ is $-1/9$.

12) T F The function $(\cos(x)+\sin(3x))/(\sin(4x)+\cos(3x))$ can be integrated using trig substitution.

13) T F The marginal cost is the anti-derivative of the total cost.

14) T F The cumulative distribution function is the anti-derivative of the probability density function.

15) T F The function $\sqrt{1-x^2}$ can be integrated by a trig substitution $x = \cos(u)$.

16) T F The integral $\int_0^1 1/x^2\, dx$ is finite.

17) T F The chain rule tells that $d/dx f(g(x)) = f'(x)g'(x)$.

18) T F For the function $f(x) = \sin(100x)$ the hull function is constant.

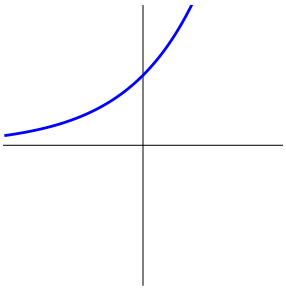
19) T F The trapezoid method is also called Simpson rule.

20) T F If $f''(x) > 0$, then the curvature of f is positive.

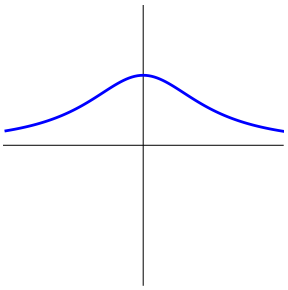
Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their derivatives.

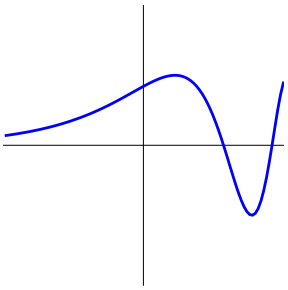
Function	Fill in the numbers 1-8
graph a)	
graph b)	
graph c)	
graph d)	
graph e)	
graph f)	
graph g)	
graph h)	



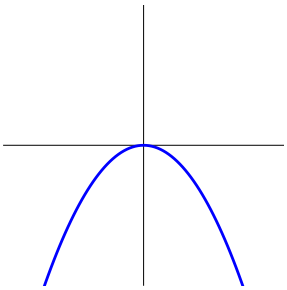
a)



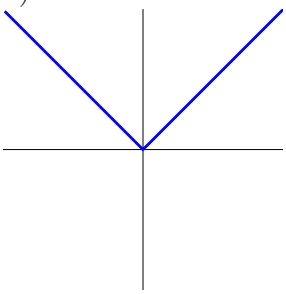
b)



c)



d)



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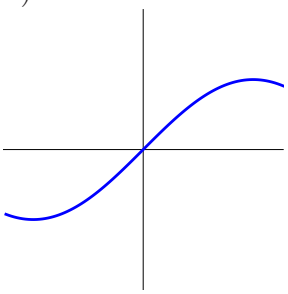

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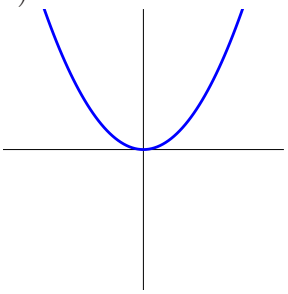
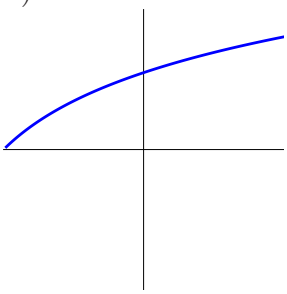


g)



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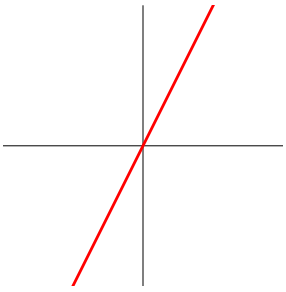
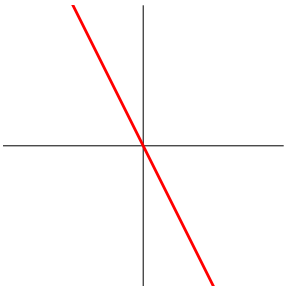


1)



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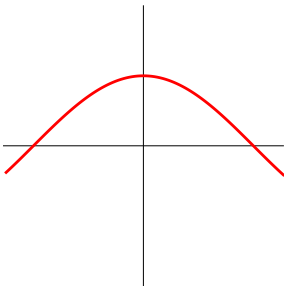

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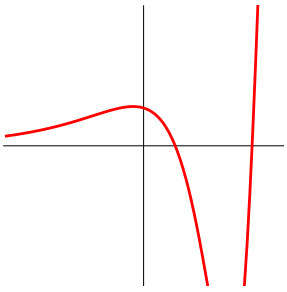

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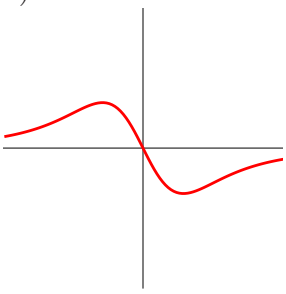

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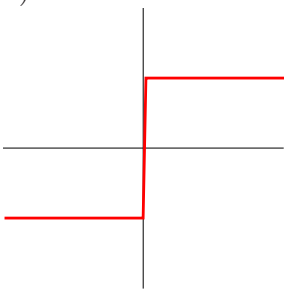

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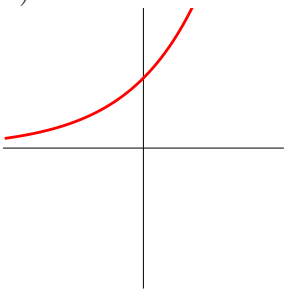

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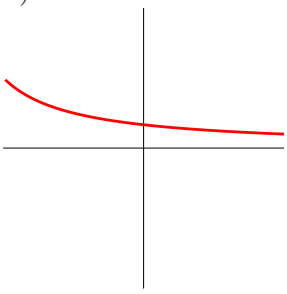
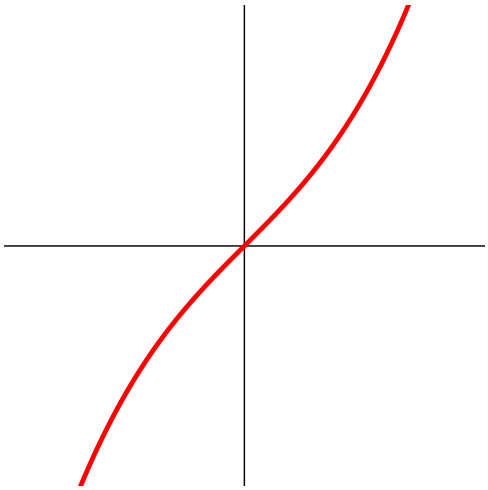


8)

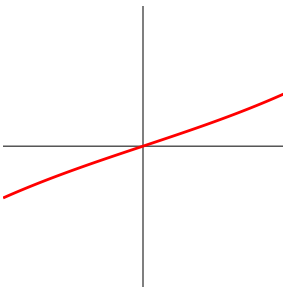
Problem 3) Matching problem (10 points) No justifications are needed.

Here is the graph of a function $f(x)$. Match the following modifications

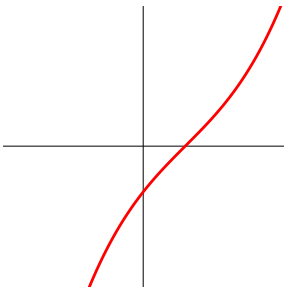


Match the following functions with their graphs.

Function	Fill in 1)-8)
$f(x-1)$	
$f'(x)$	
$f(x+1)$	
$f^{-1}(x)$	
$f(x/2)$	
$f(3x)$	
$1/f(x)$	
$f(x)+1$	



1)



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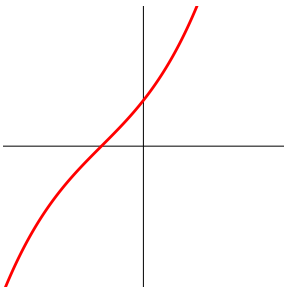

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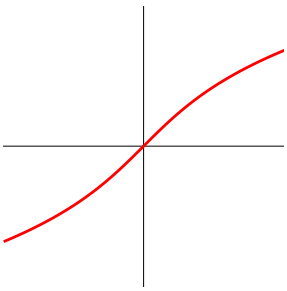

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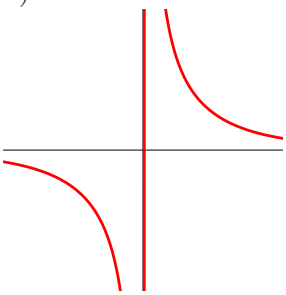

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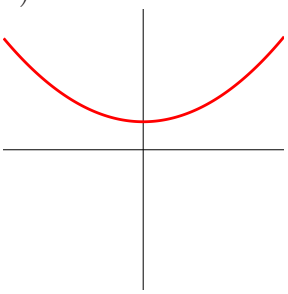

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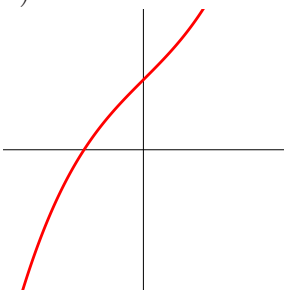

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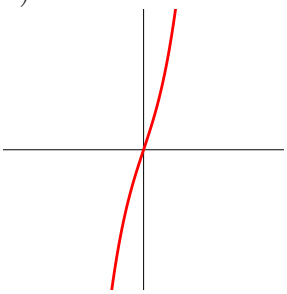
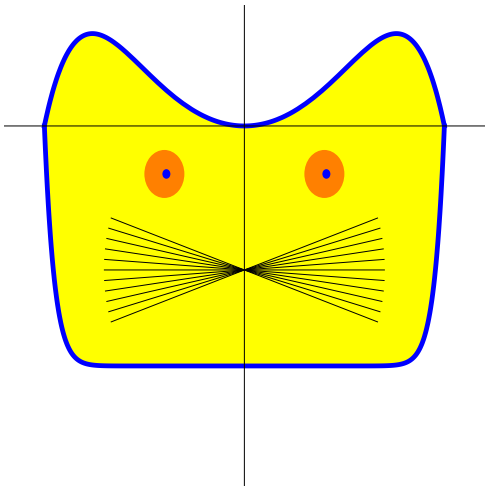


8)

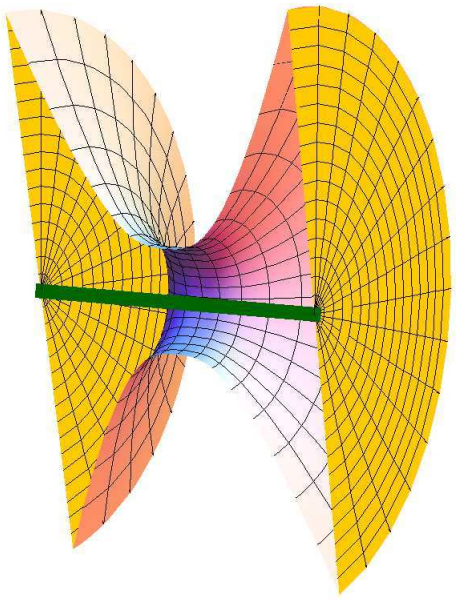
Problem 4) Area computation (10 points)

Find the area of the **cat region** which is the region enclosed by the functions $f(x) = x^{20} - 1$ and $g(x) = x^2 - x^6$. No need to count in the whiskers.



Problem 5) Volume computation (10 points)

We spin the graph of the function $f(x) = \sqrt{1 + |x|^3}$ around the x axes and get a solid of revolution. What is the volume of this solid enclosed between $x = -3$ and $x = 3$? The picture shows half of this sold.



Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals

a) $\int_{-1}^{1} \frac{1}{1+x^2}\, dx$

b) $\int_1^2 x^2 + \sqrt{x}\, dx$

c) $\int_0^{\sqrt{\pi}} \sin(x^2) 2x\, dx$.

d) $\int_0^1 \log(4+x)\, dx$.

Problem 7) Extrema (10 points)

a) (7 points) Analyse the local extrema of the function

$$f(x) = \frac{x}{1+x^2}$$

on the real axes using the second derivative test.

b) (3 points) Are there any global extrema?

Problem 8) Integration by parts (10 points)

a) (5 points) Find the anti-derivative of

$$f(x) = \sin(4x)\cos(3x)\ .$$

b) (5 points) Find the anti-derivative of

$$f(x) = (x-1)^2 \sin(1+x)\ .$$

Problem 9) Substitution (10 points)

a) (3 points) Find the integral $\int 3x\sqrt{5x^2-5}\, dx$.

b) (3 points) What is the anti-derivative of $\int \exp(x^2-x)(4x-2)$?

c) (4 points) Evaluate the definite integral

$$\int_0^{\pi/2} \sqrt{1-\cos(x)}\sin(x)\, dx\ .$$

Problem 10) Partial fractions, Trig substitution (10 points)

a) Solve the integral

$$\int \frac{2-x+x^2}{(1-x)(1+x^2)}$$

by writing

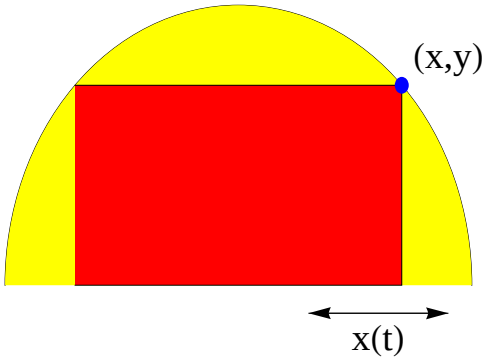
$$\frac{2-x+x^2}{(1-x)(1+x^2)} = \frac{A}{1+x^2} + \frac{B}{1-x} .$$

b) Evaluate the integral $\int \sqrt{1-x^2}x \, dx$.

Problem 11) Related rates (10 points)

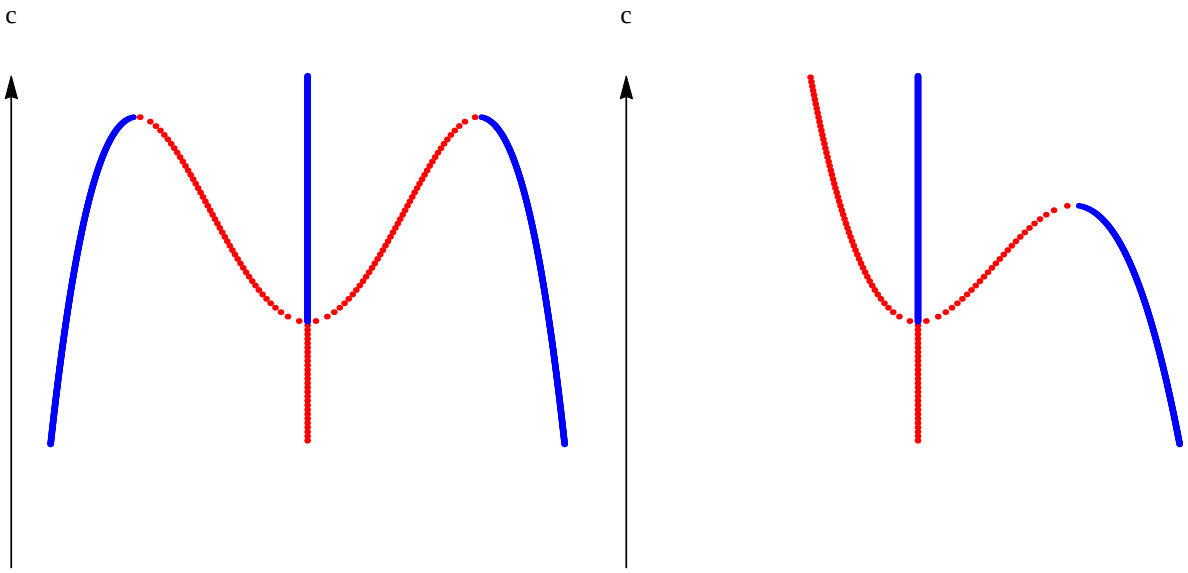
a) (7 points) A rectangle with corners at $(-x, 0), (x, 0), (x, y), (-x, y)$ is inscribed in a half circle $x^2 + y^2 = 1$ where $y \geq 0$ is in the upper half plane. Assume we move x as $x(t) = t^2$. Find the rate of change of $y(t)$.

b) (3 points) Find the rate of change of the area $A(t) = 2x(t)y(t)$ of the rectangle.



Problem 12) Catastrophes (10 points)

The following two pictures show bifurcation diagrams. The vertical axes is the deformation parameter c. On the left hand side, we see the bifurcation diagram of the function $f(x) = x^6 - x^4 + cx^2$, on the right hand side the bifurcation diagram of the function $f(x) = x^5 - x^4 + cx^2$. As done in class and homework, the bolder continuously drawn graphs show the motion of the local minima and the lighter dotted lines show the motion of the local maxima. In both cases, determine the catastrophe for the criticaal point $x = 0$.



Problem 13) Applications (10 points)

The **Laplace distribution** is a distribution on the entire real line which has the probability density $f(x) = e^{-|x|}/2$. The variance of this distribution is the integral

$$\int_{-\infty}^{\infty} x^2 f(x)\, dx \ .$$

Find it.

5/17/2014: Final Exam Practice D

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points) No justifications are needed.

1) T F If a function $f(x)$ has a critical point 0 and $f''(0) = 0$ then 0 is neither a maximum nor minimum.

2) T F If $f' = g$ then $\int_0^x g(x) = f(x)$.

3) T F The function $f(x) = 1/x$ has the derivative $\log(x)$.

4) T F The function $f(x) = \arctan(x)$ has the derivative $1/\cos^2(x)$.

5) T F The fundamental theorem of calculus implies that $\int_a^b f'(x)\ dx = f(b) - f(a)$.

6) T F $\lim_{x\to 8} 1/(x-8) = \infty$ implies $\lim_{x\to 3} 1/(x-3) = \omega$.

7) T F A continuous function which satisfies $\lim_{x\to-\infty} f(x) = 3$ and $\lim_{x\to\infty} f(x) = 5$ has a root.

8) T F The function $f(x) = (x^7 - 1)/(x - 1)$ has a limit at $x = 1$.

9) T F If $f_c(x)$ is an even function with parameter c and $f'(0) = 0$ and for $c < 3$ the function is concave up at $x = 0$ and for $c > 3$ the function is concave down at $x = 0$, then $c = 3$ is a catastrophe.

10) T F The function $f(x) = +\sqrt{x^2}$ has a continuous derivative 1 everywhere.

11) T F A rower rows on the Charles river leaving at 5 PM at the Harvard boat house and returning at 6 PM. If $f(t)$ is the distance of the rower at time t to the boat house, then there is a point where $f'(t) = 0$.

12) T F A global maximum of a function $f(x)$ on the interval $[0, 1]$ is a critical point.

13) T F A continuous function on the interval $[2, 3]$ has a global maximum and global minimum.

14) T F The intermediate value theorem assures that if f is continuous on $[a, b]$ then there is a root of f in (a, b).

15) T F On an arbitrary floor, a square table can be turned so that it does not wobble any more.

16) T F The derivative of $\log(x)$ is $1/x$.

17) T F If f is the marginal cost and $F = \int_0^x f(x)\ dx$ the total cost and $g(x) = F(x)/x$ the average cost, then points where $f = g$ are called "break even points".

18) T F At a function party, Log talks to Tan and the couple Sin and Cos, when she sees her friend Exp alone in a corner. Log: "What's wrong?" Exp: "I feel so lonely!" Log: "Go integrate yourself!" Exp sobbs: "Won't change anything." Log: "You are so right".

19) T F If a car's position at time t is $f(t) = t^3 - t$, then its acceleration at $t = 1$ is 6.

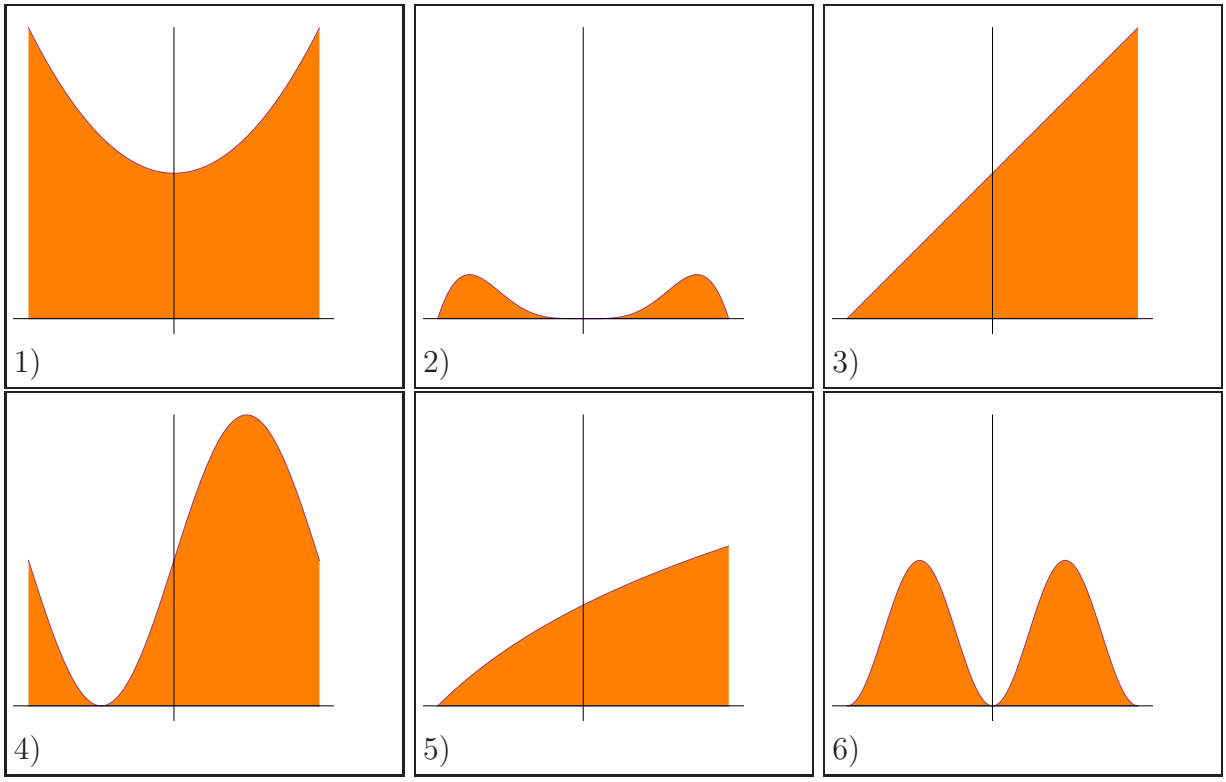
20) T F For trig substitution, the identities $u = \tan(x/2)$, $dx = \frac{2du}{(1+u^2)}$, $\sin(x) = \frac{2u}{1+u^2}$, $\cos(x) = \frac{1-u^2}{1+u^2}$ are useful.

Problem 2) Matching problem (10 points) No justifications are needed.

a) Match the following integrals with the graphs and (possibly signed) areas.

Integral	Enter 1-6
$\int_{-1}^{1} \sin(\pi x) x^3 \, dx.$	
$\int_{-1}^{1} \log(x+2) \, dx.$	
$\int_{-1}^{1} x + 1 \, dx.$	

Integral	Enter 1-6
$\int_{-1}^{1} (1 + \sin(\pi x)) \, dx.$	
$\int_{-1}^{1} \sin^2(x) \, dx.$	
$\int_{-1}^{1} x^2 + 1 \, dx.$	



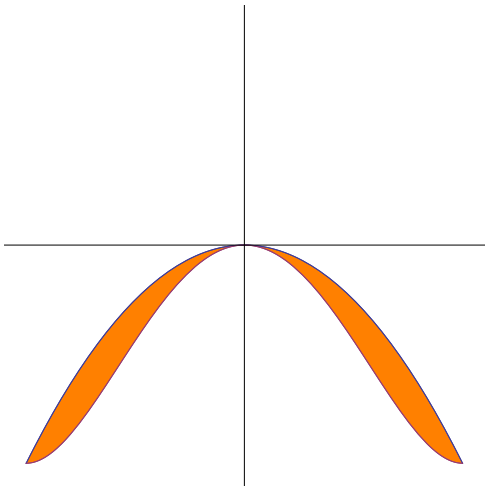
Problem 3) Matching problem (10 points) No justifications are needed.

Determine from each of the following functions, whether discontinuities appears at $x = 0$ and if, which of the three type of discontinuities it is at 0.

Function	Jump discontinuity	Infinity	Oscillation	No discontinuity
$f(x) = \log(\lvert x \rvert^5)$				
$f(x) = \cos(5/x))$				
$f(x) = \cot(1/x)$				
$f(x) = \sin(x^2)/x^3$				
$f(x) = \arctan(\tan(x - \pi/2))$				
$f(x) = 1/\tan(x)$				
$f(x) = 1/\sin(x)$				
$f(x) = 1/\sin(1/x)$				
$f(x) = \sin(\exp(x))/\cos(x)$				
$f(x) = 1/\log\lvert x \rvert$				

Problem 4) Area computation (10 points)

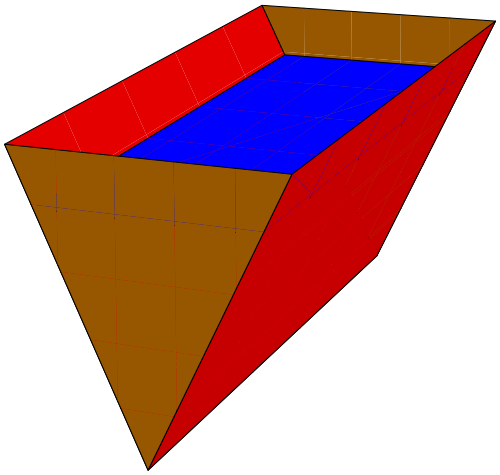
Find the area of the region enclosed by the graphs of the function $f(x) = x^4 - 2x^2$ and the function $g(x) = -x^2$.



Problem 5) Volume computation (10 points)

A farmer builds a bath tub for his warthog "Tuk". The bath has triangular shape of length 10 for which the width is $2z$ at height z. so that when filled with height z the surface area of the water is $20z$. If the bath has height 1, what is its volume?

P.S. Don't ask how comfortable it is to soak in a bath tub with that geometry. The answer most likely would be "Noink Muink".



Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (3 points) $\int_1^2 \sqrt{x} + x^2 - 1/\sqrt{x} + 1/x \; dx$.

b) (3 points) $\int_1^2 2x\sqrt{x^2-1} \; dx$

c) (4 points) $\int_1^2 2/(5x-1) \; dx$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (3 points) $\int \frac{3}{1+x^2} + x^2 \; dx$

b) (3 points) $\int \frac{\tan^2(x)}{\cos^2(x)} \; dx$

c) (4 points) $\int \log(5x) \; dx$.

Problem 8) Implicit differentiation/Related rates (10 points)

A juice container of volume $V = \pi r^2 h$ changes radius r but keeps the height $h = 2$ fixed. Liquid leaves at a constant rate $V'(t) = -1$. At which rate does the radius of the bag shrink when $r = 1/2$?

Problem 9) Global extrema (10 points)

We build a chocolate box which has 4 cubical containers of dimension $x \times x \times h$. The total material is $f(x,h) = 4x^2 + 12xh$ and the volume is $4x^2h$. Assume the volume is 4, what geometry produces the minimal cost?

Problem 10) Integration techniques (10 points)

Which integration technique works? It is enough to get the right technique and give the first step, not do the actual integration:

a) (2 points) $\int (x^2 + x + 1)\sin(x) \; dx$.

b) (2 points) $\int x/(1+x^2) \; dx$.

c) (2 points) $\int \sqrt{4-x^2}\, dx$.

d) (2 points) $\int \sin(\log(x))/x$.

e) (2 points) $\int \frac{1}{(x-6)(x-7)}\, dx$.

Problem 11) Hopital's rule (10 points)

Find the following limits as $x \to 0$ or state that the limit does not exist.

a) (2 points) $\frac{\tan(x)}{x}$

b) (2 points) $\frac{x}{\cos(x)-x}$.

c) (2 points) $x \log(1+x)/\sin(x)$.

d) (2 points) $x \log(x)$.

e) (2 points) $x/(1-\exp(x))$.

Problem 12) Applications (10 points)

The cumulative distribution function on $[0,1]$

$$F(x) = \frac{2}{\pi} \arcsin(\sqrt{x})$$

defines the **arc-sin** distribution.

a) Find the probability density function $f(x)$ on $[0,1]$.

b) Verify that $\int_0^1 f(x)\, dx = 1$.
Remark. The arc sin distribution is important chaos theory and probability theory.

3/4/2014: First Midterm Exam

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] The function $f(x) = \exp(-x^2) - 1$ has the root $x = 0$.

Solution:
The exp function is 1 at $x = 0$.

2) [T] [F] If f is continuous function and odd then 0 is a root of f.

Solution:
Odd means that the graph can be reflected at the origin and stays the same. This forces the graph to go through the origin.

3) [T] [F] $\log(\log(e)) = 0$, if log is the natural log.

Solution:
Yes, $\log(e) = 1$ and $\log(1) = 0$.

4) [T] [F] The chain rule assures that $\frac{d}{dx}\sin^2(x) = 2\cos(x)$.

Solution:
This is not the correct application.

5) [T] [F] The function $f(x) = x^2/(1-x^2)$ is continuous everywhere on the real axes.

Solution:
It has a pole at $x = 1$ and $x = -1$.

6) [T] [F] The function $\arctan(x)$ is the inverse function of the function $\tan(x)$.

Solution:
Yes, by definition

7) [T] [F] The Newton method is $T(x) = x - f'(x)/f''(x)$.

Solution:
Too many derivatives

8) [T] [F] $\cos(3\pi/2) = 0$.

Solution:
Draw the circle. The angle $3\pi/2$ corresponds to 270 degrees which is indeed a root

9) [T] [F] If a function f is continuous on $[-1, 1]$ and $f(1) = 1, f(-1) = -1$, then there is $-1 < x < 1$, where $f(x) = 0$.

Solution:
By the intermediate value theorem

10) [T] [F] The chain rule assures that $\frac{d}{dx}g(1/x) = -g'(1/x)/x^2$.

Solution:
This is the correct application of the formula!

11) [T] [F] We have $\lim_{x\to\infty}(2x + 1)/(3x - 1) = 2/3$.

Solution:
This is a consequence of l'Hopital's rule when applied twice.

12) [T] [F] If 1 is a root of f, then $f'(x)$ changes sign at 1.

Solution:
It is a point, where the second derivative changes sign

13) [T] [F] If $f''(0) < 0$ and $f''(1) > 0$ then there is a point in $(0, 1)$, where f has an inflection point.

Solution:
By the intermediate number theorem.

14) [T] [F] The intermediate value theorem assures that the equation $f(x) = x^2 - \cos(x) = 0$ has a root.

Solution:
$f(0) = -1, f(\pi) > 0$.

15) [T] [F] The function $f(x) = x/\sin(x)$ is continuous everywhere if $f(0)$ is suitably defined.

Solution:
We can define the value to be 1.

16) [T] [F] $f'(x) = 0$ and $f'''(0) < 0$ at $x = 0$ assures that f has a maximum at $x = 0$.

Solution:
It is the second derivative test, not the third one

17) [T] [F] If f is constant, then $f(x + h) - f(x)/h = 0$ for all $h > 0$.

Solution:
$Df(x) = 2x - 1$.

18) [T] [F] The quotient rule is $\frac{d}{dx}(f/g) = (f'(x)g'(x) - f(x)g(x))/(g'(x))^2$.

Solution:
No, it is not related to l'Hopital.

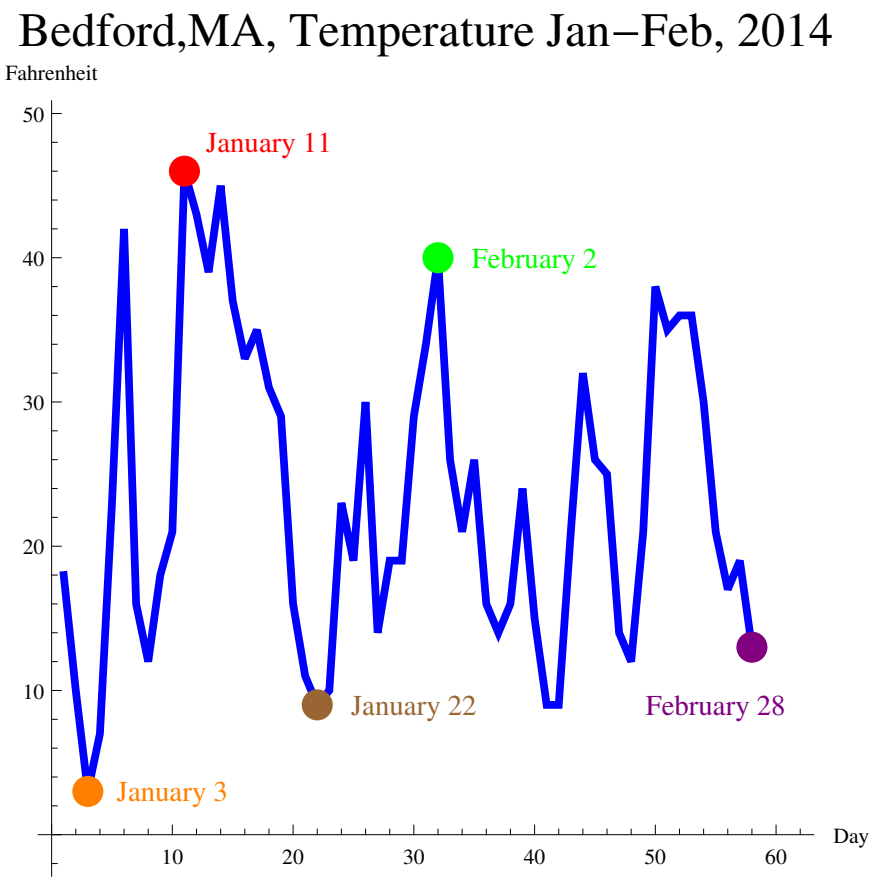
19) [T] [F] $\sin(2\pi) + \tan(2\pi) = 0$.

Solution:
$\sin(0) = \sin(2\pi) = 0$.

20) [T] [F] It is true that $e^{x \log(5)} = x^5$.

Solution:
This is no identity.

Problem 2) Matching problem (10 points) No justifications are needed.



In this winter, the **polar vortex** ruled the weather in Boston. The above graph shows the temperatures of the first two months of 2014 measured at the **Hanscom field** in Bedford, MA. While temperatures are measured hourly, you can assume that temperature is a continuous function of time. Remember that "global maximum" includes being local too so that only one entry in each line of the table below needs to be checked.

a) (5 points) Check what applies, by checking one entry in each of the 5 dates.

Date	local maximum	local minimum	global maximum	global minimum
January 3				
January 11				
January 22				
February 2				
February 28				

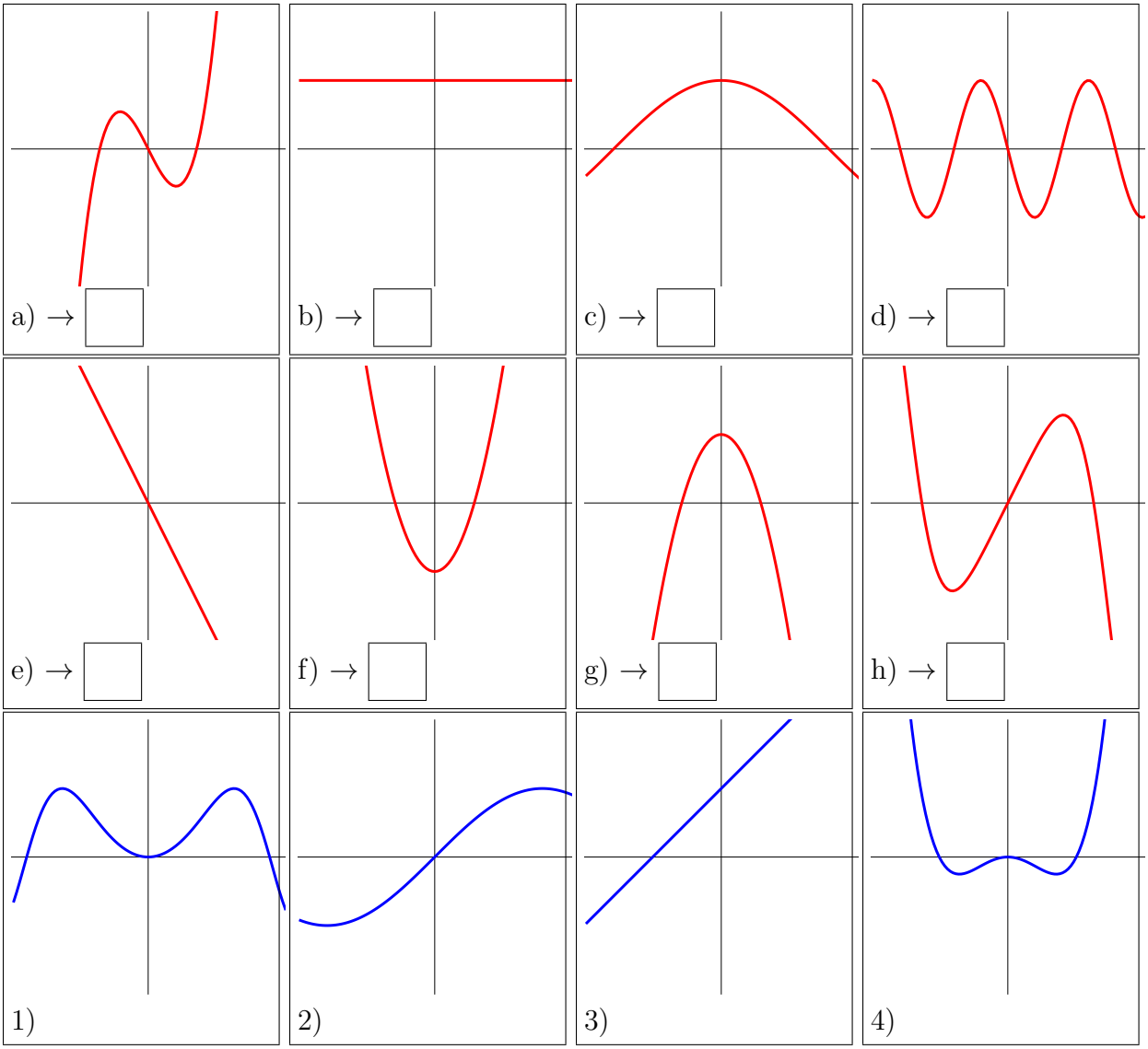
b) (2 points) Which theorem assures that on the closed interval $[0, 59]$ of 59 days, there is a global maximal temperature?

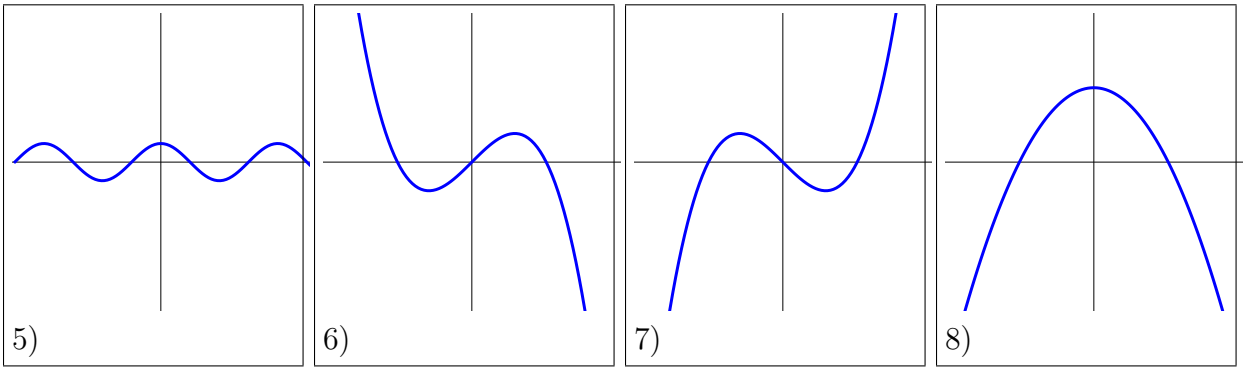
c) (3 points) Argue by citing a theorem why there is a time at which the temperature at Bedford was exactly 25 degree Fahrenheit.

Solution:
a) global min, global max, local min, local max, local min
b) Bolzano's theorem
c) Intermediate value theorem

Problem 3) Matching problem (10 points) No justifications are needed.

In the first pictures, we see the first derivatives f'. Match them with the functions f in 1-8. Note that the functions above are the derivative and the functions below are the functions.





Solution:
4325
8761

Problem 4) Continuity (10 points)

Each of the following functions has a point x_0, where the function is not defined. Find the limit $\lim_{x\to x_0} f(x)$ or state that the limit does not exist.

a) (2 points) $f(x) = \frac{1-2x^3}{1-x}$, at $x_0 = 1$.

b) (2 points) $f(x) = \sin(\sin(5x))/\sin(7x)$, at $x_0 = 0$.

c) (2 points) $f(x) = \frac{\exp(-3x)-1}{\exp(2x)-1}$, at $x_0 = 0$.

d) (2 points) $f(x) = \frac{2x}{\log(x)}$, at $x_0 = 0$.

e) (2 points) $f(x) = \frac{(x-1)^{10}}{(x+1)^{10}}$, at $x_0 = -1$.

Solution:
a) No Limit.
b) Use Hopital: 5/7.
c) Use Hopital: $-3/2$
d) Use Hopital: 0
e) No limit.

Problem 5) Derivatives (10 points)

Find the derivatives of the following functions. If you use a differentiation rule, note which one you use.

a) (2 points) $f(x) = \sqrt{\log(x+1)}$.

b) (3 points) $f(x) = 7\sin(x^3) + \frac{\log(5x)}{x}$.

c) (3 points) $f(x) = \log(\sqrt{x}) + \arctan(x^3)$.

d) (2 points) $f(x) = e^{5\sqrt{x}} + \tan(x)$.

Solution:
a) $1/[2(1+x)\sqrt{\log(1+x)}]$
b) $21x^2\cos(x^3) + [1 - \log(5x)]/x^2$. c) $1/(2x) + 3x^2/(1+x^6)$
d) $\frac{5}{2\sqrt{x}}e^{5\sqrt{x}} + 1/\cos^2(x)$

Problem 6) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions:

a) (2 points) $f(x) = \frac{\exp(3x)-\exp(-3x)}{\exp(5x)-\exp(-5x)}$.

b) (3 points) $f(x) = \frac{\cos(3x)-1}{\sin^2(x)}$.

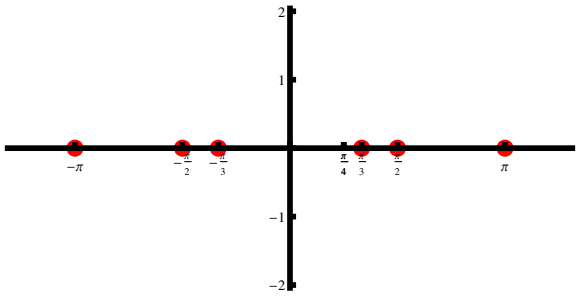
c) (3 points) $f(x) = [\arctan(x) - \arctan(0)]/x$.

d) (2 points) $f(x) = \frac{\log(7x)}{\log(11x)}$.

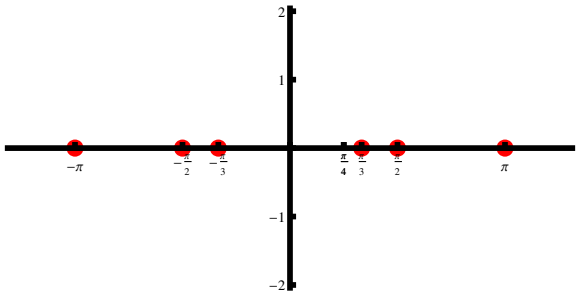
Solution:
a) Use Hopital: $\frac{3\exp(3x)+3\exp(-3x)}{5\exp(5x)+5\exp(-5x)}$. Now take the limits: $3/5$.
b) Use Hopital once $-3\sin(3x)/(2\sin(x)\cos(x))$ and then a second time $9\cos(3x)/2\cos^2(x) - 2\sin^2(x) \to -9/2$
c) This is just the derivative of arctan at 0 which is 1. Hopital gives the same.
d) Use Hopital using $d/dx\log(ax) = 1/x$ for all constants a and get 1.

Problem 7) Trig functions (10 points)

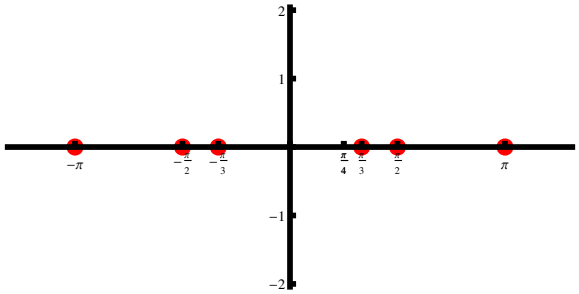
a) Draw the sin function and mark the values of $\sin(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



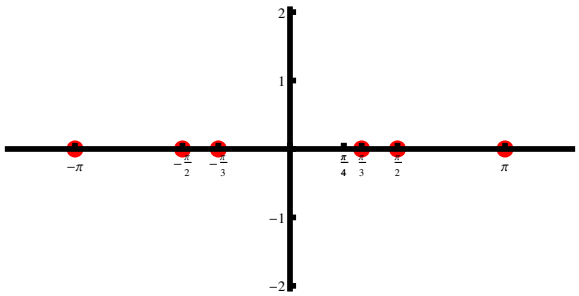
b) Draw the cos function and mark the values of $\cos(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



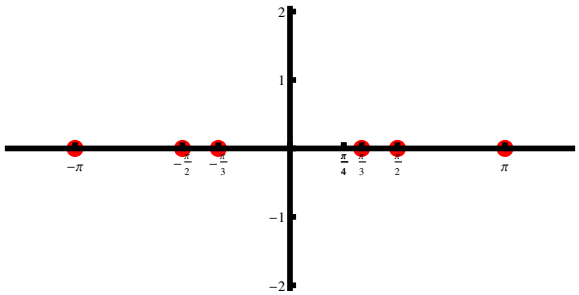
c) Draw the tan function and mark the values of $\tan(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



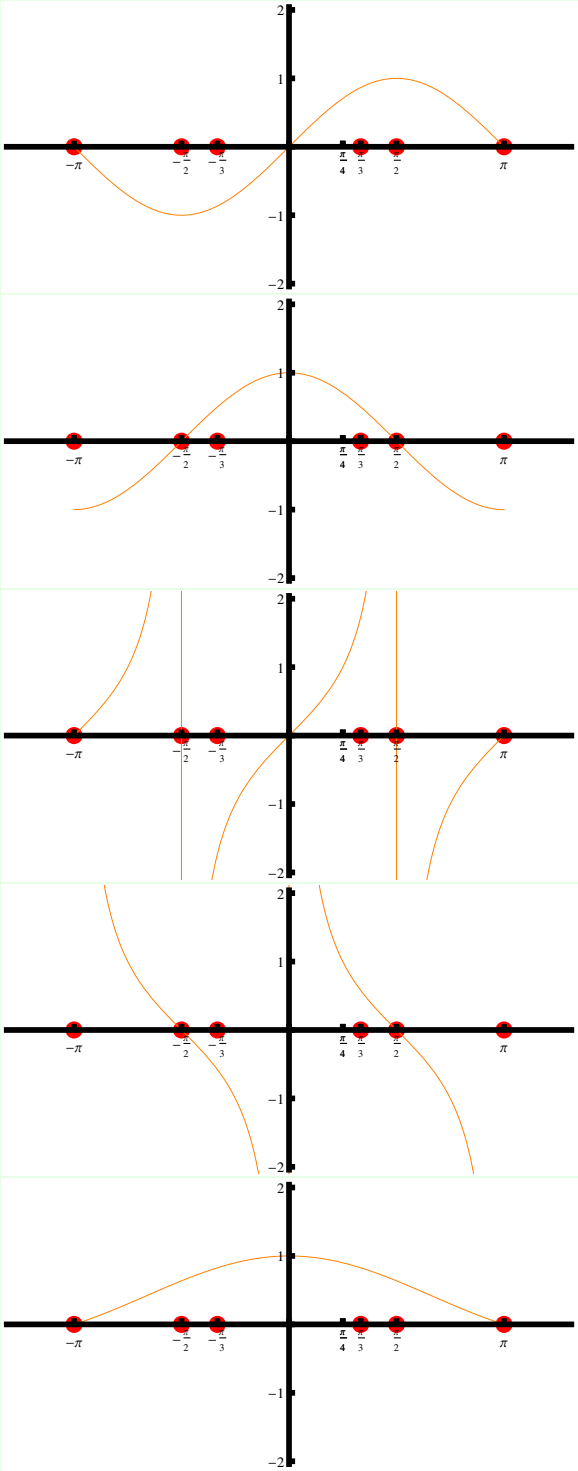
d) Draw the cot function and mark the values of $\cot(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



e) Draw the sinc function $f(x) = \sin(x)/x$ and mark the points $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2,$.



Solution:



Problem 8) Extrema (10 points)

Last week, Oliver got a new batch of strong **Neodym magnets**. They are ring shaped. Assume the inner radius is x, the outer radius y is 1 and the height is $h = x$, we want to maximize the surface area $A = 2\pi(y-x)h + 2\pi(y^2 - x^2)$. This amount of maximizing

$$f(x) = 2\pi(1-x)x + 2\pi(1-x^2)$$

a) (2 points) Using that $f(x)$ is a surface area, on what interval $[a, b]$ needs f to be considered?

b) (3 points) Find the local maxima of f inside the interval.

c) (3 points) Use the second derivative test to verify it is a maximum.

d) (2 points) Find the global maximum on the interval.

Solution:
a) The surface area is nonnegative on $[0, 1]$.
b) $f'(x) = 2\pi(1-4x)$. It is zero at $x = 1/4$. This is the only critical point.
c) $f''(x) = -8\pi$ is negative. By the second derivative test, the critical point is a local maximum.
d) To find the global maximum, compare $f(0) = 2\pi = 6.28..$, $f(1/4) = 2\pi/16 + 2\pi(3/4) = 7.07..$ and $f(1) = 0$. Since the graph of f is quadratic and is everywhere concave down, one could see also that $1/4$ is a global maximum also without evaluating the end points.

Problem 9) Trig and Exponential functions (10 points)

Simplify the following terms. log denotes the natural log and $\log_{10}$ the log to the base 10. Each result in a)-c) is an integer or a fraction

a) (2 points) $\exp(\log(2)) + e^{3\log(2)}$

b) (2 points) $\log(1/e) + \exp(\log(2)) + \log(\exp(3))$.

c) (2 points) $\log_{10}(1/100) + \log_{10}(10000)$

d) (4 points) Produce the formula for $\arccos'(x)$ by taking the derivative of the identity

$$\cos(\arccos(x)) = x\ .$$

Your answer should be simplified as we did when deriving the derivatives of arcsin, arctan in class or when you derived the derivative of arccot and arcsinh, arccosh in the homework.

Solution:
a) $\boxed{10}$
b) -1+2+3= $\boxed{4}$
c) -2+4 = $\boxed{2}$
d) Differentiate $\cos(\arccos(x)) = x$ and simplify.

3/4/2014: First hourly Practice A

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] If f is concave up on $[0,1]$ and concave down on $[1,2]$ then 1 is an inflection points.

Solution:
Indeed, f'' changes sign there.

2) [T] [F] The function $f(x) = \exp(x)$ has the root $x = 1$.

Solution:
exp does not have any root.

3) [T] [F] $\log(\exp(1)) = 1$, if log is the natural log and $\exp(x) = e^x$ is the exponential function.

Solution:
Yes, exp is the inverse of log.

4) [T] [F] The chain rule assures that $d/dx f(f(x)) = f'(f(x))f'(x)$.

Solution:
Yes, this is a special case

5) [T] [F] The function $x^2/(1+x^2)$ is continuous everywhere on the real axes.

Solution:
The denominator is never zero so that there is no pole.

6) [T] [F] The function $\cot(x)$ is the inverse of the function $\tan(x)$.

Solution:
No, it is $\operatorname{arccot}(x)$ which is the inverse.

7) [T] [F] The Newton method is $T(x) = x + f(x)/f'(x)$.

Solution:
The sign is off.

8) [T] [F] $\cos(\pi/2) = 1/2$.

Solution:
The cosine has a root at $x = \pi/2$.

9) [T] [F] If a function f is differentiable on $[-1, 1]$, then there is a point x in that interval where $f'(x) = 0$.

Solution:
take $f(x) = x$.

10) [T] [F] The chain rule assures that $d/dx(g(x^2)) = 2xg'(x^2)$.

Solution:
Yup. This is the formula!

11) [T] [F] We have $\lim_{x\to\infty}((x^2 + 1)/x^2) = 1$

Solution:
This is a consequence of l'Hopital's rule when applied twice.

12) [T] [F] An inflection point is a point, where the function $f'(x)$ changes sign.

Solution:
It is a point, where the second derivative changes sign

13) [T] [F] If $f''(-2) > 0$ then f is concave up at $x = -2$.

Solution:
The slope of the tangent increases which produces a concave up graph. One can define concave up with the property $f''(x) > 0$

14) [T] [F] The intermediate value theorem assures that the continuous function $x + \sin(x) = 0$ has a root.

Solution:
The intermediate value theorem deals with roots.

15) [T] [F] We can find a value b and define $f(0) = b$ such that the function $f(x) = (x^{28} - 1)/(x^2 - 1)$ is continuous everywhere.

Solution:
We divide by zero at $z = 1$.

16) [T] [F] If the third derivative $f'''(x)$ is negative and $f''(x) = 0$ then f has a local maximum at x.

Solution:
It is the second derivative test, not the third one

17) [T] [F] If $f(x) = x^2$ then $Df(x) = f(x + 1) - f(x)$ has a graph which is a line.

Solution:
$Df(x) = 2x - 1$.

18) [T] [F] The quotient rule is $d/dx(f/g) = f'(x)/g'(x)$.

Solution:
No, it is not related to the l'Hopital rule.

19) [T] [F] With $Df(x) = f(x + 1) - f(x)$, we have $D(1 + a)^x = a(1 + a)^x$.

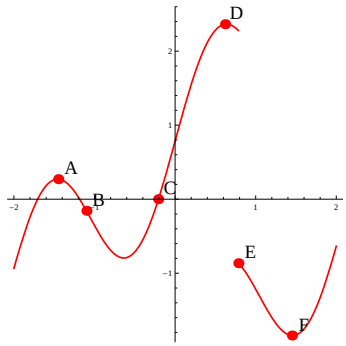
Solution:
Compound interest

20) [T] [F] It is true that $\log(5)e^x = e^{x \log(5)}$ if $\log(x)$ is the natural log.

Solution:
This is no identity.

Problem 2) Matching problem (10 points) No justifications are needed.

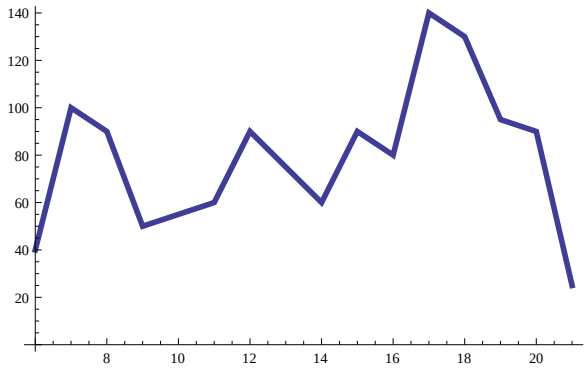
a) (6 points) You see the graph of a function $f(x)$ defined on $[-2, 3]$. Various points $(x, f(x))$ are marked. Match them:



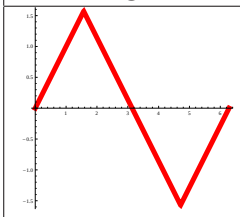
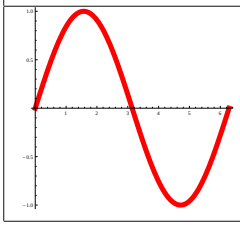
Point x is	Fill in A-F
Local maximum	
Root	
Inflection point	
Discontinuity	
Global maximum	
Local minimum	

b) (2 points) Last week, the **Harvard recreation** published a graph of a function $f(x)$ which shows the number of people at the **Mac** gym as a function of time. At 5 o'clock, there are in average 120 visitors, at 9 in the morning, there are 60 people working out. By the intermediate value theorem, there must be a moment at which exactly $\pi^4 = 97.5...$ visitors are present. This is obviously nonsense. Where is the flaw?

Reason	Check one
No differentiability	
Statistical glitch	
No Continuity	
Inaccurate data	



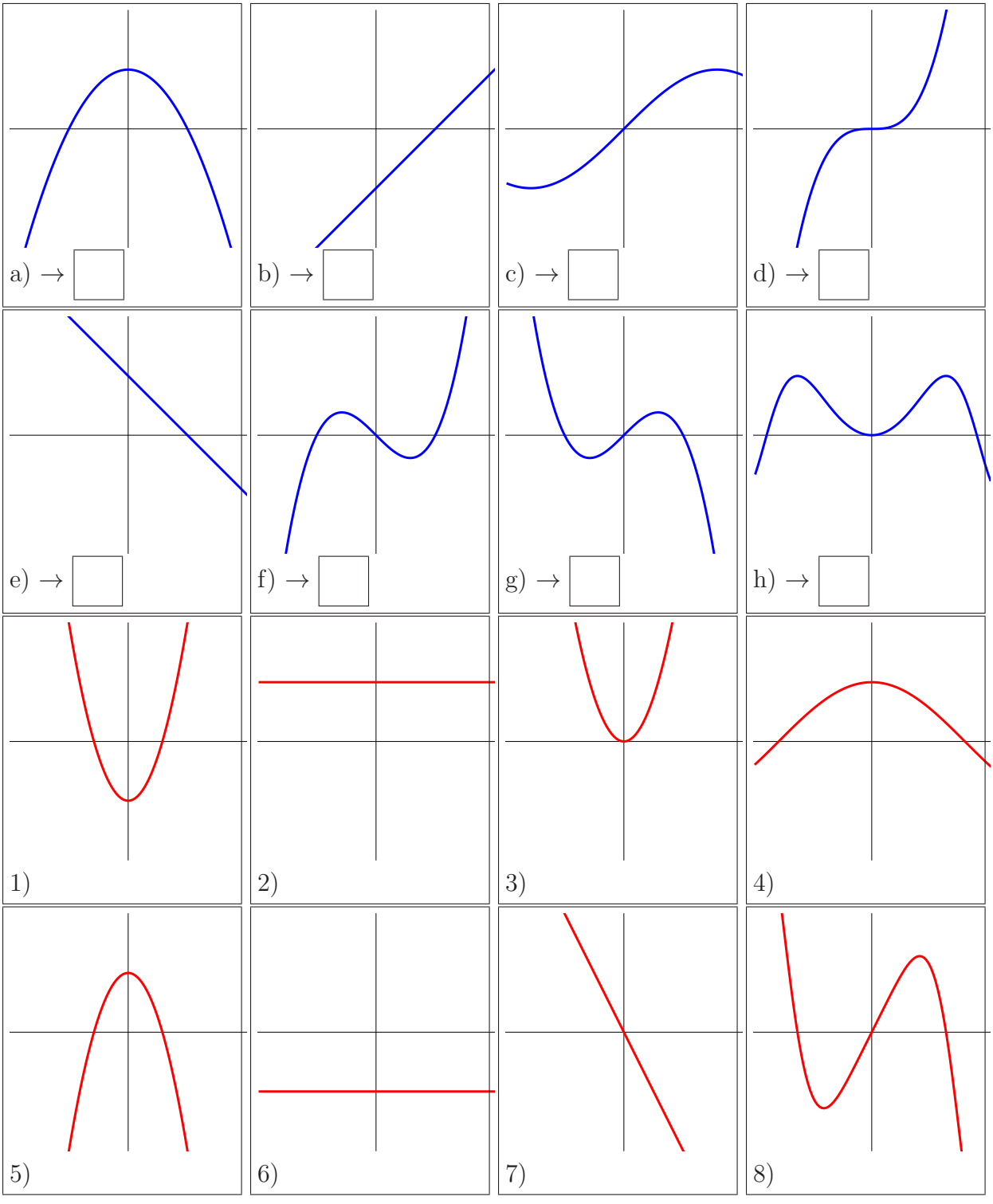
c) (2 points) In front of the **"Class of 1959 Chapel"** at the Harvard business school is an amazing clock: a marble tower contains a steel pole and a large bronze ball which moves up and down the pole indicating the time of the day. As the ball moves up and down the pole, lines with equal distance on the tower indicate the time. At noon, the sphere is at the highest point. At midnight it is at the bottom. It moves the same distance in each hour. If we plot the hight of the sphere as a function of time, which graph do we see?

The height function	Check which applies
	
	

Solution:
a) ACBEDF
b) Lack of continuity. The number of people is an integer.
c) It is the piecewise linear function because the clock moves with constant speed.

Problem 3) Matching problem (10 points) No justifications are needed.

Match the graph of the functions f in $a) - h)$ with the derivatives f' in 1)-8).



Solution:
7243, 6158

Problem 4) Continuity (10 points)

Each of the following functions has a point x_0, where the function is not defined. Find the limit $\lim_{x\to x_0} f(x)$ or state that the limit does not exist.

a) (2 points) $f(x) = \frac{x^3-8}{x-2}$, at $x_0 = 2$

b) (2 points) $f(x) = \sin(\sin(\frac{1}{x})) - \tan(x)$, at $x_0 = 0$

c) (2 points) $f(x) = \frac{\cos(x)-1}{x^2}$, at $x_0 = 0$

d) (2 points) $f(x) = \frac{\exp(x)-1}{\exp(5x)-1}$, at $x_0 = 0$

e) (2 points) $f(x) = \frac{x-1}{x}$, at $x_0 = 0$

Solution:
a) We can use Hôpital's rule to see that the limit is $\lim_{x\to 2} 3x^2/1 = \boxed{12}$.
b) There is no way that we can save the oscillatory singularity. $\boxed{\text{No limit exists}}$.
c) Apply Hopital twice to see that the limit is $\boxed{-1/2}$.
d) Apply l'Hopital to see that the limit is $\lim_{x\to 0} e^x/(5e^{5x}) = \boxed{1/5}$.
e) This can not be saved at $x = 0$. There is a pole there. $\boxed{\text{No limit exists.}}$

Problem 5) Derivatives (10 points)

Find the derivatives of the following functions. If you use a differentiation rule, note which one you use.

a) (2 points) $f(x) = \sin(7x) + (1 + x^2)$.

b) (2 points) $f(x) = \frac{\sin(7x)}{(1+x^2)}$.

c) (2 points) $f(x) = \sin(7 + x^2)$.

d) (2 points) $f(x) = \sin(7x)(1 + x^2)$.

e) (2 points) $f(x) = \sin(7x)^{(1+x^2)}$

Solution:
a) Chain rule $7\cos(7x) + 2x$.
b) Quotient rule $[(1+x^2)7\cos(7x) - \sin(7x)2x]/(1+x^2)^2$.
c) Chain rule. $2x\cos(7+x^2)$.
d) Product and chain rule. $7\cos(7x)(1+x^2) + 2x\sin(7x)$.
e) First write as $\exp((1+x^2)\log(\sin(7x)))$ and differentiate now with the chain rule: $\exp((1+x^2)\log(\sin(7x)))[2x\log(\sin(7x)) + (1+x^2)7\cos(7x)/\sin(7x)]$.

Problem 6) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions f at $x = 0$:

a) (2 points) $f(x) = \frac{(1-\exp(11x))}{(1-\exp(3x))}$

b) (2 points) $f(x) = \frac{\sin(\sin(5x))}{\sin(7x)}$

c) (2 points) $f(x) = \frac{\log(x)}{\log(5x)}$

d) (2 points) $f(x) = \frac{x^2\cos(x)}{\sin^2(x)}$

e) (2 points) $f(x) = \frac{(1+1/x^2)}{(1-1/x^2)}$

Solution:
All with Hopital.
a) 11/3
b) 5/7
c) 1
d) 1
e) -1

Problem 7) Trig functions (10 points)

A triangle with side lengths $3, 4, 5$ has a right angle. Let $\alpha < \beta < \gamma$ denote the angles ordered by size.

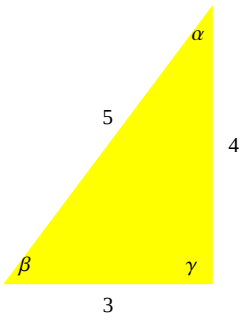
a) (4 points) What are the numerical values of $\cos(\alpha), \cos(\beta), \cos(\gamma), \sin(\gamma)$?

b) (2 points) Find the numerical value of $\tan(\alpha)$ and $\cot(\alpha)$.

The next problem is independent of the previous two.

c) (4 points) Find the derivative of the inverse function of $\arcsin(x)$ by starting with the identity

$x = \sin(\arcsin(x))$. Your derivation of $\arcsin'(x)$ should convince somebody who does not know the identity already.



Solution:
a) $\cos(\alpha) = 4/5, \cos(\beta) = 3/5 \cos(\gamma) = 0, \sin(\gamma) = 1$ b) $3/4, 4/3$
c) Differentiating $x = \sin(\arcsin(x))$ gives $1 = \cos(\arcsin(x)) \arcsin'(x)$ solve for $\arcsin'(x)$ and replace $\cos(u)$ with $\sqrt{1 - \sin^2(u)}$ to get $\arcsin'(x) = 1/\sqrt{1 - x^2}$.

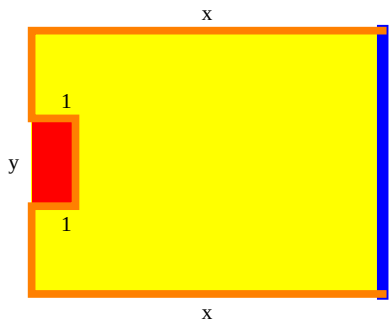
Problem 8) Extrema (10 points)

A tennis field of width x and length y contains a fenced referee area of length 2 and width 1 within the field and an already built wall. The circumference a fence satisfies $2x + y + 2 = 10$, (an expression which still can be simplified). We want to maximize the area $xy - 2$.

a) (2 points) On which interval $[a, b]$ does the variable x make sense? Find a function $f(x)$ which needs to be maximized.

b) (6 points) Find the local maximum of x and check it with the second derivative test.

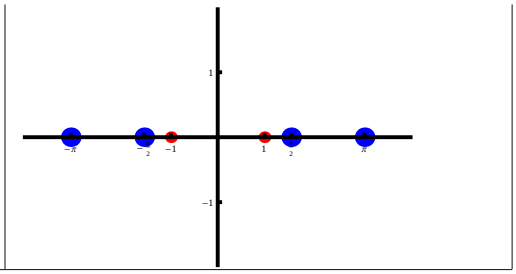
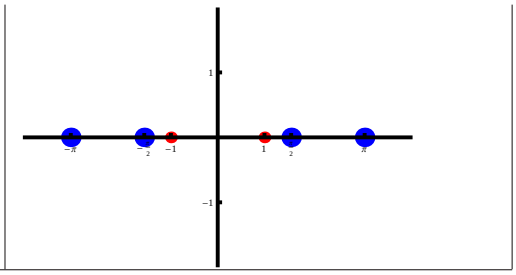
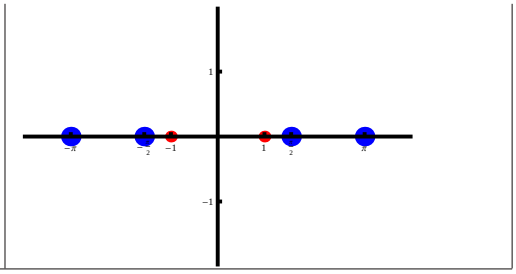
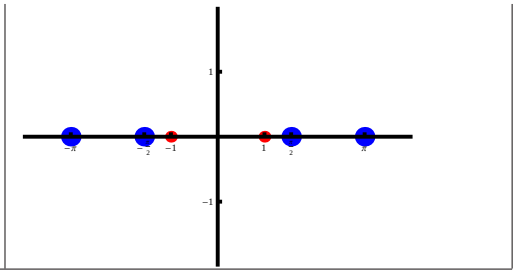
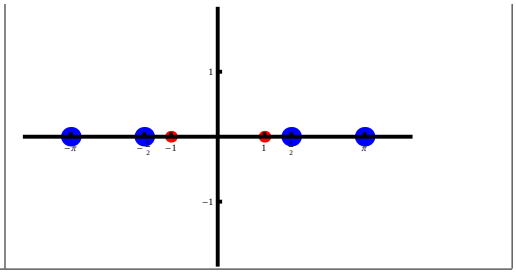
c) (2 points) What is the global maximum of f on $[a, b]$?



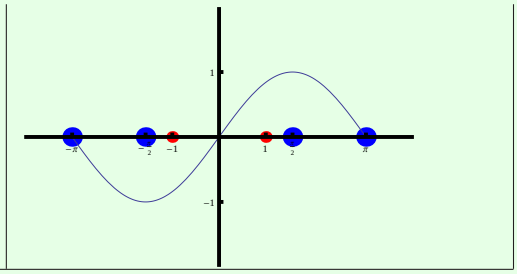
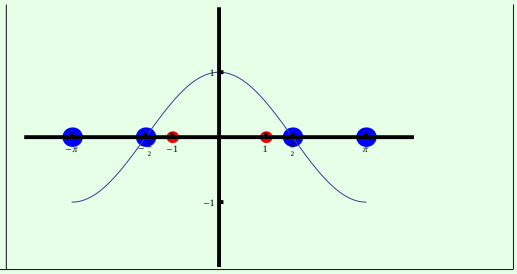
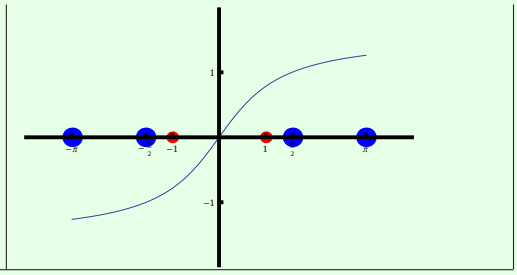
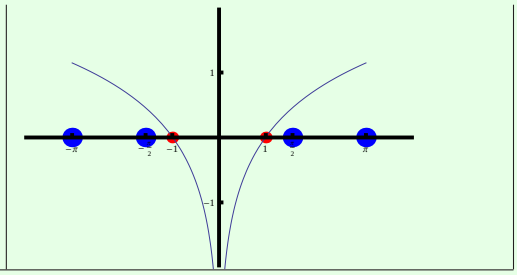
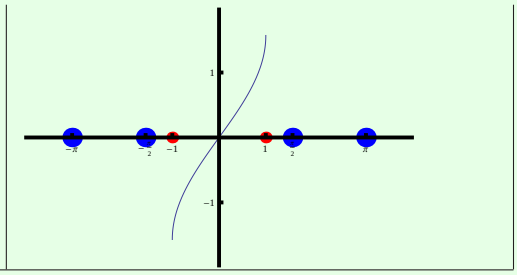
Solution:
a) We must have $x \geq 0$ and $y = 8 - 2x > 0$ so that $x \leq 4$ The interval on which f is defined is therefore $[0, 4]$. Actually, since $y \geq 2$ we must even have $x \in [0, 3]$ but both bounds were considered correct. Since the area has to be positive, one could even shave off a bit more from the lower bound but that was not necessary.
b) The function to extremize is $f(x) = x(8 - 2x) - 2$. Its derivative is $8 - 4x$. This is zero at $x = 2$. We have $y = 4$. The maximal area is $xy - 2 = 8 - 2 = 6$.
c) Compare $f(0) = -2, f(4) = -2$ and $f(2) = 6$ to see that 6 is the maximal value.

Problem 9) Trig functions (10 points)

In the following five problems, find the numerical value and then draw the graph of the function.

Problem	Answer	Graph
a) (2 points) What is $\sin(\pi/3)$? Plot $\sin(x)$.		
b) (2 points) What is $\cos(5\pi/2)$? Plot $\cos(x)$.		
c) (2 points) Find $\arctan(1)$ Plot $\arctan(x)$.		
d) (2 points) What is $\log(1)$ Plot $\log\|x\|$.		
e) (2 points) What is $\arcsin(\sqrt{3}/2)$. Plot $\arcsin(x)$		

Solution:
a) $\sqrt{3}/2$.
b) 0
c) $\pi/4$
d) 0
e) $\pi/3$ The functions needed to be plotted with roots at the correct place.

Problem	Answer	Graph
a) (2 points) What is $\sin(\pi/3)$? Plot $\sin(x)$.		
b) (2 points) What is $\cos(5\pi/2)$? Plot $\cos(x)$.		
c) (2 points) Find $\arctan(1)$ Plot $\arctan(x)$.		
d) (2 points) What is $\log(1)$ Plot $\log\|x\|$.		
e) (2 points) What is $\arcsin(\sqrt{3}/2)$. Plot $\arcsin(x)$		

Problem 10) Exponential functions (10 points)

Simplify the following terms. log denotes the natural log and $\log_{10}$ denotes the log to the base 10. All answers are integers.

a) (2 points) $\exp(\log(2))$
b) (2 points) $e^{\log(2)3}$
c) (2 points) $\log(\log(e))$
d) (2 points) $\exp(\log(2) + \log(3))$
e) (2 points) $\log_{10}(10000)$

Solution:
a) 2
b) 8
c) 0
d) 6
e) 4

3/4/2014: First hourly Practice B

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for multiple choice problems, give computations. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.
- All unspecified functions are assumed to be smooth: one can differentiate arbitrarily.
- The actual exam has a similar format: TF questions, multiple choice and then problems where work needs to be shown.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points) No justifications are needed.

1) [T] [F] The function $\cot(x)$ is the inverse of the function $\tan(x)$.

Solution:
No, it is $\operatorname{artan}(x)$ which is the inverse.

2) [T] [F] We have $\cos(x)/\sin(x) = \cot(x)$

Solution:
That is the definition.

3) [T] [F] $\sin(3\pi/2) = -1$.

Solution:
Draw the circle. The angle $3\pi/2$ corresponds to 270 degrees.

4) [T] [F] The function $f(x) = \sin(x)/x$ has a limit at $x = 0$.

Solution:
Yes, it is called the sinc function.

5) [T] [F] For the function $f(x) = \sin(\sin(\exp(x)))$ the limit $\lim_{h\to 0}[f(x+h)-f(x)]/h$ exists.

Solution:
The function is differentiable. The limit is the derivative.

6) [T] [F] If a differentiable function $f(x)$ satisfies $f'(3) = 3$ and is f' is odd then it has a critical point.

Solution:
We have $f'(3) = 3$ and $f'(-3) = -3$. The intermediate value theorem assures that $f'(x) = 0$ for some $x \in [-3, 3]$.

7) [T] [F] The l'Hopital rule assures that the derivative satisfies $(f/g)' = f'/g'$.

Solution:
The l'Hopital rule tells something about a limit $f(x)/g(x)$ as $x \to p$ but does not compute derivatives.

8) [T] [F] The intermediate value theorem assures that a continuous function has a derivative.

Solution:
This is false. The intermediate value theorem deals with continuous functions.

9) [T] [F] After healing, the function $f(x) = (x+1)/(x^2-1)$ is continuous everywhere.

Solution:
We divide by zero at $z = 1$.

10) [T] [F] If f is concave up on $[1, 2]$ and concave down on $[2, 3]$ then 2 is an inflection point.

Solution:
Indeed, f'' changes sign there.

11) [T] [F] There is a function f which has the property that its second derivative f'' is equal to its negative f.

Solution:
It is the sin or cos function.

12) [T] [F] The function $f(x) = [x]^4 = x(x-h)(x-2h)(x-3h)$ has the property that $Df(x) = 4[x]^3 = 4x(x-h)(x-2h)$, where $Df(x) = [f(x+h) - f(x)]/h$.

Solution:
Yes, this is a cool property of the polynomials $[x]^n$.

13) [T] [F] The quotient rule is $d/dx(f/g) = (f'g - fg')/g^2$ and holds whenever $g(x) \neq 0$.

Solution:
This is an important rule to know.

14) [T] [F] The chain rule assures that $d/dx f(g(x)) = f'(g(x)) + f(g'(x))$.

Solution:
This is not true. We have $f'(g(x))g'(x)$.

15) [T] [F] If f and g are differentiable, then $(3f + g)' = 3f' + g'$.

Solution:
This is called linearity of the differentiation.

16) [T] [F] For any function f, the Newton step $T(x)$ is continuous.

Solution:
There is trouble at critical points $f'(x) = 0$.

17) [T] [F] One can rotate a four legged table on an arbitrary surface such that all four legs are on the ground.

Solution:
We have seen this in class

18) [T] [F] The fundamental theorem of calculus relates integration S with differentiation D. The result is $DSf(x) = f(x), SDf(x) = f(x) - f(0)$.

19) [T] [F] The product rule implies $d/dx(f(x)g(x)h(x)) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

Solution:
This was checked in a homework.

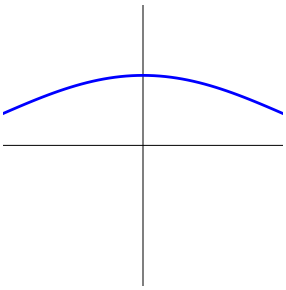
20) [T] [F] Euler and Gauss are the founders of infinitesimal calculus.

Solution:
It was Newton and Leibniz who are considered the founders, Euler and Gauss came later

Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their graphs.

Function	Fill in 1-8
$x^2 - x$	
$\exp(-x)$	
$\sin(3x)$	
$\log(\lvert x\rvert)$	
$\tan(x)$	
$1/(2+\cos(x))$	
$x - \cos(6x)$	
$\sin(3x)/x$	



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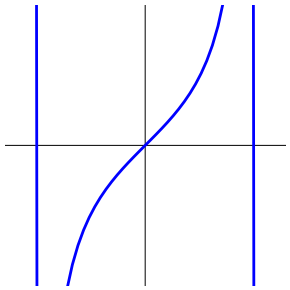

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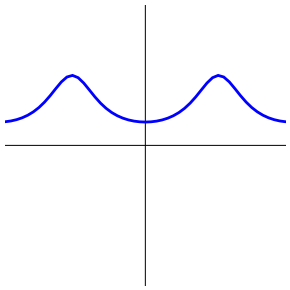

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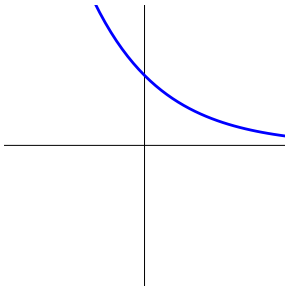

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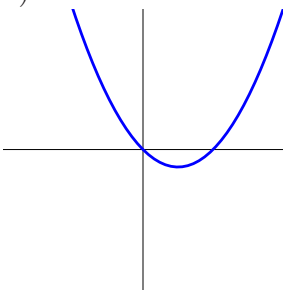

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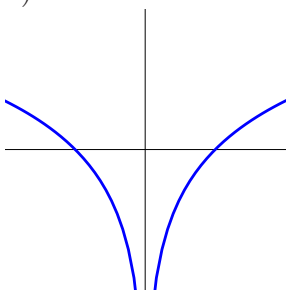

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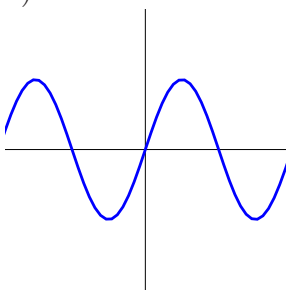

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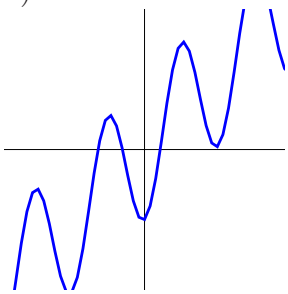


8)

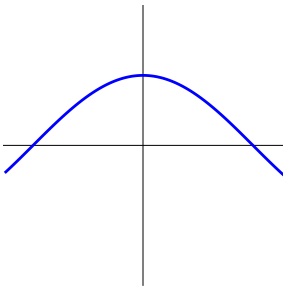
Solution:

Function	Fill in 1-8
$x^2 - x$	5
$\exp(-x)$	4
$\sin(3x)$	7
$\log(\lvert x\rvert)$	6
$\tan(x)$	2
$1/(2+\cos(x))$	3
$x - \cos(6x)$	8
$\sin(3x)/x$	1

Problem 3) Matching problem (10 points) No justifications are needed.

Match the following functions with their derivatives.

Function	Fill in the numbers 1-8
graph a)	
graph b)	
graph c)	
graph d)	
graph e)	
graph f)	
graph g)	
graph h)	



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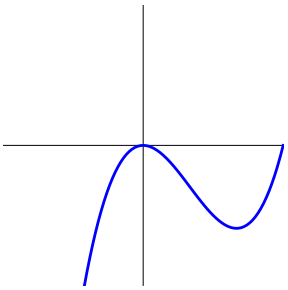

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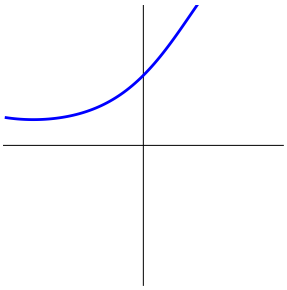

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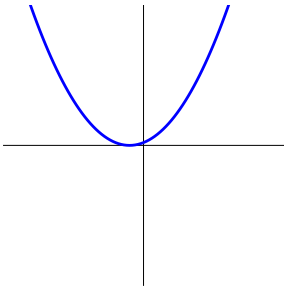

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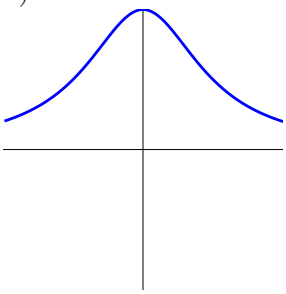

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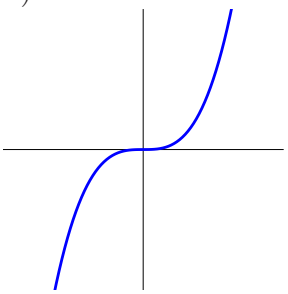

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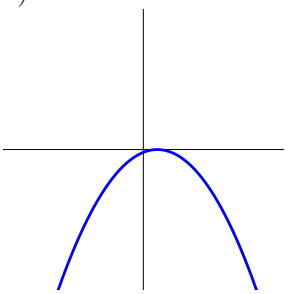

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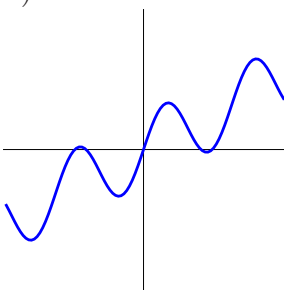

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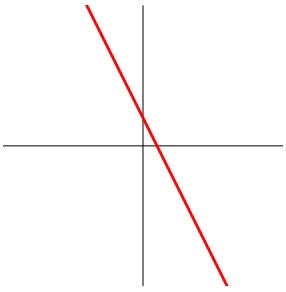

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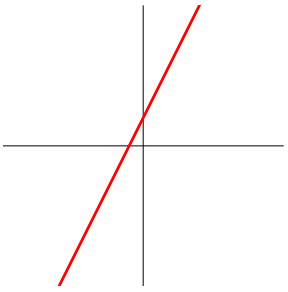

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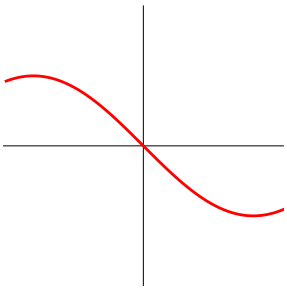

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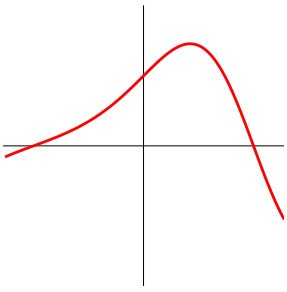

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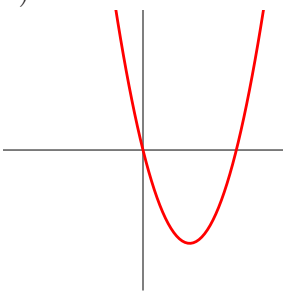
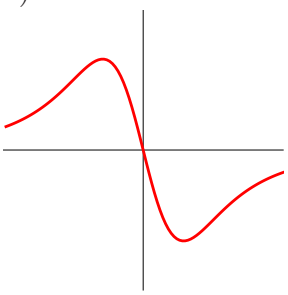
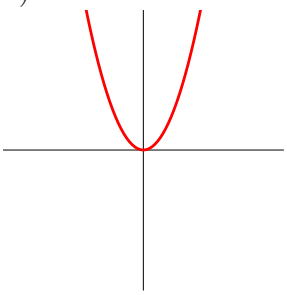


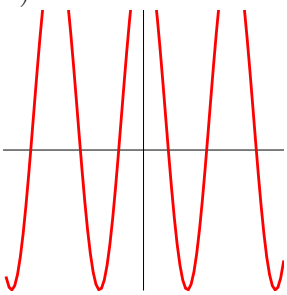
5)



6)



7)



8)

Solution:

Function	Fill in the numbers 1-8
graph a)	3
graph b)	5
graph c)	4
graph d)	2
graph e)	6
graph f)	7
graph g)	1
graph h)	8

Problem 4) Functions (10 points) No justifications are needed

Match the following functions with simplified versions. In each of the rows, exactly one of the choices A-C is true.

Function	Choice A	Choice B	Choice C	Enter A-C
$\frac{x^4-1}{x-1}$	$1+x+x^2+x^3$	$1+x+x^2$	$1+x+x^2+x^3+x^4$	
2^x	$e^{2\log(x)}$	$e^{x\log(2)}$	$2^{e\log(x)}$	
$\sin(2x)$	$2\sin(x)\cos(x)$	$\cos^2(x)-\sin^2(x)$	$2\sin(x)$	
$(1/x)+(1/(2x))$	$1/(x+2x)$	$3/(2x)$	$1/(x+2x)$	
e^{x+2}	$e^x e^2$	$2e^x$	$(e^x)^2$	
$\log(4x)$	$4\log(x)$	$\log(4)\log(x)$	$\log(x)+\log(4)$	
$\sqrt{x}^3$	$x^{3/2}$	$x^{2/3}$	$3\sqrt{x}$	

Solution:
A,B,A,B,A,C,A

Problem 5) Roots (10 points)

Find the roots of the following functions

a) (2 points) $7\sin(3\pi x)$

b) (2 points) $x^5 - x$.

c) (2 points) $\log|ex|$.

d) (2 points) $e^{5x} - 1$

e) (2 points) $8x/(x^2+4) - x$.

Solution:
a) The function is zero if $3x$ is an integer. The solutions are $\boxed{n/3}$, where n is an integer.
b) The function is zero if x is zero or if $x^4 = 1$. The later has the solutions $x = 1, -1$. The roots are $\boxed{0, 1, -1}$.
c) $\log(x) = 0$ if $x - 1$. Therefore, the root is $\boxed{x = 1/e}$ or $\boxed{x = -1/e}$. You might have been tempted to try $x = 1$ which gives $\log(e) = 1$.
d) $e^{5x} = 1$ for $\boxed{x = 0}$.
e) One solution is $x = 0$. We can factor x out. We need to solve then $1 = 8/(x^2+4)$ which means $x^2 + 4 = 8$ or $x^2 = 4$. We have two more solutions, in total $\boxed{x = 0, 2, x = -2}$.

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \cos(3x)/\cos(10x)$

b) (2 points) $f(x) = \sin^2(x)\log(1+x^2)$

c) (2 points) $f(x) = 5x^4 - 1/(x^2+1)$

d) (2 points) $f(x) = \tan(x) + 2^x$

e) (2 points) $f(x) = \arccos(x)$

Solution:
a) Use the quotient rule

$$\frac{-3\sin(3x)\cos(10x) + 10\sin(10x)\cos(3x)}{\cos^2(10x)} .$$

b) Use the product rule

$$2\sin(x)\cos(x)\log(1+x^2) + \sin^2(x)2x/(1+x^2) .$$

c) $20x^3 + 2x/(x^2+1)^2$
d) $1/\cos^2(x) + e^{x\log(2)}\log(2)$.
e) Use the chain rule on $\cos(\arccos(x)) = x$. This is $-\sin(\arccos(x))\arccos'(x) = 1$ Since $\sin(x) = \sqrt{1-\cos^2(x)}$ we get the derivative $-1/\sqrt{1-x^2}$.

Problem 7) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ of the following functions:

a) (2 points) $f(x) = (x^6 - 3x^2 + 2x)/(1 + x^2 - \cos(x))$.

b) (2 points) $f(x) = (\cos(3x) - 1)/(\cos(7x) - 1)$.

c) (2 points) $f(x) = \tan^3(x)/x^3$.

d) (2 points) $f(x) = \sin(x)\log(x^6)$

e) (2 points) $f(x) = 4x(1-x)/(\cos(x) - 1)$.

Solution:
a) The limit does $\boxed{\text{not exist}}$. After applying l'Hopital once we get a denominator which is zero and a nominator which is nonzero.
b) Use l'Hopital twice. After applying it once, we get $(-3\sin(3x))/(-7\sin(7x))$. Applying l'Hopital again gives $\boxed{9/49}$.
c) First compute the limit $\tan(x)/x$ which is 1 by l'Hopital. The expression is 1^3 which is $\boxed{1}$.
d) We can write this as $\sin(x)6\log(x)$. We can either apply l'Hopital to $6\log(x)/(1/\sin(x))$ and get $6\sin^2(x)/(x\cos(x))$. Since $\sin(x)/x$ goes to 1 we get the limit $\boxed{0}$.
e) like in a), the limit $\boxed{\text{does not exist}}$.

Problem 8) Extrema (10 points)

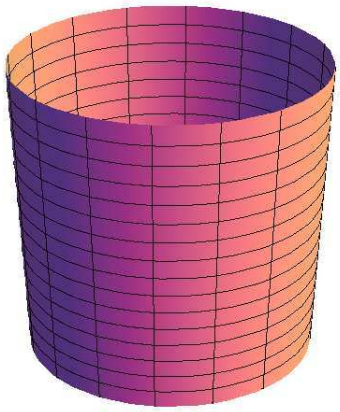
a) (5 points) Find all local extrema of the function $f(x) = 30x^2 - 5x^3 - 15x^4 + 3x^5$ on the real line.

b) (5 points) Find the global maximum and global minimum of the function $f(x) = \exp(x) - \exp(2x)$ on the interval $[-2, 2]$.

Solution:
a) The function $f'(x) = 15x^4 - 60x^3 - 15x^2 + 60x$ has roots at $-1, 0, 1, 4$. We can find them by trying with integers. These are candidates for local extrema. WE can compute the second derivative at these 4 points to get $-150, 60, -90, 900$. The points $\boxed{1, -1 \text{ are local maxima}}$, the points $\boxed{0, 4 \text{ are local minima}}$.
b) The derivative $\exp(x) - 2\exp(2x)$ has the only root $x = -\log(2)$. In order to find the maxima or minima, we also have to look at the boundary points too. The function satisfies $f(-\log(2)) = 1/4 = 0.25, f(-2) = e^{-2} - e^{-4} = 0.117..., f(2) = e^2 - e^4 = -47.209...$
$\boxed{\text{The minimum is } f(2), \text{ the maximum is } f(-\log(2)) = 1/4.}$

Problem 9) Extrema (10 points)

A cup of height h and radius r has the volume $V = \pi r^2 h$. Its surface area is $\pi r^2 + \pi r h$. Among all cups with volume $V = \pi$ find the one which has minimal surface area. Find the global minimum.



Solution:
We need to find the minimum of the function $f(r) = \pi(r^2 + 1/r)$. Compute the derivative: $f'(x) = \pi 2r - \pi/r^2$. If this is zero then $2r^3 = 1$ and $\boxed{r = 2^{-1/3}}$. For $r \to 0$ the function goes to infinity as it does for $r \to \infty$. Therefore, the function has a global minimum at $1/2^{1/3}$.

Problem 10) Newton method (10 points)

a) (3 points) Produce the first Newton step for the function $f(x) = e^x - x$ at the point $x = 1$.

b) (4 points) Produce a second Newton step.

c) (3 points) Find the Newton step map $T(x)$ if the function $f(x)$ is replaced by the function $3f(x)$.

Solution:
a) We have $f'(x) = e^x - 1$. The Newton map is $T(x) = x - (e^x - x)/(e^x - 1)$. Applying this map at $x = 1$ gives $1 - (e-1)/(e-1) = \boxed{0}$.
b) $T(0) = 0 - 1/0$ pulls us to $\boxed{\text{infinity}}$. The step can not be done. This always happens if we apply the Newton map at a local extremum.
c) The map T does $\boxed{\text{not change}}$ because $f(x)/f'(x)$ does not change if we replace f with $3f$.

3/4/2014: First hourly Practice C

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] The function arcsin(x) is defined as $1/\sin(x)$.

Solution:
The arcsin function is the inverse not the reciprocal of $\sin(x)$.

2) [T] [F] The function $f(x) = \sin(1/x^2)$ can be defined at 0 so that it becomes a continuous everywhere on the real line.

Solution:
This is the prototype oscillatory singularity.

3) [T] [F] The function $x/\sin(x)$ can be defined at $x = 0$ so that it becomes a continuous function on the real line.

Solution:
The value is 1 at 0 by the fundamental theorem of trigonometry. It can not be made continuous on the entire real line because at $\pi, 2\pi$ etc the function can not be saved.

4) [T] [F] The function $f(x) = \sin^2(x)/x^2$ has the limit 1 at $x = 0$.

Solution:
Yes, it is the square or the *sinc* function.

5) [T] [F] The function $f(x) = 1/\log|x|$ has the limit 1 at $x = 0$.

Solution:
l'Hopital gives 0, not 1.

6) [T] [F] The function $f(x) = (1+h)^{x/h}$ has the property that $Df(x) = [f(x+h) - f(x)]/h = f(x)$.

Solution:
We have seen that several time in this course and done in the homework.

7) [T] [F] $\cos(3\pi/2) = 1$.

Solution:
Draw the circle. The angle $3\pi/2$ corresponds to 270 degrees. The cosine is the x value and so zero.

8) [T] [F] If a function f is continuous on the interval $[3, 10]$, then it has a global maximum on this interval.

Solution:
This is a consequence of the extreme value theorem.

9) [T] [F] The reciprocal rule assures that $d/dx(1/g(x)) = 1/g(x)^2$.

Solution:
The minus sign is missing as well as the factor $g'(x)$.

10) [T] [F] If $f(0) = g(0) = f'(0) = g'(0) = 0$ and $g''(0) = f''(0) = 1$, then $\lim_{x\to 0}(f(x)/g(x)) = 1$

Solution:
This is a consequence of l'Hopital's rule when applied twice.

11) [T] [F] An inflection point is a point where the function $f''(x)$ changes sign.

Solution:
This is a definition.

12) [T] [F] If $f''(x) > 0$ then f is concave up at x.

Solution:
The slope of the tangent increases which produces a concave up graph. One can define concave up with the property $f''(x) > 0$

13) [T] [F] The chain rule assures that $d/dxf(g(x)) = f'(x)g'(x)$.

Solution:
This is not true. We have $f'(g(x))$ in the first factor.

14) [T] [F] The function $f(x) = 1/x + \log(x)$ is continuous on the interval $[1, 2]$.

Solution:
While there is a problem at 0, everything is nice and dandy at $[1, 2]$.

15) [T] [F] If we perform the Newton step for the function $\exp(x)$, we get the map $T(x) = x - 1$.

Solution:
The exponential function is its own derivative so that $f(x)/f'(x) = 1$.

16) [T] [F] The graph of the function $f(x) = x/(1 + x^2)$ has slope 1 at 0.

Solution:
$f'(x) = 1/(1 + x^2) - 2x^2/(1 + x^2)^2$. This is 1 for $x = 0$.

17) [T] [F] There is a differentiable function for which $f'(0) = 0$ but for which 0 is not a local extremum.

Solution:
Take $f(x) = x^3$.

18) [T] [F] The second derivative test assures that $x = p$ is a local minimum if $f'(p) = 0$ and $f''(p) < 0$.

Solution:
It is $f''(x) > 0$.

19) [T] [F] The identity $(x^7 - 1)/(x - 1) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ holds for all $x \neq 1$.

Solution:
Multiply out to see it.

20) [T] [F] The slope of the tangent at a point $(x, f(x))$ of the graph of a differentiable function f is equal to $1/f'(x)$.

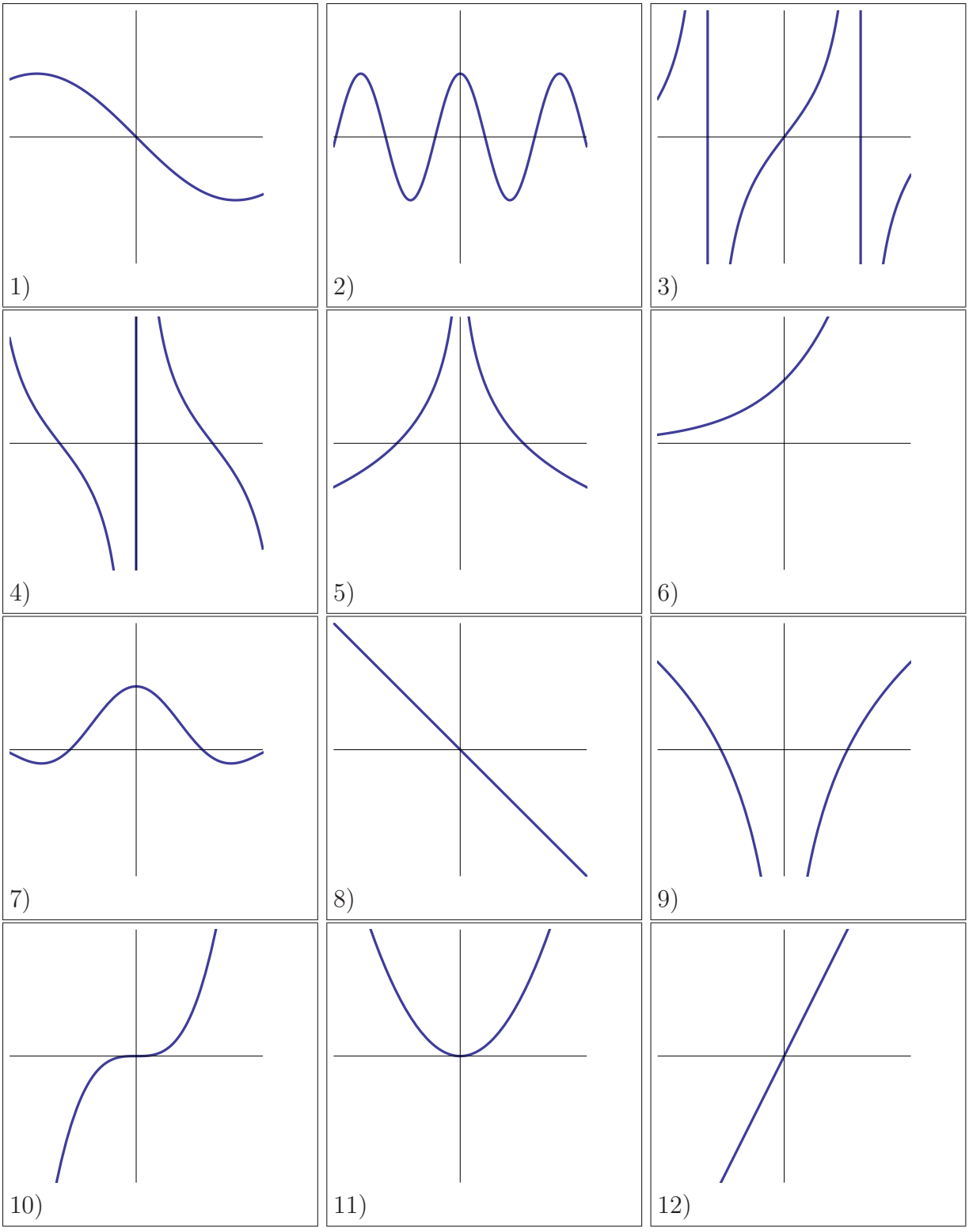
Solution:
The slope is $f'(x)$ not $1/f'(x)$.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the functions with the graphs. Naturally, only 10 of the 12 graphs will appear.

Function	Enter 1-12
$\cot(x)$	
$\cos(2x)$	
$2x$	
$\tan(x)$	
$\log(1/\lvert x\rvert)$	

Function	Enter 1-12
x^2	
$\exp(x)$	
$-\sin(x)$	
x^3	
$\operatorname{sinc}(x)$	



Solution:

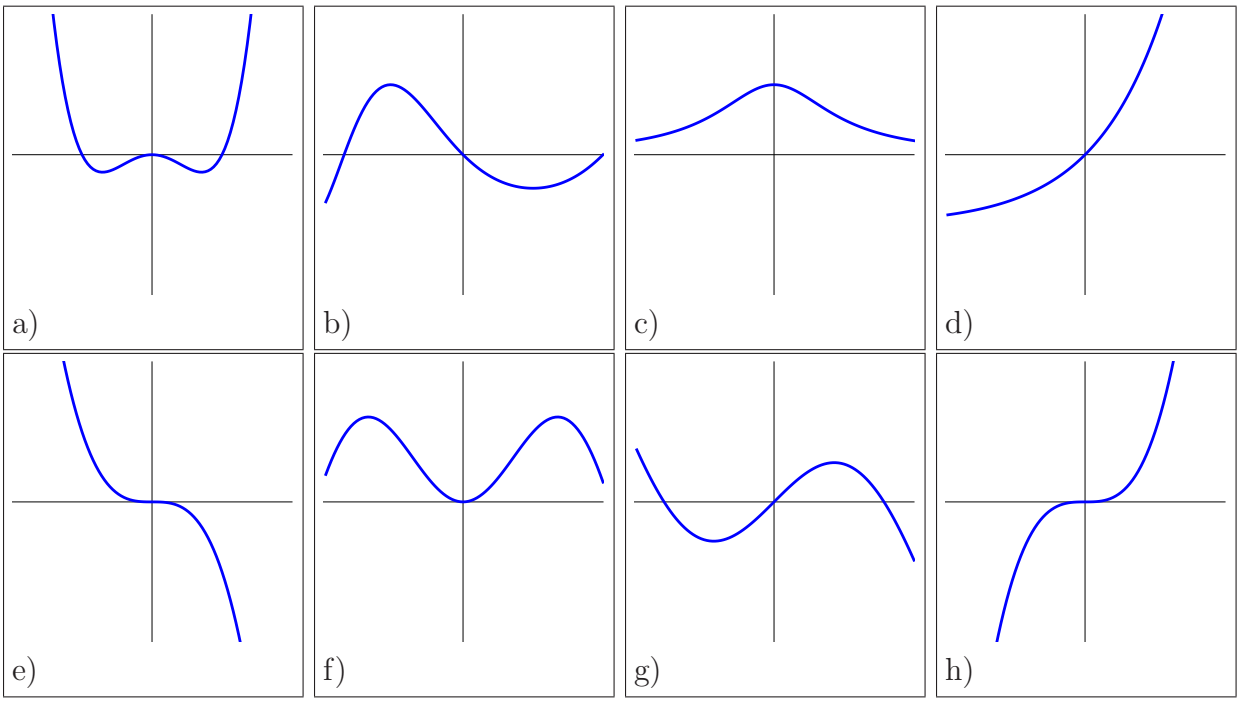
Function	Enter 1-12
$\cot(x)$	4
$\cos(2x)$	2
$2x$	12
$\tan(x)$	3
$\log(1/\lvert x\rvert)$	5

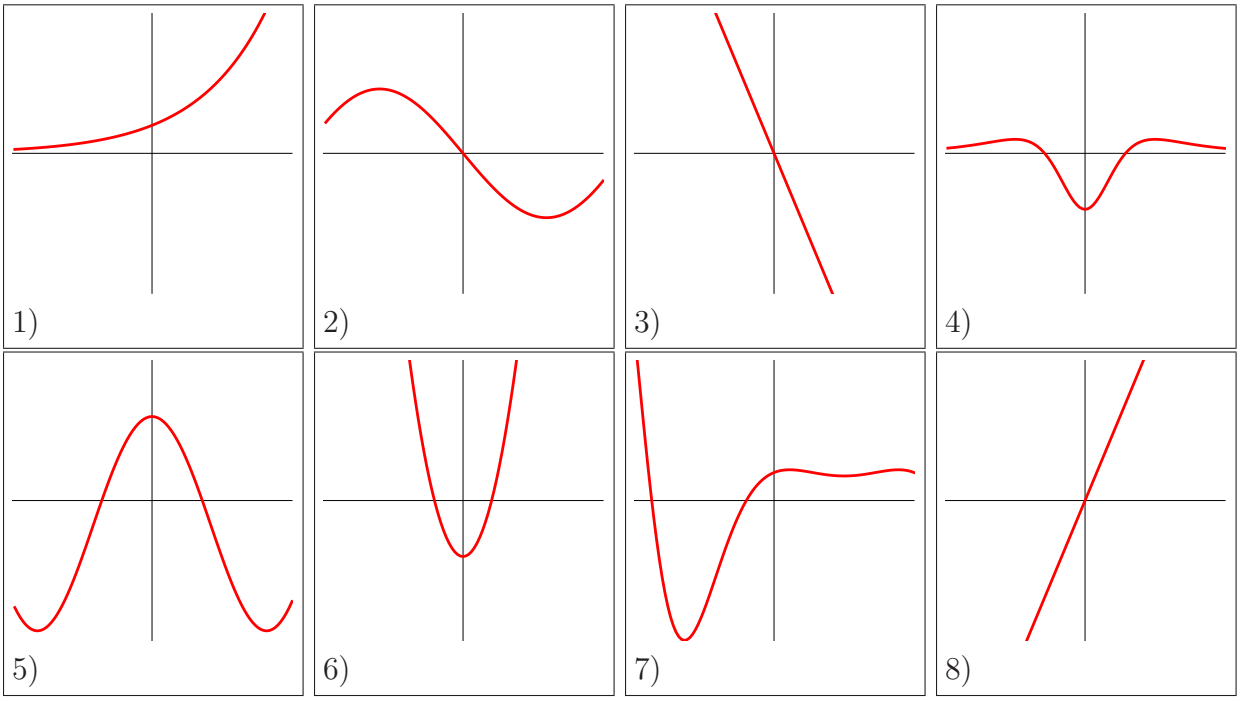
Function	Enter 1-12
x^2	11
$\exp(x)$	6
$-\sin(x)$	1
x^3	10
$\mathrm{sinc}(x)$	7

Problem 3) Matching problem (10 points) No justifications are needed.

Match the graph of the functions f in $a) - h)$ with the second derivatives f'' in 1)-8).

Function	Second derivative (Enter 1- 8 here)
a)	
b)	
c)	
d)	
e)	
f)	
g)	
h)	





Solution:

Function	Solution
a)	6 or 4
b)	7
c)	4 or 6
d)	1
e)	3
f)	5
g)	2
h)	8

Both 6,7,4 and 4,7,6 are possible.

Problem 4) Continuity (10 points)

Some of the following functions might a priori not be defined yet at the point a. In each case, decide whether f can be made a continuous function by assigning a value $f(a)$ at the point a. If no such value exist, state that the function is not continuous.

a) (2 points) $f(x) = \frac{(x^3-1)}{(x-1)}$, at $x = 1$

b) (2 points) $f(x) = \sin(\frac{1}{x}) + \cos(x)$, at $x = 0$

c) (2 points) $f(x) = \sin(\frac{1}{\log(|x|)})$, at $x = 0$

d) (2 points) $f(x) = \log(|\sin(x)|)$, at $x = 0$

e) (2 points) $f(x) = \frac{(x-1)}{x}$, at $x = 0$

Solution:
a) Heal the function by dividing out $(x-1)$. For $x \neq 1$ we get $x^2 + x + 1$. At $x = 0$ we have $\boxed{3}$.
b) The function contains the prototype $\sin(1/x)$ function, which has $\boxed{\text{no limit}}$ at $x = 0$.
c) We had seen in class that $\lim_{x\to 0} 1/\log|x| = 0$ because $\log|x| \to -\infty$. Therefore $\sin(1/\log|x|) \to 0$. Assigning the value $f(0) = 0$ makes the function continuous.
d) The function can is hopelessly discontinuous at $x = 0$. For $|x| \to 0$ we have $\sin(x) \to 0$ and $\log|\sin(x)| \to -\infty$.
e) We can write this as $1 - 1/x$. This is a prototype case $1/x$ where the function converges to ∞.

Problem 5) Chain rule (10 points)

a) (2 points) Write $1 + \cot^2(x)$ as an expression which only involves the function $\sin(x)$.

b) (3 points) Find the derivative of the function $\text{arccot}(x)$ by using the chain rule for

$$\cot(\text{arccot}(x)) = x \ .$$

c) (2 points) Write $1 + \tan^2(x)$ as an expression which only involves the function $\cos(x)$.

d) (3 points) Find the derivative of the function $\arctan(x)$ by using the chain rule for

$$\tan(\arctan(x)) = x \ .$$

Remark: even if you should know the derivatives of arccot or arctan, we want to see the derivations in b) and d).

Solution:
We have done a),b) in homework problem 3) of Lecture 10.
a) $1 + \cot^2(x) = 1 + \cos^2(x)/\sin^2(x) = (\sin^2(x) + \cos^2(x))/\sin^2(x) = 1/\sin^2(x)$.
b) $\cot'(x) = -1/\sin^2(x) = 1 + \cot^2(x)$ implies $(1 + \cot^2(\text{arccot}(x)))\text{arccot}'(x) = 1$ and so $\text{arccot}'(x) = -1/(1 + x^2)$.
We have done c),d) in class (on the side blackboard).
c) $1 + \tan^2(x) = 1 + \sin^2(x)/\cos^2(x) = 1/\cos^2(x)$.
d) $d/dx \cot(\arctan(x)) = (1/\cos^2(\arctan(x)))\arctan'(x) = (1 + \tan^2(\arctan(x)))\arctan'(x) = 1$. Therefore, $\arctan'(x) = 1/(1 + x^2)$.
We will come back to this later in the course.

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \frac{\cos(3x)}{\cos(x)}$

b) (2 points) $f(x) = \sin^2(x)\log(1+x^2)$

c) (2 points) $f(x) = 5x^4 - \frac{1}{x^2+1}$

d) (2 points) $f(x) = \tan(x) + \exp(-\sin(x^2))$

e) (2 points) $f(x) = \frac{x^3}{(1+x^2)}$

Solution:
a) Use the quotient rule $[-3\sin(3x)\cos(x) + \sin(x)\cos(3x)]/\cos^2(x)$.
b) Use the chain rule and the product rule $2\sin(x)\cos(x)\log(1+x^2) + 2x\sin^2(x)/(1+x^2)$.
c) Use the quotient and chain rule for the second summand $20x^3 + (2x)/(x^2+1)^2$.
d) The second sum uses the chain rule twice $1/\cos^2(x) + e^{-\sin(x^2)}(-\cos(x^2))2x$.
e) Use the quotient rule $(3x^2(1+x^2) - x^3(2x))/(1+x^2)^2$.

Problem 7) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions f at $x = 0$ or state (providing reasoning as usual) that the limit does not exist.

a) (2 points) $f(x) = \frac{\sin(3x)}{\sin(x)}$

b) (2 points) $f(x) = \frac{\sin^2(x)}{x^2}$

c) (2 points) $f(x) = \sin(\log(|x|))$

d) (2 points) $f(x) = \tan(x)\log(x)$

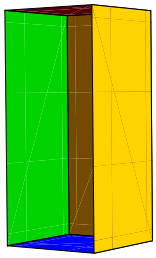
e) (2 points) $f(x) = \frac{(5x^4-1)}{(x^2+1)}$

Solution:
a) Apply l'Hopital once, to get the limit $\boxed{3}$.
b) $\boxed{1}$ since it is the square of $\sin(x)/x$ which has limit 1. One could also use l'Hopital twice.
c) There is $\boxed{\text{no limit}}$ because $log(|x|)$ goes to $-\infty$ and $\sin(\log|x|)$ oscillates indefinitely.
d) $\boxed{0}$ as we have done in class. Write as $[\sin(x)\log(x)]\cos(x)$ and $\cos(x)$ has no problem at $x = 0$. The limit $\sin(x)\log(x)$ is the same as $x\log(x)$ which we have done in class.
e) $\boxed{-1}$. There is no problem at this point because the nominator is not zero. We can just plug in $x = 0$ and get the value.

Problem 8) Extrema (10 points)

A rectangular shoe-box of width x, length x and height y is of volume 2 so that $x^2y = 2$. The surface area adds up three rectangular parts of size $(x \times y)$ and 2 square parts of size $(x \times x)$ and leads to

$$f = 2x^2 + 3xy \ .$$



a) (2 points) Write down the function $f(x)$ of the single variable x you want to minimize.

b) (6 points) Find the value of x for which the surface area is minimal.

c) (2 points) Check with the second derivative test, whether the point you found is a local minimum.

Solution:
a) Solve for $y = 2/x^2$ and substitute it into f. The function is $f(x) = 2x^2 + 6/x$.
b) $f'(x) = 4x - 6/x^2 = 0$ for $2x^3 = 3$ so that $x = (3/2)^{(1/3)}$.
c) $f''(x) = 4 + 12/x^3$. The second derivative is positive at the critical point. The critical point is a local minimum.

Problem 9) Global extrema (10 points)

In this problem we study the function $f(x) = 3x^5 - 5x^3$ on the interval $[-2, 2]$.

a) (2 points) Find all roots of f.

b) (3 points) Find all local extrema of the function.

c) (3 points) Use the second derivative test to analyze the critical points, where applicable.

d) (2 points) Find the **global** maximum and minimum of f on the interval $[-2, 2]$.

Solution:
a) The roots are $\boxed{0, 0, 0, -\sqrt{5/3}, \sqrt{5/3}}$ as you can see by factoring x^3 out. The root 0 is a triple root.
b) The derivative is $15x^4 - 15x^2 = 15x^2(x^2 - 1)$ which has roots at $\boxed{0 \text{ and } 1 \text{ and } -1}$. These are candidates for local extrema.
c) The second derivative is $30x(2x^2 - 1)$. At $x = 0$, the second derivative is zero. The second derivative test does not apply at this point. At $x = 1$, the second derivative is positive at $x = -1$ it is negative. $\boxed{x = 1 \text{ is a local min, and } x = -1 \text{ is a local max.}}$
d) We compare $(f(2), f(-2), f(1), f(-1), f(0)) = (56, -56, -2, 2, 0)$ to see that the global extrema are located at the boundary. The point $\boxed{2}$ is the global maximum and the point $\boxed{-2}$ is the global minimum.

Problem 10) Newton (10 points)

Perform one Newton step for the function $f(x) = x^5 - x$ starting at the point $x = 3$.

Solution:
The map is $T(x) = x - f(x)/f'(x) = x - (x^5 - z)/(5z^4 - 1)$. We have $T(3) = 3 - (3^5 - 3)/(5 \cdot 3^4 - 1) = 243/101$.

3/4/2014: First hourly Practice D

Your Name:

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- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] 1 is the only root of the log function on the interval $(0, \infty)$.

Solution:
Yes, log is monotone and has no other root.

2) [T] [F] $\exp(\log(5)) = 5$, if log is the natural log and $\exp(x) = e^x$ is the exponential function.

Solution:
Yes, exp is the inverse of log.

3) [T] [F] The function $\cos(x)+\sin(x)+x^2$ is continuous everywhere on the real axes.

Solution:
It is the sum of three functions for which we know it to be true.

4) [T] [F] The function $\sec(x) = 1/\cos(x)$ is the inverse of the function $\cos(x)$.

Solution:
No, it is $\arccos(x)$ which is the inverse.

5) [T] [F] The Newton method allows to find the roots of any continuous function.

Solution:
The function needs to be differentiable.

6) [T] [F] $\sin(3\pi/2) = -1$.

Solution:
Draw the circle. The angle $3\pi/2$ corresponds to 270 degrees. The sin is the y value and so -1.

7) [T] [F] If a function f is continuous on $[0, \infty)$, then it has a global maximum on this interval.

Solution:
$\exp(x)$ is a counter example. We would need a finite interval.

8) [T] [F] The reciprocal rule assures that $d/dx(1/g(x)) = -1/g(x)^2$.

Solution:
We have no g'

9) [T] [F] If $f(0) = g(0) = f'(0) = g'(0) = 0$ and $g''(0) = f''(0) = 1$, then $\lim_{x\to 0}(f(x)/g(x)) = 1$

Solution:
This is a consequence of l'Hopital's rule when applied twice.

10) [T] [F] An inflection point is a point, where the function $f''(x)$ changes sign.

Solution:
This is a definition.

11) [T] [F] If $f''(3) > 0$ then f is concave up at $x = 3$.

Solution:
The slope of the tangent increases which produces a concave up graph. One can define concave up with the property $f''(x) > 0$

12) [T] [F] The intermediate value theorem assures that a continuous function has a maximum on a finite interval.

Solution:
The intermediate value theorem deals with roots.

13) [T] [F] We can find a value b and define $f(1) = b$ such that the function $f(x) = (x^6 - 1)/(x^3 - 1)$ is continuous everywhere.

Solution:
We divide by zero at $z = 1$.

14) [T] [F] Single roots of the second derivative function f'' are inflection points.

Solution:
Indeed, f'' changes sign there.

15) [T] [F] If the second derivative $f''(x)$ is negative and $f'(x) = 0$ then f has a local maximum at x.

Solution:
This is part of the second derivative test

16) [T] [F] The function $f(x) = [x]^3 = x(x+h)(x+2h)$ satisfies $Df(x) = 3[x]^2 = 4x(x+h)$, where $Df(x) = [f(x+h) - f(x)]/h$.

Solution:
Yes, this is a cool property of the polynomials $[x]^n$ but only if $[x]^3 = x(x-h)(x-2h)$ is chosen.

17) [T] [F] The quotient rule is $d/dx(f/g) = (fg' - f'g)/g^2$.

Solution:
This is an important rule to know but the sign is off!

18) [T] [F] The chain rule assures that $d/dxf(g(x)) = f'(g(x))f'(x)$.

Solution:
This is not true. We have $f'(g(x))g'(x)$.

19) [T] [F] With $Df(x) = f(x+1) - f(x)$, we have $D2^x = 2^x$.

Solution:
So cool.

20) [T] [F] Hôpital's rule applied to the function $f(x) = \text{sinc}(x) = \sin(x)/x$ gives us the fundamental theorem of trigonometry.

Solution:
Yes, this is the ultimate way to verify that.

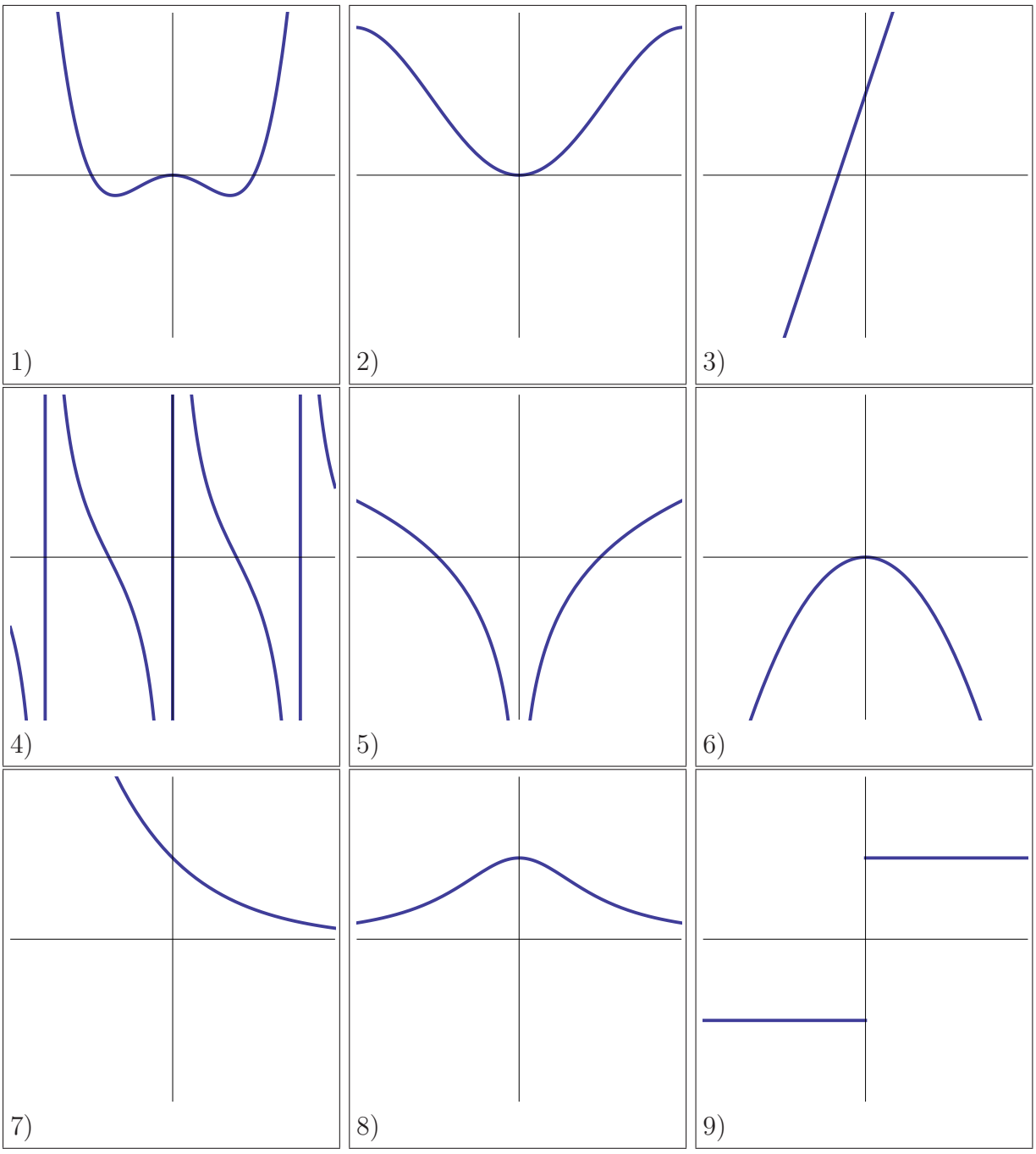
Problem 2) Matching problem (10 points) No justifications are needed.

Match the functions with the graphs.

Function	Enter 1-9
$1/(1+x^2)$	
$\cot(2x)$	
$3x+1$	

Function	Enter 1-9
$x\sin(x)$	
$\exp(-x)$	
$\log(\lvert x\rvert)$	

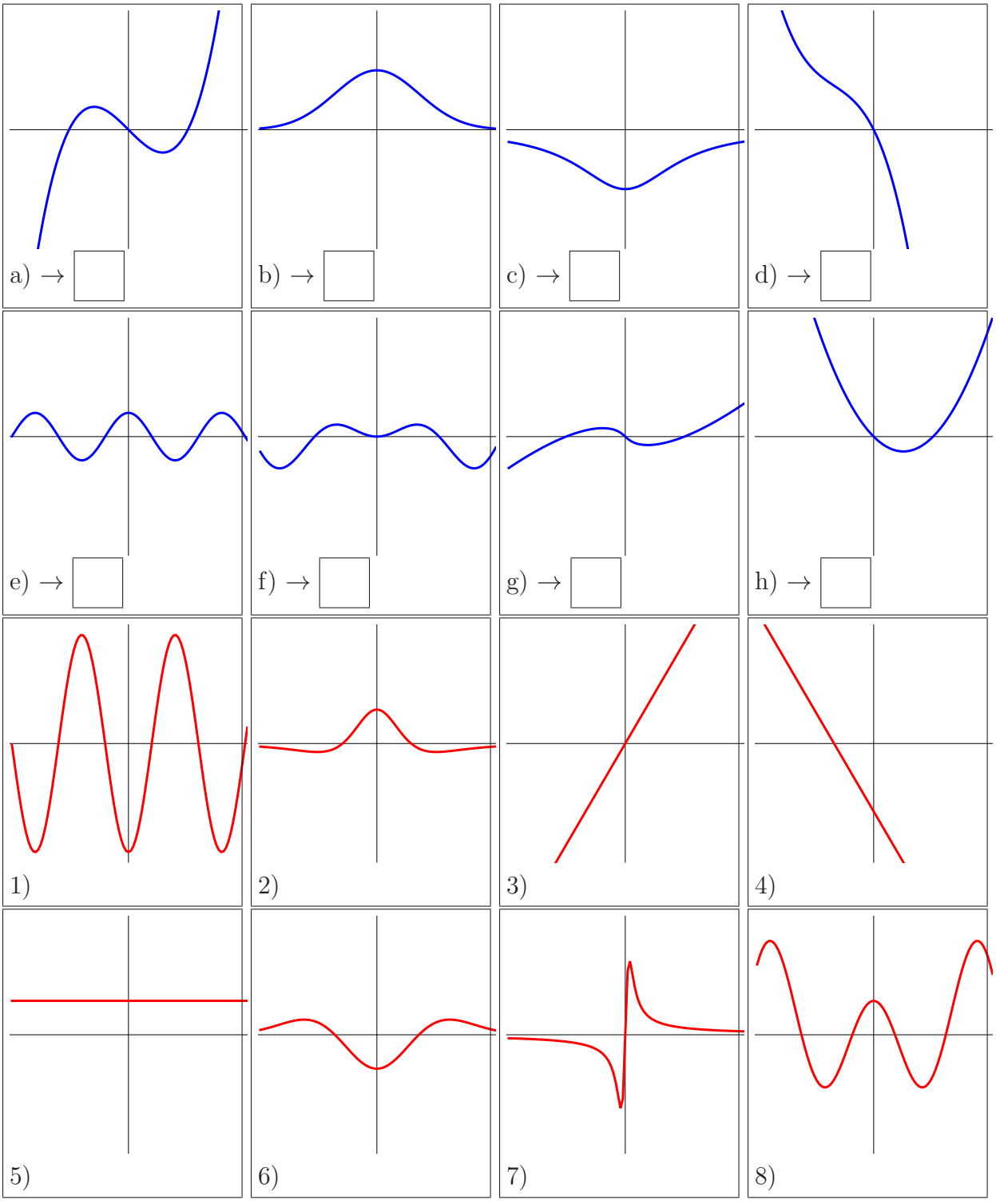
Function	Enter 1-9
$\operatorname{sign}(x)$	
x^4-x^2	
$-x^2$	



Solution:
8) 2) 9)
4) 7) 1)
3) 5) 6)

Problem 3) Matching problem (10 points) No justifications are needed.

Match the graph of the functions f in $a) - h)$ with the second derivatives f'' in 1)-8).



Solution:
3) 6) 2) 4)
1) 8) 7) 5)

Problem 4) Continuity (10 points)

Decide whether the function can be healed at the given point in order to be continuous everywhere on the real line. If the function can be extended to a continuous function, give the value at the point.

a) (2 points) $f(x) = \frac{(x^3-8)}{(x-2)}$, at $x = 2$

b) (2 points) $f(x) = \sin(\sin(1/x)) - \tan(x)$, at $x = 0$

c) (2 points) $f(x) = \frac{\cos(x)-1}{x^2}$, at $x = 0$

d) (2 points) $f(x) = (\exp(x) - 1)/(\exp(5x) - 1)$, at $x = 0$

e) (2 points) $f(x) = \frac{(x-1)}{x}$, at $x = 0$

Solution:
a) We can use Hôpital's rule to see that the limit is $\lim_{x\to 2} 3x^2/1 = 12$.
b) There is no way that we can save the oscillatory singularity.
c) Apply Hopital twice to see that the limit is $-1/2$.
d) Apply l'Hopital to see that the limit is $\lim_{x\to 0} e^x/(5e^{5x}) = 1/5$.
e) This can not be saved at $x = 0$. There is a pole there.

Problem 5) Chain rule (10 points)

In the following cases, we pretend not to know the formula for the derivative of log or arctan and again recover it using the chain rule.

b) (2 points) Rederive the derivative of the square root function $\text{sqrt}(x) = \sqrt{x}$ by differentiating

$$(\text{sqrt}(x))^2 = x$$

and solving for $\text{sqrt}'(x)$.

b) (4 points) Rederive the derivative of the logarithm function $\log(x)$ by differentiating

$$\exp(\log(x)) = x$$

and solving for $\log'(x)$.

c) (4 points) Rederive the formula for the derivative of the arctan function $\arctan(x)$ by differentiating the identity

$$\tan(\arctan(x)) = x$$

and using $1 + \tan^2(x) = 1/\cos^2(x)$ to solve for $\arctan'(x)$.

Solution:
a) Differentiate $(\sqrt{x})^2 = x$ to get $2\sqrt{x}\frac{d}{dx}\sqrt{x} = 1$ so that $\frac{d}{dx}\sqrt{x} = 1/(2\sqrt{x})$.
b) Differentiate $\exp(\log(x)) = x$ to get $\exp(\log(x))\log'(x) = 1$ and solve for $\log'(x) = 1/\exp(\log(x)) = 1/x$.
c) Differentiate $\tan(\arctan(x)) = x$ to get $\sec^2(\arctan(x))\arctan'(x) = 1$ and use $\sec^2(x) = 1 + \tan^2(x)$ to see $\arctan'(x) = 1/(1 + x^2)$.

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \frac{5\sin(x^6)}{x}$ for $x > 0$

b) (2 points) $f(x) = \tan(x^2) + \cot(x^2)$ for $x > 0$

c) (2 points) $f(x) = \frac{1}{x} + \log(x^2)$ for $x > 0$

d) (2 points) $f(x) = x^6 + \sin(x^4)\log(x)$ for $x > 0$

e) (2 points) $f(x) = \log(\log(x))$ for $x > 1$

Solution:
Use the product, quotient and chain rule.
a) $30x^4\cos(x^6) - \frac{5\sin(x^6)}{x^2}$
b) $2x\sec^2(x^2) - 2x\csc^2(x^2)$
c) $\frac{2}{x} - \frac{1}{x^2}$.
d) $6x^5 + \frac{\sin(x^4)}{x} + 4x^3\log(x)\cos(x^4)$.
e) $\frac{1}{x\log(x)}$.

Problem 7) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions f at $x = 0$ or state that the limit does not

exist. State the tools you are using.

a) (2 points) $f(x) = x^2 + x + \sin(1 - \cos(x))$

b) (2 points) $f(x) = \frac{x^3}{\sin(x^3)}$

c) (2 points) $f(x) = x^3 / \sin(x)^2$

d) (2 points) $f(x) = x^3 + \text{sign}(x)$

e) (2 points) $f(x) = \cos(x^4) + \cos(\frac{1}{x})$

Solution:
a) There is no problem at all at $x = 0$. We can just plug in the value 0 and get $\boxed{0}$.
b) We know that $x/\sin(x)$ has the limit 1. Therefore the limit is $\boxed{1}$ too.
c) We can write this as $x(x^2/\sin^2(x))$. The right part has the limit 1 by the fundamental theorem of trigonometry. Therefore the limit is $\boxed{0}$.
d) This function has no limit at $x = 0$ because $\text{sign}(x)$ has a jump singularity at 0.
e) Also this function has no limit at $x = 0$ because $\cos(1/x)$ is a devil comb at 0 with an oscillatory discontinuity.

Problem 8) Extrema (10 points)

In the following problem you can ignore the story if you like and proceed straight go to the question:

Story: a cone shaped lamp designed in 1995 by **Verner Panton** needs to have volume $\pi r^2 h = \pi$ to be safe. To minimize the surface area $A = \pi r\sqrt{h^2 + r^2}$, we minimize the square A^2 and so $\pi^2 r^2(h^2 + r^2)$. From the volume assumption, we get $r^2 = 1/h$ so that we have to minimize $(\pi^2/h)(h^2 + 1/h) = \pi^2 f(h)$.

Which height h minimizes the function

$$f(h) = h + \frac{1}{h^2} \ ?$$

Use the second derivative test to check that you have a minimum.

Solution:
The derivative is $f'(h) = 1 - 2/h^3 = 0$. This is zero at $2 = h^3$ which means $\boxed{h = 2^{1/3}}$. The second derivative is $f''(h) = 6/h^4$ which is positive at h. We have a **local minimum**.

Problem 9) Global extrema (10 points)

An investment problem leads to the profit function

$$f(x) = x - 2x^2 + x^3 \ ,$$

where $x \in [0, 2]$. Find the local and global maxima and minima of f on this interval and use the second derivative test.

Solution:
First find the critical points: $f'(x) = 1 - 4x + 3x^2 = 0$ for $\boxed{x = 1/3}$ and $\boxed{x = 1}$. The second derivative is $f''(x) = -4 + 6x$ which is -2 for $x = 1/3$ and 2 for $x = 1$. Therefore, $1/3$ is a local maximum and 1 is a local minimum. In order to find the global extrema, we have to evaluate the function also at the boundary $0, 2$ and compare the values

$$f(1/3) = 3/27, f(1) = 0, f(2) = 2, f(0) = 0 \ .$$

$\boxed{\text{The global maximum is at } \boxed{x = 2}}$, $\boxed{\text{the global minimum are } x = 0 \text{ and } x = 1}$.

4/8/2014: Second midterm exam

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1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] For any continuous function f we have $\int_0^1 3f(t)\,dt = 3\int_0^1 f(t)\,dt$.

Solution:
Yes this is linearity.

2) [T] [F] For any continuous function $\int_0^3 f(t)\,dt = 3\int_0^1 f(t)\,dt$.

Solution:
Looks good but is nonsense.

3) [T] [F] For any continuous function $\int_0^1 1 - f(t)\,dt = 1 - (\int_0^1 f(t)\,dt)$.

Solution:
Because the integral over 1 can be computed directly.

4) [T] [F] The anti-derivative of $\tan(x)$ is $-\log(\cos(x)) + C$.

Solution:
Differentiate the right hand side to check.

5) [T] [F] The fundamental theorem of calculus implies that $\int_1^3 f'(x)\,dx = f(3) - f(1)$.

Solution:
Yes, this is it.

6) [T] [F] The integral $\pi\int_0^1 x^2\,dx$ gives the volume of a cone of height 1.

Solution:
Yes the area of a slice is $x^2\pi$.

7) [T] [F] The anti-derivative of $1/\cos^2(x)$ is $\tan(x)$.

Solution:
Check.

8) [T] [F] The function $F(x) = \int_0^x \tan(t^2)\, dt$ has the derivative $\tan(x^2)$.

Solution:
The first derivative of F is f.

9) [T] [F] If the area $A(r(t))$ of a disk changes in a constant rate, then the radius $r(t)$ changes in a constant rate.

Solution:
This is a related rates problem.

10) [T] [F] The identity $\frac{d}{dx}\int_1^2 \log(x)\, dx = \log(2) - \log(1)$ holds.

Solution:
We differentiate a constant.

11) [T] [F] If $xy = 3$ and $x'(t) = 1$ at $(3, 1)$ then $y' = 1$.

Solution:
This is a simple example of implicit derivatives.

12) [T] [F] If $f < 1$, then $\int_0^2 f(x)\, dx$ can be bigger than 1.

Solution:
Take $f(x) = 0.6$ for example.

13) [T] [F] An improper integral is an improperly defined definite indefinite integral.

Solution:
If you marked this true, you must have been properly drunk or behaved improperly.

14) [T] [F] The anti derivative $F(x)$ of $f(x)$ satisfies $F'(x) = f(x)$.

Solution:
This is the fundamental theorem of calculus

15) [T] [F] A parameter value c for which the number of minima are different for parameters smaller or larger than c is called a catastrophe.

Solution:
This is a definition.

16) [T] [F] If f is unbounded at 0, then $\int_0^1 f(x)\,dx$ is infinite.

Solution:
The function $\sqrt{x}$ was a counter example.

17) [T] [F] If $f(-1) = 0$ and $f(1) = 1$ then $f' = 2$ somewhere on $(-1, 1)$.

Solution:
This is close to the intermediate value theorem.

18) [T] [F] The anti-derivative of $\log(x)$ is $x\log(x) - x + C$, where log is the natural log.

Solution:
You might not have known this by heart, but you can check it!

19) [T] [F] The sum $\frac{1}{n}[(\frac{0}{n})^2 + (\frac{1}{n})^2 + ... + (\frac{n-1}{n})^2]$ converges to $1/3$ in the limit $n \to \infty$.

Solution:
It is a Riemann sum.

20) [T] [F] The **improper integral** $\int_1^\infty \frac{1}{x^2}\,dx$ represents a finite area.

Solution:
We have no problem at infinity.

4/8/2014: Second midterm practice A

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4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] If f is a continuous function then $\int_0^x f(t)\ dt$ is an area and therefore positive.

Solution:
Take $f(x) = -x$ then $\int_0^x -t\ dt = -x^2/2$ is negative. Parts under the x axes contribute negatively to the integral.

2) [T] [F] The anti-derivative of $\operatorname{arccot}(x)$ is $-\log(\sin(x)) + C$.

Solution:
Differentiate the right hand side to check.

3) [T] [F] The fundamental theorem of calculus implies that $\int_0^3 f''(x)\ dx = f'(3) - f(0)$.

Solution:
Yes this is a special case of the fundamental theorem, if there were not the missing prime at the end. It had been noted on the blackboard that there is no typo in this problem.

4) [T] [F] The volume of a cylinder of height 3 and radius 5 is given by the integral $\int_0^3 \pi 5^2\ dx$.

Solution:
Yes the area of a slice is $r^2\pi$.

5) [T] [F] The antiderivative of $\tan(x)$ is $1/\cos^2(x)$.

Solution:
The derivative of $\tan(x)$ is $1/\cos(x)^2$.

6) [T] [F] The mean value theorem implies that the derivative of $\sin(x)$ in the interval $[0, \pi/2]$ is $2/\pi$ somewhere.

Solution:
This is a typical application of the mean value theorem.

7) [T] [F] The function $F(x) = \int_0^x \sin(t^2)\, dt$ has the derivative $\sin(x^2)$.

Solution:
The first derivative of F is f.

8) [T] [F] The level of wine in a parabolic glass changes with a constant rate if the volume decreases in a constant rate.

Solution:
This is a related rates problem. The balloon radius grows slower for large volumes.

9) [T] [F] The identity $\frac{d}{dx} \int_0^1 \sin(x)\, dx = \sin(1)$ holds.

Solution:
We differentiate a constant. This was the most commonly wrongly checked problem.

10) [T] [F] If a solid is scaled by a factor 2 in all directions then its volume increases by a factor 8.

Solution:
Yes, the volume goes cubic.

11) [T] [F] If $x^2 - y^2 = 3$ and $x'(t) = 1$ at $(2, 1)$ then $y' = 1$.

Solution:
This is a simple example of related rates. But the result is off by a factor.

12) [T] [F] If $f(x)$ is smaller than $g(x)$ for all x, then $\int_0^1 f(x) - g(x)\, dx$ is negative.

Solution:
Yes, we can take the 7 constant outside the integral.

13) [T] [F] Every improper integral defines an infinite area.

Solution:
No, it can be finite.

14) [T] [F] The anti derivative of $f'(x)$ is equal to $f(x) + c$.

Solution:
Yes, taking anti derivatives cancels taking derivatives, up to a constant.

15) [T] [F] Catastrophes can explain why minima can change discontinuously.

Solution:
This is a definition.

16) [T] [F] If f is discontinuous at 0, then $\int_{-1}^{1} f(x)\,dx$ is infinite.

Solution:
Think about the sign function. The integral is finite.

17) [T] [F] If $f(-\infty) = 0$ and $f(\infty) = 1$ then $f' = 1$ somewhere on $(-\infty, \infty)$.

Solution:
No, the slope can be arbitrarily small. We have only to increase the value by 1 and all the space of the world to do that. This was probably the second most commonly wrongly checked TF problem in this exam.

18) [T] [F] The anti-derivative of $1/x$ is $\log(x) + C$, where log is the natural log.

Solution:
Yes, we know that, and should never,never,never,never forget!

19) [T] [F] A catastrophe is defined as a critical point of f which is a minimum.

Solution:
No, it deals with changes of critical points.

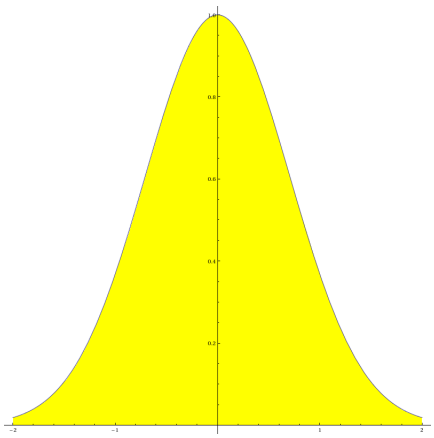
20) [T] [F] The integral $\int_0^\infty 1/x^2\,dx$ represents a finite area.

Solution:
We have no problem at infinity but a problem at $x = 0$.

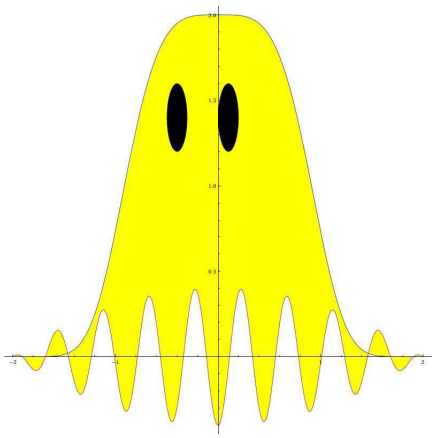
Problem 2) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the following integrals with the regions. Graphs 1) and 2) are inspired by a cartoon by Matthew Freeman (J Epidemiol. Community Health. 2006 January; 60(1): 6)

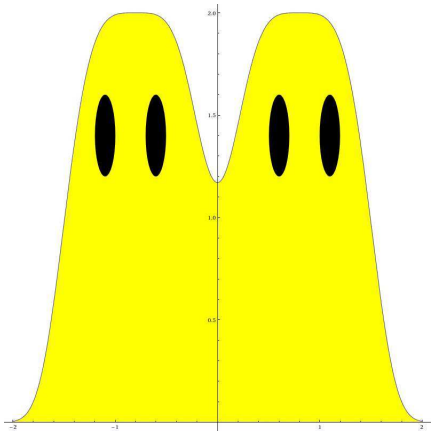
Integral	Fill in 1-4
$\int_{-2}^{2}(4-x^2)\cos^2(14x)/10-(4-x^2)\cos(14x)/15dx$	
$\int_{-2}^{2} 2\exp(-3(x+0.8)^4)+2\exp(-3(x-0.8)^4\ dx$	
$\int_{-2}^{2}\exp(-x^2)\ dx$	
$\int_{-2}^{2} 2\exp(-x^4)-(x^2-4)\cos(14x)/10\ dx$	



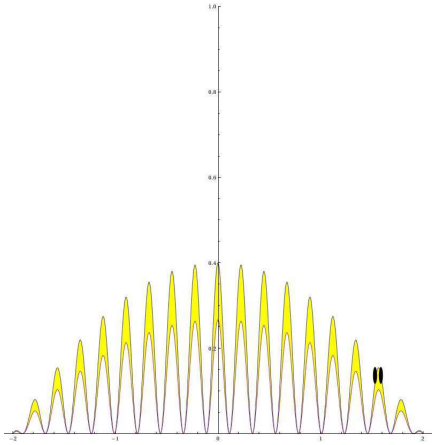
1) Normal distribution



2) Paranormal distribution



3) Abnormal distribution



4) Wormal distribution

b) (4 points) Which of the following statements follows from Rolle's theorem? Check only one.

Result	Check
If $f(0)=-1$ and $f(1)=1$ then there is x with $0\le x\le 1$ with $f'(x)=2$	
If $f(0)=1$ and $f(1)=1$ then there is a critical point x of f in $(0,1)$	
If $f(0)=1$ and $f(1)=1$ then there is point where $f(x)=2$ in $(0,1)$	
If $f(0)=1$ and $f(1)=1$ then there is point where $f''(p)=0$ in $(0,1)$	

Solution:
a) 4,3,1,2.
b) second choice.

Problem 3) (10 points)

a) (4 points) Having seen some applications of integration and differentiation, complete the table:

Function f	Antiderivative F
Probability density function	
	Total cost
	Mass
Area	
	Velocity
Power	
Velocity	

b) (2 points) We have seen two methods to find roots $f(x) = 0$ of equations. Both methods need some assumptions on the functions: Choose from the following: "differentiability", "continuity", "positivity".

Method	Assumption which f has to satisfy
Disection method	
Newton method	

c) (2 points) Which is more general? In each row, check one box.

	Related rates	Implicit differentiation	
	Rolles theorem	Intermediate value theorem	

d) (2 points) Which integral is finite? Chose one!

Integral	finite	infinite	
$\int_1^\infty 1/\sqrt{x}\, dx$			
$\int_1^\infty 1/x^2\, dx$			

Solution:

Function f	Antiderivative F
Probability density function	CDF
Marginal cost	Total cost
Density	Mass
Area	Volume
Acceleration	Velocity
Power	Work or Energy
Velocity	Position

b) (2 points) We have seen two methods to find roots $f(x) = 0$ of equations. Both methods need some assumptions on the functions: Choose from the following: "differentiability", "continuity", "positivity".

Method	Assumption which f has to satisfy
Disection method	Continuity
Newton method	Differnentiability

c) (2 points) Which is more general? In each row, check one box.

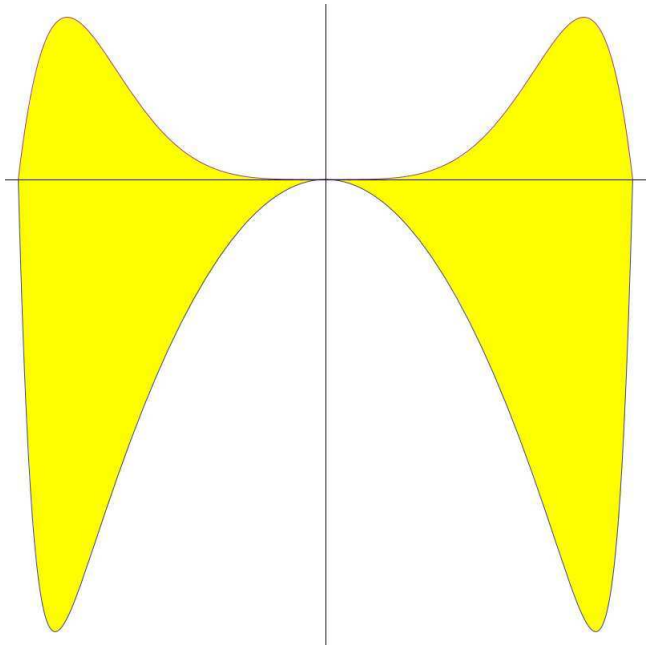
X	Related rates	Implicit differentiation	
	Rolles theorem	Intermediate value theorem	X

d) (2 points) Which integral is finite? Chose one!

Integral	finite	infinite
$\int_1^\infty 1/\sqrt{x}\, dx$		X
$\int_1^\infty 1/x^2\, dx$	X	

Problem 4) Area computation (10 points)

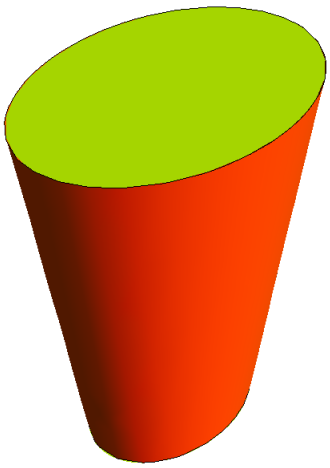
The region enclosed by the graphs of $f(x) = x^{20} - x^2$ and $g(x) = x^4 - x^8$ is a cross section for a catamaran sailing boat. Find the area.



Solution:
Note that $x^4 - x^8$ is positive on $[-1, 1]$ an d $x^{20} - x^2$ is negative:

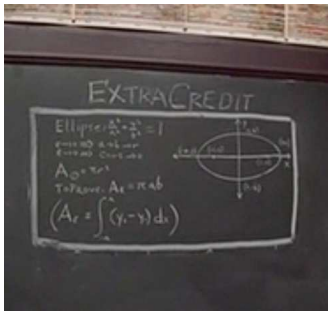
$$\int_{-1}^{1} x^4 - x^8 - (x^{20} - x^2)\, dx = 8/45 + 4/7 = 236/315 \; .$$

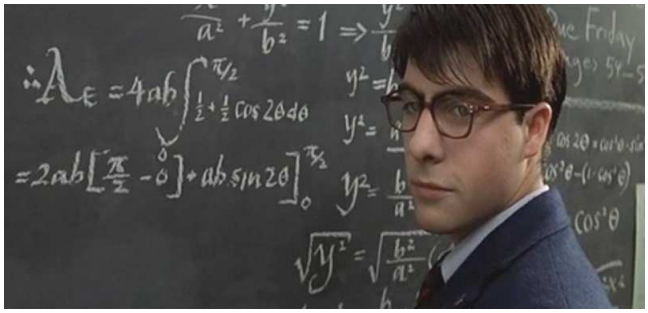
Problem 5) Volume computation (10 points)



An ellipse with diameters $2b$ and $2a$ has area πab. Find the volume of part of a cone whose height is between $z = 3$ and $z = 5$ for which the cross section at height z is an ellipse with parameters $a = 2z$ and $b = 3z$.

Remark. We will see later the area formula. In the movie "Rushmore", the teacher tells about the problem: "I put that up as a joke. It's probably the hardest geometry equation in the world".





Screen shots from the movie Rushmore shows a blackboard where the formula for the ellipse is computed using trig substitution. You might spot a double angle formula. We will come to that.

Solution:
The area at height z is $\pi 6z^2$. The answer is $\int_3^5 \pi 6z^2\, dx = 2\pi z^3|_3^5 = 196\pi$.

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. Each of the problems produces a numerical answer.

a) (2 points) $\int_0^1 (x-1)^4\, dx$

b) (2 points) $\int_0^1 x^{1/3}\, dx$.

c) (2 points) $\int_0^{\sqrt{3}} \frac{6}{1+x^2}\, dx$

d) (2 points) $\int_{-2}^{e-3} \frac{5}{3+x}\, dx$

e) (2 points) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$.

Solution:
a) 5^{-1}
b) $3/4$
c) 2π
d) 5
e) $\pi/2$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (2 points) $\int e^{7x} - \sqrt{x}\, dx$

b) (2 points) $\int \frac{5}{x+1} + 7\cos^2(x)\, dx$

c) (2 points) $\int \frac{11}{1+x^2} + 9\tan(x)\, dx$

d) (2 points) $\int \frac{4}{\cos^2(x)} + \frac{2}{\sin^2(x)}\, dx$

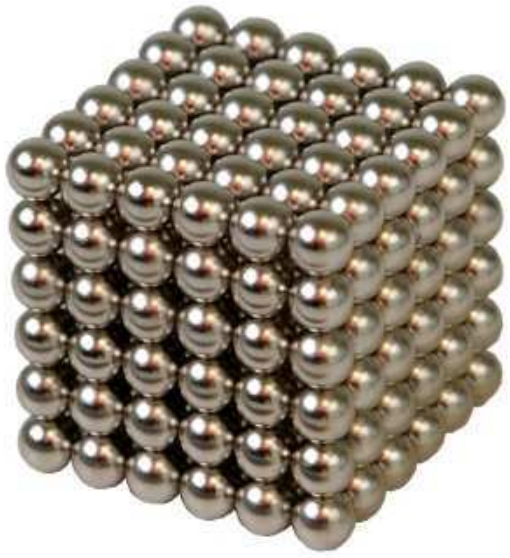
e) (2 points) $\int 2x\cos(x^2)\, dx$

Solution:
a) $\exp(7x)/7 - 2x^{3/2}/3 + C$.
b) $7x/2 + 5\log(1+x) + 7\sin(2x)/4 + C$.
c) $11\arctan(x) - 9\log(\cos(x)) + C$.
d) $4\tan(x) - 2\cot(x) + C$.
e) $\sin(x^2) + C$.

Problem 8) Implicit differentiation and related rates (10 points)

a) (5 points) Find the slope y' of the curve $x^2y = \sin(xy) + (y-1)$ at $x = \pi/2, y = 1$.

b) (5 points) A magnetic Neodym metal cube of length x is heated and changes the volume in time at a rate $V' = 1$. At which rate does the length $x(t)$ of the cube change, when the volume is $V = 27$?



Neodym magnets. Soon outlawed since kids can swallow them, leading to a change of topology of their intestines. Dangerous stuff! Gun bullets can be obtained more easily, naturally because they can not be swallowed ...

Solution:
a) $2xy + x^2y' = \cos(xy)(x'y + xy') + 1$ gives $\pi + \pi/4y' = y'$ so that $y' = \pi/(\pi^2/4 - 1)$ which is equivelent to $y' = 4\pi/(4 - \pi^2)$.
b) Since $V(x(t)) = x(t)^3$ we have $1 = V' = 3x^2x'$ so that $x' = 1/(3x^2) = 1/27$.
P.S. There is now even a recall of some magnets:
en.wikipedia.org/wiki/Neodymium_magnet_toys.

Problem 9) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = x^6 - cx^4 - cx^2$, where c is a parameter.

a) (4 points) Verify that f has a critical point 0 for all c.

b) (3 points) Determine whether 0 is a minimum or maximum depending on c.

c) (3 points) For which c does a catastrophe occur?

Solution:
a) Differentiate to see that $f' = 6x^5 - 4x^3c - 2xc$ has critical points at $x = 0$.
b) The second derivative is $f''(x) = 30x^4 - 12x^2c - 2c$ which is $f''(0) = -2c$. We see that for $c < 0$ we have a minimum and for $c > 0$ we have a maximum.
c) $c = 0$ is the catastrophe, because for this parameter the number a minimum becomes a maximum.

Problem 10) Basic integrals (10 points)

Find the anti derivatives. You have to solve in 10 seconds each. For every second over that limit, one point of the entire exam will be taken off. So, for example: if you use 62 seconds for the following 5 problems, you have used 12 seconds too much and 12 points are taken off from your exam. Don't worry, we do not assign negative points; your final score will always remain a number between 0 and 110 points. To fill a loophole in that setup: if you choose not do the problems, 50 points are taken off.

a) (2 points) e^{-2x}.
b) (2 points) $\cos(15x)$.
c) (2 points) 2^x.
d) (2 points) $1/(1 - x)$
e) (2 points) $1/(1 + x^2)$

Solution:
The optimal strategy is of course to write down something within 10 seconds, even if you are not sure. But you should not have to think: these are all integrals you have to be able to do in your sleep:
a) $e^{-2x}/(-2) + c$,
b) $\sin(15x)/15 + c$,
c) Since this is $e^{x \log(2)}$, the answer is $2^x / \log(2) + c$
d) $-\log(1-x) + c$.
e) $\arctan(x) + c$.

4/8/2014: Second midterm practice B

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] The anti-derivative of $\tan(x)$ is $-\log(\cos(x)) + C$.

Solution:
Differentiate the right hand side to check.

2) [T] [F] The fundamental theorem of calculus implies that $\int_0^1 f'(x)\ dx = f(1) - f(0)$.

Solution:
Yes this is a special case of the fundamental theorem.

3) [T] [F] The volume of truncated pyramid with a base square length 2 and top square length 3 is given by the integral $\int_2^3 x^2\ dx$.

Solution:
Yes the area of a slice is x^2.

4) [T] [F] The derivative of $\arctan(x)$ is $1/\cos^2(x)$.

Solution:
The derivative of $\tan(x)$ is $1/\cos(x)^2$.

5) [T] [F] The mean value theorem implies $\int_a^b f'(x)\ dx = f'(c)(b-a)$ for some c in the interval (a, b).

Solution:
This is a typical application of the mean value theorem.

6) [T] [F] If $F(x) = \int_0^x f(t)\ dt$ has an critical point at $x = 1$ then f has a root at $x = 1$.

Solution:
The first derivative of F is f.

7) [T] [F] The anti-derivative of the derivative of f is equal to $f + C$ where C is a constant.

Solution:
This is a consequence of the fundamental theorem.

8) [T] [F] If we blow up a balloon so that the volume V changes with constant rate, then the radius $r(t)$ changes with constant rate.

Solution:
This is a related rates problem. The balloon radius grows slower for large volumes.

9) [T] [F] The identity $\frac{d}{dx}\int_5^9 f(x)\,dx = f(9) - f(5)$ holds for all continuous functions f.

Solution:
We differentiate a constant.

10) [T] [F] Two surfaces of revolution which have the same cross section area $A(x)$ also have the same volume.

Solution:
This is Archimedes insight and true.

11) [T] [F] If $x^2 + y^2 = 2$ and $x(t), y(t)$ depend on time and $x' = 1$ at $x = 1$ then $y' = -1$ is possible.

Solution:
This is a simple example of related rates.

12) [T] [F] The identity $\int_2^9 7f(x)\,dx = 7\int_2^9 f(x)\,dx$ is true for all continuous functions f.

Solution:
Yes, we can take the 7 constant outside the integral.

13) [T] [F] The improper integral $\int_1^\infty 1/x\,dx$ in the sense that $\int_1^R 1/x\,dx$ converges for $R \to \infty$ to a finite value.

Solution:
This is what we mean with the existence. But the integral does not exist.

14) [T] [F] If $f_c(x)$ has a local minimum at $x = 2$ for $c < 1$ and no local minimum anywhere for $c > 1$, then $c = 1$ is a catastrophe.

Solution:
This is a definition.

15) [T] [F] An improper integral is an indefinite integral which does not converge.

Solution:
These two terms are easy to mix up. Improper means that we either have a discontinuity of f or integrate over an infinite interval. Indefinite means that we do not specify bounds.

16) [T] [F] If $f(-5) = 0$ and $f(5) = 10$ then $f' = 1$ somewhere on the interval $[-5, 5]$.

Solution:
Yes this is the mean value theorem.

17) [T] [F] The sum $\frac{1}{n}\sum_{k=0}^{n-1} \frac{k}{n} = \frac{1}{n}[\frac{0}{n} + \frac{1}{n} + \cdots + \frac{n-1}{n}]$ is a Riemann sum to the integral $\int_0^1 x\, dx$.

Solution:
Yes, this is the Riemann sum.

18) [T] [F] The anti-derivative of $\text{sinc}(x) = \sin(x)/x$ is equal to $\sin(\log(x)) + C$.

Solution:
Differentiate the right hand side to see that this is not true

19) [T] [F] The anti-derivative of $\log(x)$ is $1/x + C$.

20) [T] [F] We have $\int_0^x tf(t)\, dt = x \int_0^x f(t)\, dt$ for all functions f.

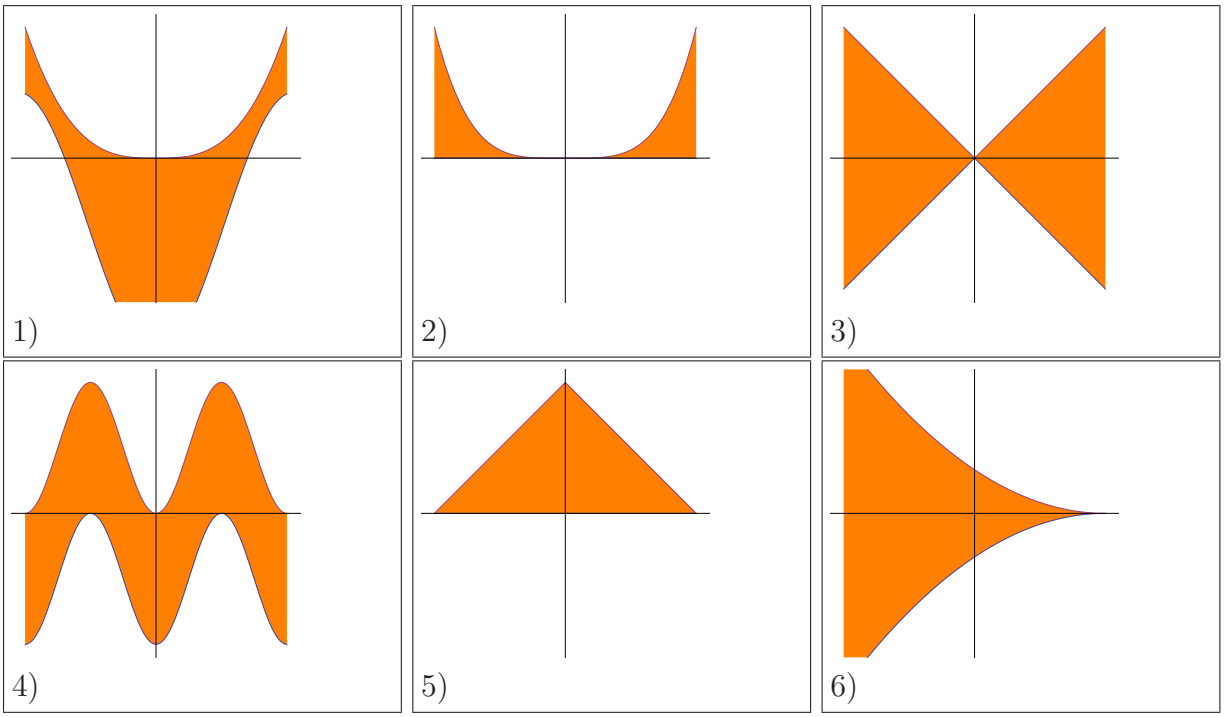
Solution:
It is already false for the constant function $f(t) = 1$.

Problem 2) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the integrals with the pictures.

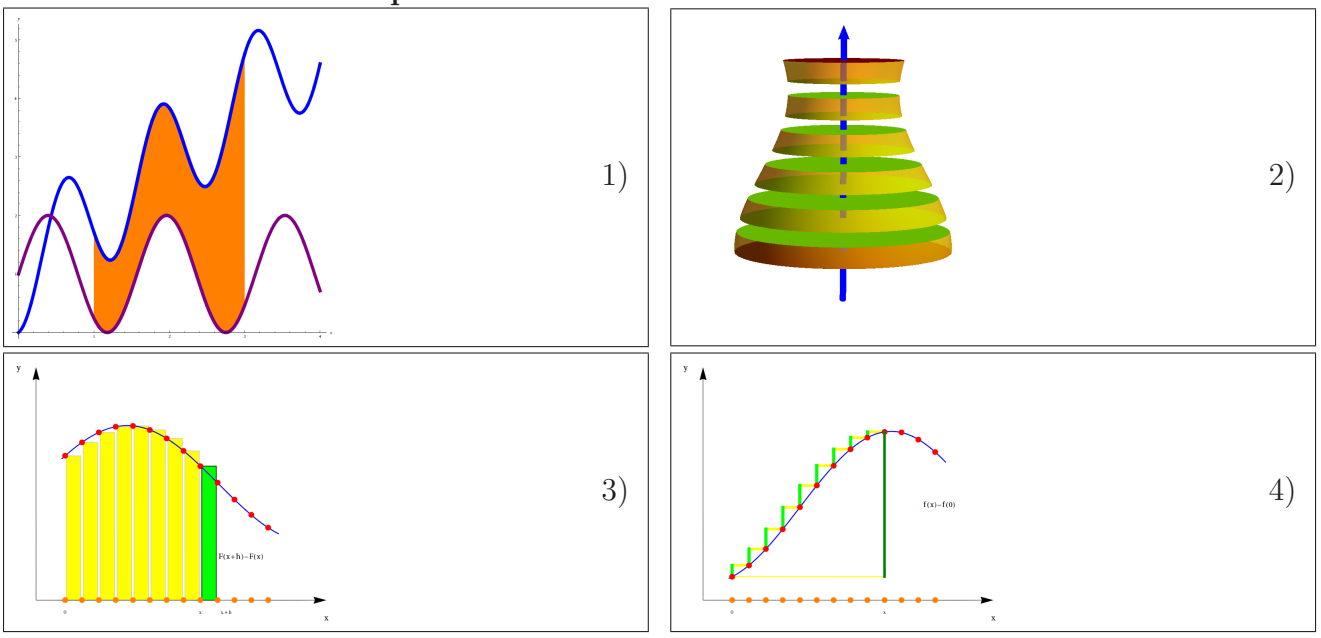
Integral	Enter 1-6
$\int_{-1}^{1}(1-x)^2\,dx$	
$\int_{-1}^{1}\lvert x\rvert\,dx$	
$\int_{-1}^{1}x^4\,dx$	

Integral	Enter 1-6
$\int_{-1}^{1}\lvert x\rvert^3-\cos(3x)\,dx$	
$\int_{-1}^{1}[\sin^2(\pi x)-\cos^2(\pi x)]\,dx$	
$\int_{-1}^{1}1-\lvert x\rvert\,dx$	



Solution:
6,3,2,1,4,5.

b) (4 points) Match the concepts: each of the 4 figures illustrates one of the formulas which are the centers of the **mind map** we have drawn for this exam:



Formula	Enter 1-4
$\int_a^b A(z)\,dz$	
$\int_a^b g(x) - f(x)\,dx$	

Formula	Enter 1-4
$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$	
$\int_0^x f'(t)\,dt = f(x) - f(0)$	

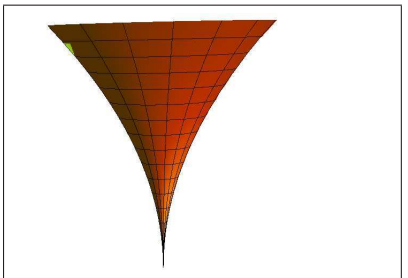
Solution:
2,1 and 3,4

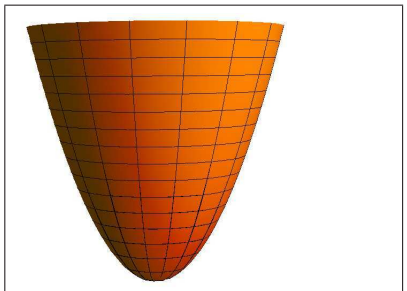
Problem 3) Matching problem (10 points) No justifications are needed.

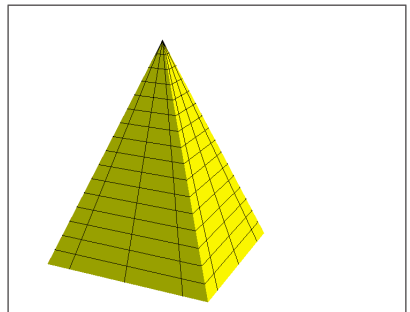
a) (6 points) Match the volumes of solids.

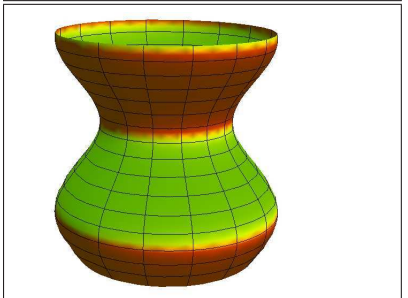
Integral	Enter 1-6
$\int_0^1 \pi z^4\,dz$	
$\int_0^1 \pi z\,dz$	
$\int_0^1 \pi(4+\sin(4z))\,dz$	

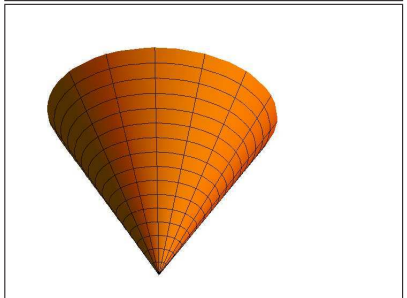
Integral	Enter 1-6
$\int_{-1}^1 \pi e^{-4z^2}\,dz$	
$\int_0^1 \pi z^2\,dz$	
$\int_0^1 (1-z)^2\,dz$	

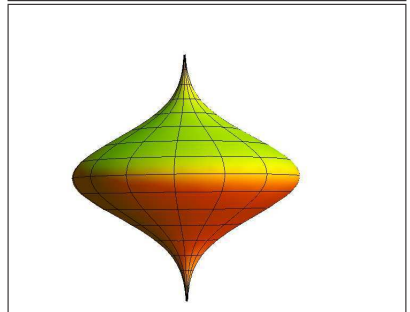

1)


2)


3)


4)


5)


6)

Solution:
1,2,4,6,5,3

b) (4 points) Fill in the missing word which links **applications** of integration.

The probability density function is the		of the cumulative distribution function.
The total cost is the		of the marginal cost.
The volume of a solid is the		of the cross section area function.
The velocity of a ball is the		of the acceleration of the ball.

Solution:
derivative, antiderivative, antiderivative, antiderivative

Problem 4) Area computation (10 points)

Find the area of the region enclosed the graphs of $y = x^4 - 12$ and $y = 8 - x^2$.

Solution:
The two curves intersect at $x = 2$ and $x = -2$.

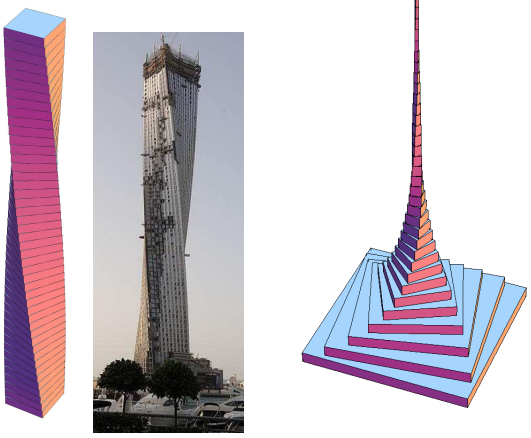
$$\int_{-2}^{2} 8 + x^2 - (x^4 - 12)\, dx = ... = \frac{928}{15}$$

Problem 5) Volume computation (10 points)

The **infinity tower** in Dubai of height 330 meters has floors which can rotate. After much delay, it is expected to be completed this year. Inspired by the name "infinity", we build a new but twisted science center for which the side length of the square floor is

$$l(z) = \frac{1}{1+z} .$$

Find the volume of this new **Harvard needle building** which extends from 0 to ∞. We are the best!



Solution:
The cross section area is $A(z) = 1/(1+z)^2$. The integral is

$$\int_0^\infty \frac{1}{(1+z)^2}\, dz = -\frac{1}{1+z}|_0^\infty = 1$$

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. You should get a definite real number in each case.

a) (2 points) $\int_0^\infty e^{-x}\, dx$

b) (3 points) $\int_0^1 x^{1/5} + x^3\, dx$.

c) (3 points) $\int_{-1}^1 \frac{1}{1+x^2}\, dx$

d) (2 points) $\int_0^{e-1} \frac{2}{1+x}\, dx$

Solution:
a) 1
b) 13/12
c) $\pi/2$
d) 2

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (2 points) $\int \frac{3}{\sqrt{1+3x}} + \cos(x)\, dx$

b) (3 points) $\int e^{x/5} - 7x^6 + \frac{4}{x^2+1}\, dx$

c) (2 points) $\int \frac{4}{e^{4x+5}} + 3\sin(x)\, dx$

d) (3 points) $\int \frac{1}{\sin^2(x)} + \frac{4}{x}\, dx$

Solution:
a) $2\sqrt{1+3x} + \sin(x) + C$.
b) $5e^{x/5} - x^7 + 4\arctan(x) + C$.
c) $-e^{-4x-5} - 3\cos(x) + C$.
d) $-\cot(x) + 4\log(x) + C$.

Problem 8) Implicit differentiation and related rates (10 points)

a) (5 points) The implicit equation

$$x^3 + y^4 = y + 1$$

defines a function $y = y(x)$ near $(x, y) = (-1, -1)$. Find the slope $y'(x)$ at $x = -1$.

b) (5 points) An ice cube of side length x melts and changes volume V with a rate $V' = -16$. What is the rate of change of the length x at $x = 4$?

Solution:
a) Differentiate the equation to get

$$3x^2x' + 4y^3y' = y'$$

and solve for y' using $x = -1, y = -1$ to get $\boxed{-3/5}$.
b) Differentiate $V = x^3$ to get $-16 = V' = 3x^2x'$ which gives for $x = 4$ the solution $x' = -16/(3 \cdot \cdot 4^2) = -1/3$.

Problem 9) Catastrophes (10 points)

Verify first for each of the following functions that $x = 0$ is a critical point. Then give a criterium for stability of $x = 0$. The answer will depend on c.

a) (3 points) $f(x) = x^5 + 2x^2 - cx^2$.

b) (3 points) $f(x) = x^4 + cx^2 - x^2$.

Determine now in both examples for which parameter c the catastrophe occurs

c) (2 points) in the case $f(x) = x^5 + 2x^2 - cx^2$.

d) (2 points) in the case $f(x) = x^4 + cx^2 - x^2$.

Solution:
a) $f'(x) = 5x^4+4x-2xc$ has $x = 0$ as a root. Its stability is determined by $f''(0) = 4-2c$.
b) $f'(x) = 4x^3+2cx-2x$ has $x = 0$ as a root. Its stability is determined by $f''(0) = 2c-2$.
c) For $c < 2$ the point $x = 0$ is a local minimum. For $c > 2$ it is a local maximum. $c = 2$ is a catastrophe.
d) For $c < 1$ the point $x = 0$ is a local minimum. For $c > 1$ it is a local maximum. $c = 1$ is a catastrophe.

4/8/2014: Second midterm practice C

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] The formula $\int_0^x f''(x)\,dx = f'(x) - f'(0)$ holds.

Solution:
Apply the fundamental theorem to the derivative.

2) [T] [F] The area of the upper half disc is the integral $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

Solution:
The circle has the equation $x^2 + y^2 = 1$. Solving for y gives $y = \sqrt{1-x^2}$.

3) [T] [F] If the graph of the function $f(x) = x^2$ is rotated around the interval $[0, 1]$ in the x axes we obtain a solid with volume $\int_0^1 \pi x^4\,dx$.

Solution:
Yes, the cross section area is $A(x) = \pi x^4$. Integrate this from 0 to 1.

4) [T] [F] The function $f(x) = e^x$ is the only anti derivative of e^x.

Solution:
We can add a constant c.

5) [T] [F] If f has a critical point at 1, then $F(x) = \int_0^x f(t)\,dt$ has an inflection point at 1.

Solution:
By the fundamental theorem of calculus, $F' = f$ and $F'' = f'$. If f has the critical point 1, then $f'(1) = 0$ and so $F''(1) = 0$ which by definition means that F has an inflection point. An inflection point for F is a point, where $F'' = f'$ changes sign. That means $f' = 0$.

6) [T] [F] Catastrophes are parameter values c for a family of functions $f_c(x)$, for which a local minimum of f_c disappears.

Solution:
This is the definition of a catastrophe.

7) [T] [F] The volume of a cylinder of height and radius 1 minus the volume of a cone of height and radius 1 is half the volume of a sphere of radius 1.

Solution:
This was Archimedes insight and you have discovered yourself in a homework.

8) [T] [F] Rolle's theorem tells that if $0 < c < 1$ is a critical point of f and $f(0) = f(1)$, then the critical point is in the interval $[0, 1]$.

Solution:
There could also be critical points outside the interval.

9) [T] [F] Rolle also introduced the notation $|x|^{1/3}$ for roots.

Solution:
Yes, this was mentioned in class and on the handout.

10) [T] [F] Integrals are linear: $\int_0^x f(t) + g(t)\, dt = \int_0^x f(t)\, dt + \int_0^x g(t)\, dt$.

Solution:
You have verified this in a homework.

11) [T] [F] The function $\mathrm{Li}(x) = \int_2^x dt/\log(t)$ has an anti-derivative which is a finite construct of trig functions.

Solution:
No, it is known that this logarithmic integral has no elementary anti-derivative.

12) [T] [F] There is a region enclosed by the graphs of x^5 and x^6 which is finite and positive.

Solution:
Yes, there is a finite region enclosed. It is between $x = 0$ and $x = 1$.

13) [T] [F] The integral $\int_{-1}^1 1/x^4\, dx = -1/(5x^5)|_{-1}^1 = -1/5 - 1/5 = -2/5$ is defined and negative.

Solution:
It is not defined at 0 and can not even be saved using the Cauchy principal value.

14) [T] [F] Gabriel's trumpet has finite volume but infinite surface area.

Solution:
Yes, we have seen this in class.

15) [T] [F] A function $f(x)$ is a probability density function, if $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Solution:
This is the definition of a probability density function.

16) [T] [F] The mean of a function on an interval $[a, b]$ is $\int_a^b f(x)\, dx$.

Solution:
For the mean, we would have to divide by $(b - a)$.

17) [T] [F] The cumulative probability density function is an antideriative of the probability density function.

Solution:
Yes, by definition. It is a particular antiderivative which has the property that $\lim_{x\to-\infty} F(x) = 0$.

18) [T] [F] The integral $\int_{-\infty}^{\infty}(x^2 - 1)\, dx$ is finite.

Solution:
No way. The function does not even go to zero at infinity. $\int_{-R}^{R}(x^2-1)\, dx = (x^3/3-x)|_{-R}^{R} = 2R^3/3 - 2R$ does not converge for $R \to \infty$.

19) [T] [F] The total prize is the derivative of the marginal prize.

Solution:
Its reversed. The marginal prize is the derivative of the total prize.

20) [T] [F] The acceleration is the anti-derivative of the velocity.

Solution:
It is reversed. The acceleration is the derivative of the velocity.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their anti derivatives. Of course only 6 of the 30 functions will appear.

Function	Antiderivative Enter 1-30
$\cos(3x)$	
$\sin(3x)$	
$3x$	

Function	Antiderivative Enter 1-30
$1/(3x)$	
$\tan(3x)$	
$1/(1+9x^2)$	

1) $\sin(3x)$
2) $-\sin(3x)/3$
3) $\sin(3x)/3$
4) $-3\sin(3x)$
5) $3\sin(3x)$
6) $\cos(3x)$
7) $-\cos(3x)/3$
8) $\cos(3x)/3$
9) $-3\cos(3x)$
10) $3\cos(3x)$
11) $\log(x)/3$
12) $1/(3-x)$
13) $1/(3x)$
14) $\log(x/3)$
15) $-1/(3x^2)$
16) $3x^2$
17) $x^2/2$
18) $3x^2/2$
19) 3
20) x^2
21) $\arctan(3x)/3$
22) $3\arctan(3x)$
23) $1/(1+9x^2)$
24) $3/(1+9x^2)$
25) $-3/(1+x^2)$
26) $1/\cos^2(3x)$
27) $\log(\cos(3x))$
28) $-\log(\cos(3x))/3$
29) $\log(\cos(3x))/3$
30) $3/\cos^3(3x)$

Solution:
The magic numbers are $3, 7, 18, 11, 28, 21$.

Problem 3) Matching problem (10 points) No justifications are needed.

Which of the following formulations is a Riemann sum approximating the integral $\int_0^3 f(x)\,dx$ of $f(x) = x^2$ over the interval $0, 3$.

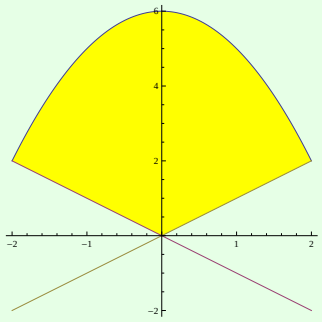
Sum	Check if this is the Riemann sum.
$n\sum_{k=0}^{n-1}(3k/n)^2$	
$\frac{1}{n}\sum_{k=0}^{n-1}(3k/n)^2$	
$n\sum_{k=0}^{3n-1}(k/n)^2$	
$\frac{1}{n}\sum_{k=0}^{3n-1}(k/n)^2$	

Solution:
It is the fourth entry.

Problem 4) Area computation (10 points)

Find the area of the region enclosed by the three curves $y = 6 - x^2, y = -x$ and $y = x$ which is above the x axes.

Solution:
Make a picture:



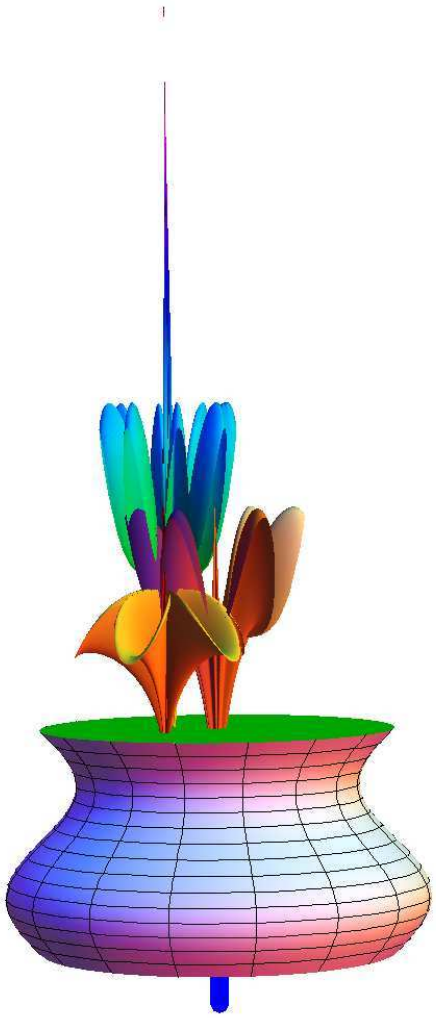
We can look at the region on the right of the y-axis and have therefore the area between the graph of $f(x) = x$ and $f(x) = 6 - x^2$. These graphs intersect at $x = 2$. The integral is

$$\int_0^2 (6 - x^2 - x)\, dx = 22/3 \; .$$

Since we have only computed half, we have to multiply this with 2. The area is 44/3.

Problem 5) Volume computation (10 points)

Emma R. grows magical plants in a pot which is a rotationally symmetric solid for which the radius at position x is $5 + \sin(x)$ and $0 \leq x \leq 2\pi$. Find the volume of the pot.



Solution:
The area of the cross section at hight x is $A(x) = \pi(5 + \sin(x))^2$. The volume is

$$\int_0^{2\pi} \pi(5 + \sin(x))^2 \, dx = \pi \int_0^{2\pi} 25 + 10\sin(x) + \sin^2(x) \, dx = (50\pi + \pi)\pi = 51\pi^2 \, .$$

To find the anti-derivative of $\sin^2(x)$ we use the **double angle formula** $1 - 2\sin^2(x) = \cos(2x)$ leading to $\sin^2(x) = [1 - \cos(2x)]/2$ so that $\int \sin^2(x) \, dx = x/2 - \sin(2x)/4$.

Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (5 points) $\int_1^2 x^{1/5} + x^4 + 1/x \, dx$.

b) (5 points) $\int_1^3 2x + \sin(x-1) + \cos(x+2)\, dx$

Solution:
a) We get $((5/6)x^{6/5} + x^5/5 + \log(x)|_1^2 = 161/30 + (5/3)2^{1/5} + \log(2)$.
b) We get $x^2 - \cos(x-1) + \sin(x+2)|_1^3 = 9 - \cos(2) - \sin(3) + \sin(5)$.

Problem 7) Anti-derivatives (10 points)

Find the following anti-derivatives

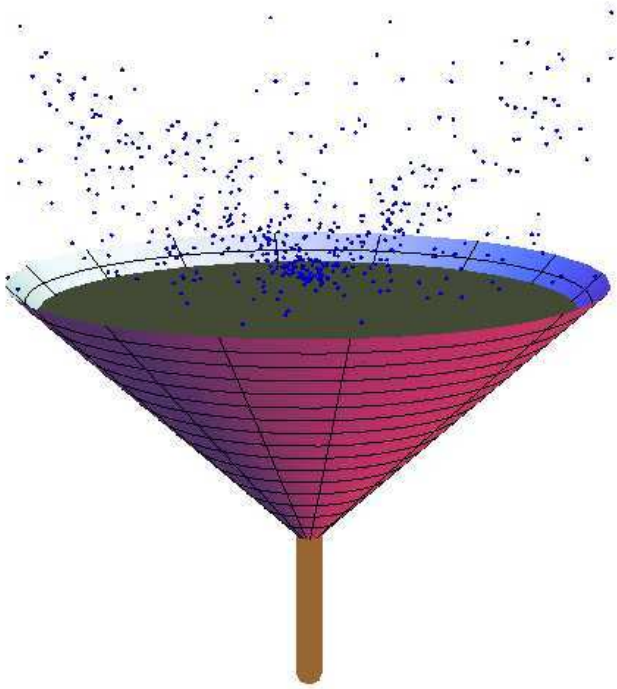
a) (5 points) $\int \frac{3}{\sqrt{1-x^2}} + x^4 + \frac{1}{1+x^2}\, dx$

b) (5 points) $\int \frac{1}{x-2} + \frac{1}{x+4} + \frac{2}{x-1}\, dx$

Solution:
a) $3\arcsin(x) + x^5/5 + \arctan(x) + C$.
b) $\log(x-2) + \log(x+4) + 2\log(x-1) + C$.

Problem 8) Related rates (10 points)

A coffee machine has a filter which is a cone of radius z at height z. Coffee drips out at a rate of 1 cubic centimeter per second. How fast does the water level sink at height $z = 10$?



Solution:
We first compute the area $A(z) = z^2\pi$ and then the volume $V(z) = \int_0^z s^2\pi \, ds = z^3\pi/3$. Therefore, $1 = d/dtV(z(t)) = z^2\pi z'(t)$ so that $z'(t) = 1/(z^2\pi) = 1/(100\pi)$.

Problem 9) Implicit differentiation (10 points)

Find the derivatives $y' = dy/dx$ of the following implicitly defined functions:

a) (5 points) $x^5 + 3x + y = e^y$.

b) (5 points) $\sin(x^2 - 2y) = y - x$.

Solution:
a) $5x^2 + 3 + y' = e^y y'$. Solve for $y' = (5x^4 + 3)/(e^y - 1)$.
b) $\cos(x^2 - 2y)(2x - 2y') = y' - 1$. Solve for $y' = [1 + 2x(\cos(x^2 - 2y)]/(1 + 2\cos(x^2 - 2y)$.

Problem 10) Improper integrals (10 points)

Evaluate the following improper integrals or state that they do not exist

a) (3 points) $\int_1^\infty 1/\sqrt{x}\, dx$.

b) (2 points) $\int_0^1 \sqrt{x}\, dx$.

c) (3 points) $\int_0^\infty 2xe^{-x^2}\, dx$.

d) (2 points) $\int_0^\infty \frac{1}{x}\, dx$

Solution:
a) $\int_1^R 1/\sqrt{x}\, dx = 2\sqrt{x}|_1^R$ does not have a limit for $R \to \infty$.
b) $\int_0^1 \sqrt{x}\, dx = x^{3/2}(2/3)|_0^1 = 2/3$.
c) $-e^{-x^2}|_0^\infty = 1$.
d) $\log(x)|_a^R = \log(R) - \log(a)$ does not exist on both ends. For $a \to 0$ and for $R \to \infty$.

4/8/2014: Second midterm practice D

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] The formula $\int_0^x f''(x)\,dx = f'(x) - f'(0)$ holds.

Solution:
Apply the fundamental theorem to the derivative.

2) [T] [F] The area of the lower half disc is the integral $\int_{-1}^1 -\sqrt{1-x^2}\,dx$

Solution:
The area is positive. The integral given is negative.

3) [T] [F] If the graph of the function $f(x) = x^2$ is rotated around the interval $[0,1]$ we obtain a solid with volume $\int_0^1 \pi x^4\,dx$.

Solution:
Indeed the area is $A(x) = \pi x^4$.

4) [T] [F] The identity $d/dx \int_0^x f''(t)\,dt = f'(x)$ holds.

Solution:
The result is $f''(x)$.

5) [T] [F] There is a point in $[0,1]$, where $f'(x) = 0$ if $f(x) = x^3 - x^2 + 1$.

Solution:
Since $f(0) = f(1) = 1$, Rolle's theorem assures this.

6) [T] [F] The fundamental theorem of calculus assures that $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Solution:
Yes, this is one of the important reformulations.

7) [T] [F] If f is differentiable on $[a,b]$, then $\int_a^b f(x)\,dx$ exists.

Solution:
Yes, we have seen the proof in class.

8) [T] [F] The integral $\int_0^{\pi/2} \sin(\sin(x))\, dx$ is positive.

Solution:
$\sin(\sin(x)) > 0$ there so that the integral is positive

9) [T] [F] The anti-derivative of an anti-derivative of f is equal to the derivative of f.

Solution:
This is just total nonsense. We would have to differentiate three times the anti derivative of the anti derivative to get to the derivative of f.

10) [T] [F] If a function is positive everywhere, then $\int_a^b f(x)\, dx$ is positive too.

Solution:
Yes, the integral has the meaning of an area under the curve.

11) [T] [F] If a differentiable function is odd, then $\int_{-1}^1 f(x)\, dx = 0$.

Solution:
We have a cancellation to the left and to the right.

12) [T] [F] If $f_c(x)$ is a function with a local minimum at 0 for all $c < 0$ and no local minimum in $[-1, 1]$ for $c > 0$, then $c = 0$ is called a catastrophe.

Solution:
Yes this is a pretty precise definition of a catastrophe.

13) [T] [F] The term "improper integral" is a synonym for "indefinite integral".

Solution:
Improper means that we either integrate a function which has a discontinuity or that we integrate over an infinite interval.

14) T [F] The function $F(x) = x\sin(x)$ is an antiderivative of $\sin(x)$.

Solution:
Too good to be true. Just differentiate and you see that the derivative of F is not f.

15) T [F] The mean value theorem holds for every continuous function.

Solution:
No, only for differentiable functions.

16) T [F] Newton and Leibniz were best buddies all their life. Leibniz even gave once the following famous speech: "You guys might not know this, but I consider myself a bit of a loner. I tend to think of myself as a one-man wolf pack. But when my sister brought Isaac home, I knew he was one of my own. And my wolf pack ... it grew by one.

Solution:
This line is from the movie "hangover". No, Newton and Leibniz had a dispute about who discovered calculus.

17) T [F] Any function $f(x)$ satisfying $f(x) > 0$ is a probability density function.

Solution:
What is missing is that the integral over the real line is equal to 1.

18) [T] F The moment of inertia integral I can be used to compute energy with the relation $E = \omega^2 I/2$ where ω is the angular velocity.

Solution:
Yes, this is why the moment of inertia is an important quantity in physics.

19) [T] F If $0 \leq f(x) \leq g(x)$ then $0 \leq \int_0^1 f(x)\,dx \leq \int_0^1 g(x)\,dx$.

Solution:
Yes, just look at the areas.

20) [T] [F] The improper integral $\int_0^\infty 1/(x^4+1)\,dx$ is finite.

Solution:
It works well at 0 because the function does not have a pole there. The integral is smaller than $\int 1/x^4\,dx$ which has the anti-derivative $-1/(3x^3)$. The improper integral exists.

Problem 2) Matching problem (10 points) No justifications are needed.

From the following functions there are two for which no elementary integral is found. Find them. You can find them by spotting the complement set of functions which you can integrate.

Function	Antiderivative is not elementary	Function	Antiderivative is not elementary
e^{-x^2}		$1/\log(x)$	
$\sin(3x)$		$\tan(3x)$	
$1/x$		$\arctan(3x)$	

Solution:
The $1/\log(x)$ and e^{-x^2} have no elementary integral.

Problem 3) Matching problem (10 points) No justifications are needed.

Which of the following problems are related rates problems? Several answers can apply.

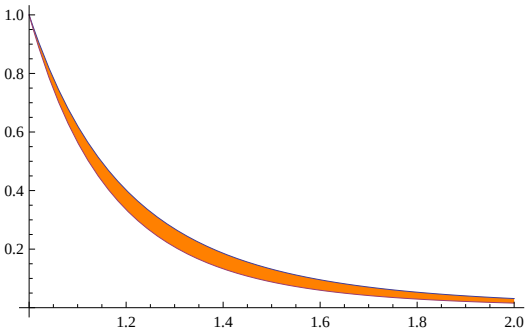
Problem	Related rates?
Find the volume of a sphere in relation to the radius.	
Relate the area under a curve with value of the curve.	
If $x^3 + y^3 = 5$ and $x' = 3$ at $x = 1$, find y'.	
Find the rate of change of the function $f(x) = \sin(x)$ at $x = 1$	
Find r' for a sphere of volume V satisfying $d/dtV(r(t)) = 15$.	
Find the inflection points of $f(x) = x^3 + 3x + 4$.	
Find the global maxima of $f(x) = x^4 + x^3 - x$.	

Solution:
The third and fifth are related rates problems.

Problem 4) Area computation (10 points)

a) (5 points) Find the area of the region enclosed by the curves $3 - x^4$ and $3x^2 - 1$.

b) (5 points) Find the area of the region between $1/x^6$ and $1/x^5$ from $x = 1$ to $x = \infty$.



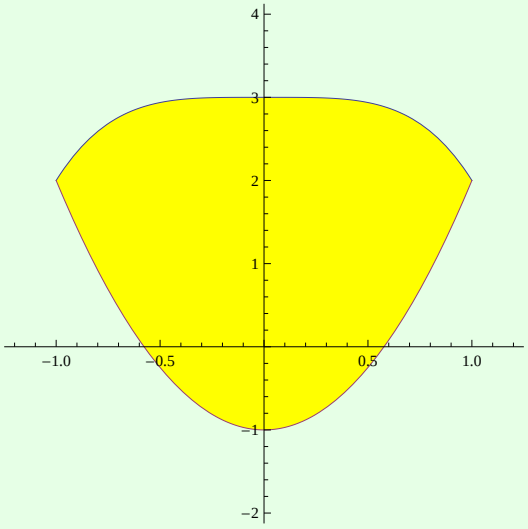
Solution:

a) The two curves intersect at $x = -1$ and $x = 1$ so that the area is

$$\int_{-1}^{1} (3 - x^4) - (3x^2 - 1)\, dx = 3x - x^5/5 - x^3 + x|_{-1}^{1} = 28/5\ .$$

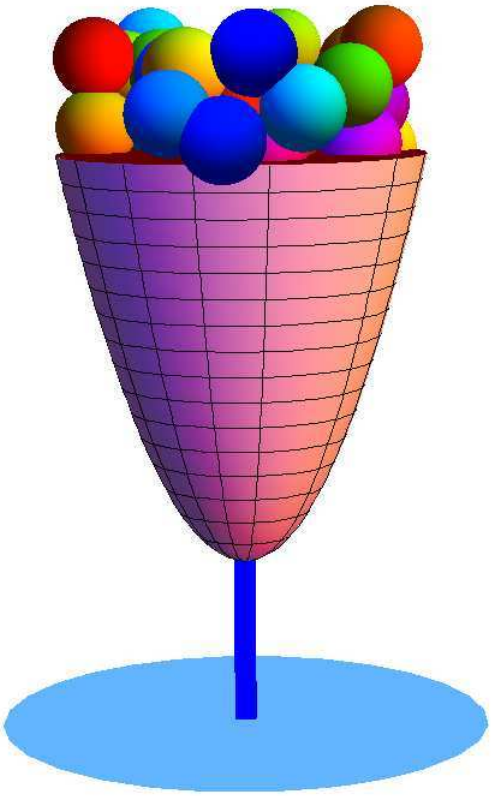
b) The graph of $1/x^5$ is above $1/x^6$. We have

$$\int_{1}^{\infty} x^{-5} - x^{-6}\, dx = -x^{-4}/4 + x^{-5}/5|_{1}^{\infty} = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}\ .$$



Problem 5) Volume computation (10 points)

Cody eats some magic "Bertie Botts Every Flavor Beans" from a cup which is a rotationally symmetric solid, for which the radius at position x is $\sqrt{x}$ and $0 \leq x \leq 4$. Find the volume of Cody's candy cup.



Solution:
The area at height x is $\sqrt{x}^2\pi = x\pi$ so that the volume is

$$\int_0^4 x\pi \, dx = \pi x^2/2|_0^4 = 8\pi \, .$$

Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (5 points) $\int_1^2 x + \tan(x) + \sin(x) + \cos(x) + \log(x) \, dx$.

b) (5 points) $\int_1^3 (x+1)^3 \, dx$

Solution:
a) The anti-derivative is $x^2/2 - \log(\cos(x)) - \cos(x) + \sin(x) + (x\log(x) - x)$. (The $\log(x)$ integral is a bit out of line at this stage, but we have seen it at some point).
b) $(x+1)^4/4|_1^3 = (4^4 - 2^4)/4 = 60$.

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

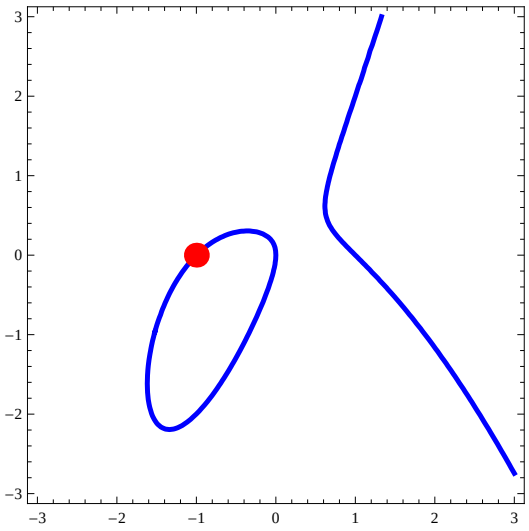
a) (5 points) $\int \sqrt{x^3}\, dx$

b) (5 points) $\int 4/\sqrt{x^5}\, dx$

Solution:
a) The function is $x^{3/2}$. Integration gives $x^{5/2}(2/5) + C$.
b) The function is $4x^{-5/2}$. Integration gives $4x^{-3/2}(-2/3) + C$.

Problem 8) Implicit differentiation (10 points)

The curve $y^2 = x^3 + 2xy - x$ is an example of an **elliptic curve**. Find dy/dx at the point $(-1, 0)$ without solving for y first.



Solution:
Differentiate, $2yy' = 3x^2 + 2y + 2xy' - 1$ and solve for $y' = (3x^2 + 2y - 1)/(2y - 2x)$. At the point $(-1, 0)$ this is 1.

Problem 9) Applications (10 points)

The probability density of the exponential distribution is given by $f(x) = (1/2)e^{-x/2}$.
The probability to wait for for time x (hours) to get an idea for a good calculus exam problem is

$\int_0^x f(x)\,dx$. What is the probability to get a good idea if we wait for $T = 10$ (hours)?

Solution:
$\int_0^{10}(1/2)e^{-x/2}\,dx = (1 - e^{-5})$ which is almost certain. Indeed, we did not have to wait so long to get this great problem.

Problem 10) Applications (10 points)

What is the **average value** of the function

$$f(x) = 4 + 1/(1 + x^2)$$

on the interval $[-1, 1]$?

Solution:
It is $\int_{-1}^{1} f(x)\,dx/2 = (8 - 2\arctan(1))/2 = 4 + \arctan(1) = 4 + \pi/4$.

5/17/2014: Final Exam

Your Name:

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- All functions f if not specified otherwise can be assumed to be smooth so that arbitrary many derivatives can be taken.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) [T] [F] $\cos(17\pi/4) = \sqrt{2}/2$.

Solution:
Yes, it is $\cos(\pi/4)$.

2) [T] [F] The tangent function is monotonically increasing on the open interval $(-\pi/2, \pi/2)$.

Solution:
Indeed, its derivative is $1/\cos^2(x)$.

3) [T] [F] The arccot function is monotonically increasing from $\pi/4$ to $3\pi/4$.

Solution:
It is decreasing.

4) [T] [F] If f is a probability density function, then $\int_{-\infty}^{\infty} f(x)\, dx = 0$

Solution:
It is 1.

5) [T] [F] $\frac{d}{dx} e^{\log(x)} = 1$.

Solution:
First simplify.

6) [T] [F] If $f''(0) = -1$ then f has a local maximum at $x = 0$.

Solution:
We need the first derivative to be zero

7) [T] [F] The improper integral $\int_{-1}^{1} 1/|x|\, dx$ is finite.

Solution:
This is an improper integral which does not exist

8) [T] [F] The function $-\cos(x) - x$ has a root in the interval $(-100, 100)$.

Solution:
Use the intermediate value theorem.

9) [T] [F] If a function f has a local maximum in $(0, 1)$ then it also has a local minimum in $(0, 1)$.

Solution:
Take $f(x) = -x^2$. It does not have a local minimum.

10) [T] [F] The anti derivative of $1/(1 - x^2)$ is equal to $\arctan(x)$.

Solution:
It is an other sign.

11) [T] [F] The function $f(x) = (e^x - e^{2x})/(x - x^2)$ has the limit 1 as x goes to zero.

Solution:
Use Hopital's rule to see that it is the same than the limit $(e^x - 2e^{2x})/(1 - 2x)$ for for $x \to 0$ but this is -1.

12) [T] [F] If you listen to the sound $e^{-x}\sin(10000x)$, then it gets louder and louder as time goes on.

Solution:
The amplitude decays like e^{-x}.

13) [T] [F] The function $f(x) = e^{x^2}$ has a local minimum at $x = 0$

Solution:
The function is positive near 0 but equal to zero at 0.

14) [T] [F] The function $f(x) = (x^{55} - 1)/(x - 1)$ has the limit 1 for $x \to 1$.

Solution:
Use Hopital's rule, or heal the function. The limit is 55.

15) [T] [F] If the total cost $F(x)$ of an entity is extremal at x, then we have a break even point $f(x) = g(x)$.

Solution:
This is not the strawberry theorem.

16) [T] [F] The value $\int_{-\infty}^{\infty} xf(x)\,dx$ is called the expectation of the PDF f.

Solution:
Yes this is true

17) [T] [F] The trapezoid rule is an integration method in which the left and right Riemann sum are averaged.

Solution:
This is a good description

18) [T] [F] $\tan(\pi/3) = \sqrt{3}$.

Solution:
Yes, it is equal to $\sin(\pi/6)$.

19) [T] [F] A Newton step for the function f is $T(x) = x + \frac{f(x)}{f'(x)}$.

Solution:
Wrong. The sign is off.

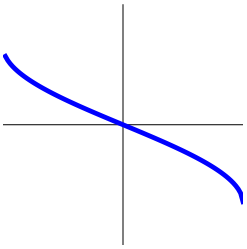
20) [T] [F] $\sin(\arctan(1)) = \sqrt{3}$.

Solution:
We have $\arctan(1) = \pi/4$ and so $\sin(\arctan(1)) = \sqrt{3}$.

Problem 2) Matching problem (10 points) No justifications needed

(5 points) Match the functions names with their graphs (1-4) their derivatives (A-D) (middle row) and second derivatives (a-d) (last row).

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$\sin(x)/x$			
$\tan(x)$			
$\arcsin(x)$			
$1/(1+x^2)$			



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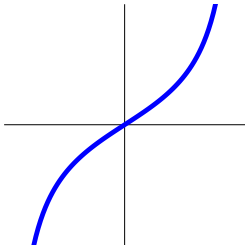

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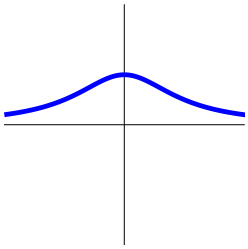

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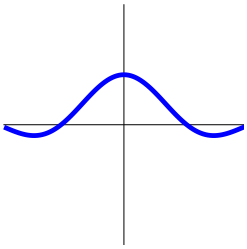

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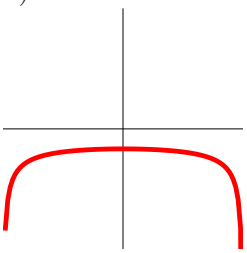

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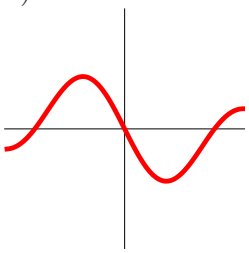

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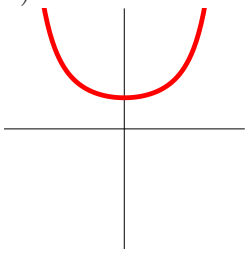
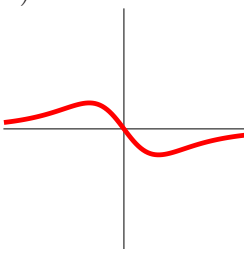


C)



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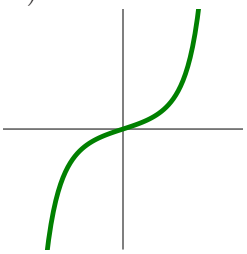
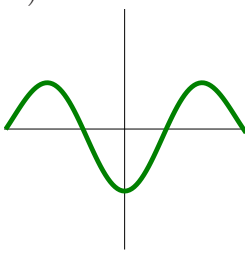


a)



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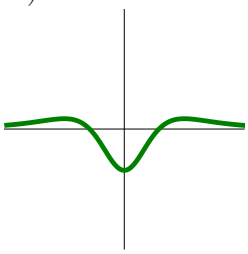

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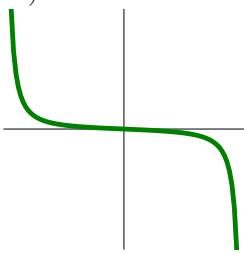


d)

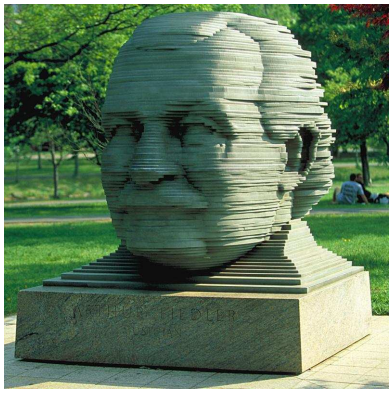
(5 points) Which of the following limits exists in the limit $x \to 0$.

Function	exists	does not exist
$\sin^4(x)/x^4$		
$1/\log\|x\|$		
$\arctan(x)/x$		
$\log\|x\|/(x-1)$		
$\cos(x)/(x-1)$		
$(x^{10}-1)/(x-1)$		

Solution:
a) 4,B,b
2,C,a
1,A,d
3,D,c
b) Every limit exists except the case $\log|x|/(x-1)$.

Problem 3) Matching or short answer problem (10 points). No justifications are needed.

a) (4 points) On the Boston Esplanade is a sculpture of **Arthur Fiedler** (1894-1979) a long-time conductor of the Boston Pops Orchestra. His head is sliced into thin slices. Assume that the thickness of each level is $h = 1.5$ inch and the area of each of the 100 slices k is $A(k)$. Which formula gives the volume of the head? (One applies.)



Formula	Check if true
$1.5[A(1)+\cdots+A(100)]$	
$\frac{1}{1.5}[A(1)+\cdots+A(100)]$	

Formula	Check if true
$1.5[\frac{1}{A(1)}+\cdots+\frac{1}{A(100)}]$	
$\frac{1.5}{100}[A(1)+\cdots+A(100)]$	

b) (4 points) The summer has arrived on May 12 for a day before it cooled down again. Harvard students enjoy the **Lampoon pool** that day in front of the **Lampoon castle**. Assume the water volume at height z is $V(z) = 1+5z-\cos(z)$. Assume water evaporates at a rate of $V'(z) = -1$ gallon per day. How fast does the water level drop at $z = \pi/2$ meters? Check the right answer: (one applies)

Rate	Check if true
-6	
$-1/6$	

Rate	Check if true
-4	
$-1/4$	

c) (2 points) Speaking of weather: the temperature on May 13 in Cambridge was 52 degrees Fahrenheit. The day before, on May 12, the temperature had been 85 degrees at some point and had us all dream about **beach time**. Which of the following theorems assures that there was a moment during the night of May 12 to May 13 that the temperature was exactly 70 degrees? (One applies.)

Theorem	check if true
Mean value theorem	
Rolle theorem	

Theorem	check if true
Intermediate value theorem	
Bolzano theorem	

Solution:
$1.5[A(1) + ... + A(100)]$ is the Riemann sum because $dz = 1.5$.
b) $\boxed{z' = -1/6}$ as

$$V' = 5z' + \sin(z)z' = -1$$

gives for $z = \pi/2$ the equation $6z' = -1$ leading to the result. The other selections could made sense if some mistake was done like writing $5 + \sin(z)z' = -1$ for example which would lead to $z' = -6$ which is false.
c) It is the intermediate value theorem.

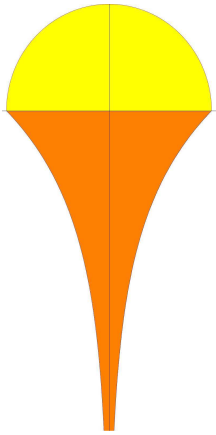
Problem 4) Area computation (10 points)

Find the area enclosed by the graphs of the functions

$$f(x) = \log|x|$$

and

$$g(x) = \sqrt{1 - x^2}\ .$$



Solution:
The integral is

$$\int_{-1}^{1} \sqrt{1-x^2} - \log|x| \, dx \, .$$

The first integral is $\pi/2$ as it is the area of half the circle. The second integral is $-2\int_0^1 \log|x| \, dx$. $2(x - x\log(x)|_{-1}^1) = 4$. The answer is $\boxed{\pi/2 + 4}$.

Problem 5) Volume computation (10 points)

The lamps near the front entrance of the **Harvard Malkin Athletic Center** (MAC) have octagonal cross sections, where at height z, the area is

$$A(z) = 2(1+\sqrt{2})(1+z)^2$$

with $0 \leq z \leq 3$. What is the volume of the lamp?

Solution:
$(2+2\sqrt{2})\int_0^3 (1+z)^2 \, dz = 21(2+\sqrt{8}) = 42 + 21\sqrt{8} = \boxed{42(1+\sqrt{2})}$.

Problem 6) Improper integrals (10 points)

Which of the following limits $R \to \infty$ exist? If the limit exist, compute it.

a) (2 points) $\int_1^R \sin(2\pi x) \, dx$

b) (2 points) $\int_1^R \frac{1}{x^2} \, dx$

c) (2 points) $\int_1^R \frac{1}{\sqrt{x}} \, dx$

d) (2 points) $\int_1^R \frac{1}{1+x^2} \, dx$

e) (2 points) $\int_1^R x \, dx$

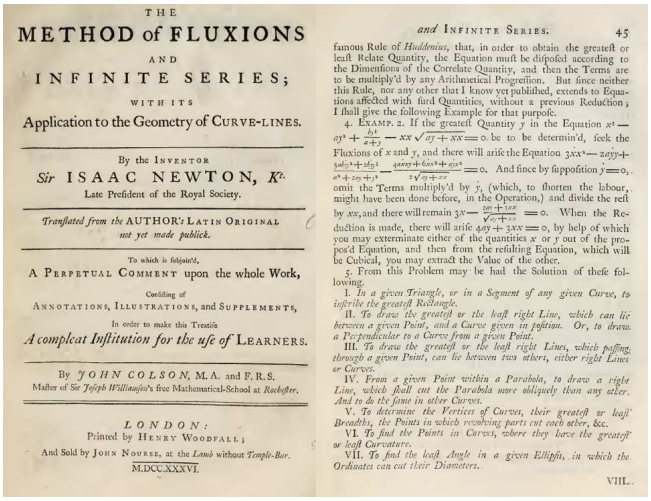
Solution:
a) $\lim_{R\to\infty} -\cos(2\pi x)|_1^R$ does not exist.
b) $\lim_{R\to\infty} -\frac{1}{x}|_1^R = 1$ exists.
c) $\lim_{R\to\infty} 2\sqrt{x}|_1^R$ does not exist.
d) $\lim_{R\to\infty} \arctan(x)_1^R = \pi/2 - \pi/4 = \pi/4$.
e) $\lim_{R\to\infty} R^2/2$ does not exist.

Problem 7) Extrema (10 points)

In Newton's masterpiece **"The Method of Fluxions"** on the bottom of page 45, Newton asks: "In a given triangle or in a segment of any given curve, to inscribe the greatest rectangle." Lets be more specific and find rectangle with largest area

$$A = xy$$

in the triangle given by the x-axes, y-axes and line $y = 2 - 2x$. Use the second derivative test to make sure you have found the maximum.



THE
METHOD of FLUXIONS
AND
INFINITE SERIES;
WITH ITS
Application to the Geometry of CURVE-LINES.

By the INVENTOR
Sir ISAAC NEWTON, *Kt.*
Late Prefident of the Royal Society.

Tranſlated from the AUTHOR'S LATIN ORIGINAL
not yet made publick.

To which is ſubjoin'd,
A PERPETUAL COMMENT upon the whole Work,
Conſiſting of
ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,
In order to make this Treatiſe
A compleat Inſtitution for the uſe of LEARNERS.

By JOHN COLSON, M.A. and F.R.S.
Maſter of Sir *Joſeph Williamſon's* free Mathematical-School at *Rocheſter*.

LONDON:
Printed by HENRY WOODFALL;
And Sold by JOHN NOURSE, at the *Lamb* without *Temple-Bar*.
M.DCC.XXXVI.

and INFINITE SERIES. 45

famous Rule of *Huddenius*, that, in order to obtain the greateſt or leaſt Relate Quantity, the Equation muſt be diſpoſed according to the Dimenſions of the Correlate Quantity, and then the Terms are to be multiply'd by any Arithmetical Progreſſion. But ſince neither this Rule, nor any other that I know yet publiſhed, extends to Equations affected with ſurd Quantities, without a previous Reduction; I ſhall give the following Example for that purpoſe.

4. EXAMP. 2. If the greateſt Quantity y in the Equation $x^3 - ay^3 + \frac{by^4}{a+y} - xx\sqrt{ay+xx} = 0$ be to be determin'd, ſeek the Fluxions of x and y, and there will ariſe the Equation $3\dot{x}x^2 - 2a\dot{y}y + \frac{3a\dot{y}y^2+2b\dot{y}y^3}{a^2+2ay+y^2} - \frac{4a\dot{x}xy+6\dot{x}x^2+\dot{y}y^3}{2\sqrt{a+xx}} = 0$. And ſince by ſuppoſition $\dot{y} = 0$, omit the Terms multiply'd by $\dot{y}$, (which, to ſhorten the labour, might have been done before, in the Operation,) and divide the reſt by $\dot{x}x$, and there will remain $3x - \frac{2a+2xx}{\sqrt{a+xx}} = 0$. When the Reduction is made, there will ariſe $4ay + 3xx = 0$, by help of which you may exterminate either of the quantities x or y out of the propoſed Equation, and then from the reſulting Equation, which will be Cubical, you may extract the Value of the other.

5. From this Problem may be had the Solution of theſe following.

I. *In a given Triangle, or in a Segment of any given Curve, to inſcribe the greateſt Rectangle.*

II. *To draw the greateſt or the leaſt right Line, which can lie between a given Point, and a Curve given in poſition. Or, to draw a Perpendicular to a Curve from a given Point.*

III. *To draw the greateſt or the leaſt right Lines, which paſſing through a given Point, can lie between two others, either right Lines or Curves.*

IV. *From a given Point within a Parabola, to draw a right Line, which ſhall cut the Parabola more obliquely than any other. And to do the ſame in other Curves.*

V. *To determine the Vertices of Curves, their greateſt or leaſt Breadths, the Points in which revolving parts cut each other,* &c.

VI. *To find the Points in Curves, where they have the greateſt or leaſt Curvature.*

VII. *To find the leaſt Angle in a given Ellipſis, in which the Ordinates can cut their Diameters.*

VIII.

Solution:
The function to extremize is $f(x) = x(2-2x) = 2x - 2x^2$. Its derivative is $f'(x) = 2 - 4x$. It is zero if $\boxed{x = 1/2}$. The second derivative is $f''(x) = -4$. As it is negative, the extremum is a **maximum**.

Problem 8) Integration by parts (10 points)

a) (5 points) Find

$$\int (1 + x + x^2 + x^3 + x^4)(\sin(x) + e^x)\, dx\ .$$

b) (5 points) Find

$$\int \log(x) \frac{1}{x^2}\, dx\ .$$

Solution:
a) Use TicTacToe:

$1+x+x^2+x^3+x^4$	$\sin(x)+e^x$	
$1+2x+3x^2+4x^3$	$-\cos(x)+e^x$	$\oplus$
$2+6x+12x^2$	$-\sin(x)+e^x$	$\ominus$
$6+24x$	$\cos(x)+e^x$	$\oplus$
24	$\sin(x)+e^x$	$\ominus$
0	$-\cos(x)+e^x$	$\oplus$

Collecting together, we could write $(4x^3+3x^2-22x-5)\sin(x)+(x^4-3x^3+10x^2-19x+20)e^x+(-x^4-x^3+11x^2+5x-23)\cos(x)$.
b) As we know from LIATE, we differentiate the log $\log(x)$. We get

$$-\log(x)\frac{1}{x}+\int_x \frac{1}{x^2}=-\log(x)/x-1/x+C\ .$$

Problem 9) Substitution (10 points)

a) (5 points) **"One,Two,Three,Four Five, once I caught a fish alive!"**

$$\int \frac{(1+2x+3x^2+4x^3+5x^4)}{(1+x+x^2+x^3+x^4+x^5)}\,dx\ .$$

b) (5 points) A **"Trig Trick-or-Treat"** problem:

$$\int (1-x^2)^{-3/2}+(1-x^2)^{-1/2}+(1-x^2)^{1/2}\,dx\ .$$

Solution:
a) Substitute $u=1+x+x^2+x^3+x^4+x^5$ so that we get $\int du/u=\log(u)+c=\log(1+x+x^2+x^3+x^4+x^5)+C$.
b) Use trig substitution $x=\sin(u)$ in all cases. We get

$$\int \frac{1}{\cos^2(u)}+1+\cos^2(u)\,du=\tan(u)+u+(1+\sin(2u)/2)/2+C$$

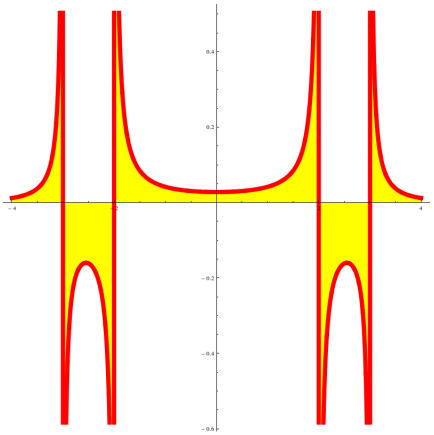
which is $\tan(\arcsin(x))+\arcsin(x)+(1+\sin(2\arcsin(x))/2)/2+C$.

Problem 10) Partial fractions (10 points)

Integrate

$$\int_{-1}^{1} \frac{1}{(x+3)(x+2)(x-2)(x-3)}\, dx\ .$$

The graph of the function is shown to the right.



Lets call it the **friendship graph**.

Solution:
Use partial fraction with the Hopital method of course: Write

$$\frac{1}{(x+3)(x+2)(x-2)(x-3)} = \frac{A}{x+3} + \frac{B}{x+2} + \frac{C}{x-2} + \frac{D}{x-3}$$

To get A, multiply the entire equation with $x+3$, simplify and take the limit $x \to 3$. This gives $A = \frac{1}{(3+3)(3+2)(3-2)} = -1/30$. Similarly, we get $B = \frac{1}{(2+3))(2+2)(2-3)} = -1/20$ and $C = \frac{1}{(-2+3)(-2-2)(-2-3)} = 1/20$ and $D = \frac{1}{(-3+2)(-3-2)(-3-3)} = 1/30$. Now we can write the integral as

$$\begin{aligned} & \frac{-1}{30}\log|x+3| - \frac{1}{20}\log|x+2| + \frac{1}{20}\log|x-2| + \frac{1}{30}\log|x-3||_{-1}^{1} \\ = & \frac{-1}{30}(\log(4)-\log(2)) - \frac{1}{20}\log(3) + \frac{1}{20}\log(3) + \frac{1}{30}(\log(2)-\log(4)) \\ = & -\frac{2}{30}\log(2) + \frac{2}{20}\log(3) = -\frac{\log(2)}{15} + \frac{\log(3)}{10}\ . \end{aligned}$$

$\boxed{\log(3)/10 - \log(2)/15}$ is probably the most elegant solution. Results which were not simplified as such were also ok of course. Taking absolute values in the log would not matter neither as logarithms of negative values are technically ok (even so they can be complex, the complex parts cancel).

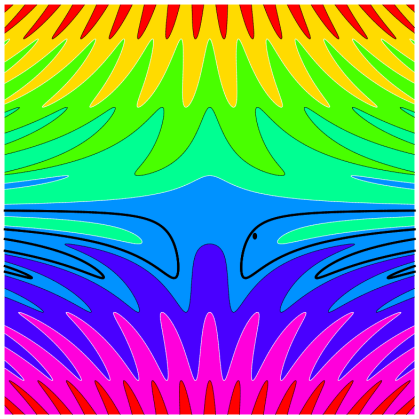
Problem 11) Related rates or implicit differentiation. (10 points)

Assume $x(t)$ and $y(t)$ are related by

$$(\cos(xy) - y) = 1 \ .$$

We know that $x' = 2$ at $(x, y) = (\pi/2, -1)$. Find y' at this point.

P.S. The figure shows other level curves of a **monster function**. The traced out curve is the curve under consideration.



Solution:
Since a third variable t appears this is a "everybody hates" related rates problem Differentiate with respect to t and solve $-\sin(xy)(x'y + xy') - y' = 0$ with respect to y'. This gives for $x = \pi/2, y = -1, x' = 2$ the answer $-2 + (\pi/2)y' = y'$ so that $y' = 2/(\pi/2 - 1) = \boxed{4/(\pi - 2)}$.

Problem 12) Various integration problems (10 points)

a) (2 points) $\int_0^{2\pi} 2\cos^2(x) - \sin(x)\ dx$

b) (2 points) $\int x^2 e^{3x}\ dx$

c) (2 points) $\int_1^{\infty} \frac{1}{(x+2)^2}\ dx$

d) (2 points) $\int \sqrt{x}\log(x)\ dx$

e) (2 points) $\int_1^e \log(x)^2\ dx$

Solution:
a) Double angle formula: 2π
b) Parts (twice) $e^{3x}(2 - 6x + 9x^2)/27$
c) Substitute $u = x + 2$ to get $1/3$
d) Parts differentiating the log $(3\log(x) - 2)2x^{3/2}/9$
e) parts writing it as $\log(x)^2 \cdot 1$ or as $\log(x) \cdot \log(x)$ (both worked when differentiating the log). The result is $e - 2$

Problem 13) Applications (10 points)

a) (2 points) [**Agnesi density**]

The CDF of the PDF $f(x) = \pi^{-1}/(1+x^2)$ is $\boxed{}$

b) (2 points) [**Piano man**]

The upper hull of $f(x) = x^2 \sin(1000x)$ is the function $\boxed{}$

c) (2 points) [**Rower's wisdom**]

If f is power, F is work and $g = F/x$ then $f = g$ if and only if $g'(x) = \boxed{}$

d) (2 points) [**Catastrophes**]

For $f(x) = c(x-1)^2$ there is a catastrophe at $c = \boxed{}$

e) (2 points) [**Randomness**]

We can use chance to compute integrals. It is called the $\boxed{}$ method.

Solution:

a) Integrate from $-\infty$ to x to get $\arctan(x)/\pi + 1/2$.

b) The function in front is x^2. It gives the amplitude.

c) This is the Strawberry theorem applied to an other situation $g' = 0$.

d) See where the critical point $x = 1$ changes nature: $c = 0$.

e) This was just a knowledge question: Monte Carlo.

5/17/2014: Final Practice A

Your Name:

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- All functions f if not specified otherwise can be assumed to be smooth so that arbitrary many derivatives can be taken.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) [T] [F] $\frac{d}{dx}e^{e^x} = e^x$.

Solution:
Use the chain rule.

2) [T] [F] A function f which is concave down at 0 satisfies $f''(0) \leq 0$.

Solution:
Yes, this can even be taken as a definition of concavity

3) [T] [F] The integral $\int_{1/2}^{1} \log(x)\ dx$ is positive. Here $\log(x) = \ln(x)$ is the natural log.

Solution:
The integrand is always negative.

4) [T] [F] The function $x + \sin(\cos(\sin(x)))$ has a root in the interval $(-10, 10)$.

Solution:
Use the intermediate value theorem.

5) [T] [F] The function $x(1-x) + \sin(\sin(x(1-x)))$ has a maximum or minimum inside the interval $(0, 1)$.

Solution:
Use Rolle's theorem

6) [T] [F] The derivative of $1/(1+x^2)$ is equal to $\arctan(x)$.

Solution:
It is the other way round.

7) [T] [F] The limit of $\sin^{100}(x)/x^{100}$ for $x \to 0$ exists and is equal to 100.

Solution:
It is equal to 1.

8) [T] [F] The function $f(x) = (1 - e^x)/\sin(x)$ has the limit 1 as x goes to zero.

Solution:
Use Hopital's rule

9) [T] [F] The frequency of the sound $\sin(10000x)$ is higher than the frequency of $\sin(\sin(3000x))$.

Solution:
Yes, about 3 times larger. The frequency is 10'000 versus 3000.

10) [T] [F] The function $f(x) = \sin(x^2)$ has a local minimum at $x = 0$

Solution:
The function is positive near 0 but equal to zero at 0.

11) [T] [F] The function $f(x) = (x^5 - 1)/(x - 1)$ has a limit for $x \to 5$.

Solution:
Use Hopital's rule, or heal the function.

12) [T] [F] The average cost $g(x) = F(x)/x$ of an entity is extremal at x for which $f(x) = g(x)$. Here, $f(x)$ denotes the marginal cost and $F(x)$ the total cost.

Solution:
This is the strawberry theorem.

13) [T] [F] The mean of a probability density function is defined as $\int f(x)\, dx$.

Solution:
No, it is equal to 1.

14) [T] [F] The differentiation rule $(f(x)^{g(x)})' = (f'(x))^{g(x)}g'(x)$ holds for all differentiable functions f, g.

Solution:
No there is no such rule.

15) [T] [F] $\sin(5\pi/6) = 1/2$.

Solution:
Yes, it is equal to $\sin(\pi/6)$.

16) [T] [F] Hôpital's rule assures that $\sin(10x)/\tan(10x)$ has a limit as $x \to 0$.

Solution:
Yes, the limit is 1

17) [T] [F] A Newton step for the function f is $T(x) = x - \frac{f'(x)}{f(x)}$.

Solution:
Wrong. The derivative is in the denominator.

18) [T] [F] A minimum x of a function f is called a catastrophe if $f'''(x) < 0$.

Solution:
Nonsense, where did he get this from?

19) [T] [F] The fundamental theorem of calculus implies $\int_{-1}^{1} g'(x)\,dx = g(1) - g(-1)$ for all differentiable functions g.

Solution:
Yes, even if the function is called g.

20) [T] [F] If f is a differentiable function for which $f'(x) = 0$ everywhere, then f is constant.

Solution:
Yes, integrating shows $f = c$.

Problem 2) Matching problem (10 points) No justifications needed

a) (2 points) One of three statements A)-C) is not the part of the fundamental theorem of calculus. Which one?

A)	$\int_0^x f'(t)\,dt = f(x) - f(0)$
B)	$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$
C)	$\int_a^b f(x)\,dx = f(b) - f(a)$

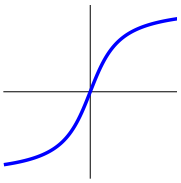
b) (3 points) Biorythms can be fascinating for small kids, giving them a first exposure to trig functions and basic arithmetic. The "theory" tells that there are three functions $p(x) = \sin(2\pi x/23)$ (Physical) $e(x) = \sin(2\pi x/28)$ (Emotional) and $i(x) = \sin(2\pi x/33)$ (Intellectual), where x is the number of days since your birth. Assume **Tuck**, the pig you know from the practice exams, is born on October 10, 2005. Today, on May 11, 2014, it is 2670 days old. Its biorythm is $E = 0.7818, P = -0.299, I = -0.5406$. It is a happy fellow, tired, but feeling a bit out of spirit, like the proctor of this exam feels right now. Which of the following statements are true?

Check if true	
	i) One day old Tuck had positive emotion, intellect and physical strength.
	ii) Among all cycles, the physical cycle takes the longest to repeat.
	iii) Comparing with all cycles, the physical increases fastest at birth.

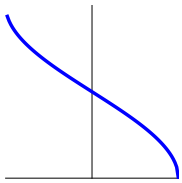
c) (4 points) Name the statements:

$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$ is called the	.
Rule $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$ is called	.
$\int_0^x f'(t)\,dt = f(x) - f(0)$ is called	.
The PDF $f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ is called the	

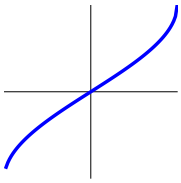
d) (1 point) Which of the following graphs belongs to the function $f(x) = \arctan(x)$?



1)



2)



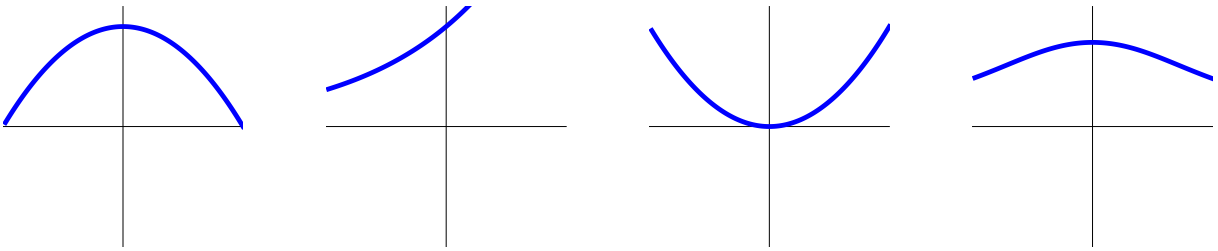
3)

Solution:
a) C
b) i) and iii)
c) Fundamental Theorem of trigonometry, Hopital's rule, Fundamental theorem of calculus, Normal distribution.
d) The first picture 1).

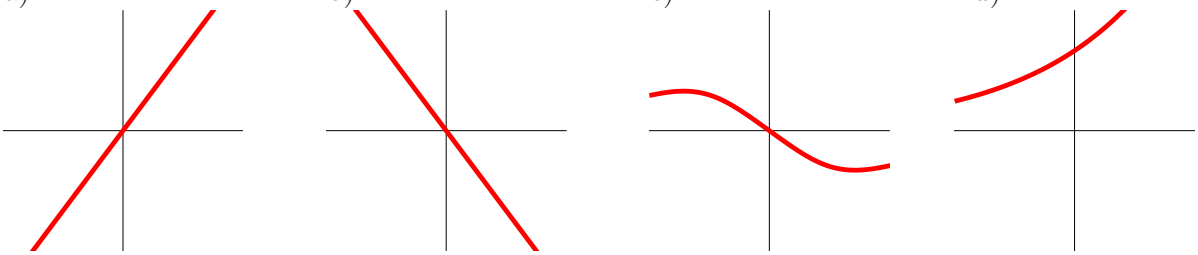
Problem 3) Matching or short answer problem (10 points). No justifications are needed.

a) (4 points) Match the functions (a-d) (top row) with their derivatives (1-4) (middle row) and second derivatives (A-D) (last row).

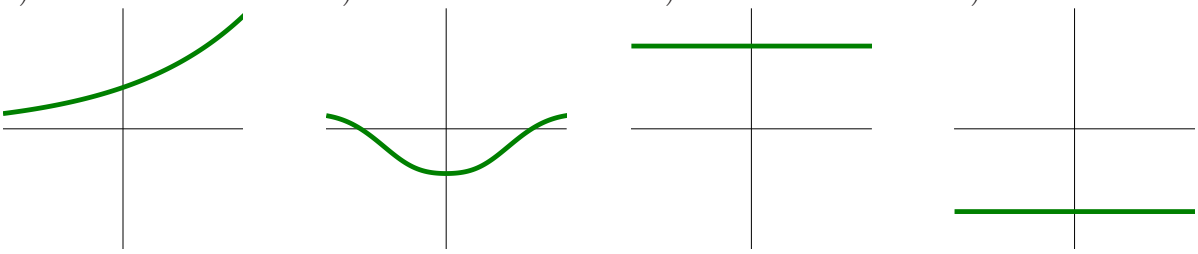
Function a)-d)	Fill in 1)-4)	Fill in A)-D)
graph a)		
graph b)		
graph c)		
graph d)		



a) b) c) d)



1) 2) 3) 4)



A) B) C) D)

b) (4 points) Match the following integrals with the areas in the figures:

Integral	Enter 1-4
$\int_{-\pi}^{\pi} x \sin(x)\ dx.$	
$\int_{-\pi}^{\pi} \exp(-x^2)\ dx.$	

Integral	Enter 1-4
$\int_{-\pi}^{\pi} \pi + x\ dx.$	
$\int_{-\pi}^{\pi} 1 - \sin(x^3/\pi^3)\ dx.$	

1) 2) 3) 4)

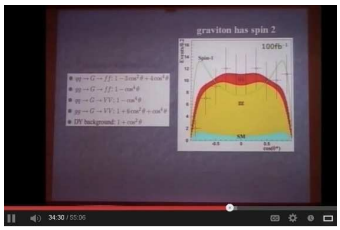
c) (2 points) Name two different numerical integration methods. We have seen at least four.

Your first method	
Your second method	

Solution:
a) 2D,4A,1C,3B
b) The middle two graphs are even functions and belong to the left box.
$\int_{-\pi}^{\pi} x \sin(x)\, dx.$ matches 2
$\int_{-\pi}^{\pi} \exp(-x^2)\, dx.$ matches 3
$\int_{-\pi}^{\pi} \pi + x\, dx.$ matches 1
$\int_{-\pi}^{\pi} 1 - \sin(x^3/\pi^3)\, dx.$ matches 4 c) We have seen Simpson, Trapezoid, MonteCarlo

Problem 4) Area computation (10 points)

A slide in a lecture of Harvard physicist **Lisa Randall** shows the area between two functions. Lisa is known for her theory of "branes" which can explain why gravity is so much weaker than electromagnetism. Assist Lisa and write down the formula for the area between the graphs of $1 - \cos^2(x)$ and $1 - \cos^4(x)$, where $0 \leq x \leq \pi$. Find the area.



Lisa Randall - Extra Dimensional Particle Resonances at the LHC (1/3)

Hint. Lisa already knows the identity

$$\cos^2(x) - \cos^4(x) = \cos^2(x)(1 - \cos^2(x)) = \cos^2(x)\sin^2(x)\ .$$

Solution:
By the hint, we are led to the integral

$$\int_0^\pi \cos^2(x)\sin^2(x)\,dx = \frac{1}{4}\int_0^\pi \sin^2(2x)\,dx = \frac{1}{8}\int_0^\pi (1-\cos(4x))\,dx = \frac{\pi}{8}\,.$$

The answer is $\boxed{\pi/8}$. The key were double angle formulas. [An other possibility was to use $\cos^2(x) = (1+\cos(2x))/2, \sin^2(x) = (1-\cos(2x))/2$ and then use the double angle formula $\sin(4x) = 2\cos(2x)\sin(2x)$ later.]

Problem 5) Volume computation (10 points)

Find the volume of the solid of revolution for which the radius at height z is

$$r(z) = \sqrt{z\log(z)}$$

and for which z is between 1 and 2. Here, log is the natural log. Naturalmente!

Solution:
The integral $\pi\int z\log(z)\,dz$ can be done using integration by parts by differentiating log first (LIATE,LIPTE). We have $\pi(z^2\log(z)/2 - \int z/2\;dz) = \pi(\log(z)/2 - 1/4)z^2$. The definite integral is $\boxed{\pi(\log(4) - 3/4)}$.

Problem 6) Improper integrals (10 points)

a) (5 points) Find the integral or state that it does not exist

$$\int_1^\infty \frac{7}{x^{3/4}}\,dx\,.$$

b) (5 points) Find the integral or state that it does not exist

$$\int_1^\infty \frac{13}{x^{5/4}}\,dx\,.$$

Solution:
a)

$$7\int_1^\infty \frac{1}{x^{3/4}}\,dx = 28x^{1/4}|_1^\infty = \infty\,.$$

b)

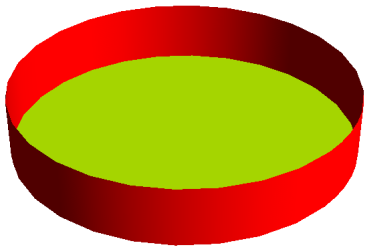
$$13\int_1^\infty \frac{1}{x^{5/4}}\,dx = -52x^{-1/4}|_1^\infty = 52\,.$$

Problem 7) Extrema (10 points)

A **candle holder** of height y and radius x is made of aluminum. Its total surface area is $2\pi xy + \pi x^2 = \pi$ (implying $y = 1/(2x) - x/2$). Find x for which the volume

$$f(x) = x^2 y(x)$$

is maximal.



Solution:
Substituting $y(x)$ in gives the function $f(x) = x/2 - x^3/2$ which has the derivative $f'(x) = 1/2 - 3x^2/2$. It is zero for $\boxed{x = 1/\sqrt{3}}$. The second derivative $-3x$ is negative there so that this is a maximum.

Problem 8) Integration by parts (10 points)

a) (5 points) Find

$$\int (x+5)^3 \sin(x-4)\, dx\ .$$

b) (5 point) Find the indefinite integral

$$\int e^x \cos(2x)\, dx\ .$$

Don't get dizzy when riding this one.

Solution:
a) Use the Tic-Tac-Toe integration method:

$(x+5)^3$	$\sin(x-4)$	
$3(x+5)^2$	$-\cos(x-4)$	$\oplus$
$6(x+5)^1$	$-\sin(x-4)$	$\ominus$
6	$\cos(x-4)$	$\oplus$
0	$\sin(x-4)$	$\ominus$

We can read off the answer $-(x-5)^3\cos(x-4)+3(x+5)^2\sin(x-4)+6(x+5)\cos(x-4)-6\sin(x-4)+C$.
b) We use the merry go round by using integration by parts twice calling the integral I. We have

$$I = \cos(2x)e^x + \int 2\sin(2x)e^x\,dx = \cos(2x)+2\sin(2x)-4I\ .$$

Solving for I gives $\boxed{(\cos(x)+2\sin(2x))e^x/5+C}$.

Problem 9) Substitution (10 points)

a) (3 points) Solve the integral $\int \log(x^3)x^2\,dx$.

b) (4 points) Solve the integral $\int x\cos(x^2)\exp(\sin(x^2))\,dx$.

c) (3 points) Find the integral $\int \sin(\exp(x))\exp(x)\,dx$.

Solution:
These are all standard substitution problems:
a) Substitute $u=x^3$ to get $(x^3\log(x^3)-x^3)/3+C$
b) Substitute $u=\sin(x^2)$ to get $\exp(\sin(x^2))+C$.
c) Substitute $u=\exp(x)$ to get $-\cos(\exp(x))+C$.

Problem 10) Partial fractions (10 points)

a) (5 points) Find the definite integral

$$\int_1^5 \frac{1}{(x-2)(x-3)(x-4)}\,dx\ .$$

(Evaluate the absolute values $\log|\cdot|$ in your answer. The improper integrals exist as a Cauchy principal value).

b) (5 points) Find the indefinite integral

$$\int \frac{1}{x(x-1)(x+1)(x-2)}\,dx\ .$$

Solution:
a) We use the hopital method to find the constants $A = 1/2, B = -1, C = 1/2$ in

$$\frac{1}{(x-2)(x-3)(x-4)} = \frac{A}{(x-2)} + \frac{B}{(x-3)} + \frac{C}{(x-4)}\ .$$

The answer is $\log(x-2)/2 - \log(x-3) + \log(x-4)/2$. The definite integral is $\boxed{\text{zero}}$.
b) Again use Hopital to get $A = 1/2, B = -1/2, C = -1/6, D = 1/6$ in

$$\frac{1}{x(x-1)(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{D}{x-2}\ .$$

The answer is

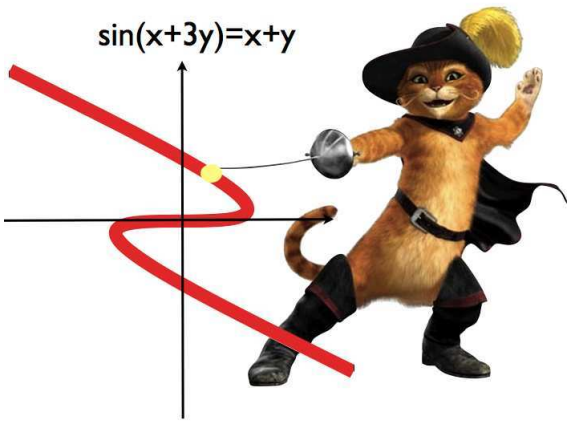
$$\log(x)/2 - \log(x-1)/2 - \log(x+1)/6 + \log(x-2)/6 + C\ .$$

Problem 11) Related rates or implicit differentiation. (10 points)

a) (5 points) Assume

$$x^4(t) + 3y^4(t) = 4y(t)$$

and $x'(t) = 5$ at $(1,1)$. What is y' at $(1,1)$?



b) (5 points) What is the derivative $y'(x)$ at $(0,0)$ if

$$\sin(x+3y) = x + y\ .$$

Solution:
a) This is a related rates problem. Differentiate to get $4x^3 \cdot 5 + 12y^3 \cdot y' = 4y'$. Now fill in $x = y = 1$ to get $y' = -20/8$ which is $\boxed{-5/2}$.
b) This is an implicit differentiation problem. We differentiate with respect to x and get $\cos(x+3y)(1+3y') = 1 + y'$. At $x = y = 0$ we have $1 + 3y' = 1 + y'$ so that $\boxed{y' = 0}$.

Problem 12) Various integration problems (10 points)

Find the anti-derivatives of the following functions:

a) (2 points) $f(x) = x\log(x) + \frac{1}{1+x^2}$.

b) (3 points) $f(x) = \frac{2x}{x^2+1} + \frac{1}{x^2-4}$.

c) (2 points) $f(x) = \sqrt{16-x^2} + \frac{1}{\sqrt{1-x^2}}$.

d) (3 points) $f(x) = \log(x) + \frac{1}{x\log(x)}$.

Solution:
1) The first part appeared in the volume problem before. We have a) $x^2\log(x)/2 - x^2/4 + \arctan(x) + C$.
b) $\log(x^2+1) - \log(x-2)/4 + \log(x+2)/4 + C$.
c) $\arcsin(x/4) + (1 - \cos(2\arcsin(x/4)))/4 + C$.
d) $x\log(x) - x + \log(\log(x)) + C$.

Problem 13) Applications (10 points)

a) (3 points) Find the CDF $\int_0^x f(t)\,dt$ for the PDF which is $f(x) = \exp(-x/3)/3$ for $x \geq 0$ and 0 for $x < 0$.

b) (2 points) Perform a single Newton step for the function $f(x) = \sin(x)$ starting at $x = \pi/3$.

c) (3 points) Check whether the function $f(x) = 1/(2x^2)$ on the real line $(-\infty, \infty)$ is a probability density function.

d) (2 points) A rower produces the power $P(t)$ is $\sin^2(10t)$. Find the energy when rowing starting at time $t = 0$ and ending at $t = 2\pi$.

Solution:
a) The antiderivative is $\boxed{(1 - e^{-x/3}}$.
b) $\pi/3 - \tan(\pi/3) = \boxed{\pi/3 - \sqrt{3}}$.
c) The improper integral does not exist. It is **not** a probability density function.
d) Integrate $\int_0^{2\pi} \sin^2(10t)\,dt = \int_0^{2\pi}(1 - \cos(20t))/2\,dx = x/2 - \sin(20t)/40|_0^{2\pi} = \boxed{\pi}$.

5/17/2014: Final Practice B

Your Name:

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- All functions f if not specified otherwise can be assumed to be smooth so that arbitrary many derivatives can be taken.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) [T] [F] The definite integral $\int_0^{2\pi} \sin^2(5x)\, dx$ is zero.

Solution:
The integrand is never negative and almost everywhere positive.

2) [T] [F] The intermediate value theorem assures that the function $\exp(\sin(x))$ has a root in the interval $(0, 2\pi)$.

Solution:
The function $\exp(\sin(x))$ is never zero.

3) [T] [F] $\frac{d}{dx}\cos(4x) = -4\sin(4x)$.

Solution:
differentiate

4) [T] [F] If $f''(1) < 0$ then 1 is a local maximum of f.

Solution:
It also has to be a critical point.

5) [T] [F] The derivative of $1/x$ is $\log(x)$ for all $x > 0$.

Solution:
It is the anti-derivative, not the anti derivative

6) [T] [F] The limit of $\sin(3x)/(5x)$ for $x \to 0$ exists and is equal to $3/5$.

Solution:
Use Hôpital

7) [T] [F] The function $(e^t - 1)/t$ has the limit 1 as t goes to zero.

Solution:
Use Hopital

8) [T] [F] The derivative of $f(f(x))$ is $f'(f'(x))$ for any differentiable function f.

Solution:
This is not the chain rule

9) [T] [F] A monotonically increasing function f has no point x, where $f'(x) < 0$.

Solution:
Increasing means that the derivative is positive.

10) [T] [F] The function $f(x) = \exp(-x^2)$ has an inflection point x somewhere on the real line.

Solution:
The second derivative can be zero. One can see this by looking at the graph.

11) [T] [F] The function $f(x) = (1 - x^3)/(1 + x)$ has a limit for $x \to -1$.

Solution:
The top $1 - x^3$ is not zero at $x = -1$ so that the function has a pole

12) [T] [F] If we know the marginal cost for all quantities x as well as the total cost for $x = 1$ we know the total cost for all x.

Solution:
We can form the anti derivative and fix the constant from $F(1)$.

13) [T] [F] The function f which satisfies $f(x) = 0$ for $x < 0$ and $f(x) = e^{-x}$ for $x \geq 0$ is a probability density function.

Solution:
True, it is nonnegative every where and the total integral is 1.

14) [T] [F] The differentiation rule $(f \cdot g)' = f'(g(x)) \cdot g'(x)$ holds for all differentiable functions f, g.

Solution:
We would need the Leibniz product rule, not the chain rule.

15) [T] [F] Hôpital's rule assures that $\cos(x)/\sin(x)$ has a limit as $x \to 0$.

Solution:
The nominator does not go to zero for $x \to 0$.

16) [T] [F] A Newton step for the function f is $T(x) = x - \frac{f(x)}{f'(x)}$.

Solution:
By definition

17) [T] [F] The family of functions $f_c(x) = cx^2$ where c is a parameter has a catastrophe at $x = 0$.

Solution:
For $c < 0$ we have a local max, for $c > 0$ we have a local min.

18) [T] [F] The fundamental theorem of calculus implies $\int_{-x}^{x} f'(t)\, dt = f(x) - f(-x)$ for all differentiable functions f.

Solution:
Yes, this is the most important result in this course.

19) [T] [F] If f is a smooth function for which $f''(x) = 0$ everywhere, then f is constant.

Solution:
It can be linear

20) [T] [F] The function $f(x) = \sin(x)/(1 - \cos(x))$ can be assigned a value $f(0)$ such that $f(x)$ is continuous at 0.

Solution:
Use l'Hopital to see that the limit is the same as $\lim_{x\to 0} \cos(x)/\sin(x)$ which has no limit at $x = 0$.

Problem 2) Matching problem (10 points) Only short answers are needed.

We name some important concepts in this course. To do so, please complete the sentences with one or two words. Each question is one point.

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ is called the		of f.
$f'(x) = 0, f''(x) > 0$ implies that x is a		of f.
The sum $\frac{1}{n}[f(0) + f(1/n) + f(2/n) + \ldots + f((n-1)/n) + f(1)]$ is called a		sum.
If $f(0) = -3$ and $f(4) = 8$, then f has a root on the interval $(0, 4)$ by the		theorem.
There is a point $x \in (0, 1)$ where $f'(x) = f(1) - f(0)$ by the		theorem.
The expansion rate $r'(t)$ can be obtained from $d/dtV(r(t)) = -5$ by the method of		rates.
The anti derivative $\int_{-\infty}^{x} f(t)\, dt$ of a probability density function f is called the		function.
A point x for which $f(x) = 0$ is called a		of f.
A point x for which $f''(x) = 0$ is called an		of f.
At a point x for which $f''(x) > 0$, the function is called		up.

Solution:
Derivative
Local minimum
Riemann sum
Intermediate value
Mean value
Related
Cumulative distribution
Root
Inflection Point
Concave

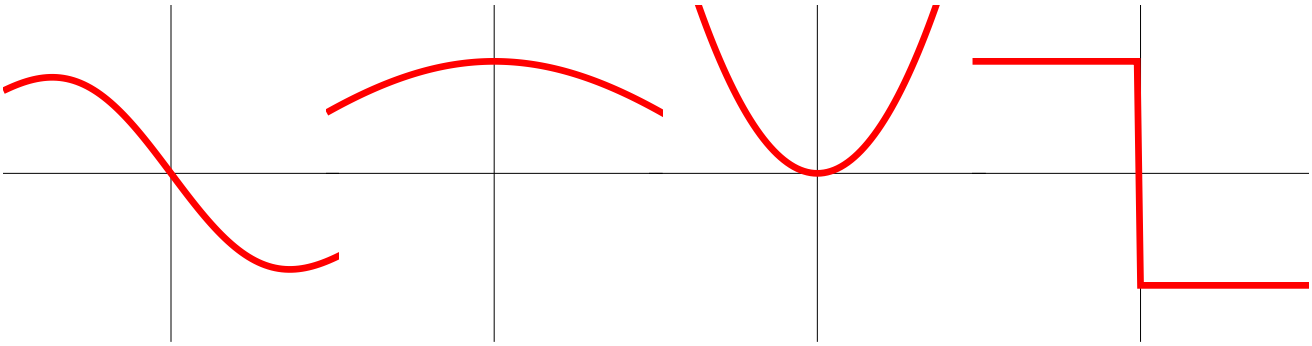
Problem 3) Matching or short answer problem (10 points). No justifications are needed.

a) (4 points) Find the relation between the following functions:

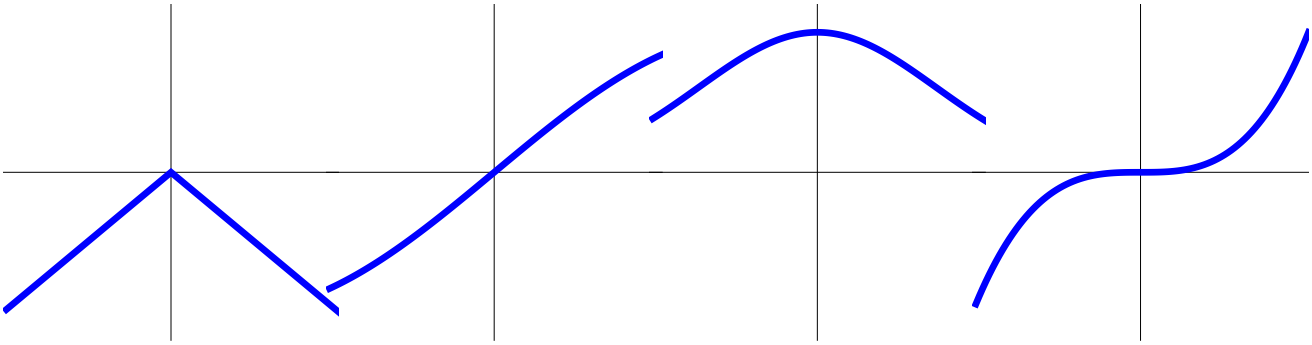
function f	function g	$f = g'$	$g = f'$	none
$\log \lvert \sin(x) \rvert$	$\cot(x)$			
$1/\cos^2(x)$	$\tan(x))$			
x^5	$5x^4$			
$1/x^2$	$-1/x$			
$\sin(\log(x))$	$\cos(\log(x))/x$			

b) (3 points) Match the following functions (a-d) with a choice of **anti-derivatives** (1-4).

Function a)-d)	Fill in 1)-4)
graph a)	
graph b)	
graph c)	
graph d)	



a) b) c) d)



1) 2) 3) 4)

c) (3 points) Find the limits for $x \to 0$

Function f	$\lim_{x\to 0} f(x)$
$x/(e^{2x}-1)$	
$(e^{2x}-1)/(e^{3x}-1)$	
$\sin(3x)/\sin(5x)$	

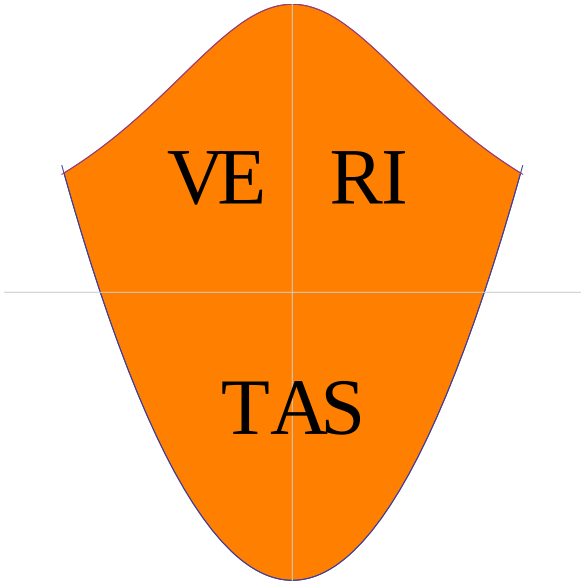
Solution:
a)

function f	function g	$f = g'$	$g = f'$	none
$\log\lvert\sin(x)\rvert$	$\cot(x)$		*	
$1/\cos^2(x)$	$\tan(x))$	*		
x^5	$5x^4$		*	
$1/x^2$	$-1/x$	*		
$\sin(\log(x))$	$\cos(\log(x))/x$		*	

b) 3,2,4,1
c) Use l'Hopital: 1/2,2/3,3/5

Problem 4) Area computation (10 points)

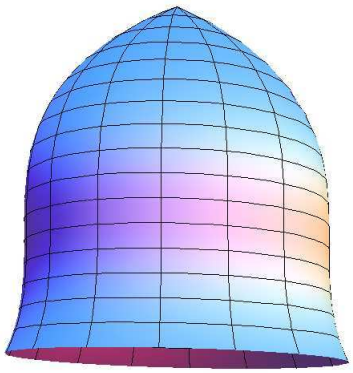
Find the area of the shield shaped region bound by the two curves $1/(1+x^2)$ and x^2-1.



Solution:
The two curves intersect at $x = \pm 2^{1/4}$.
$\int_{-2^{1/4}}^{2^{1/4}} \frac{1}{1+x^2} - x^2 + 1 \, dx = \arctan(x) - x^3/3 + x|_{-2^{1/4}}^{2^{1/4}} = 2\arctan(2^{1/4}) - (2/3)2^{3/4} + 2 \cdot 2^{1/4}$.

Problem 5) Volume computation (10 points)

Did you know that there is a scaled copy of the **liberty bell** on the campus of the Harvard business school? Here we compute its volume. Find the volume of the rotationally symmetric solid if the radius $r(z)$ at height z is $r(z) = 8 - (z-1)^3$ and the height z of the bell is between 0 and 3.



Solution:

$$\pi \int_0^3 \pi(8-(z-1)^3)^2\, dz = \pi \int_{-1}^2 (8-u^3)^2\, du = \pi \int_{-1}^2 64-16u^3+u^6\, du = \pi(64z-16u^4/4+u^7/7)|_{-1}^3 = \pi 1053/7;$$

Problem 6) Improper integrals (10 points)

a) (5 points) Find the integral or state that it does not exist

$$\int_1^\infty \frac{1}{x^4}\, dx\ .$$

b) (5 points) Find the integral or state that it does not exist

$$\int_1^\infty \frac{1}{x^{3/2}}\, dx\ .$$

Solution:

a)

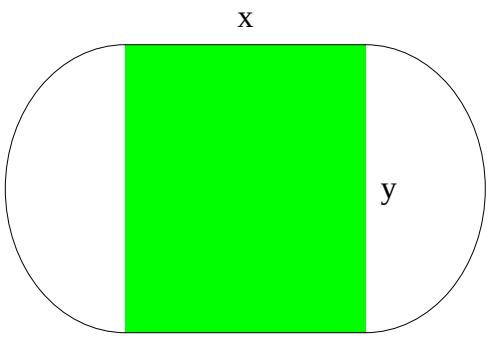
$$\int_1^\infty \frac{1}{x^4}\, dx = \frac{-x^{-3}}{3}\Big|_1^\infty = \frac{1}{3}\ .$$

b)

$$\int_1^\infty \frac{1}{x^{3/2}}\, dx = -2x^{-1/2}|1^\infty = 2\ .$$

Problem 7) Extrema (10 points)

The Harvard stadium has a track which encloses a rectangular field of dimensions x, y. The circumference of the track is $400 = 2\pi y + 2x$ and is fixed. We want to maximize the area xy for a play field. Which x achieves this?



Solution:
Solve for $y = (200 - x)/\pi$ and plug this into the function to get

$$f(x) = xy = x(200 - x)/\pi \ .$$

To find the maximum of this function, we differentiate with respect to x and look where the derivative is zero:

$$f'(x) = (200 - 2x)/\pi = 0$$

showing that $x = 100$ is the maximum.

Problem 8) Integration by parts (10 points)

Find the antiderivative:

$$\int (x-1)^4 \exp(x+1)\, dx \ .$$

Solution:
Use the Tic-Tac-Toe integration method:

$(x-1)^4$	$\exp(x+1)$	
$4(x-1)^3$	$\exp(x+1)$	$\oplus$
$12(x-1)^2$	$\exp(x+1)$	$\ominus$
$24(x-1)$	$\exp(x+1)$	$\oplus$
24	$\exp(x+1)$	$\ominus$
0	$\exp(x+1)$	$\oplus$

Adding things up gives

$$e^{x+1}[(x-1)^4 - 4(x-1)^3 + 12(x-1)^2 - 24(x-1) + 24] \, .$$

Problem 9) Substitution (10 points)

a) (3 points) Solve the integral $\int e^{x^2} 2x \, dx$.

b) (3 points) Solve the integral $\int 2x \log(x^2) \, dx$.

c) (4 points) Find the integral $\int e^{-2e^x} e^x \, dx$.

Solution:
These are all standard substitution problems:
a) $e^{x^2} + c$
b) $x^2 \log(x^2) - x^2 + c$
c) $-e^{-2x^2}/2 + c$

Problem 10) Partial fractions (10 points)

a) (5 points) Find the definite integral

$$\int_1^5 \frac{1}{(x-4)(x-2)} \, dx \, .$$

b) (5 points) Find the indefinite integral

$$\int \frac{1}{(x-1)(x-3)(x-5)} \, dx \, .$$

Solution:
In both problems we can find the coefficients quickly with the l'Hopital method: a) $\int_1^5 \frac{1}{(x-4)(x-2)}\,dx = \frac{1}{2}\int_1^5 [\frac{1}{x-4} - \frac{1}{x-2}]\,dx = \frac{1}{2}[\log|x-4| - \log|x-2|]|_1^5 = -\log(3)$. b) The factorization

$$\frac{1}{(x-1)(x-3)(x-5)} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{x-5}$$

can be obtained quickly from l'Hopital: $A = \lim_{x\to 1} \frac{1}{(x-3)(x-5)} = \frac{1}{8}$ and $B = \lim_{x\to 3} \frac{1}{(x-1)(x-5)} = -\frac{1}{4}$ and $C = \lim_{x\to 5} \frac{1}{(x-1)(x-3)} = \frac{1}{8}$ so that the result is

$$[\log|x-1| - 2\log|x-3| + \log|x-5|]/8\ .$$

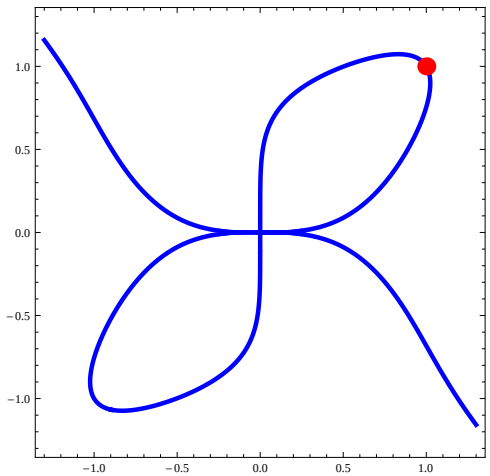
[P.S. As in the homeworks, we do not worry in a) that these are improper integrals, integrating over the logarithmic singularity. They are no problem because the integral of $\log|x|$ is $x\log|x| - x$ which has a limit 0 for $x \to 0$.]

Problem 11) Related rates (10 points)

The coordinates of a car on a freeway intersection are $x(t)$ and $y(t)$. They are related by

$$x^7 + y^7 = 2xy^2\ .$$

We know $x' = 3$ at $x = 1, y = 1$. Find the derivative y'.



Solution:
Differentiate the relation with respect to t and solve for y':

$$7x^6x' + 6y^2y' = 2x'y^2 + 4xyy'\ .$$

Therefore,

$$y' = (7x^6x' - 7y^6x')/(4xy - 6y^2)\ .$$

The final answer is $\boxed{-5}$.

Problem 12) Various integration problems (10 points)

Find the anti-derivatives of the following functions:

a) (2 points) $f(x) = \sin^5(x)\cos(x)$.

b) (3 points) $f(x) = \frac{1}{x^2+1} + \frac{1}{x^2-1}$.

c) (2 points) $f(x) = \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}}$.

d) (3 points) $f(x) = \log(x) + \frac{1}{\log(x)}$.

Solution:
a) $\sin^6(x)/6 + c$
b) $\arctan(x) + \log(x-1)/2 + \log(x+1)/2 + c$
c) $\arcsin(x) + \arcsin(x)/2 + \sin(2\arcsin(x)))/4$
d) $x\log(x) - x + li(x)$ the second integral is a nonelementary integral. Was a freebe. You got 3 points even without solving that...

Problem 13) Applications (10 points)

a) (5 points) We know the total cost $F(x) = -x^3 + 2x^2 + 4x + 1$ for the quantity x. In order to find the positive **break-even point** x satisfying $f(x) = g(x)$, where $g(x) = F(x)/x$ is the total cost and $f(x) = F'(x)$ is the marginal cost, we do - how sweet it is - find the maximum of the average cost $g(x) = F(x)/x$. Find the maximum!

b) (5 points) We know the **"velocity"**, **"acceleration"** and **"jerk"** as the first second and third derivative of position. The fourth, fifth and sixth derivatives of position as a function of time are called **"snap"**, **"crackle"** and **"pop"** according to characters used in a cereal add. Assume we know the snap $x''''(t) = t$. Find $x(t)$ satisfying $x(0) = x'(0) = x''(0) = 0, x'''(0) = 0$.

Solution:
a) We have to solve the equation $g(x) = 0$ by the strawberry theorem. Giving the equation $-x^2 + 2x + 4 + 1/x = 0$ was enough. The solution needs to be evaluated numerically, for example with Newton.
b) Integrate 4 times to get $x(t) = t^5/120$. All constants are zero.

5/17/2014: Final Exam Practice C

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] The quantum exponential function $\exp_h(x) = (1+h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$ for $h > 0$.

Solution:
This is an extremely important identity because it leads to the property $\exp' = \exp$.

2) [T] [F] The function $\text{sinc}(x) = \sin(x)/x$ has a critical point at $x = 0$.

Solution:
Take the derivative $[x\cos(x) - \sin(x)]/x^2$ and apply l'Hopital to get 0. This was not an easy TF problem.

3) [T] [F] The limit of $1/\log(1/|x|)$ for $x \to 0$ exists.

Solution:
We have seen this in class and in a midterm. Since $\log(1/|x|) = -log|x|$ goes to infinity for $|x| \to 0$, we know that $1/\log(1/|x|)$ converges to 0.

4) [T] [F] The strawberry theorem tells that for any $f(x)$, its anti-derivative $F(x)$ and $g(x) = F(x)/x$ the points $f = g$ are the points where $g' = 0$.

Solution:
Yes, that is the theorem. Tasty!

5) [T] [F] The function $f(x) = \tan(x)$ has a vertical asymptote at $x = \pi/2$.

Solution:
Yes, if the angle x goes to $\pi/2$, then this means the slope goes to infinity.

6) [T] [F] The function $x/(1+x)$ converges to 1 for $x \to \infty$ and has therefore a horizontal asymptote.

Solution:
We could use the Hopital rule to see that the limit $x \to \infty$ is $1/1 = 1$. It can also be seen intuitively. If x=1000 for example, we have 1000/1001.

7) [T] [F] The function $f(x) = \tan(x)$ is odd: it satisfies $f(x) = -f(-x)$.

Solution:
Yes, it appears odd but it is true: tan is odd. Even tomorrow.

8) [T] [F] The function $\sin^3(x)/x^2$ is continuous on the real line.

Solution:
It can be written as $\text{sinc}(x)^2 \sin(x)$, a product of two continuous functions.

9) [T] [F] With $Df(x) = f(x+1) - f(x)$ we have $D(fg)(x) = Dfg(x+1) + f(x)Dg(x)$.

Solution:
This is also called the quantum Leibniz rule.

10) [T] [F] If f has a critical point 0 then f has a minimum or maximum at 0.

Solution:
The function $f(x) = x^3$ is a counter example.

11) [T] [F] The limit of $[\frac{1}{3+h} - \frac{1}{3}]/h$ for $h \to 0$ is $-1/9$.

Solution:
This is the derivative of $1/x$ at $x = 3$.

12) [T] [F] The function $(\cos(x) + \sin(3x))/(\sin(4x) + \cos(3x))$ can be integrated using trig substitution.

Solution:
Yes, this works with the magic substitution $u = \tan(x/2)$, $dx = \frac{2du}{(1+u^2)}$, $\sin(x) = \frac{2u}{1+u^2}$ and $\cos(x) = \frac{1-u^2}{1+u^2}$.

13) [T] [F] The marginal cost is the anti-derivative of the total cost.

Solution:
It is the derivative of the total cost, not the anti-derivative.

14) [T] [F] The cumulative distribution function is the anti-derivative of the probability density function.

Solution:
Yes, now we are talking.

15) [T] [F] The function $\sqrt{1-x^2}$ can be integrated by a trig substitution $x = \cos(u)$.

Solution:
For example, one could also take $x = \sin(u)$.

16) [T] [F] The integral $\int_0^1 1/x^2\, dx$ is finite.

Solution:
The anti-derivative of the function inside the integral is $1/x$ which does not look good in the limit $x \to 0$. Nope, the integral does not exist. Intuitively, $1/x^2$ just goes to infinity too fast if $x \to 0$.

17) [T] [F] The chain rule tells that $d/dx f(g(x)) = f'(x)g'(x)$.

Solution:
There is a missing link in the chain. Look it up.

18) [T] [F] For the function $f(x) = \sin(100x)$ the hull function is constant.

Solution:
Yes the maxima are at 1. Connecting the maxima gives the line $x = 1$. The lower hull is $x = -1$. —

19) [T] [F] The trapezoid method is also called Simpson rule.

Solution:
No, the Simpson rules are more sophisticated and use 2 or 3 values in between the interval.

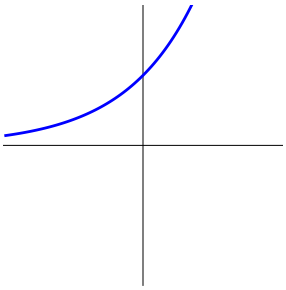
20) [T] [F] If $f''(x) > 0$, then the curvature of f is positive.

Solution:
Indeed, the curvature is defined as $\frac{f''(x)}{(1+f'(x)^2)^{3/2}}$.

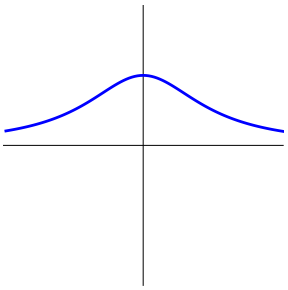
Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their derivatives.

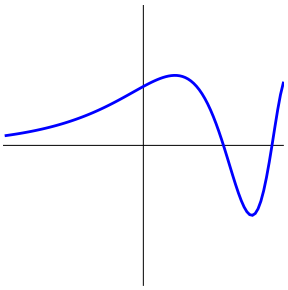
Function	Fill in the numbers 1-8
graph a)	
graph b)	
graph c)	
graph d)	
graph e)	
graph f)	
graph g)	
graph h)	



a)



b)



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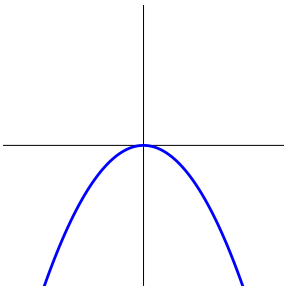

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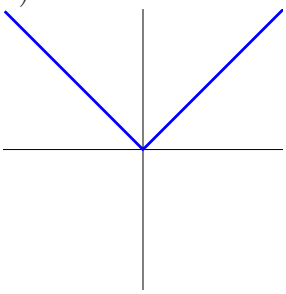

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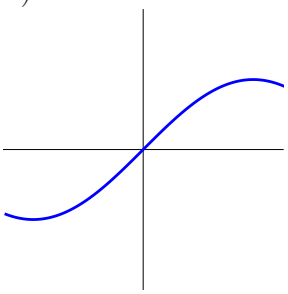

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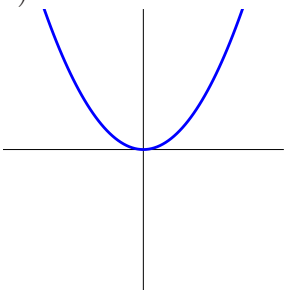
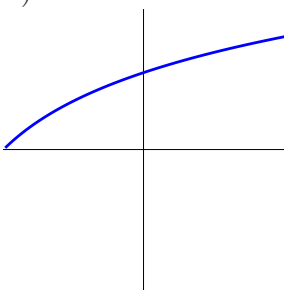


g)



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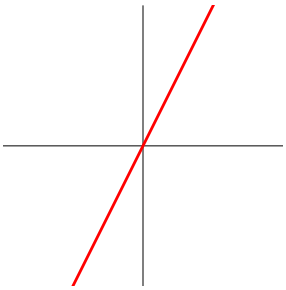
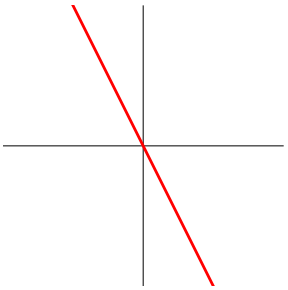


1)



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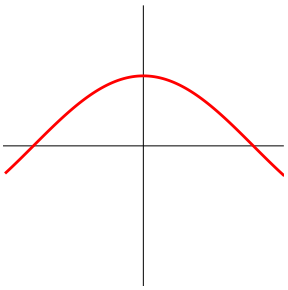

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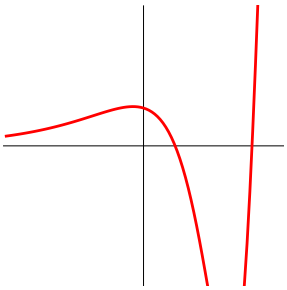

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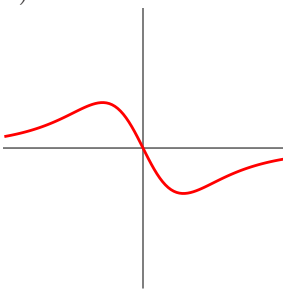

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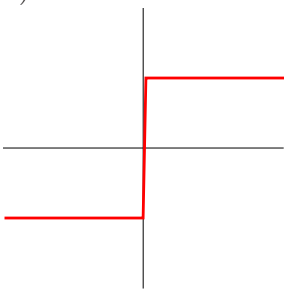

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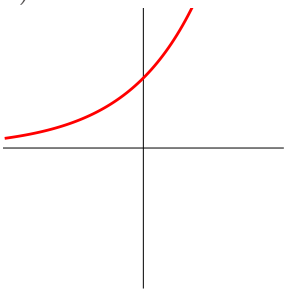

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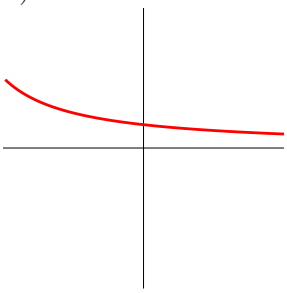


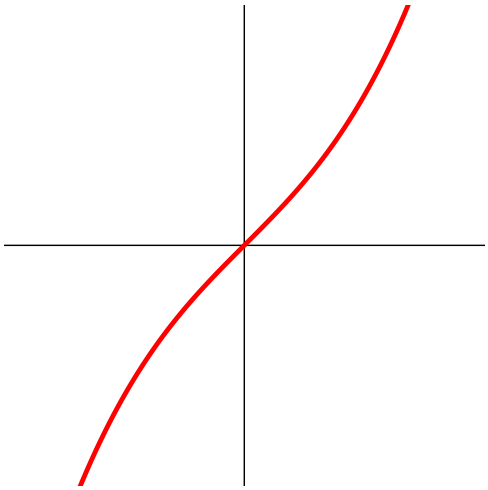
8)

Solution:

Function	Fill in the numbers 1-8
graph a)	7
graph b)	5
graph c)	4
graph d)	2
graph e)	6
graph f)	3
graph g)	1
graph h)	8

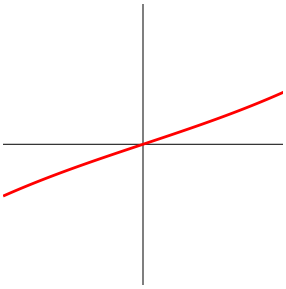
Problem 3) Matching problem (10 points) No justifications are needed.

Here is the graph of a function $f(x)$. Match the following modifications



Match the following functions with their graphs.

Function	Fill in 1)-8)
$f(x-1)$	
$f'(x)$	
$f(x+1)$	
$f^{-1}(x)$	
$f(x/2)$	
$f(3x)$	
$1/f(x)$	
$f(x)+1$	



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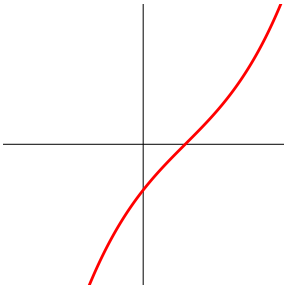

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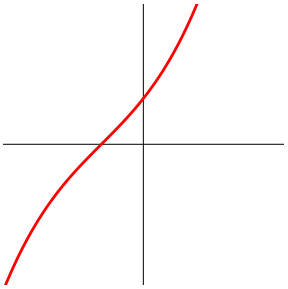

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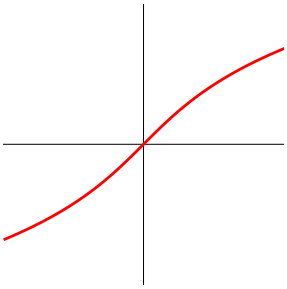

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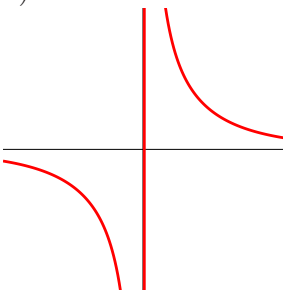
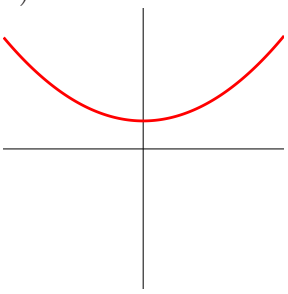


5)



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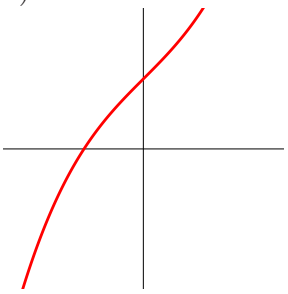
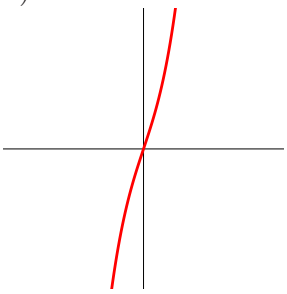


7)

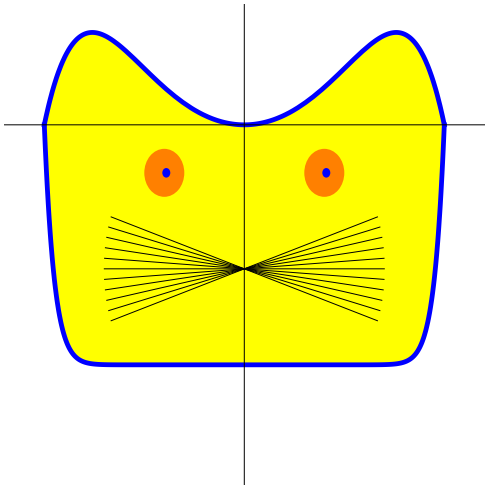


8)

Solution:
2,6,3,4,1,8,5,7

Problem 4) Area computation (10 points)

Find the area of the **cat region** which is the region enclosed by the functions $f(x) = x^{20} - 1$ and $g(x) = x^2 - x^6$. No need to count in the whiskers.

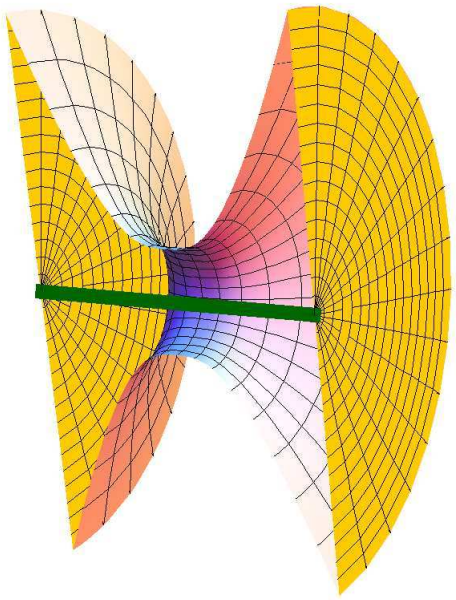


Solution:
We first have to find where the two graphs intersect to determine the integration bounds. They intersect at $x = 1$ and $x = -1$. Next, we have to know which function is above and which is below. If we look at $x = 0$, then we see that f is below. We can see this also from the fact that f is always nonpositive and g is always non-negative. Therefore,

$$\int_{-1}^{1} x^2 - x^6 - (x^{20} - 1)\, dx = (x + x^3/3 - x^7/7 - x^{21}/21)|_{-1}^{1} = 16/7\ .$$

Problem 5) Volume computation (10 points)

We spin the graph of the function $f(x) = \sqrt{1 + |x|^3}$ around the x axes and get a solid of revolution. What is the volume of this solid enclosed between $x = -3$ and $x = 3$? The picture shows half of this sold.



Solution:
The area at position x is $\pi f(x)^2 = \pi(1 + |x|^3)$. The volume of a slice of thickness dx is $\pi(1 + |x|^3)\, dx$. We have to integrate this from $x = -3$ to $x = 3$. To avoid the absolute value, we take twice the integral from 0 to 3 and have

$$2\pi \int_0^3 (1 + x^3)\, dx = 2\pi(x + x^4/4)|_0^3 = 2\pi 93/4 = \pi 93/2\ .$$

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals

a) $\int_{-1}^{1} \frac{1}{1+x^2}\, dx$

b) $\int_1^2 x^2 + \sqrt{x}\, dx$

c) $\int_0^{\sqrt{\pi}} \sin(x^2) 2x\, dx$.

d) $\int_0^1 \log(4+x)\, dx$.

Solution:
a) The anti-derivative is $\arctan(x)$. Evaluated at 1 it is $\pi/4$ at -1 it is $-\pi/4$. The integral is $\pi/2$.
b) The integral is $(x^3/3 + x^{3/2}(2/3))|_1^2 = (5+4\sqrt{2})/3$.
c) $-\cos(x^2)|_0^{\sqrt{\pi}} = 2$.
d) Write $4+x=u$ and get $\int_4^5 \log(u)\, du = u\log(u) - u|_4^5 = 5\log(5) - 4\log(4) - 1$.

Problem 7) Extrema (10 points)

a) (7 points) Analyse the local extrema of the function

$$f(x) = \frac{x}{1+x^2}$$

on the real axes using the second derivative test.

b) (3 points) Are there any global extrema?

Solution:
a) The derivative is

$$f'(x) = \frac{1-x^2}{1+x^2)^2}\,.$$

The extrema are $x=1$ and $x=-1$. The second derivative is

$$f''(x) = \frac{2x(x^2-3)}{(1+x^2)^3}\,.$$

b) Asymptotically, we have $f(x) \to 0$ for $|x| \to \infty$. This means that $x=1$ is a global maximum and $x=-1$ is a global minimum.

Problem 8) Integration by parts (10 points)

a) (5 points) Find the anti-derivative of

$$f(x) = \sin(4x)\cos(3x) \ .$$

b) (5 points) Find the anti-derivative of

$$f(x) = (x-1)^2 \sin(1+x) \ .$$

Solution:
a) This is a problem, where we have to do integration by parts twice. It is a "merry go round problem". Call I the integral. Then do integration by parts twice to isolate $I = \int \sin(4x)\cos(3x)\ dx$. We will also encounter $J = \int \cos(4x)\sin(3x)\ dx$. Then integration by parts gives

$$I = \sin(4x)\sin(3x)/3 - (4/3)J$$

$$J = -\cos(4x)\cos(3x)/3 - (4/3)I$$

so that $I = \sin(4)\sin(3x)/3 + (4/9)\cos(4x)\cos(3x)) + (16/9)I$ so that $I(1-16/9) = \sin(4)\sin(3x)(1/3) + (4/9)(\cos(4x)\cos(3x))$ and

$$I = [\sin(4x)\sin(3x)(1/3) + (4/9)(\cos(4x)\cos(3x))](-9/7) \ .$$

b) Use integration by parts twice. We end up with

$$-(x-1)^2\cos(1+x) + (2x-2)\sin(1+x) + 2\cos(1+x) \ .$$

Problem 9) Substitution (10 points)

a) (3 points) Find the integral $\int 3x\sqrt{5x^2-5}\ dx$.

b) (3 points) What is the anti-derivative of $\int \exp(x^2-x)(4x-2)$?

c) (4 points) Evaluate the definite integral

$$\int_0^{\pi/2} \sqrt{1-\cos(x)}\sin(x)\ dx \ .$$

Solution:
a) $(5x^2-5)^{3/2}/5 + C$.
b) $2\exp(x^2-x) + C$.
c) $(1-\cos(x))^{3/2}(2/3)|_0^{\pi/2} = 2/3$.

Problem 10) Partial fractions, Trig substitution (10 points)

a) Solve the integral

$$\int \frac{2-x+x^2}{(1-x)(1+x^2)}$$

by writing

$$\frac{2-x+x^2}{(1-x)(1+x^2)} = \frac{A}{1+x^2} + \frac{B}{1-x} \, .$$

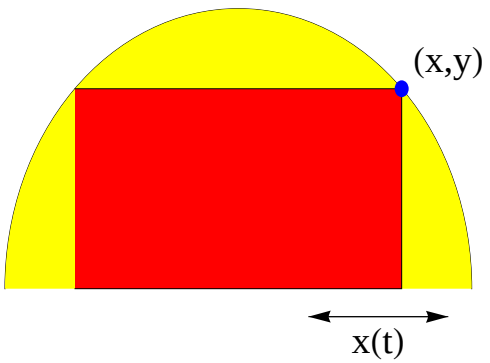
b) Evaluate the integral $\int \sqrt{1-x^2}x \, dx$.

Solution:
a) $\arctan(x) - \log(x-1)$.
b) Write $x = \sin(u)$ to get $\int \cos^2(u)\sin(u) \, du = -\cos(u)^{3/2}/3$. This can also be solved more easily by substitution: $u = 1-x^2, du = -2xdx$ etc.

Problem 11) Related rates (10 points)

a) (7 points) A rectangle with corners at $(-x,0),(x,0),(x,y),(-x,y)$ is inscribed in a half circle $x^2+y^2=1$ where $y \geq 0$ is in the upper half plane. Assume we move x as $x(t) = t^2$. Find the rate of change of $y(t)$.

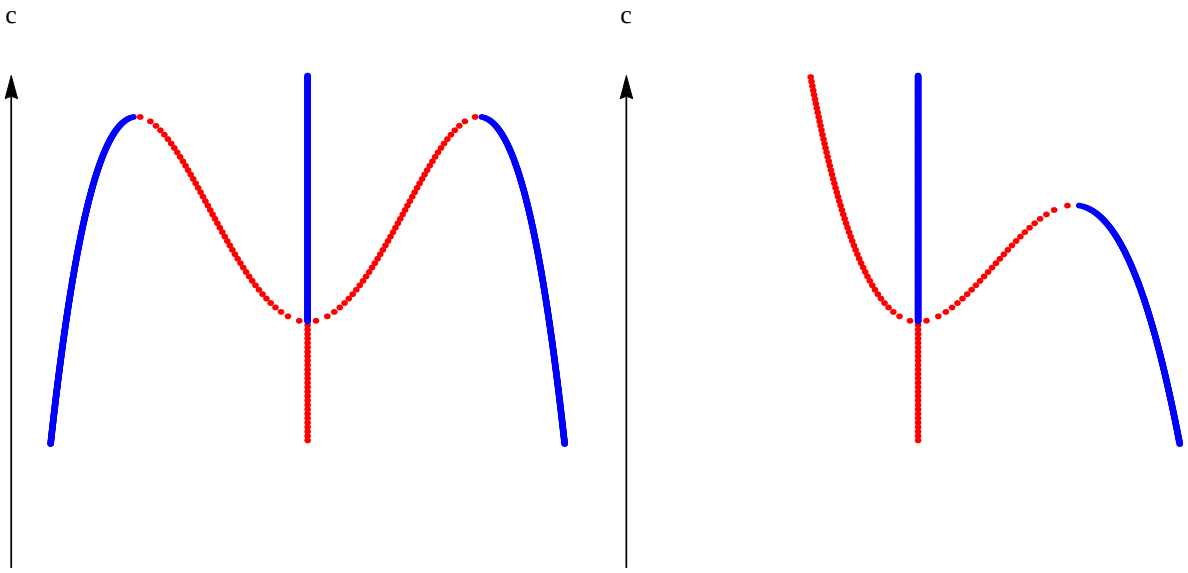
b) (3 points) Find the rate of change of the area $A(t) = 2x(t)y(t)$ of the rectangle.



Solution:
a) $2xx' + 2yy' = 0$ so that $y' = -xx'/y = -t^2(2t)/\sqrt{1-t^4}$.
b) $A' = 2x'y + 2xy' = 4ty + 2t^2y'$. We have to plug in y' from a).

Problem 12) Catastrophes (10 points)

The following two pictures show bifurcation diagrams. The vertical axes is the deformation parameter c. On the left hand side, we see the bifurcation diagram of the function $f(x) = x^6 - x^4 + cx^2$, on the right hand side the bifurcation diagram of the function $f(x) = x^5 - x^4 + cx^2$. As done in class and homework, the bolder continuously drawn graphs show the motion of the local minima and the lighter dotted lines show the motion of the local maxima. In both cases, determine the catastrophe for the criticaal point $x = 0$.



Solution:
Bifurcation parameters are parameter values where a local minimum disappears. In the first picture, the bifurcations happen at $c = 0$.

Problem 13) Applications (10 points)

The **Laplace distribution** is a distribution on the entire real line which has the probability density $f(x) = e^{-|x|}/2$. The variance of this distribution is the integral

$$\int_{-\infty}^{\infty} x^2 f(x)\, dx \ .$$

Find it.

Solution:
The integral is $\int_0^\infty x^2 e^{-x}$. We compute this using integration by parts:

x^2	$\exp(-x)$	
$2x$	$-\exp(-x)$	$\oplus$
2	$\exp(-x)$	$\ominus$
0	$-\exp(-x)$	$\oplus$

The integral is $2(-x^2-2x-2)e^{-x}|_0^\infty$. The definite integral $2\int_0^\infty$ is 4. The original function had $e^{-x}/2$. The final result is $\boxed{2}$.

5/17/2014: Final Exam Practice D

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] If a function $f(x)$ has a critical point 0 and $f''(0) = 0$ then 0 is neither a maximum nor minimum.

Solution:
We can have $f(x) = x^4$ for example.

2) [T] [F] If $f' = g$ then $\int_0^x g(x) = f(x)$.

Solution:
Not necessarily. The right answer is $\int_0^x g(x)\,dx = f(x) - f(0)$.

3) [T] [F] The function $f(x) = 1/x$ has the derivative $\log(x)$.

Solution:
No! We went in the wrong direction. The anti derivative is $\log(x)$.

4) [T] [F] The function $f(x) = \arctan(x)$ has the derivative $1/\cos^2(x)$.

Solution:
It is $\tan(x)$ which has the derivative $1/\cos^2(x)$, not arctan.

5) [T] [F] The fundamental theorem of calculus implies that $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Solution:
Right on! The most important result in calculus is the fundamental theorem of calculus. Hit me again.

6) [T] [F] $\lim_{x\to 8} 1/(x-8) = \infty$ implies $\lim_{x\to 3} 1/(x-3) = \omega$.

Solution:
This is a classical calculus joke. The fact that it is funny does not make it true.

7) [T] [F] A continuous function which satisfies $\lim_{x\to-\infty} f(x) = 3$ and $\lim_{x\to\infty} f(x) = 5$ has a root.

Solution:
It would be a consequence of the intermediate value theorem if the sign would be different but the signs are not different .

8) [T] [F] The function $f(x) = (x^7 - 1)/(x - 1)$ has a limit at $x = 1$.

Solution:
This is a classical case for healing: we can factor out $(x - 1)$ and see $f(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ for x different than 1. The limit now exists and is 7. Alternatively, we could have brought the function to Hopital and cured it there. Differentiating the top and bottom gives 7 too.

9) [T] [F] If $f_c(x)$ is an even function with parameter c and $f'(0) = 0$ and for $c < 3$ the function is concave up at $x = 0$ and for $c > 3$ the function is concave down at $x = 0$, then $c = 3$ is a catastrophe.

Solution:
Yes, an even function has a minimum at 0 if it is concave up and a maximum at 0 if it is concave down. At the parameter value $c = 3$ the nature of the critical point changes. This implies that it is a catastrophe.

10) [T] [F] The function $f(x) = +\sqrt{x^2}$ has a continuous derivative 1 everywhere.

Solution:
Agreed, thats a nasty question, but the plus sign should have taken the sting out of it. The function $f(x)$ satisfies $f(x) = |x|$ and has no derivative at $x = 0$.

11) [T] [F] A rower rows on the Charles river leaving at 5 PM at the Harvard boat house and returning at 6 PM. If $f(t)$ is the distance of the rower at time t to the boat house, then there is a point where $f'(t) = 0$.

Solution:
This is a consequence of Rolles theorem.

12) [T] [F] A global maximum of a function $f(x)$ on the interval $[0, 1]$ is a critical point.

Solution:
The extremum could be at the boundary.

13) [T] [F] A continuous function on the interval $[2, 3]$ has a global maximum and global minimum.

Solution:
Every continuous function on a closed interval has a maximum as well as a minimum somewhere.

14) [T] [F] The intermediate value theorem assures that if f is continuous on $[a, b]$ then there is a root of f in (a, b).

Solution:
One has to assume that $f(a), f(b)$ have different signs.

15) [T] [F] On an arbitrary floor, a square table can be turned so that it does not wobble any more.

Solution:
Yes, this is the greatest theorem ever.

16) [T] [F] The derivative of $\log(x)$ is $1/x$.

Solution:
Now we are talking. A previous question above had it wrong.

17) [T] [F] If f is the marginal cost and $F = \int_0^x f(x)\ dx$ the total cost and $g(x) = F(x)/x$ the average cost, then points where $f = g$ are called "break even points".

Solution:
Yes, this is precisely the definition of a break-even point. The Strawberry theorem assured that $g' = 0$ at those points.

18) [T] [F] At a function party, Log talks to Tan and the couple Sin and Cos, when she sees her friend Exp alone in a corner. Log: "What's wrong?" Exp: "I feel so lonely!" Log: "Go integrate yourself!" Exp sobbs: "Won't change anything." Log: "You are so right".

Solution:
Yes, $\exp(x)' = \exp(x)$. There was a happy end nevertheless: Exp later met Cot and had a good time too.

19) [T] [F] If a car's position at time t is $f(t) = t^3 - t$, then its acceleration at $t = 1$ is 6.

Solution:
Differentiate twice.

20) [T] [F] For trig substitution, the identities $u = \tan(x/2)$, $dx = \frac{2du}{(1+u^2)}$, $\sin(x) = \frac{2u}{1+u^2}$, $\cos(x) = \frac{1-u^2}{1+u^2}$ are useful.

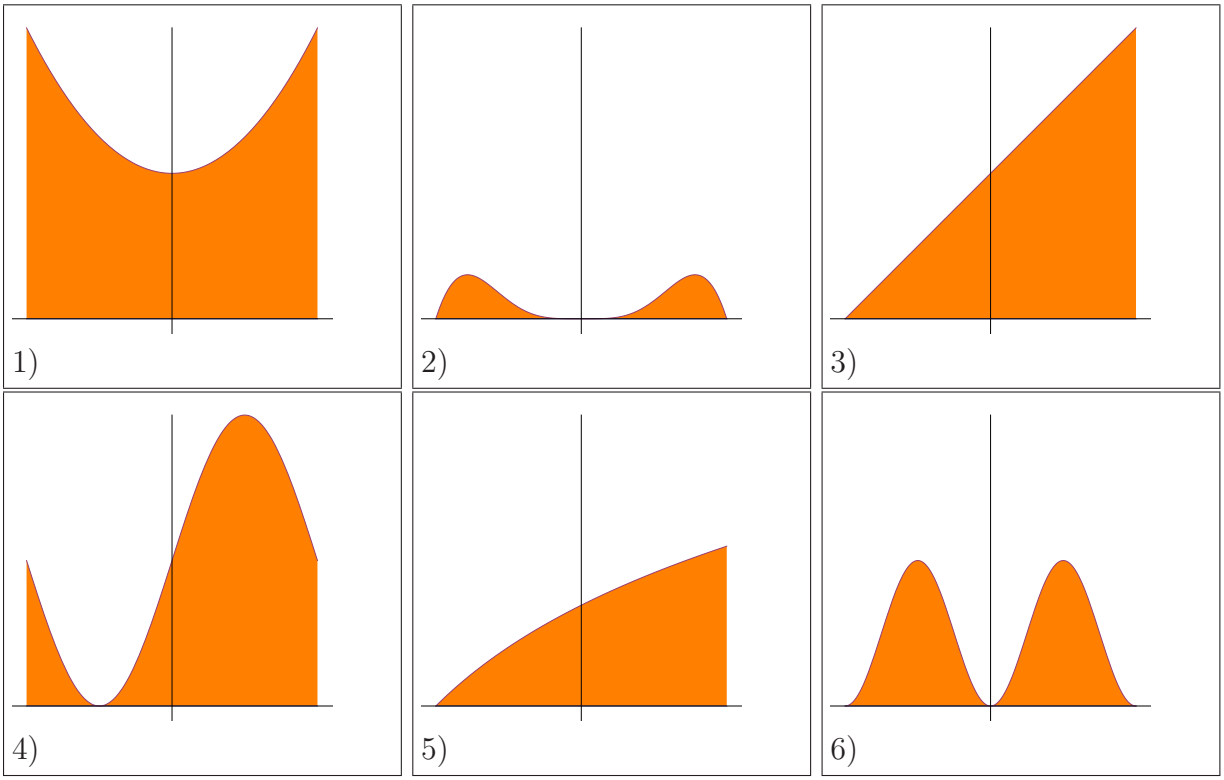
Solution:
They are and they are magic too.

Problem 2) Matching problem (10 points) No justifications are needed.

a) Match the following integrals with the graphs and (possibly signed) areas.

Integral	Enter 1-6
$\int_{-1}^{1} \sin(\pi x)x^3 \, dx.$	
$\int_{-1}^{1} \log(x+2) \, dx.$	
$\int_{-1}^{1} x+1 \, dx.$	

Integral	Enter 1-6
$\int_{-1}^{1} (1+\sin(\pi x)) \, dx.$	
$\int_{-1}^{1} \sin^2(x) \, dx.$	
$\int_{-1}^{1} x^2+1 \, dx.$	



Solution:
2) 5) 3) and 4) 6) 1)

Problem 3) Matching problem (10 points) No justifications are needed.

Determine from each of the following functions, whether discontinuities appears at $x = 0$ and if, which of the three type of discontinuities it is at 0.

Function	Jump discontinuity	Infinity	Oscillation	No discontinuity
$f(x) = \log(\lvert x\rvert^5)$				
$f(x) = \cos(5/x))$				
$f(x) = \cot(1/x)$				
$f(x) = \sin(x^2)/x^3$				
$f(x) = \arctan(\tan(x - \pi/2))$				
$f(x) = 1/\tan(x)$				
$f(x) = 1/\sin(x)$				
$f(x) = 1/\sin(1/x)$				
$f(x) = \sin(\exp(x))/\cos(x)$				
$f(x) = 1/\log\lvert x\rvert$				

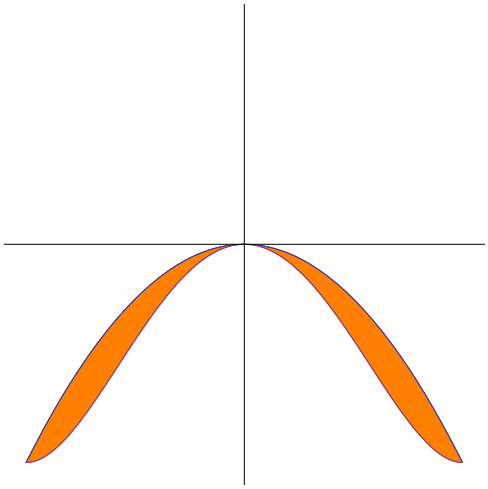
Solution:

Function	Jump discontinuity	Infinity	Oscillation	No discontinuity
$f(x) = \log(\lvert x\rvert^5)$		X		
$f(x) = \cos(5/x))$			X	
$f(x) = \cot(1/x)$			X	
$f(x) = \sin(x^2)/x^3$		X		
$f(x) = \arctan(\tan(x - \pi/2))$	X (*)			
$f(x) = 1/\tan(x)$		X		
$f(x) = 1/\sin(x)$		X		
$f(x) = 1/\sin(1/x)$			X	
$f(x) = \sin(\exp(x))/\cos(x)$				X
$f(x) = 1/\log\lvert x\rvert$				X

(*) this one is a bit tricky and be correct when marking it "no discontinuity" . The reason is that $f(x) = x - \pi/2$ But if arctan is confined to the branch $(-\pi/2, \pi/2)$, then the function will have jumps.

Problem 4) Area computation (10 points)

Find the area of the region enclosed by the graphs of the function $f(x) = x^4 - 2x^2$ and the function $g(x) = -x^2$.



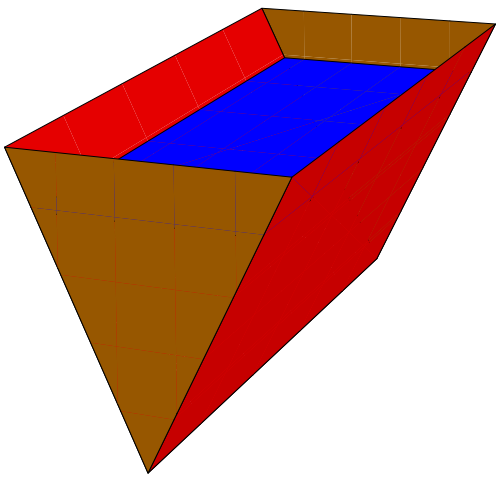
Solution:
The difficulty is to see what is above and what is below. The function $-x^2$ is above. The picture has helped us here but we could also plug in some values like $x = 1/2$ to see. Furthermore, we also have to find the intersection points which are -1 and 1. Now we can set up the integral

$$\int_{-1}^{1} -x^2 - (x^4 - 2x^2)\, dx = 8/5 \, .$$

Problem 5) Volume computation (10 points)

A farmer builds a bath tub for his warthog "Tuk". The bath has triangular shape of length 10 for which the width is $2z$ at height z. so that when filled with height z the surface area of the water is $20z$. If the bath has height 1, what is its volume?

P.S. Don't ask how comfortable it is to soak in a bath tub with that geometry. The answer most likely would be "Noink Muink".



Solution:

$$\int_0^1 20z\, dz = 20z^2/2|0^1 = 10\ .$$

Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (3 points) $\int_1^2 \sqrt{x} + x^2 - 1/\sqrt{x} + 1/x\ dx$.

b) (3 points) $\int_1^2 2x\sqrt{x^2-1}\ dx$

c) (4 points) $\int_1^2 2/(5x-1)\ dx$

Solution:
a) $x^{3/2}(2/3) + x^3/3 - x^{1/2}2 + \log(x)|_1^3 = -1/3 + 10\sqrt{2}/3 + \log(2)$.
b) $(x^2-1)^{3/2}(2/3)|_1^2 = 2\sqrt{3}$.
c) $2\log(5x-1)/5|_1^2 = 2(4/5)\log(3/2)$.

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (3 points) $\int \frac{3}{1+x^2} + x^2\ dx$

b) (3 points) $\int \frac{\tan^2(x)}{\cos^2(x)}\ dx$

c) (4 points) $\int \log(5x)\ dx$.

Solution:
a) $3\arctan(x) + x^3/3 + C$.
b) $\tan^3(x)/3 + C$.
c) Use $u = 5x$, $du = 5dx$ and get $\int \log(u)\ du = (u\log(u) - u)/5 = x\log(5x) - x$.

Problem 8) Implicit differentiation/Related rates (10 points)

A juice container of volume $V = \pi r^2 h$ changes radius r but keeps the height $h = 2$ fixed. Liquid leaves at a constant rate $V'(t) = -1$. At which rate does the radius of the bag shrink when $r = 1/2$?

Solution:
Differentiate the equation $V(r) = 2\pi r^2(t)$ and use the chain rule: $-1 = V'(r) = 4\pi r r'$. We get $r' = -1/(4\pi r) = -1/(2\pi)$.

Problem 9) Global extrema (10 points)

We build a chocolate box which has 4 cubical containers of dimension $x \times x \times h$. The total material is $f(x, h) = 4x^2 + 12xh$ and the volume is $4x^2 h$. Assume the volume is 4, what geometry produces the minimal cost?

Solution:
$4x^2 g = 4$ implies $h = 1/x^2$. We have to minimize the function $f(x) = 4x^2 + 12/x$. Its derivative is $8x - 12/x^2$ which is zero for $x = (3/2)^{1/3}$.

Problem 10) Integration techniques (10 points)

Which integration technique works? It is enough to get the right technique and give the first step, not do the actual integration:

a) (2 points) $\int (x^2 + x + 1)\sin(x)\, dx$.

b) (2 points) $\int x/(1 + x^2)\, dx$.

c) (2 points) $\int \sqrt{4 - x^2}\, dx$.

d) (2 points) $\int \sin(\log(x))/x$.

e) (2 points) $\int \frac{1}{(x-6)(x-7)}\, dx$.

Solution:
a) integration by parts. Take $u = x^2 + x + 1$ and $v = \sin(x)$.
b) substitution $1 + x^2 = u$.
c) trig substitution $x^2 = 4\sin^2(u)$.
d) substitution $\log(x) = u$.
e) partial fractions.

Problem 11) Hopital's rule (10 points)

Find the following limits as $x \to 0$ or state that the limit does not exist.

a) (2 points) $\frac{\tan(x)}{x}$

b) (2 points) $\frac{x}{\cos(x)-x}$.

c) (2 points) $x\log(1+x)/\sin(x)$.

d) (2 points) $x\log(x)$.

e) (2 points) $x/(1-\exp(x))$.

Solution:
a) Use Hopital and differentiate both sides. Leading to $1/1 = 1$. Alternatively, write it as $(1/\cos(x))\sin(x)/x$. Because $1/\cos(x) \to 1$ and $\sin(x)/x \to 1$, the limit is 1.
b) No Hopital is needed (because we do not divide by 0). The limit is 0.
c) Use Hopital, differentiate and see that the limit is 0.
d) Use Hopital, differentiate and see the limit is 0.
e) Use Hopital, the limit is -1.

Problem 12) Applications (10 points)

The cumulative distribution function on $[0, 1]$

$$F(x) = \frac{2}{\pi}\arcsin(\sqrt{x})$$

defines the **arc-sin** distribution.

a) Find the probability density function $f(x)$ on $[0, 1]$.

b) Verify that $\int_0^1 f(x)\,dx = 1$.
Remark. The arc sin distribution is important chaos theory and probability theory.

Solution:
a) the probability density function is the derivative which is

$$\frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}\,.$$

b) Since we know the antiderivative already there is no need to integrate this again. We know it is $F(1) - F(0) = 1 - 0 = 1$.