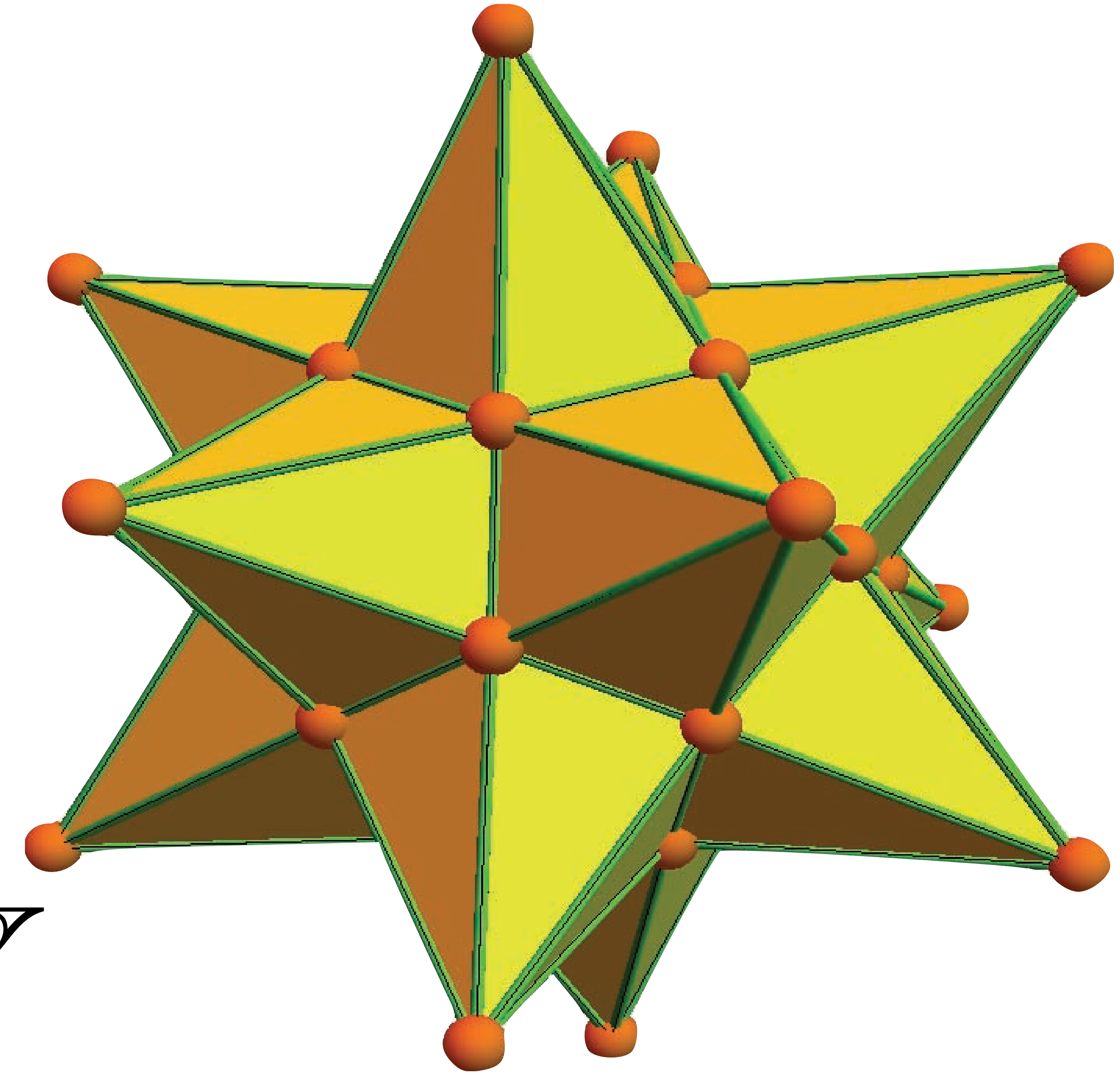
COLORING GRAPHS USING TOPOLOGY



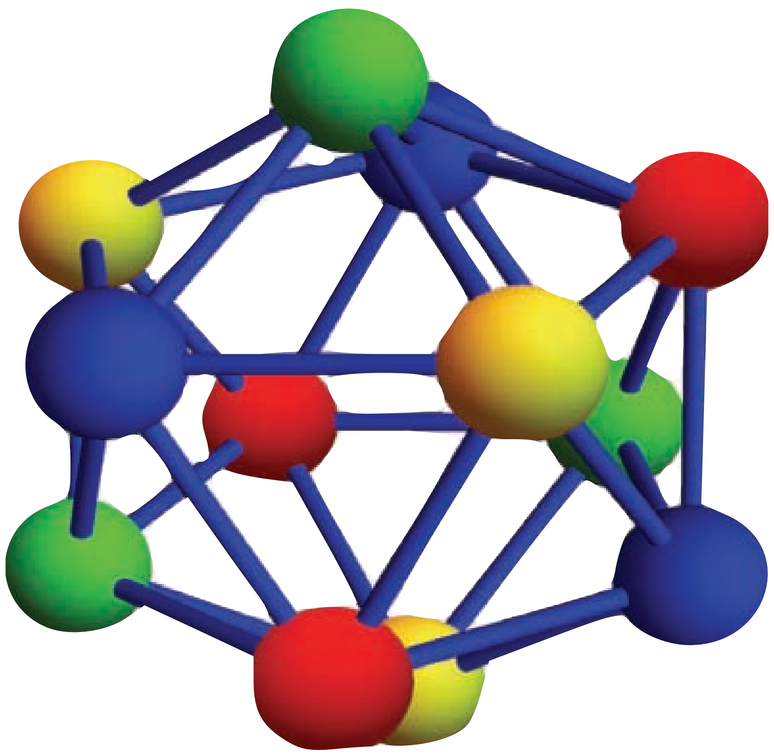
Oliver Knill Harvard University

December 27, 2014

http://arxiv.org/abs/1412.6985

Thanks to the Harvard College Research Program HCRP for supporting work with

Jenny Nitishinskaya



from June 10-August 7, 2014

work which initiated this research on graph coloring.

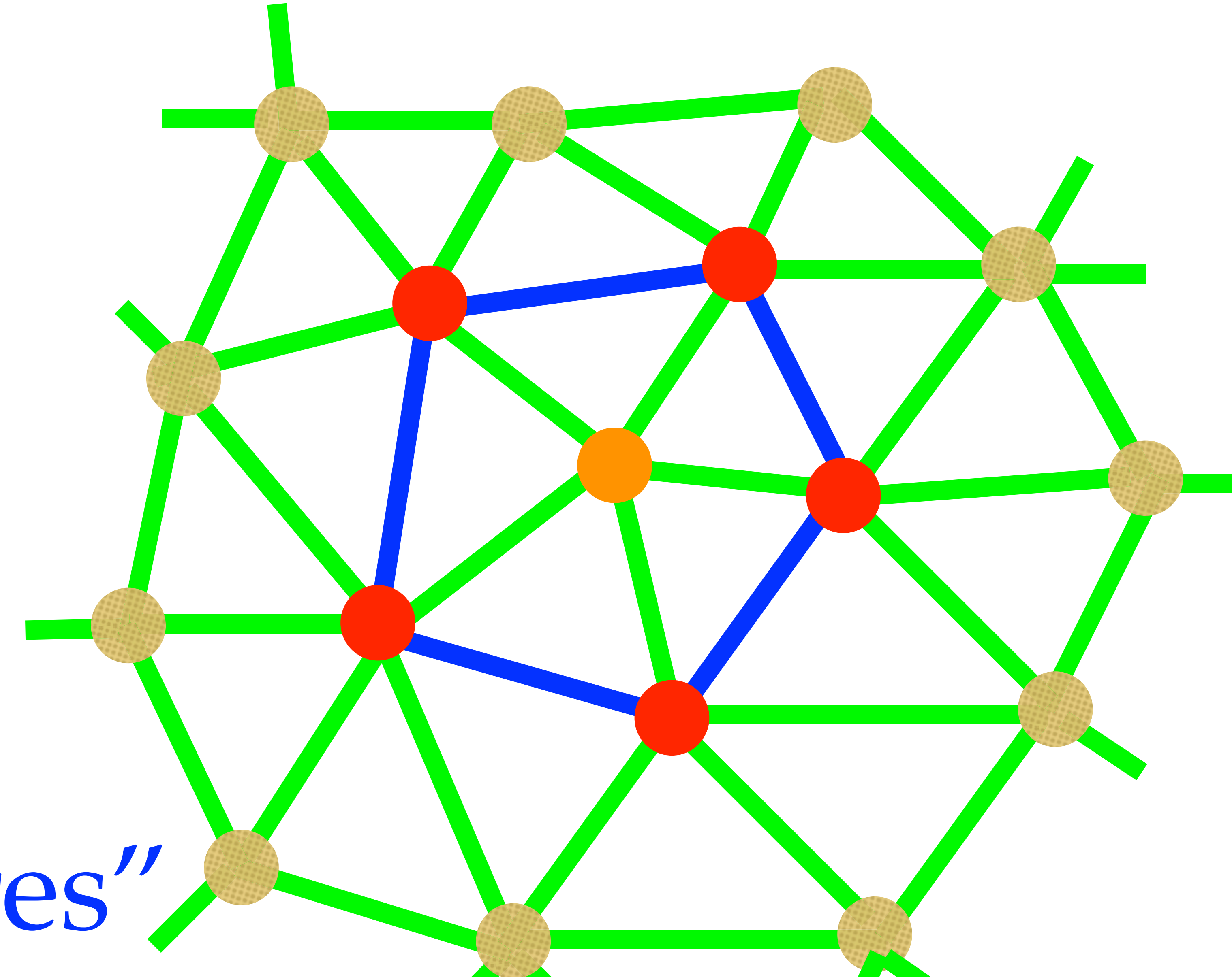
2 DIM SPHERES

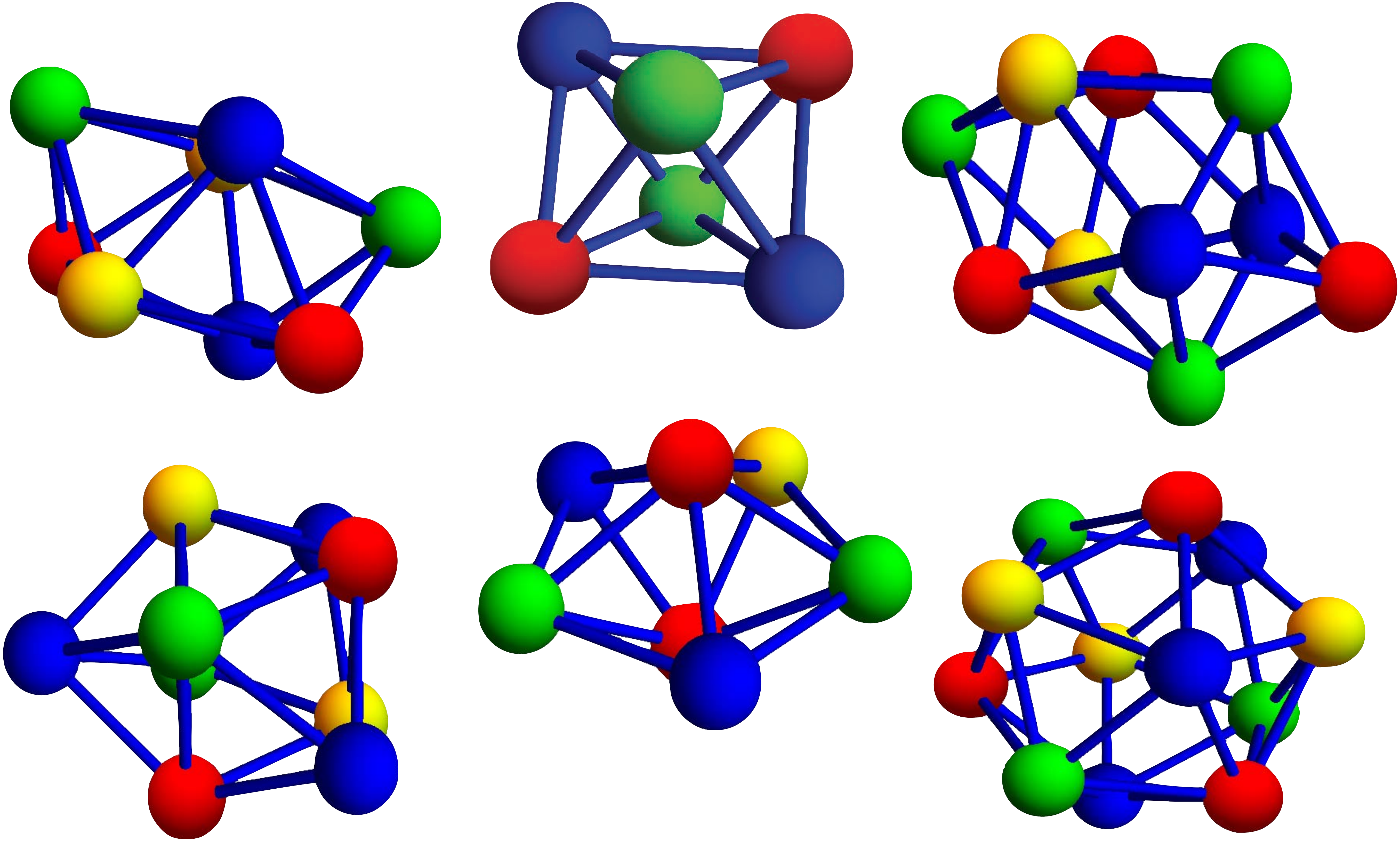
$$\mathcal{S}_2 = \{ G \mid S(x) \text{ is cyclic } C_n \text{ with every unit sphere } n(x)>3 \text{ and } \chi(G)=2 \}$$

$\chi(G)=v-e+f$

Euler characteristic in two dimensions

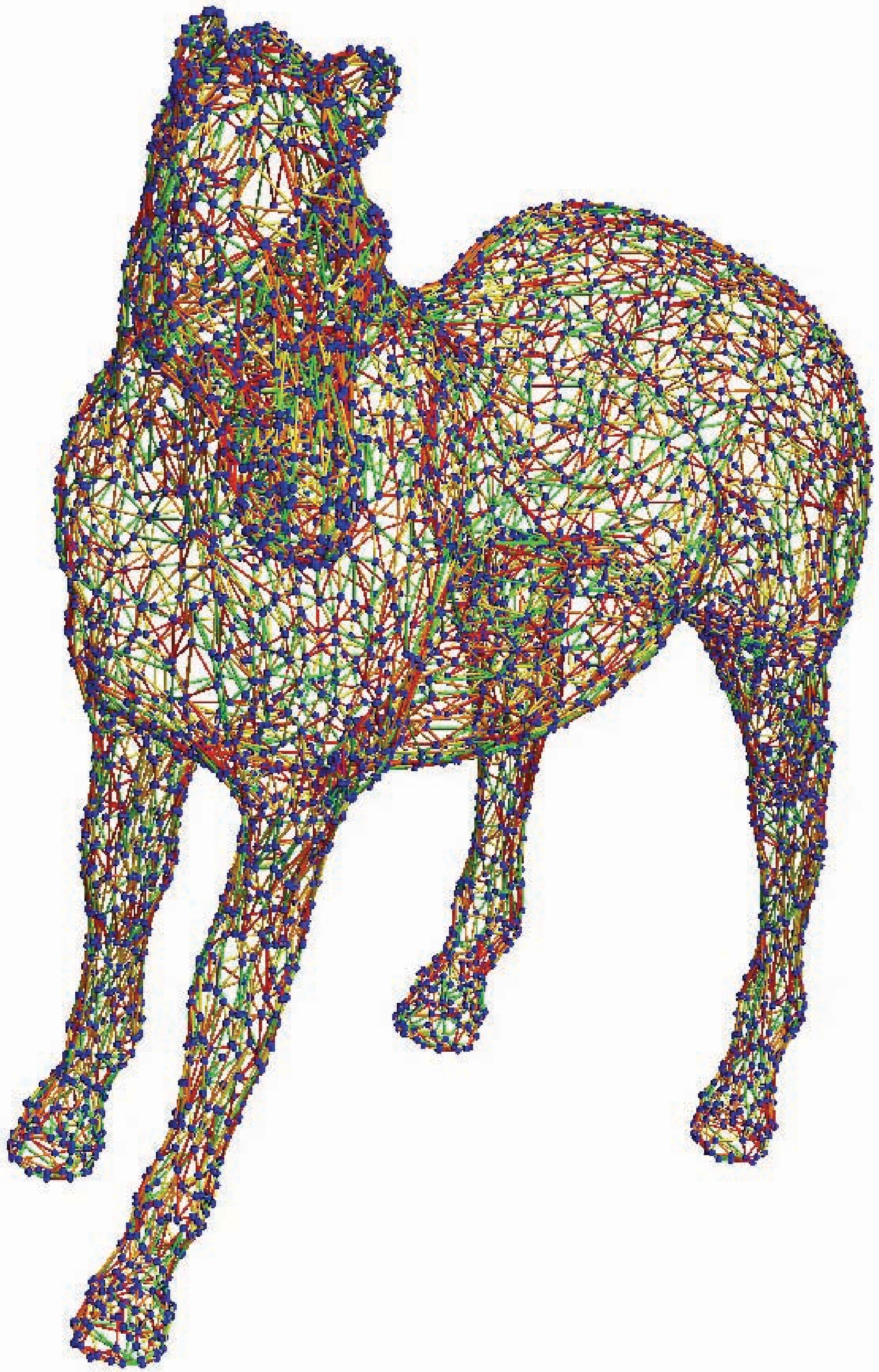
``spheres have circular unit spheres''





POSITIVE CURVATURE

AN OTHER SPHERE?



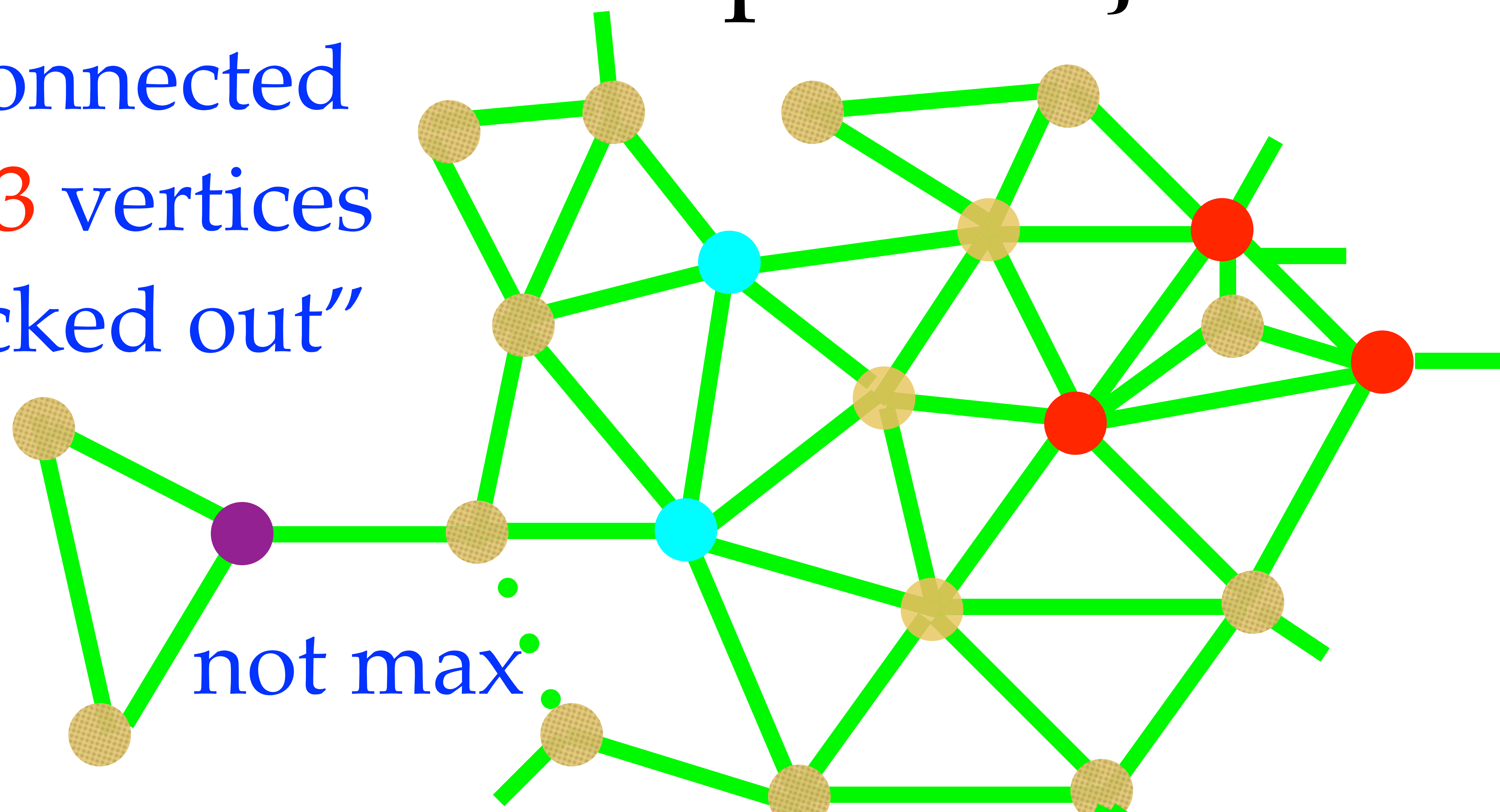
this graph is not maximal planar.

``not all triangulations are spheres!”

WHITNEY GRAPHS

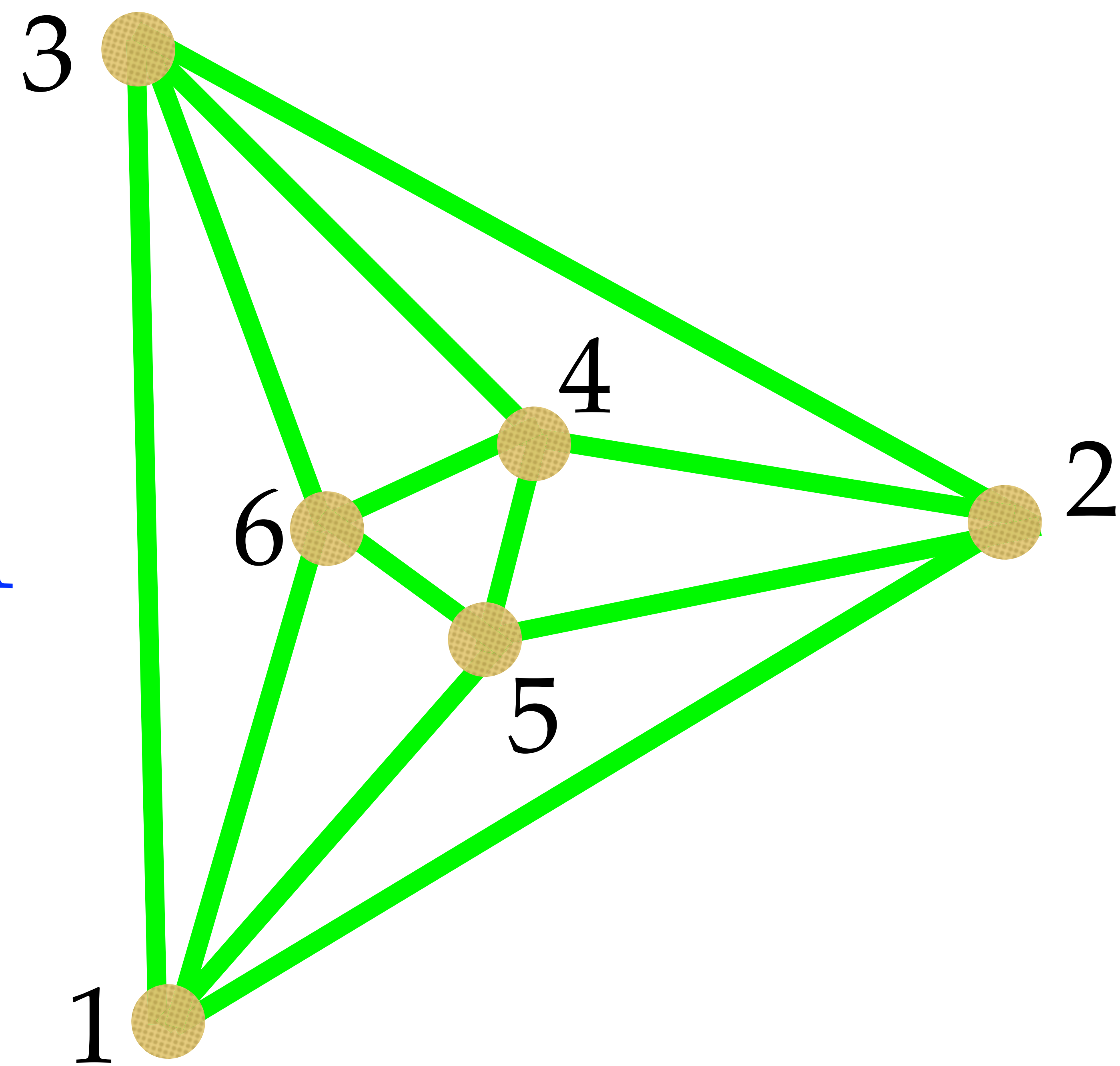
$$\mathcal{W} = \{ G \mid G \text{ is 4-connected and maximal planar } \}$$

``stay connected if 1,2 or 3 vertices are knocked out"



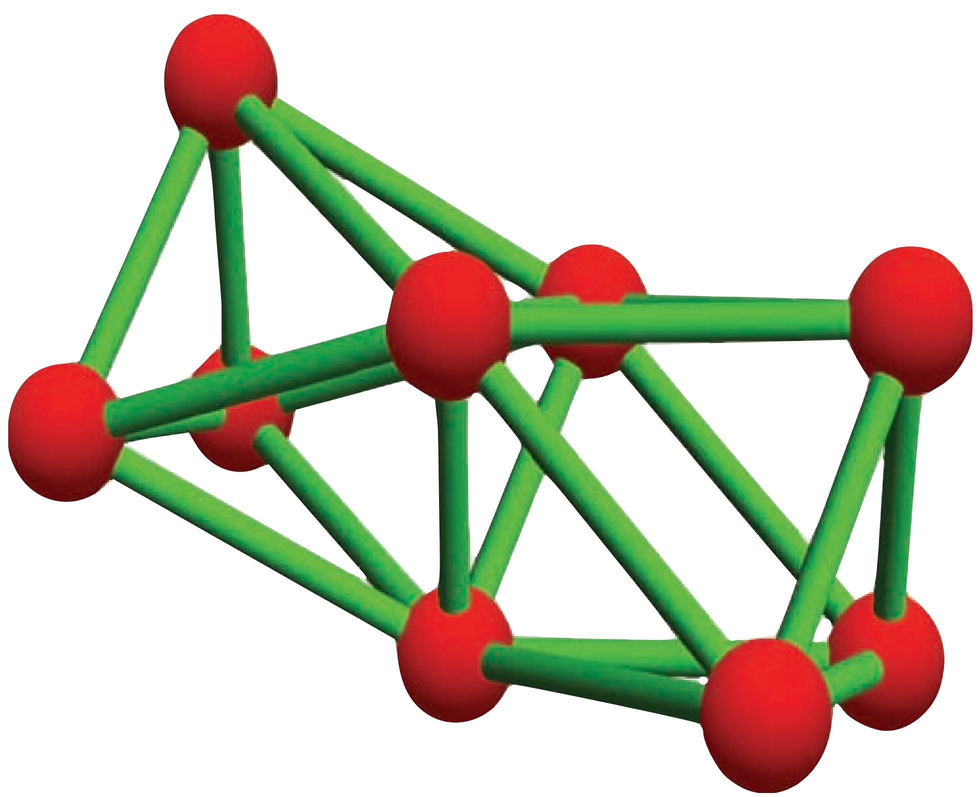
WHITNEY THEOREM

Every $G \varepsilon \mathbb{W}$ is Hamiltonian

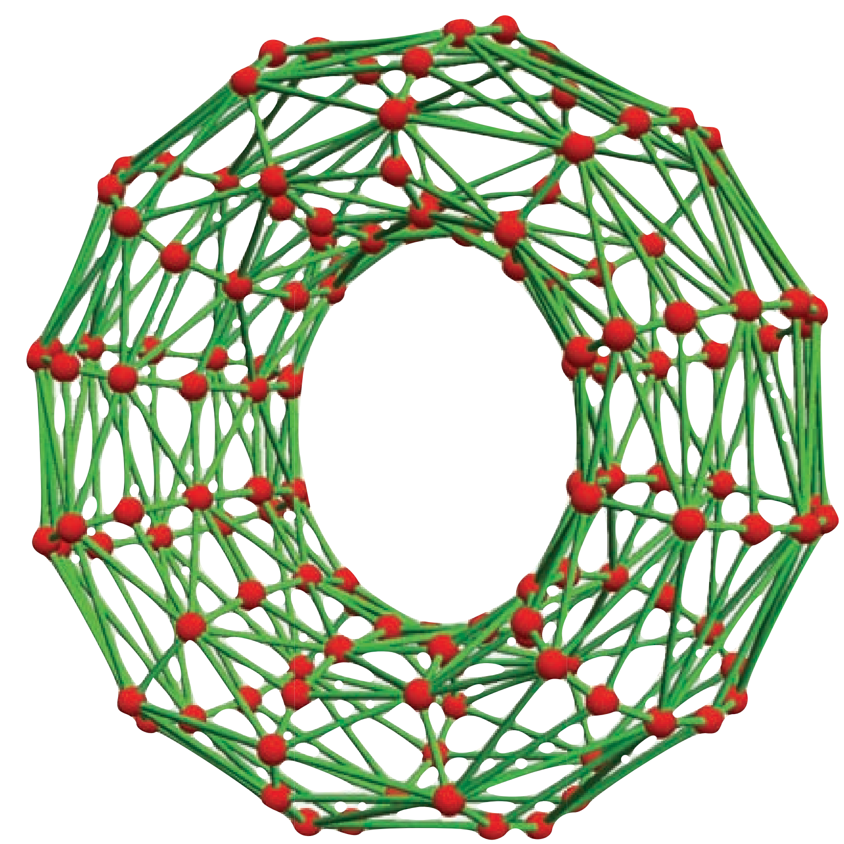


``Hamiltonian connection”

COMBINATION



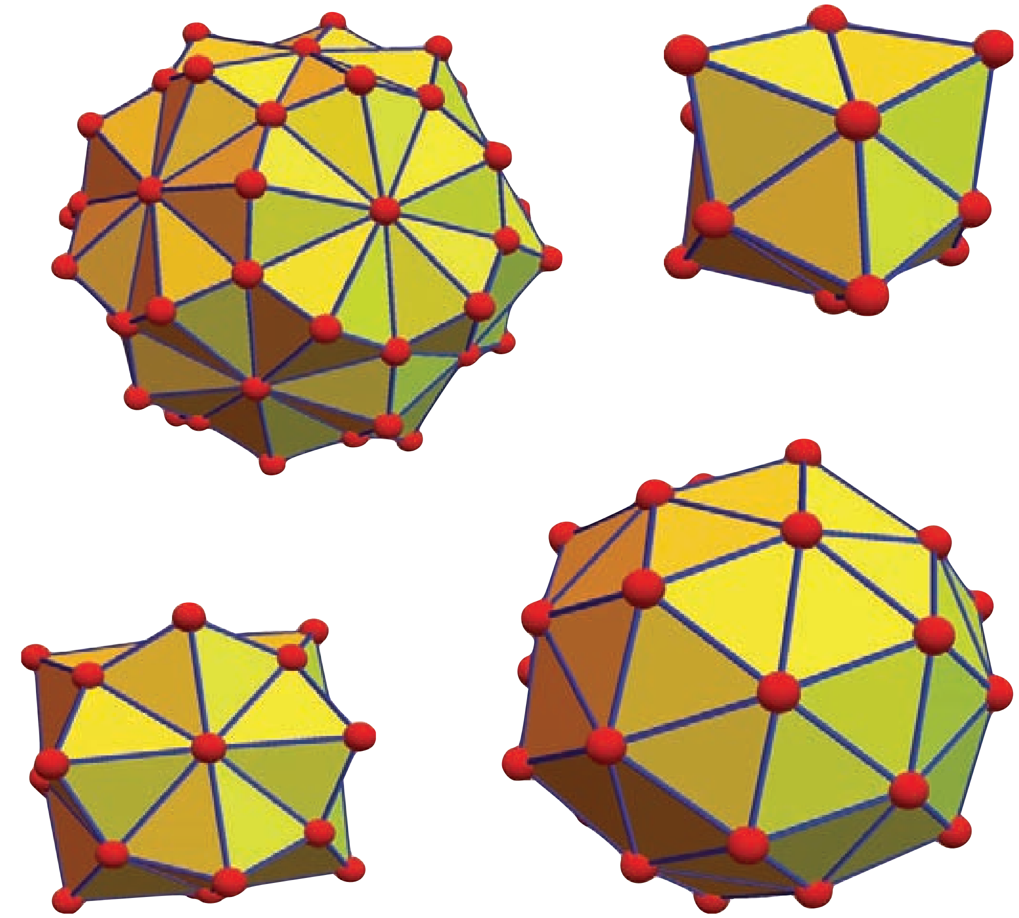
``twin octahedron is 4 -disconnected"



``torus is non-planar. with χ=0"

SPHERE LEMMA

$$\mathbb{W} = \mathcal{S}_2$$



``Whitney graphs are spheres”

4 COLOR THEOREM

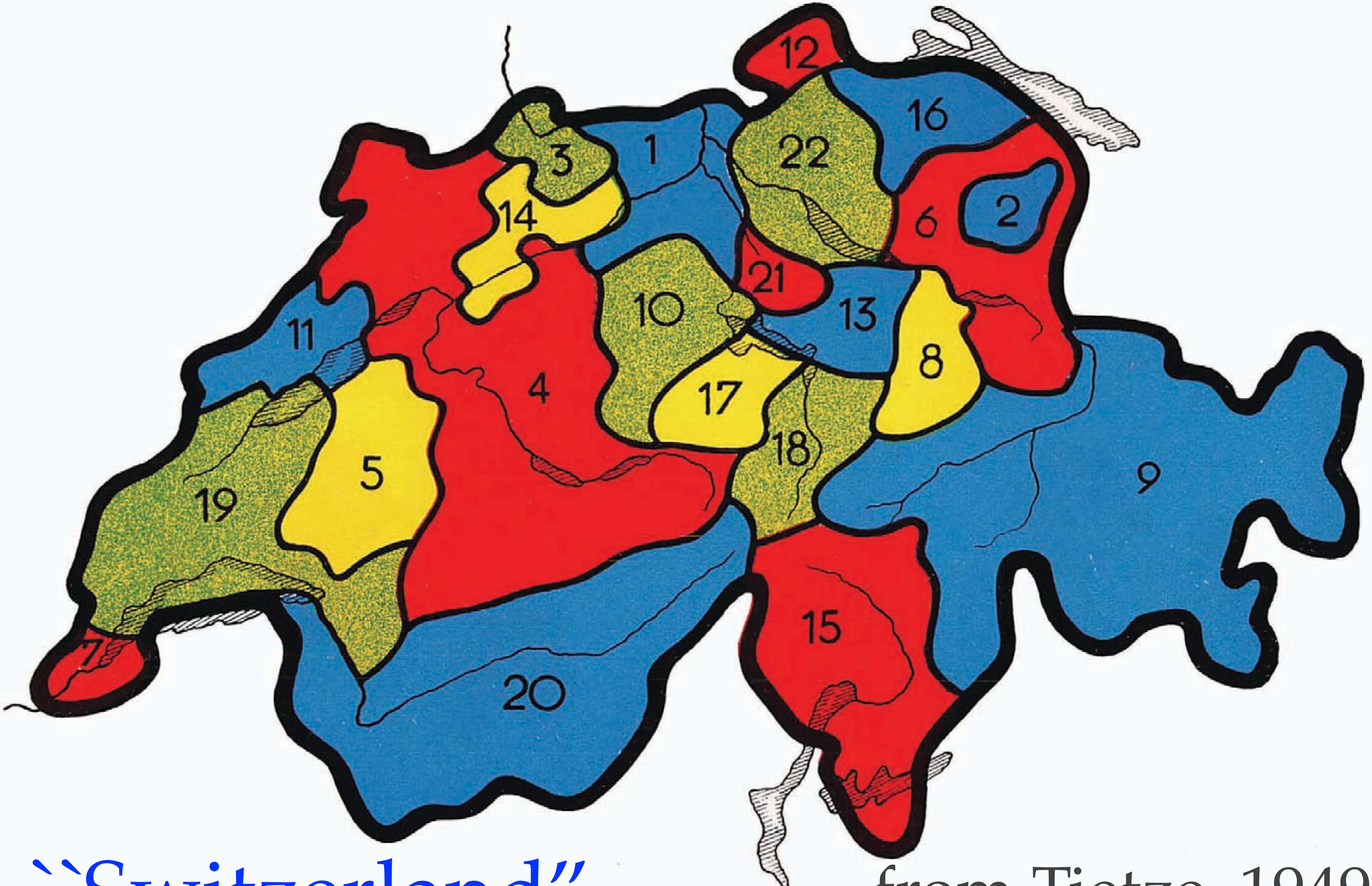
$\mathcal{P}$ = planar graphs

$\mathcal{C}_4$ = 4 -colorable graphs

$$\mathcal{P} \subset \mathcal{C}_4$$

``only computer proof so far"

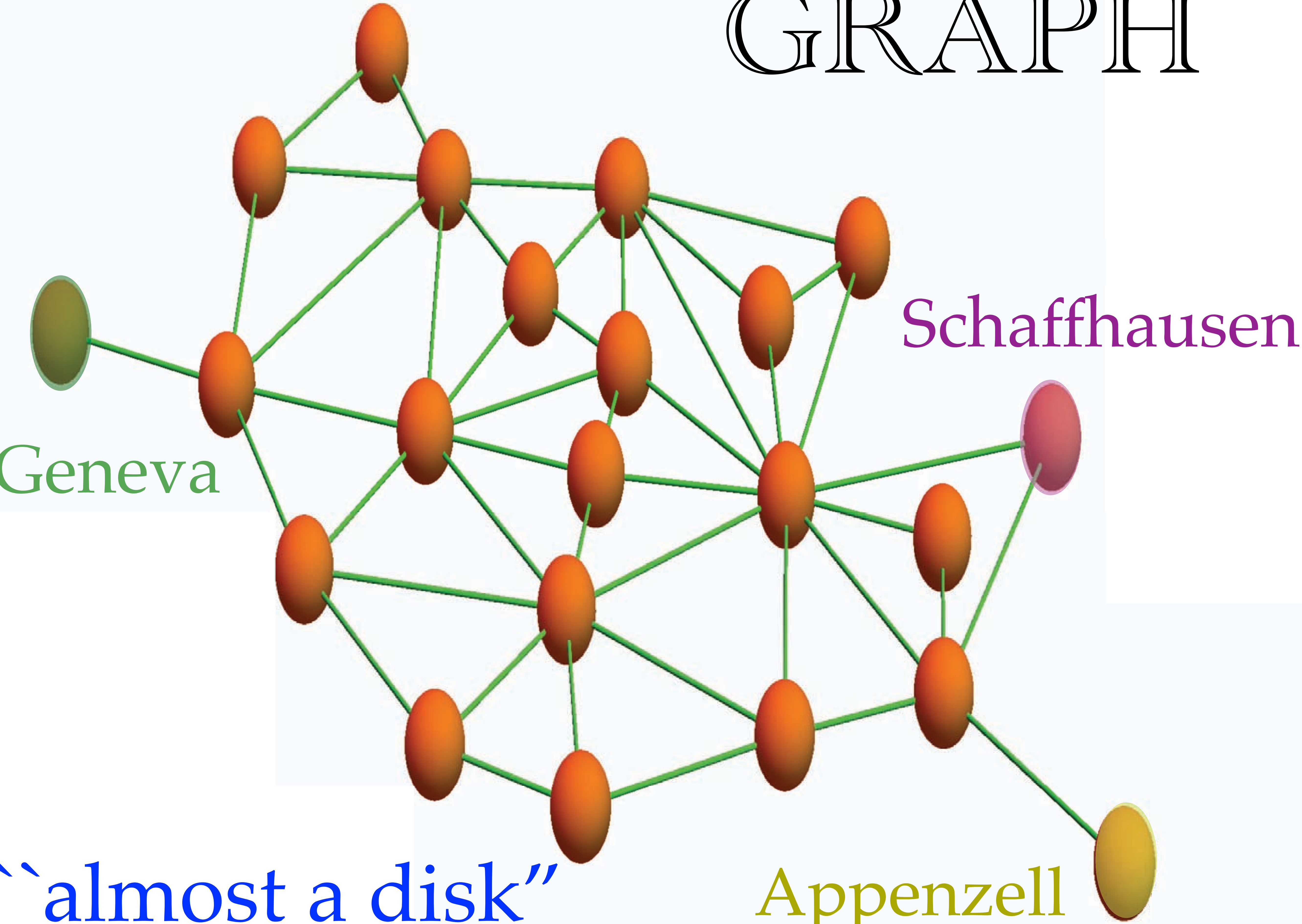
MAP COLORING



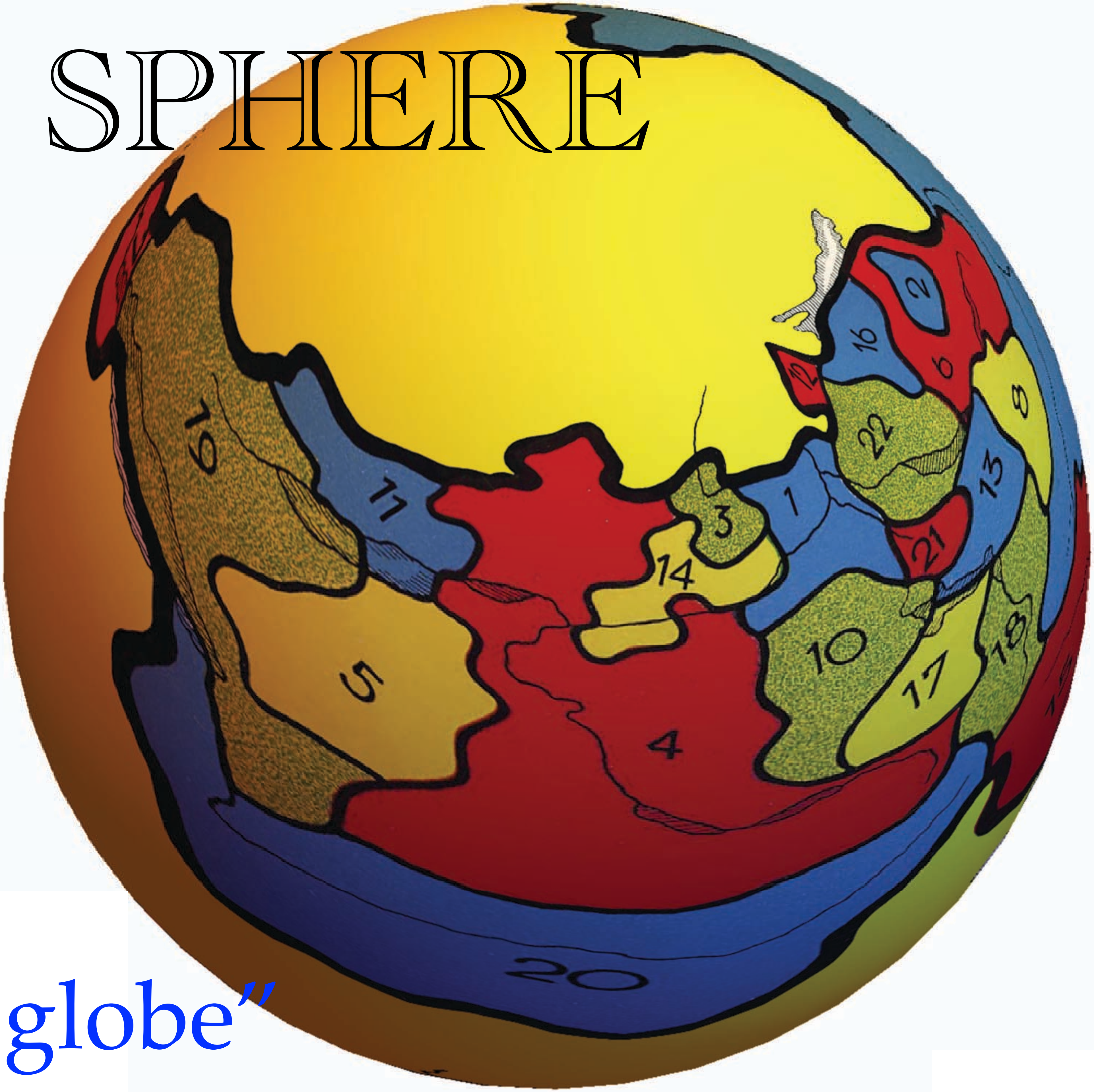
``Switzerland”

from Tietze, 1949

GRAPH



ON SPHERE



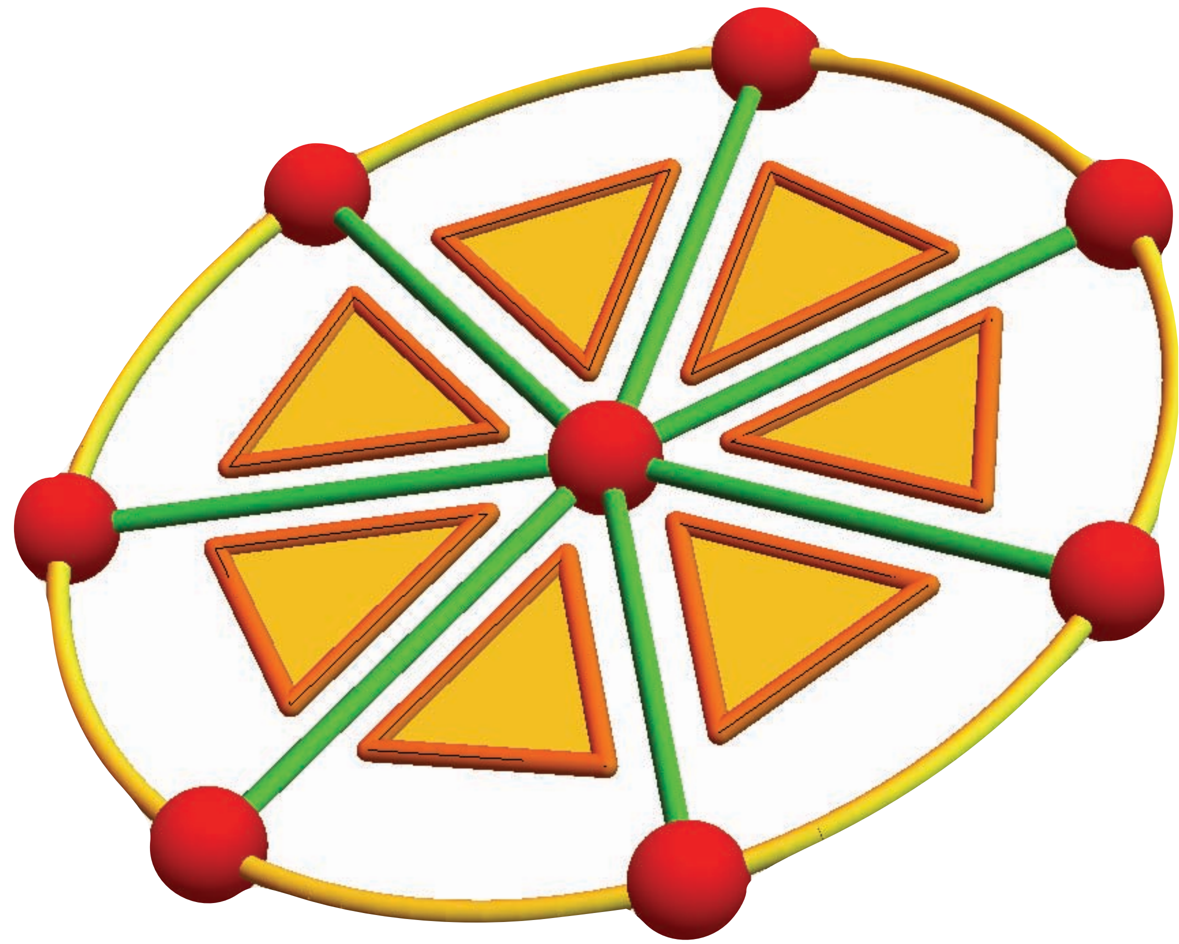
``on the globe''

REFORMULATION

$$\mathcal{P} \subset \mathcal{C}_4 \quad \Leftrightarrow \quad \mathcal{S}_2 \subset \mathcal{C}_4$$

``Need only to color spheres''

VERTEX DEGREE



``loop size in dual graph”

KEMPE-HEAWOOD

$$\mathcal{S}_2 \cap \mathcal{C}_3 = \mathcal{S}_2 \cap \mathcal{E}$$

$\mathcal{E}$ = Eulerian graphs

= {all vertex degrees are even}

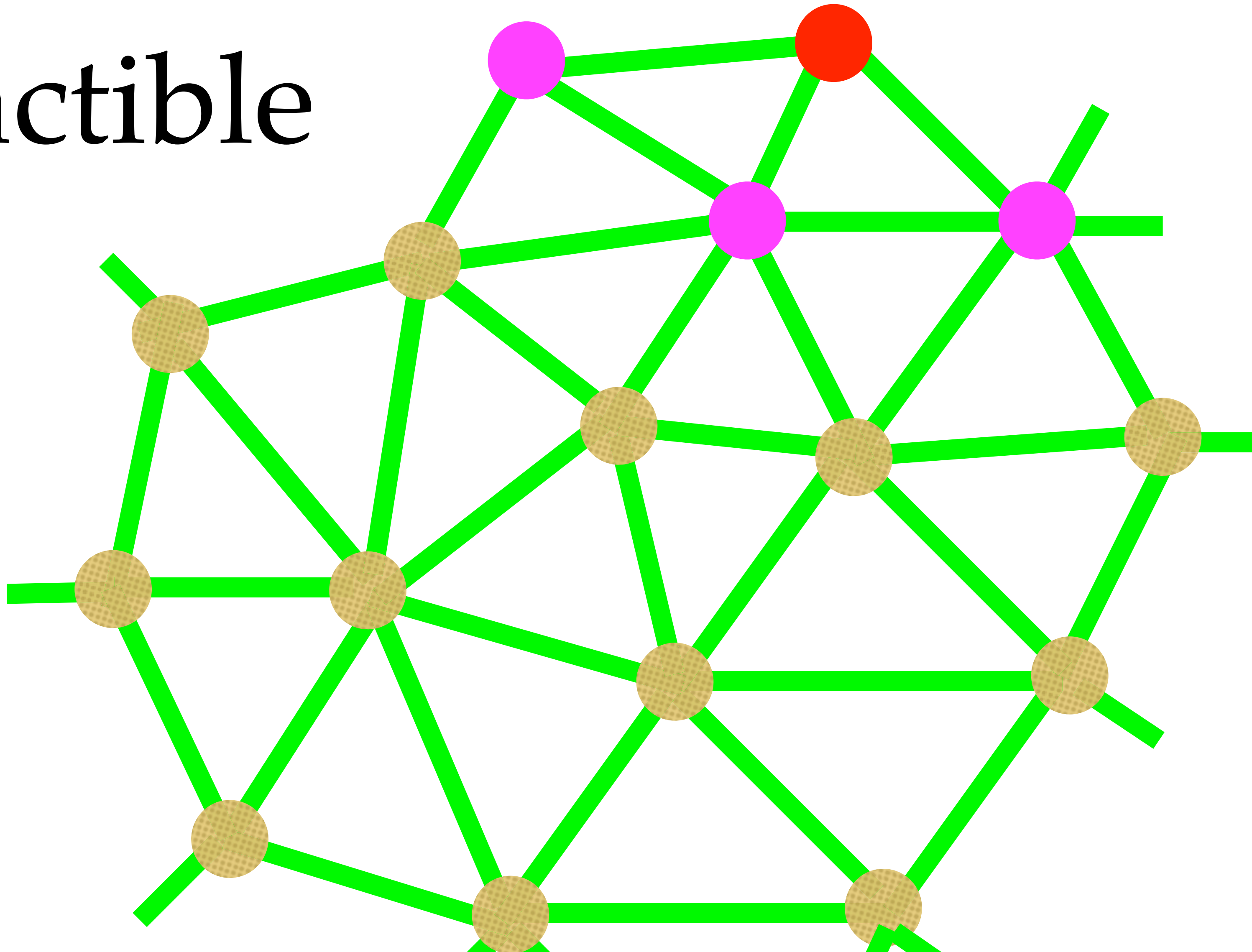
``Euler”

CONTRACTIBLE

G contractible if there is x such that S(x) and G-B(x) are contractible

∅ graph is contractible

``inductive setup''



GEOMETRIC GRAPH

$$\mathcal{S}_{-1} = \mathcal{B}_{-1} = \mathcal{G}_{-1} = \{ \emptyset \}$$

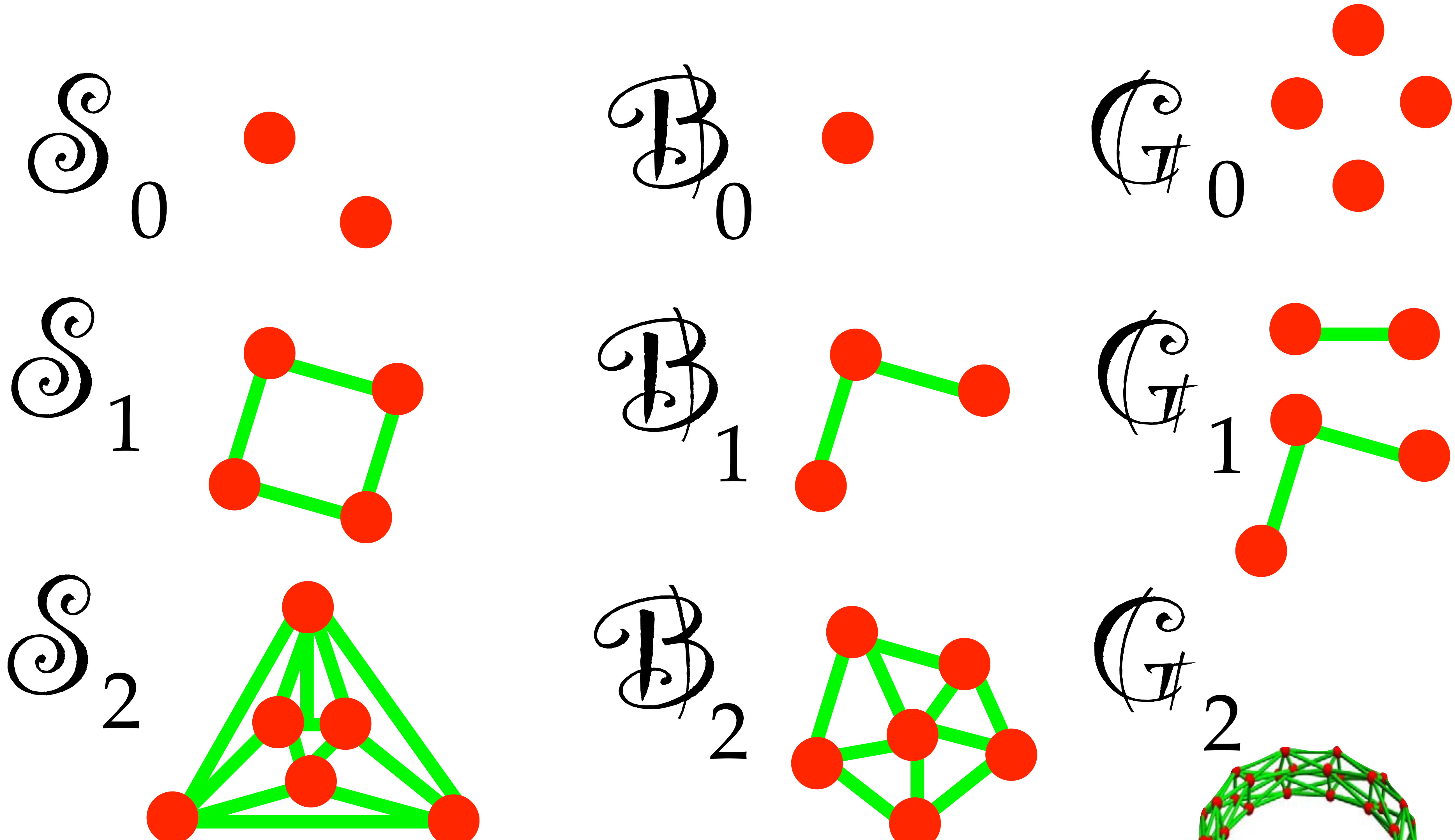
$$\mathcal{G}_d = \{ G \mid \text{all } S(x) \in \mathcal{S}_{d-1} \}$$

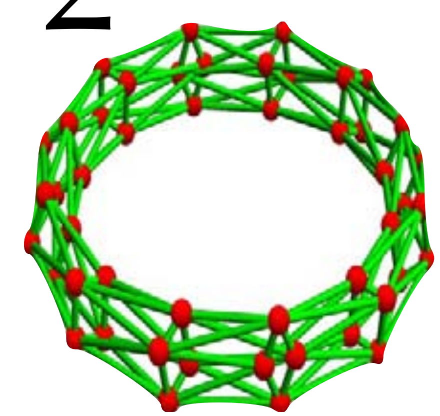
$$\mathcal{B}_d = \{ G \in \mathcal{G}_d \mid G \text{ contractible, } \delta G \in \mathcal{S}_{d-1} \}$$

$$\mathcal{S}_d = \{ G \in \mathcal{G}_d \mid G - \{x\} \in \mathcal{B}_d \}$$

``inductive definitions”

EXAMPLES

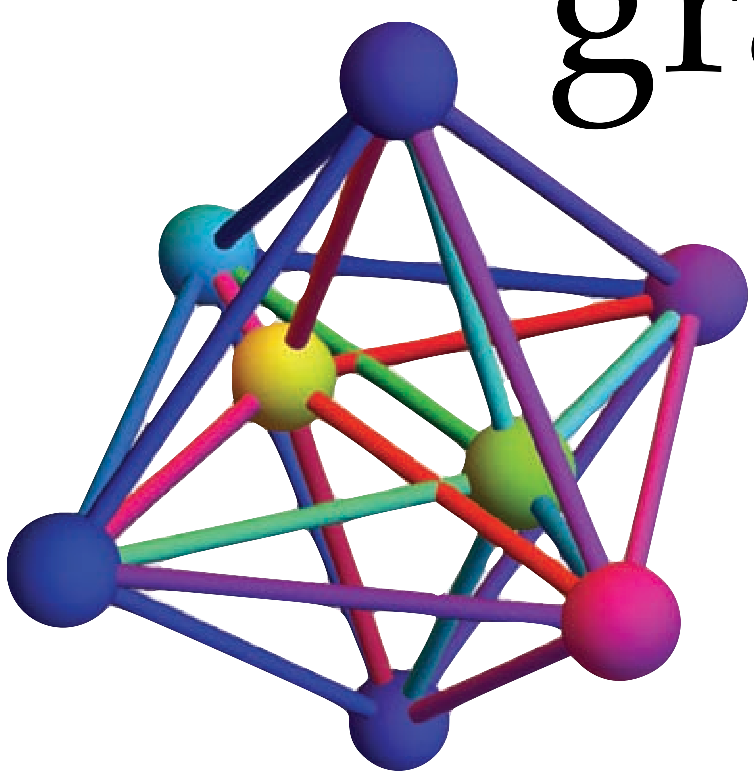




``does the right thing”

3 DIM SPHERES

$\mathcal{S}_3$ = {Unit spheres in $\mathcal{S}_2$
+ punching a hole makes
graph contractible }



``dimension +
homotopy"

DIMENSION

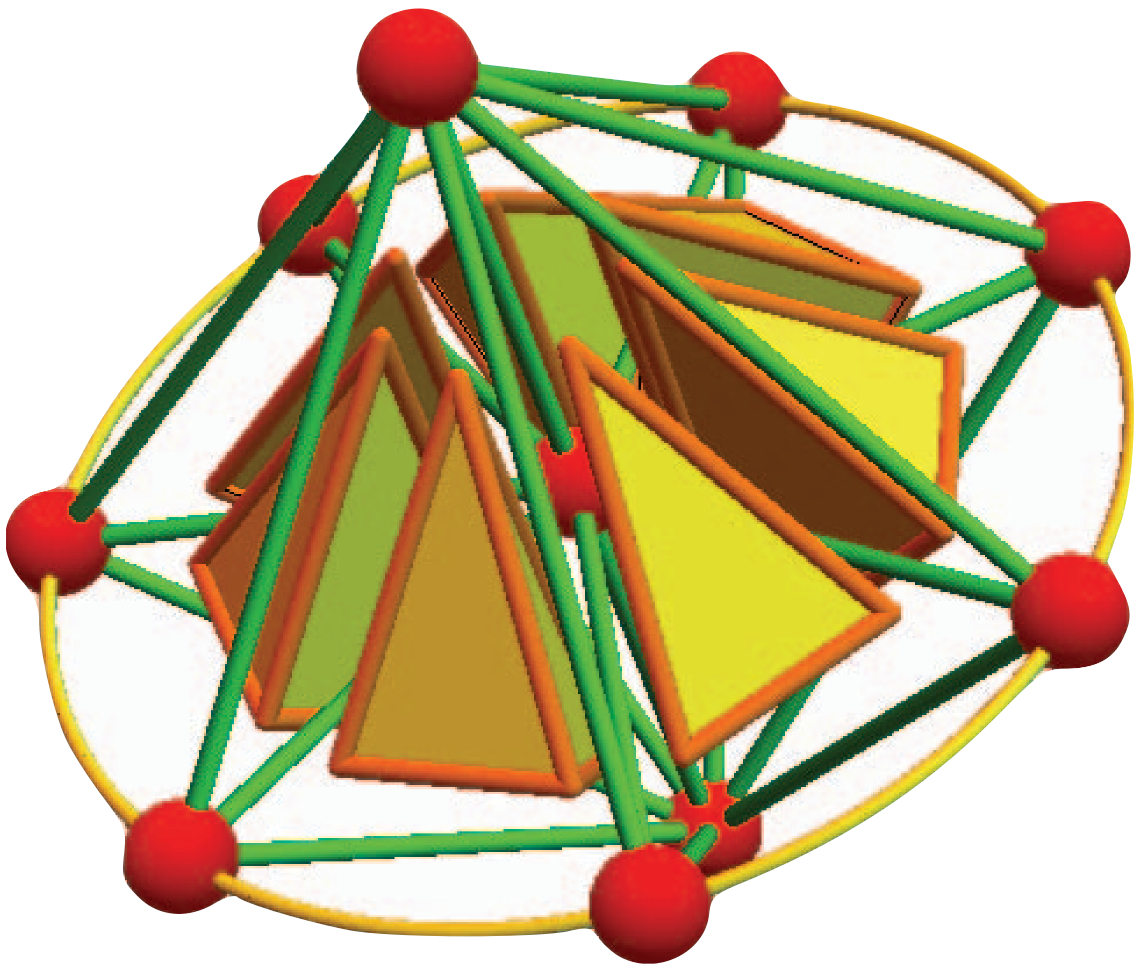
$$\dim(\emptyset) = -1$$

$$\dim(G) = 1 + E[\dim(S(x))]$$

E[X] = average over all vertices, with counting measure

``inductive dimension”

EDGE DEGREE



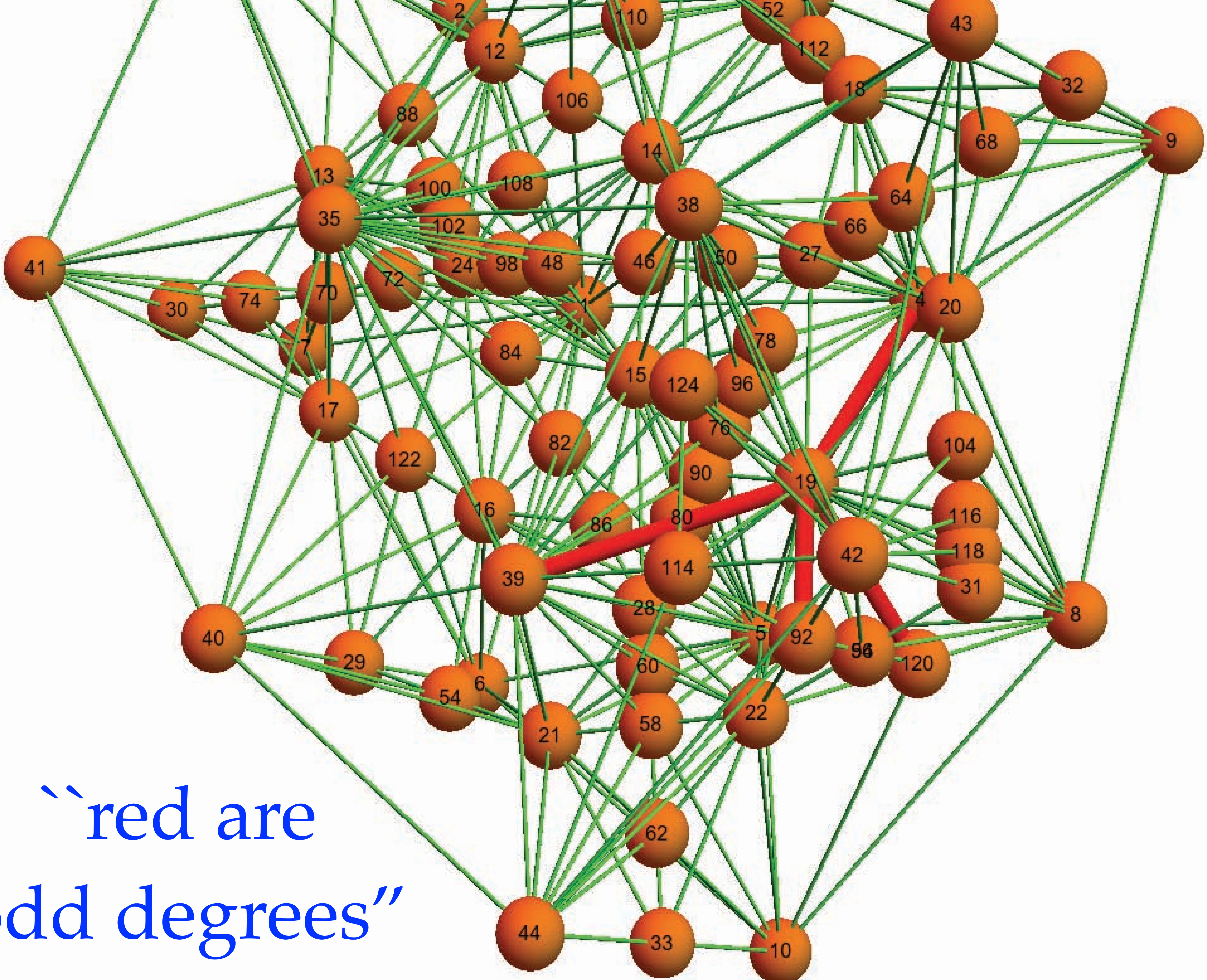
odd degree is
obstruction
to color
minimally

``loop size in dual graph”

CONSERVATION LAW

$$\sum_{x \text{ in } e} \deg(e) \quad \text{is even}$$

``is twice edge size on S(x)
if x is interior”



``red are
odd degrees''

MINIMAL COLORING

$$\mathcal{S}_3 \cap \mathcal{C}_4 = \mathcal{S}_3 \cap \mathcal{E}_3$$

$\mathcal{E}_3$ = Euler 3D graphs
= {all edge degrees are even}

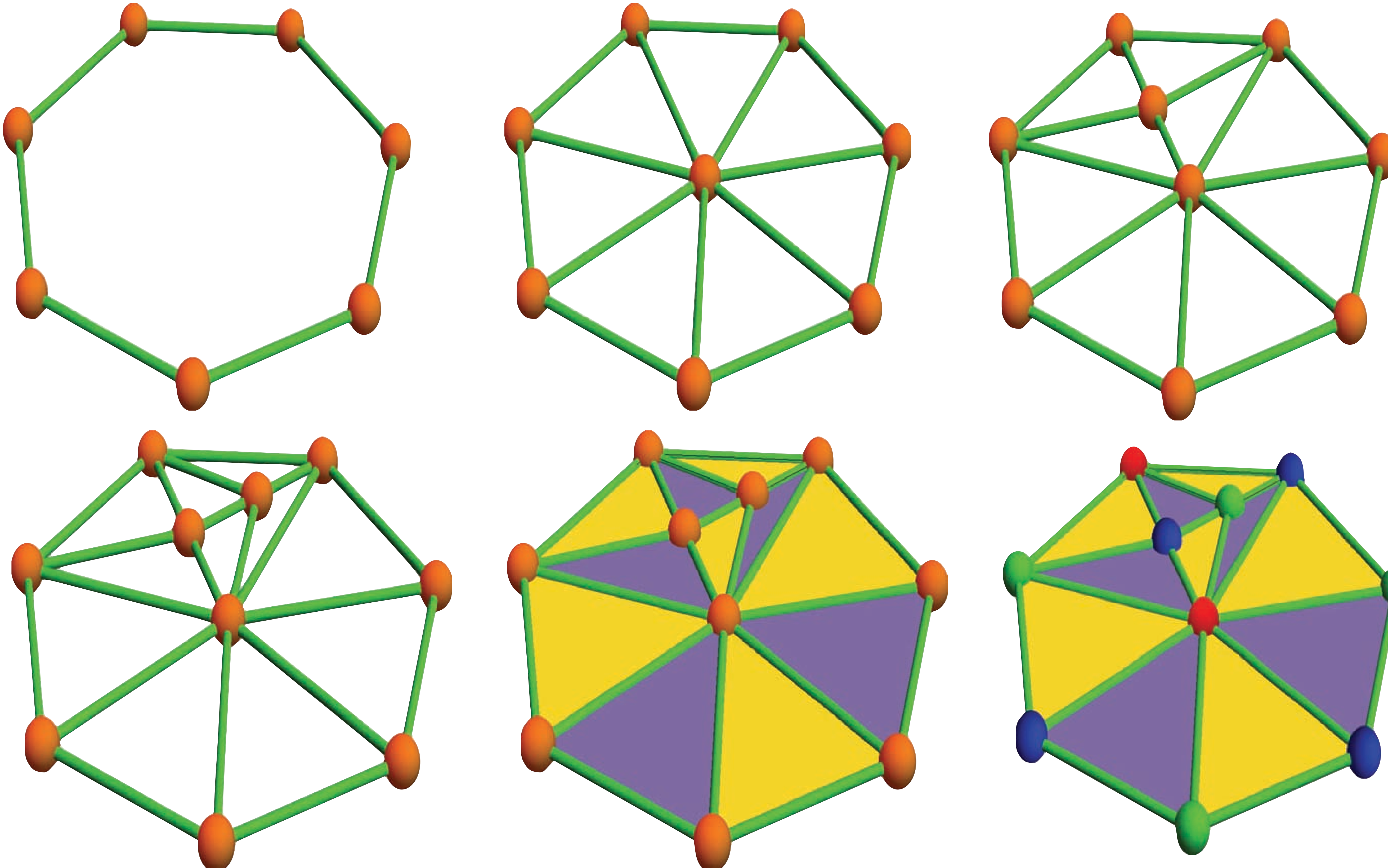
``from 1970ies"

MOTIVATION

Every $G \in \mathcal{S}_1$ is the boundary of a $H \in \mathcal{B}_2 \cap \mathbb{E}$

``silly as trivial, but it shows main idea"

PROOF



"cut until Eulerian"

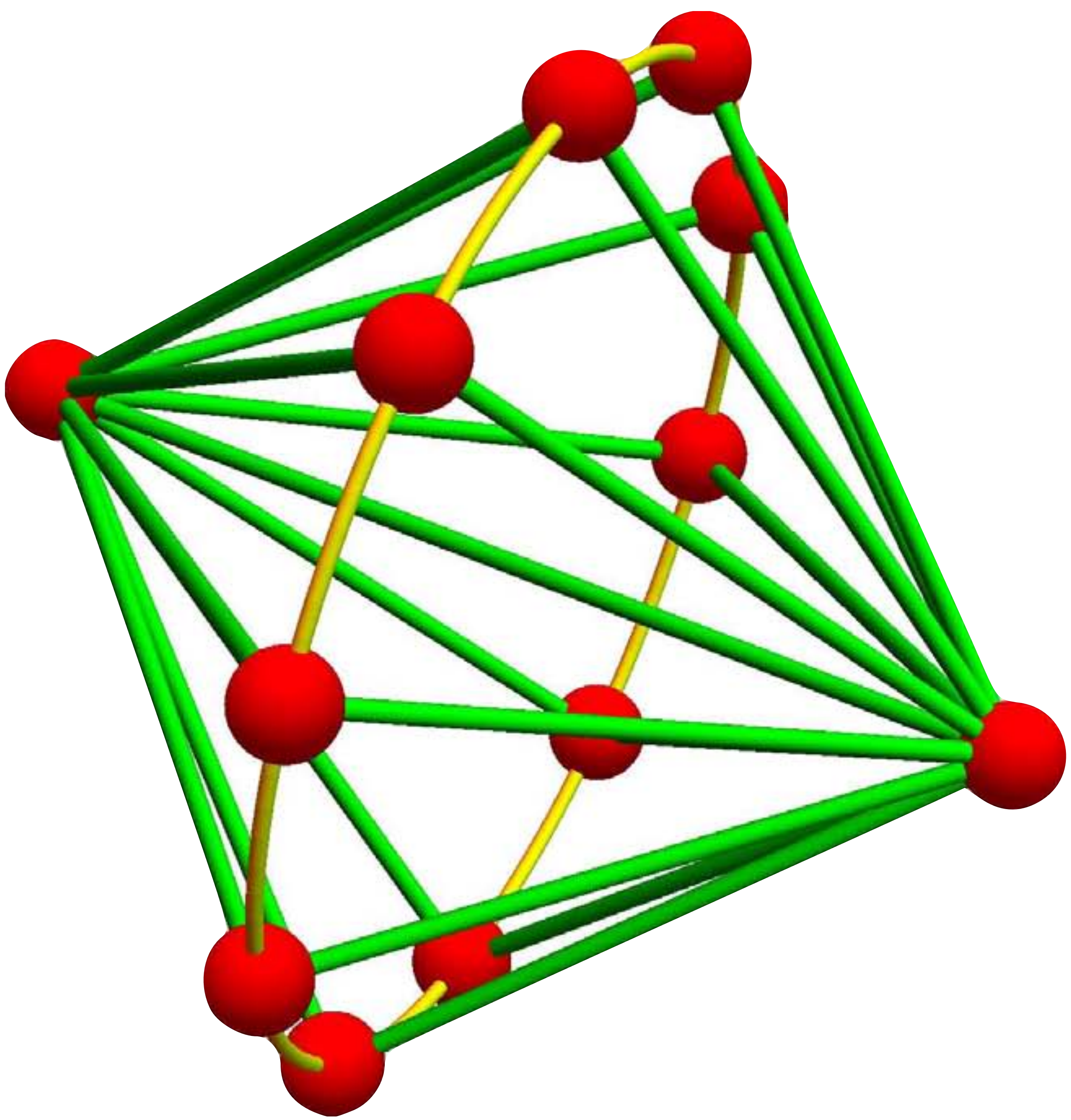
CONJECTURE

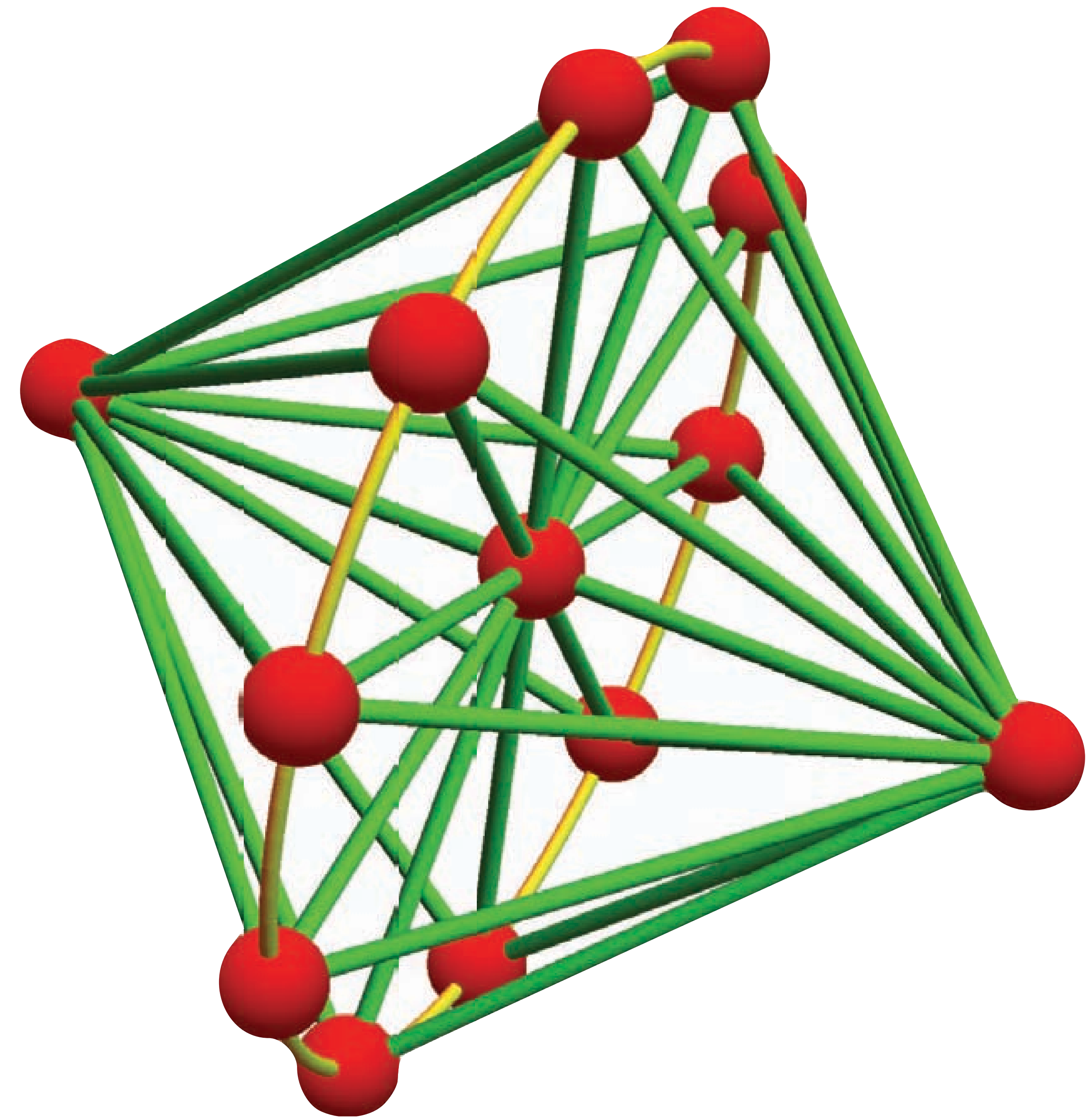
Every G ε $\mathcal{S}_2$

is boundary of H ε $\mathcal{B}_3 \cap \mathcal{E}_3$

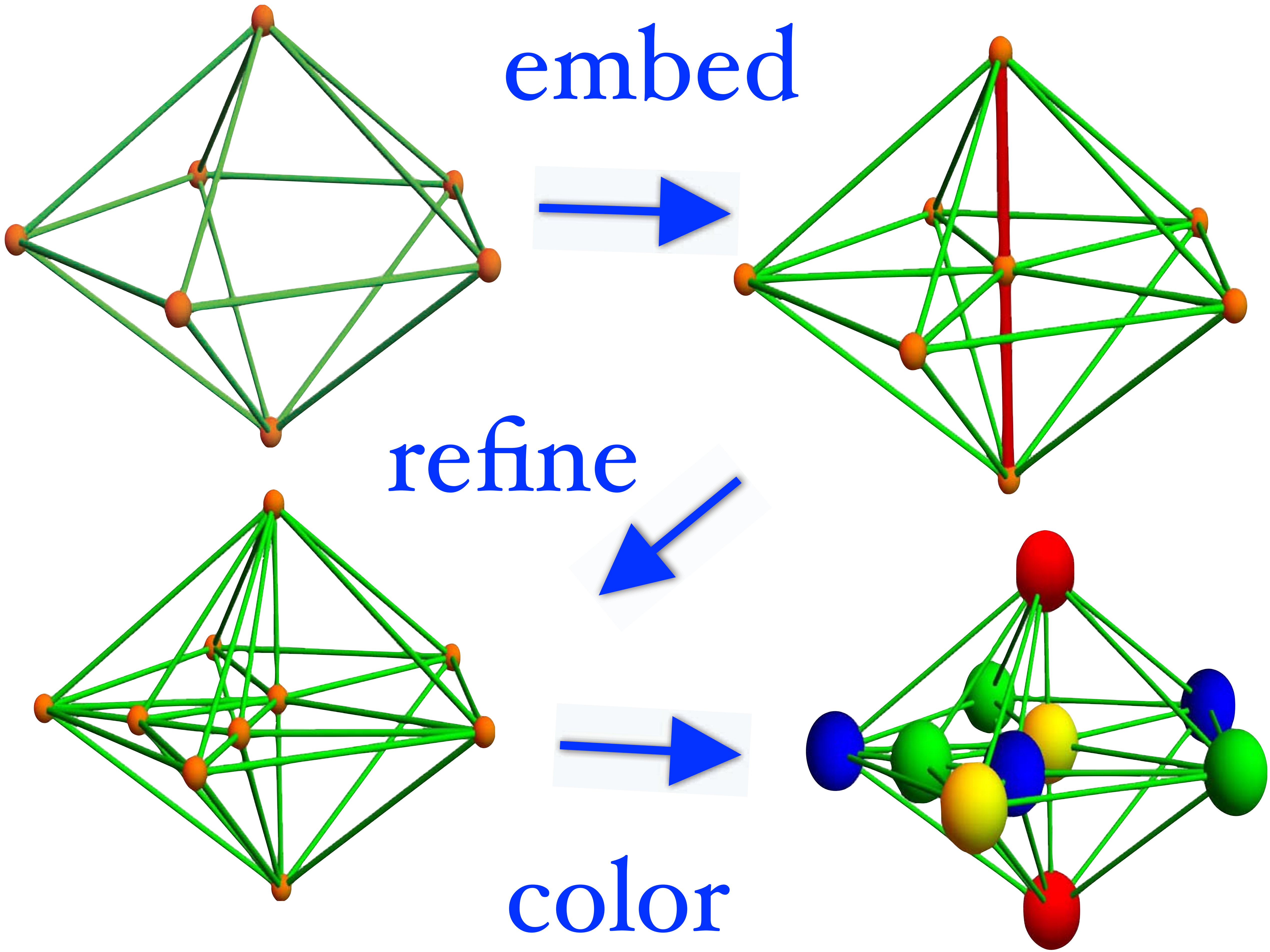
``we would see why the 4 color theorem is true"

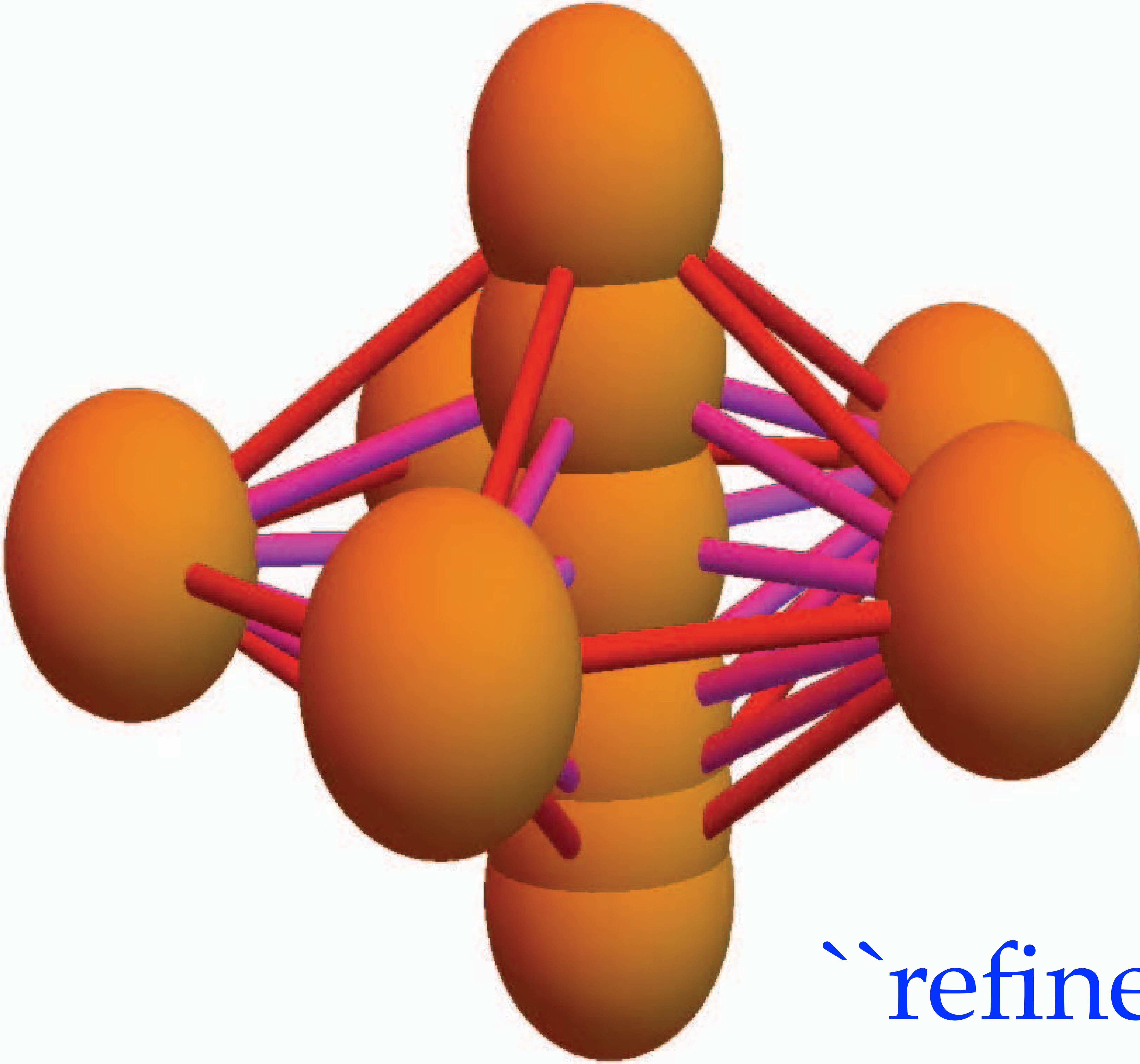
REFINEMENTS





``cut an edge”


embed
refine
color

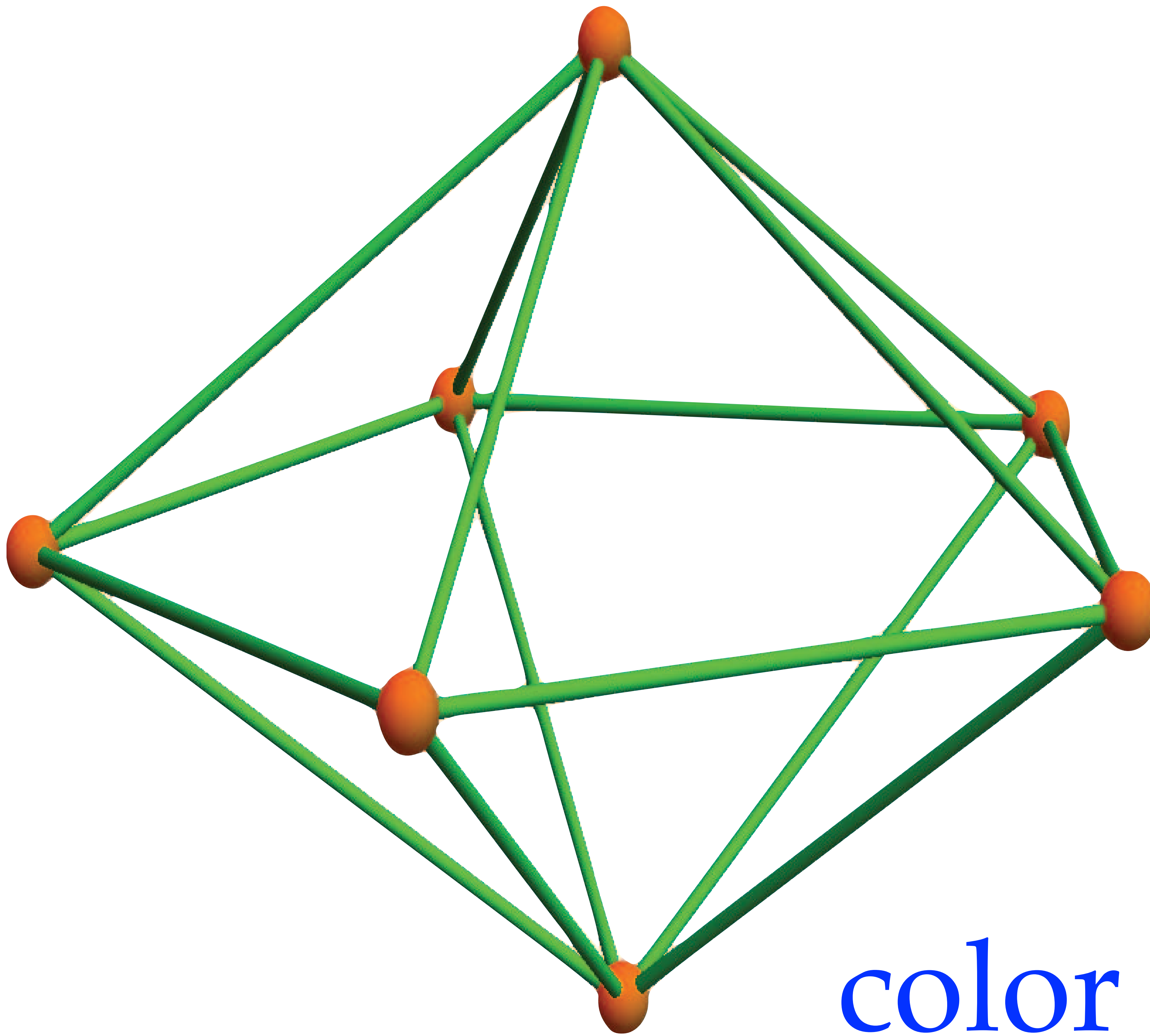


``refine!”

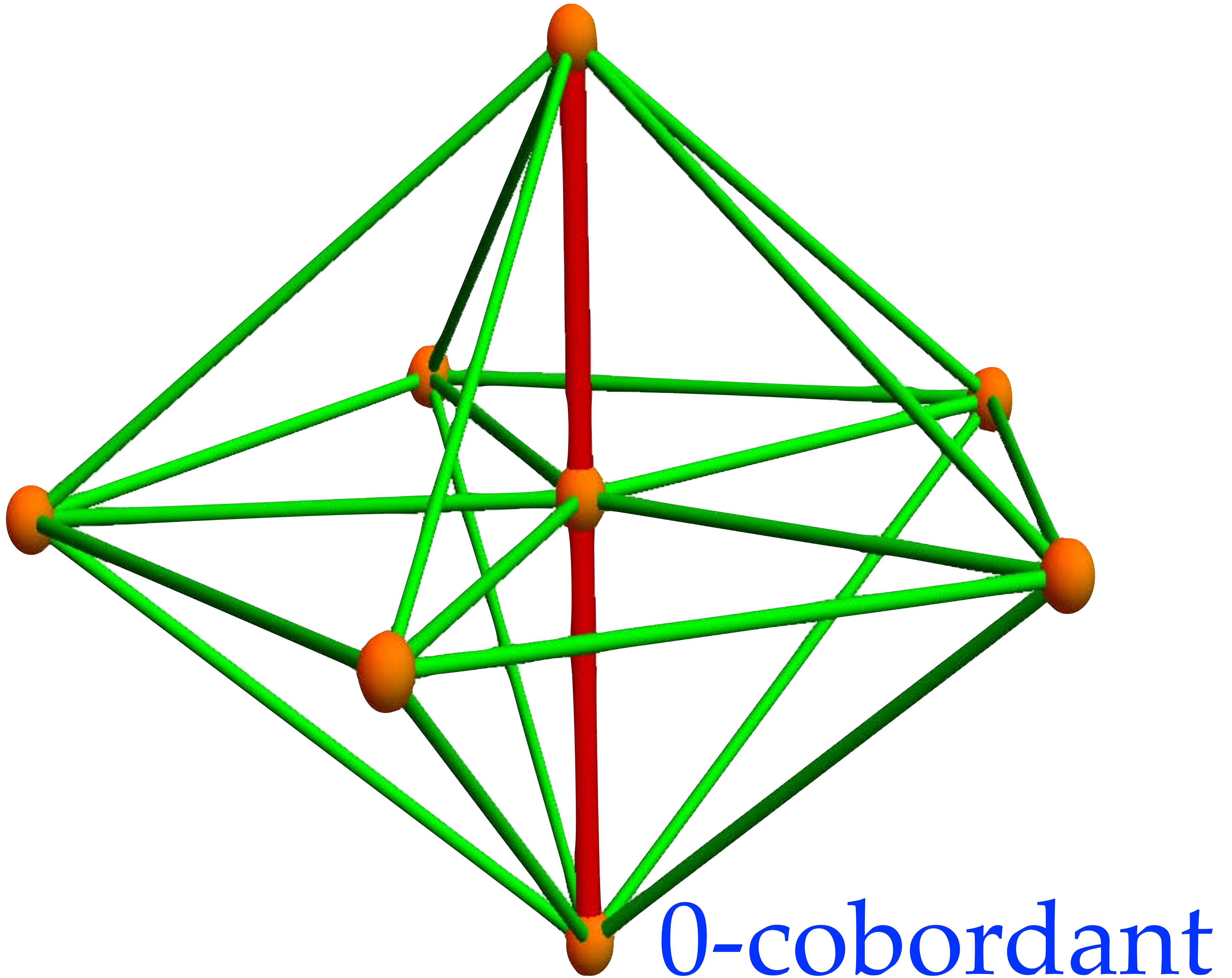
LETS TRY IT!

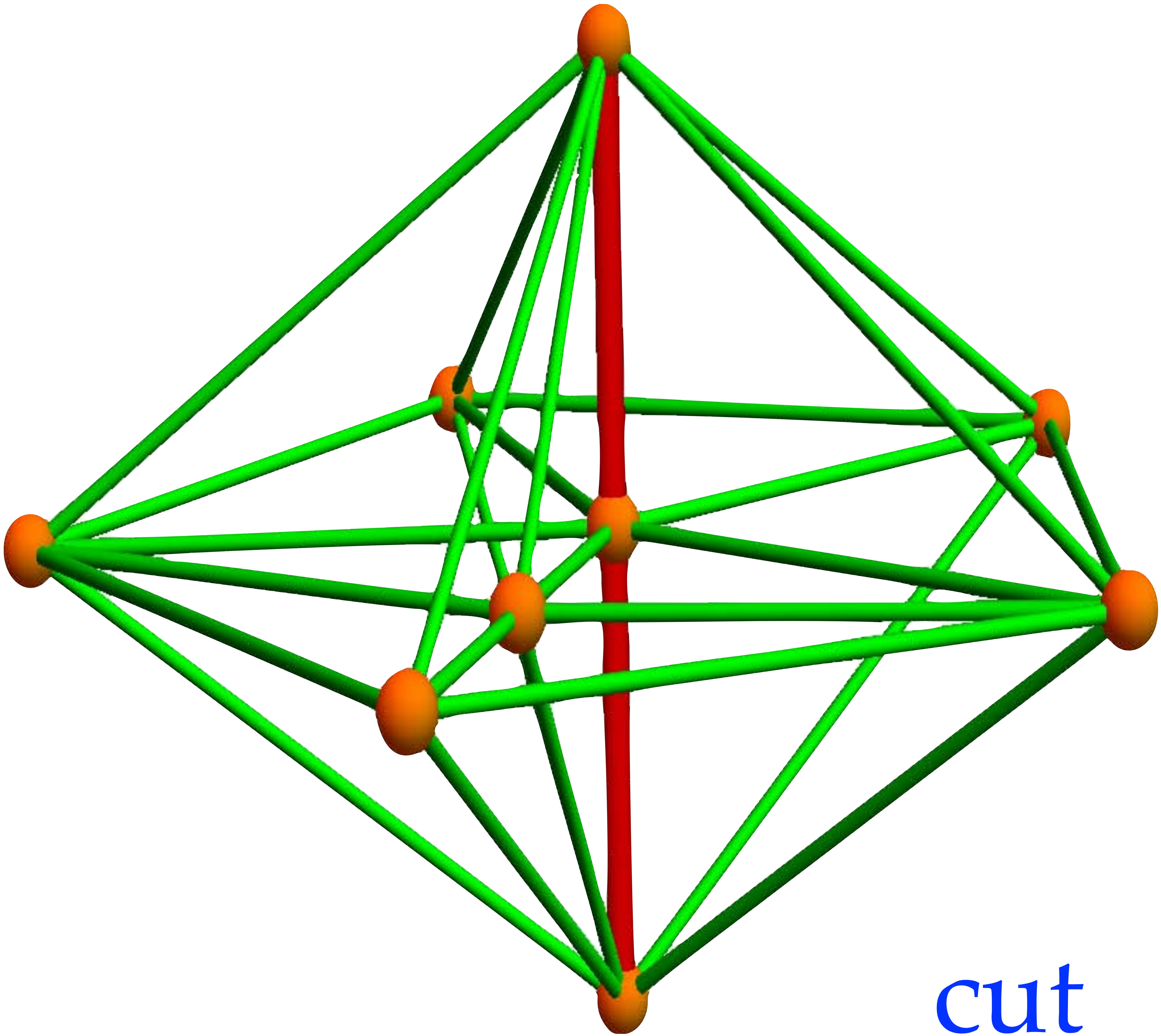
``does it work?”

DECAHEDRON

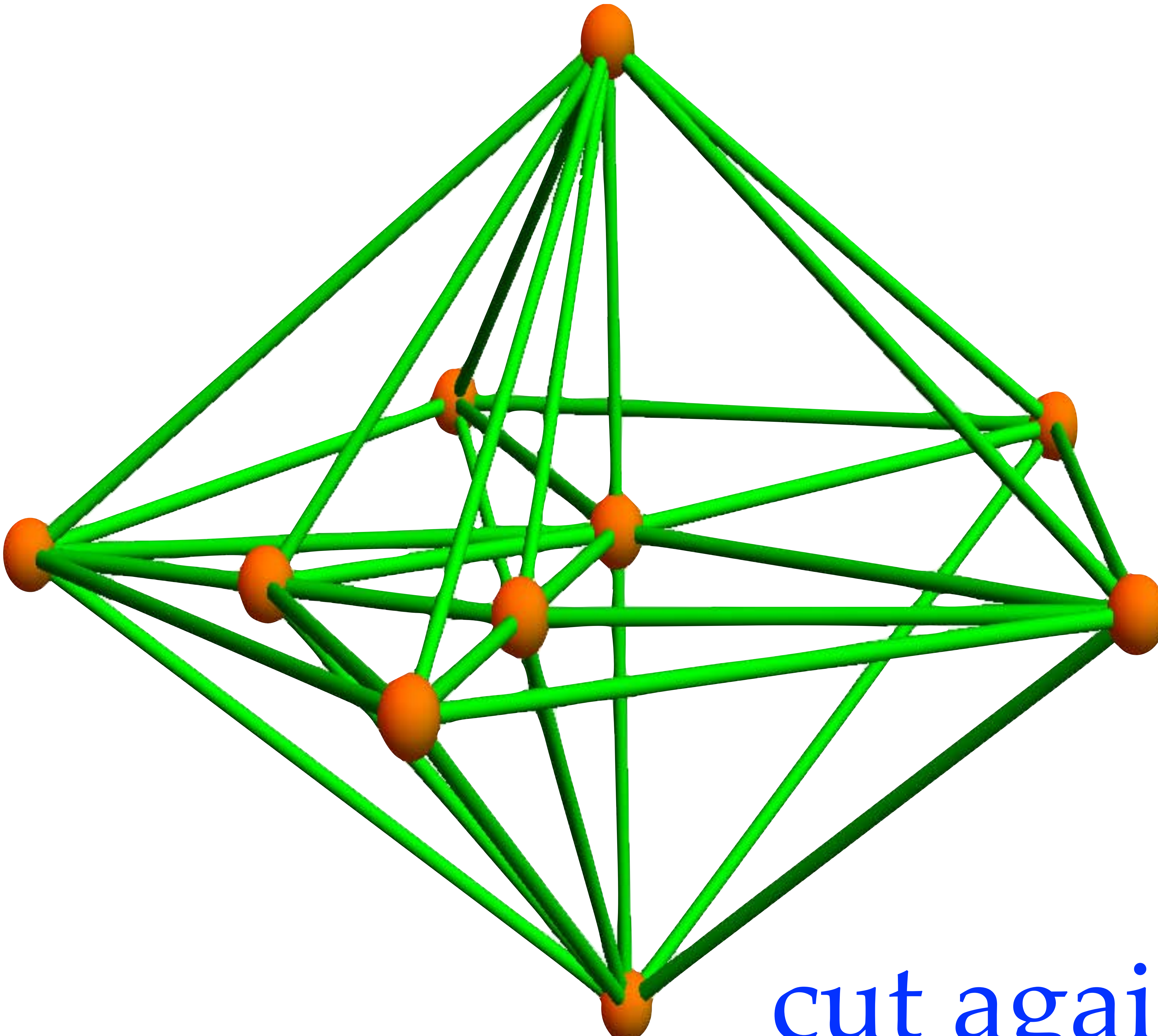


color that

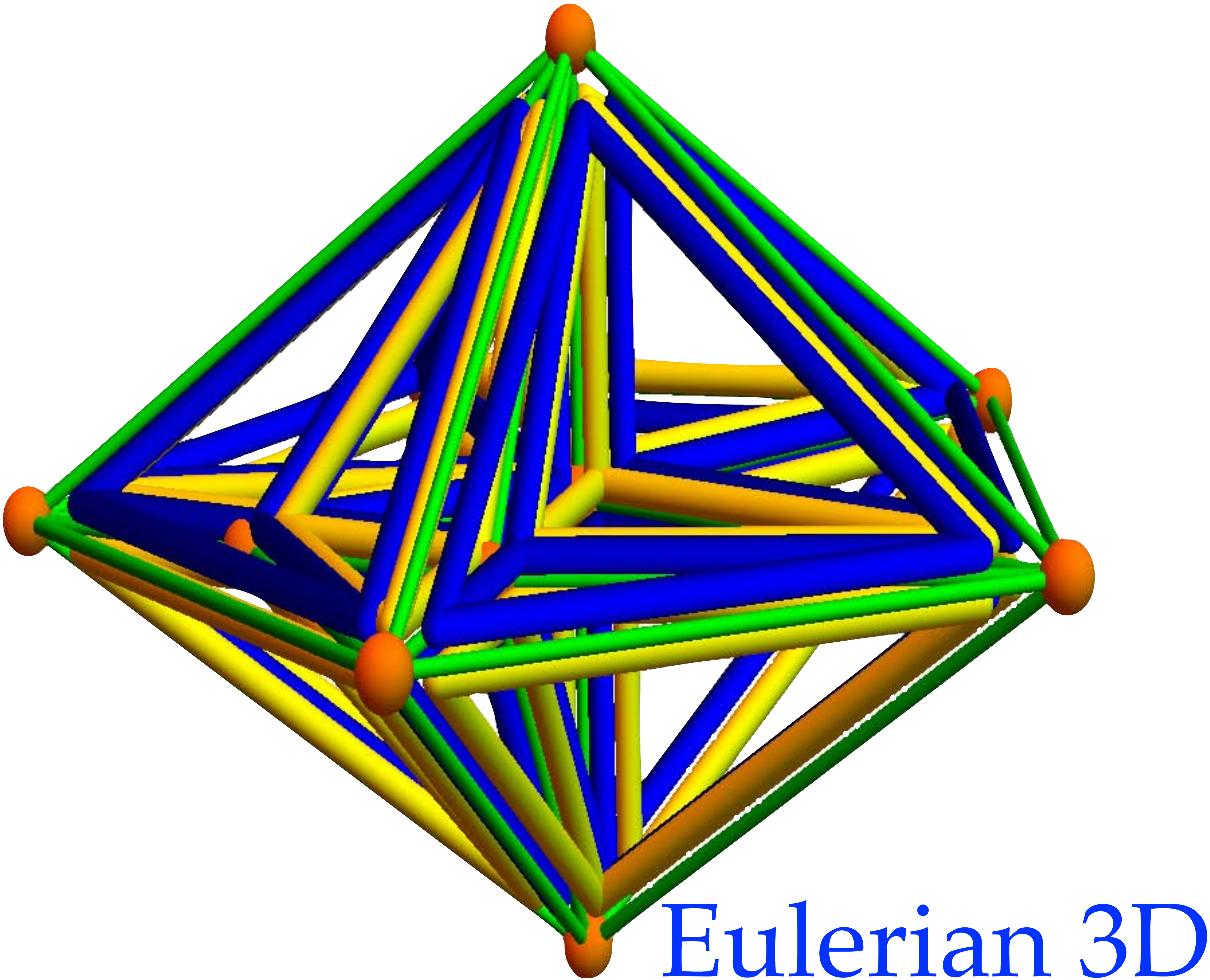

0-cobordant

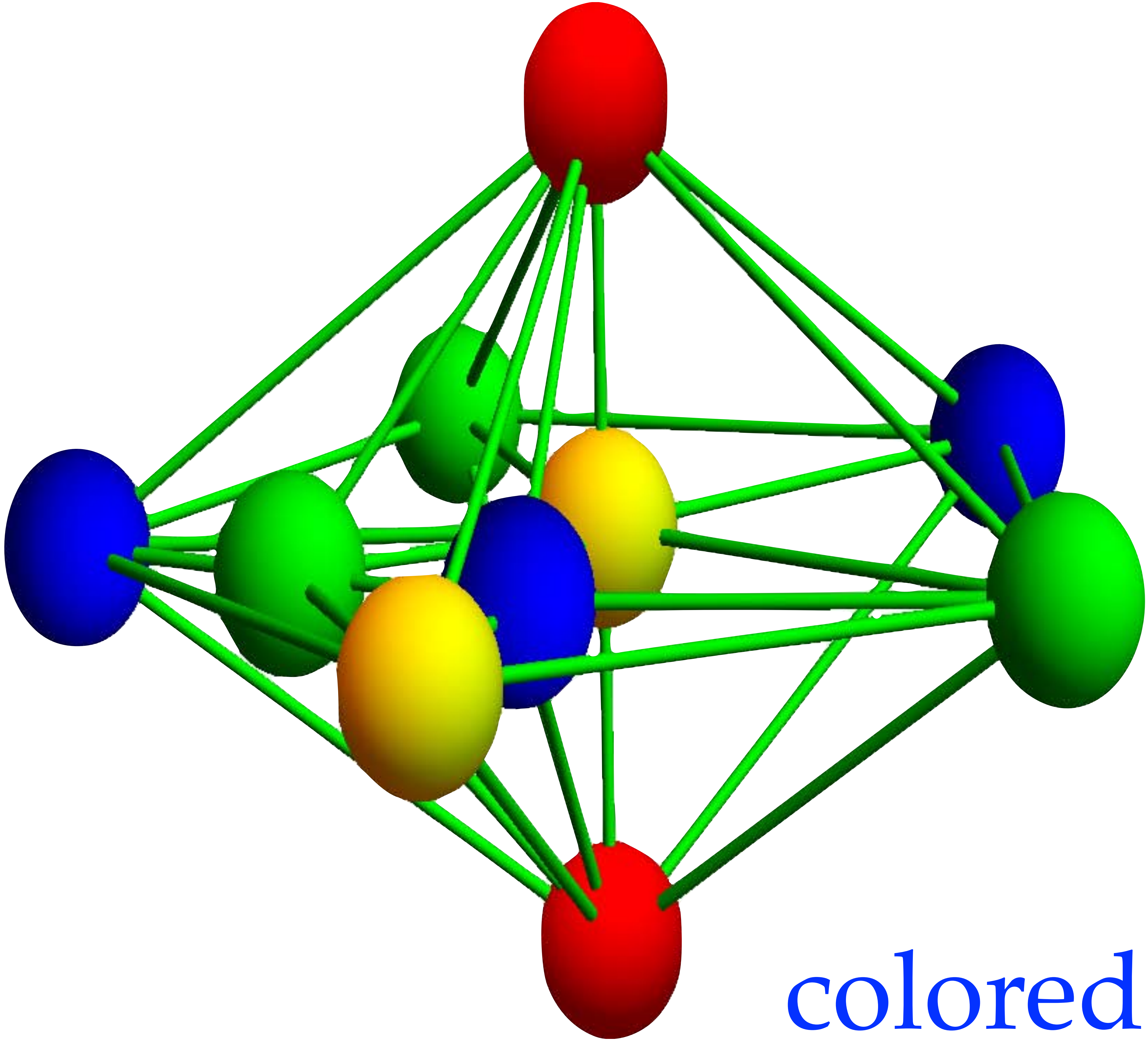


cut

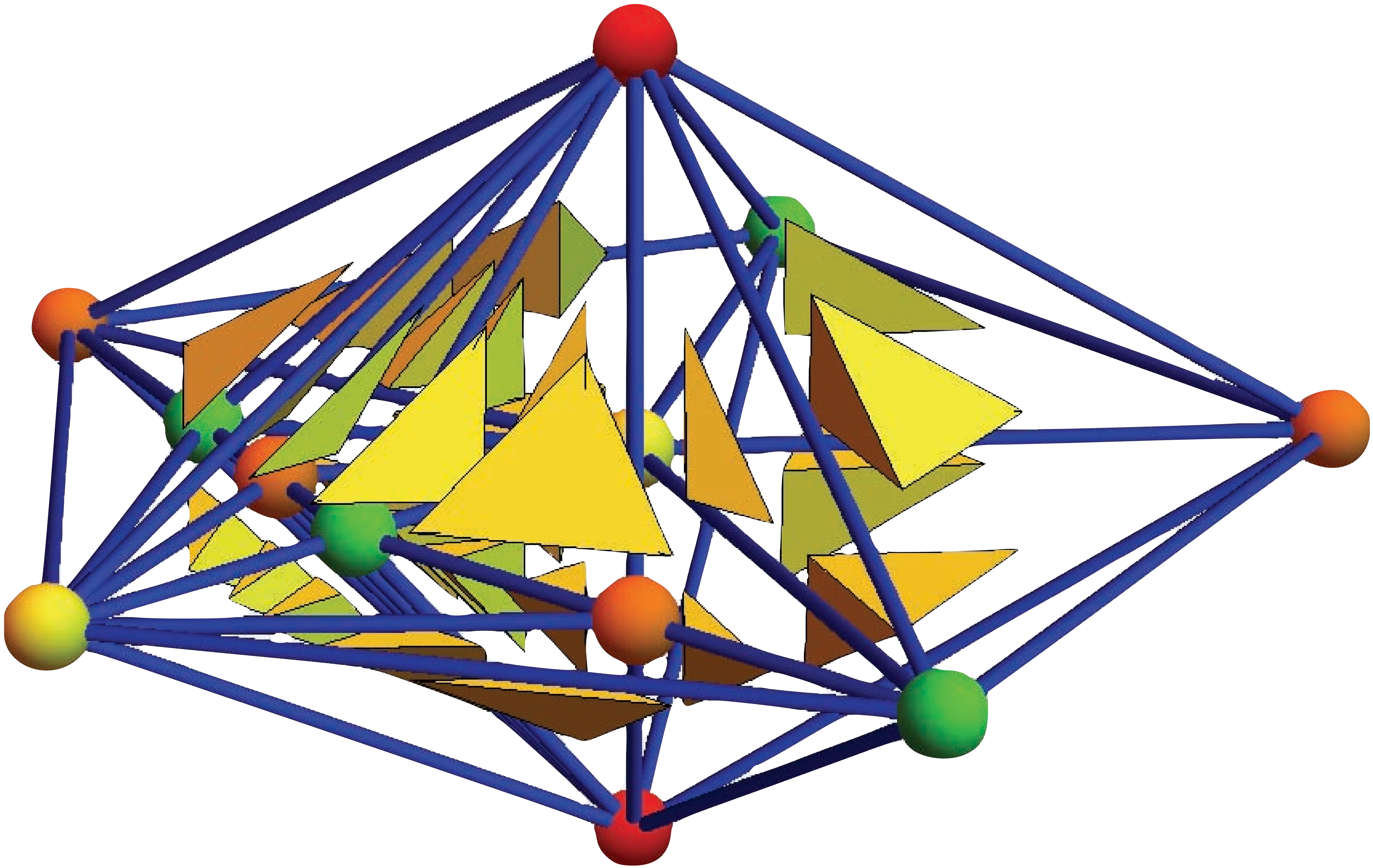


cut again


Eulerian 3D

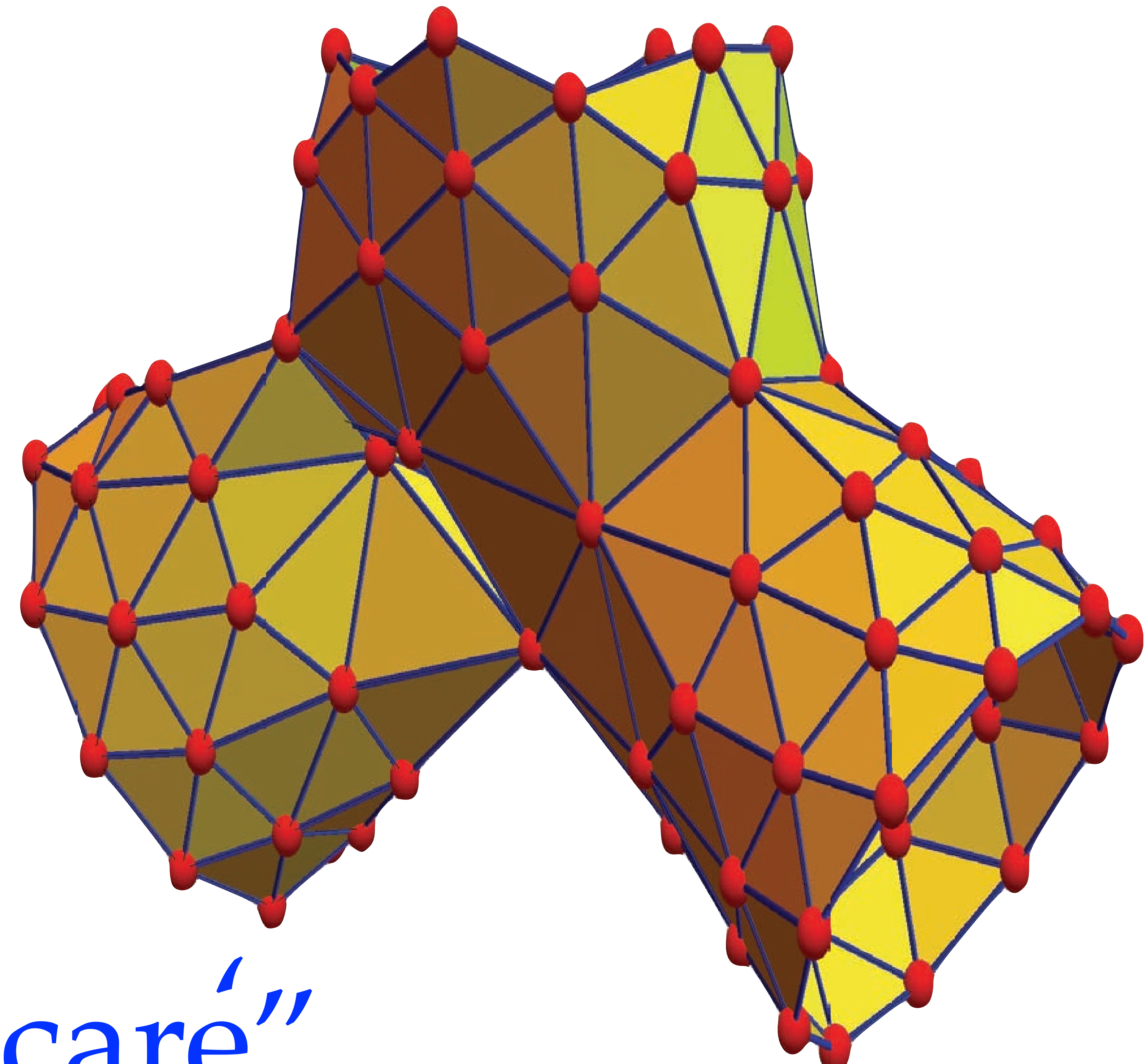


colored!



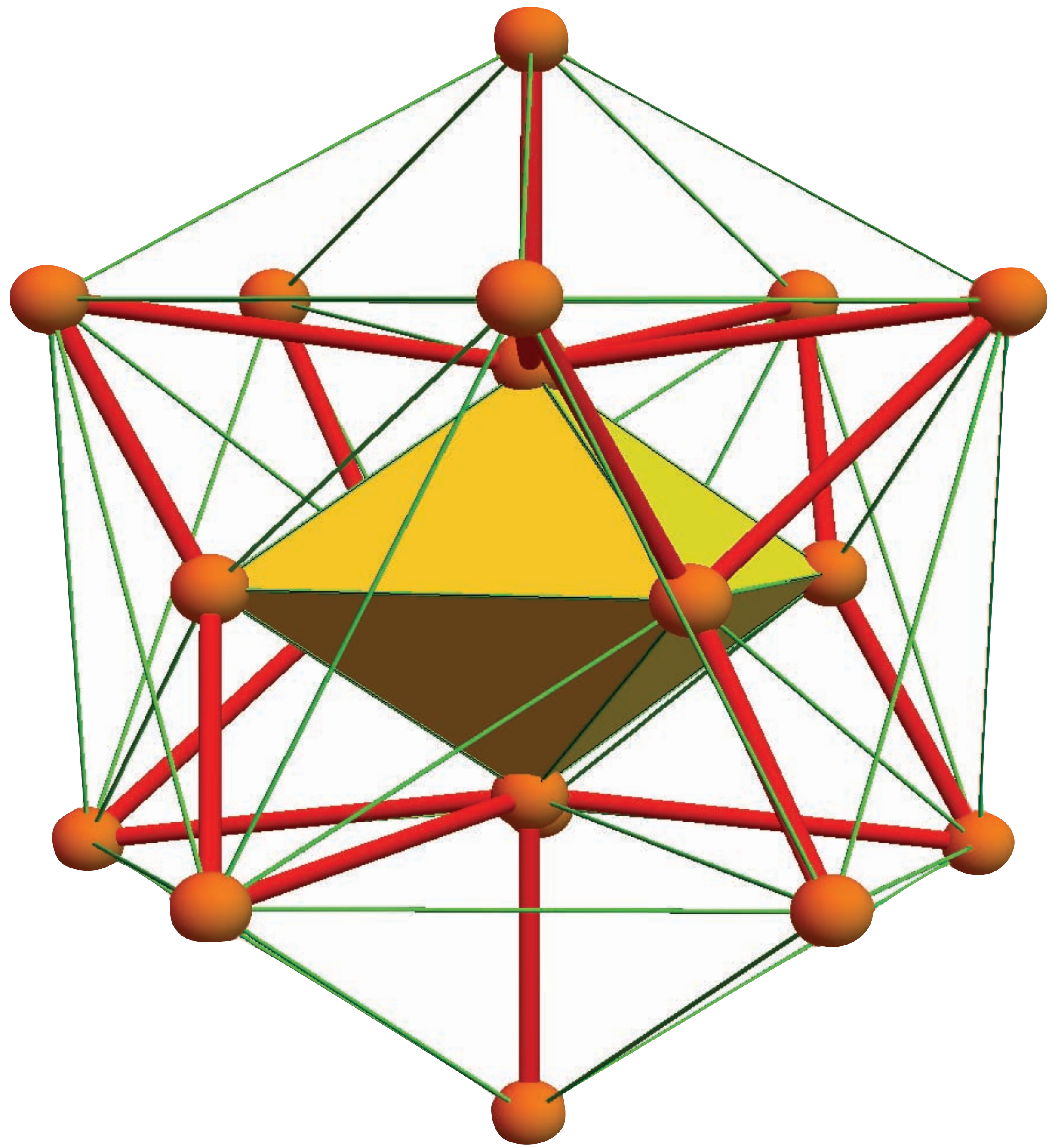
“by tetrahedra!”

COBORDISM



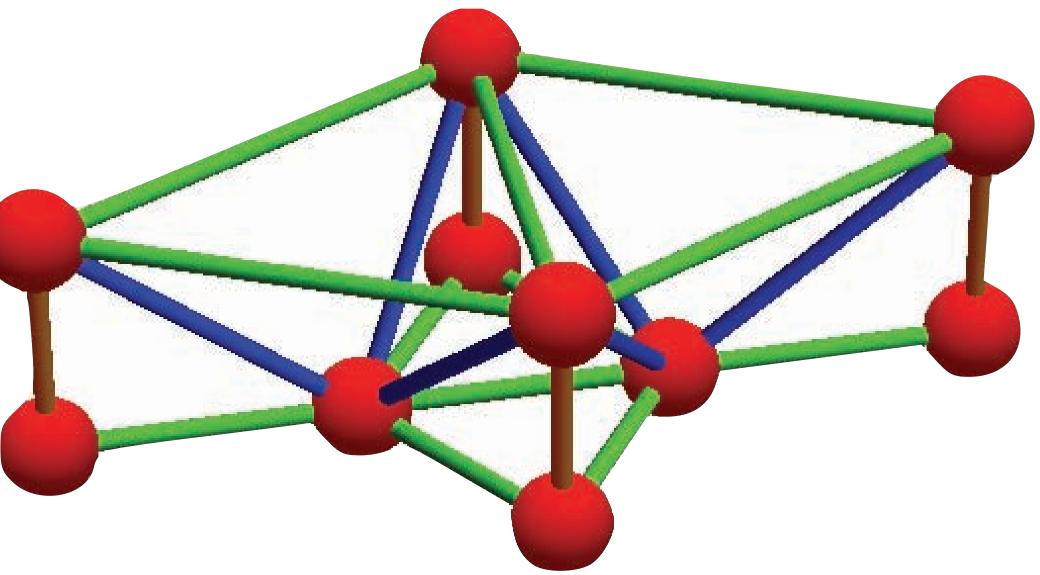
``Poincaré''

OCTA-ICOSA

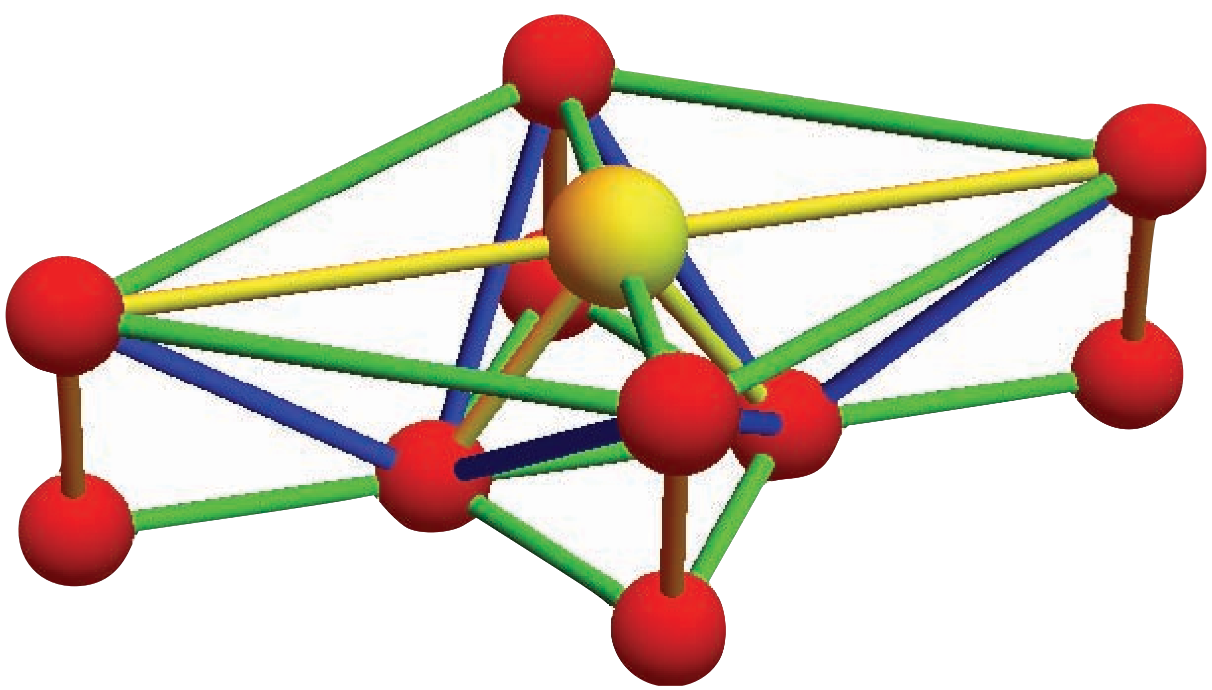


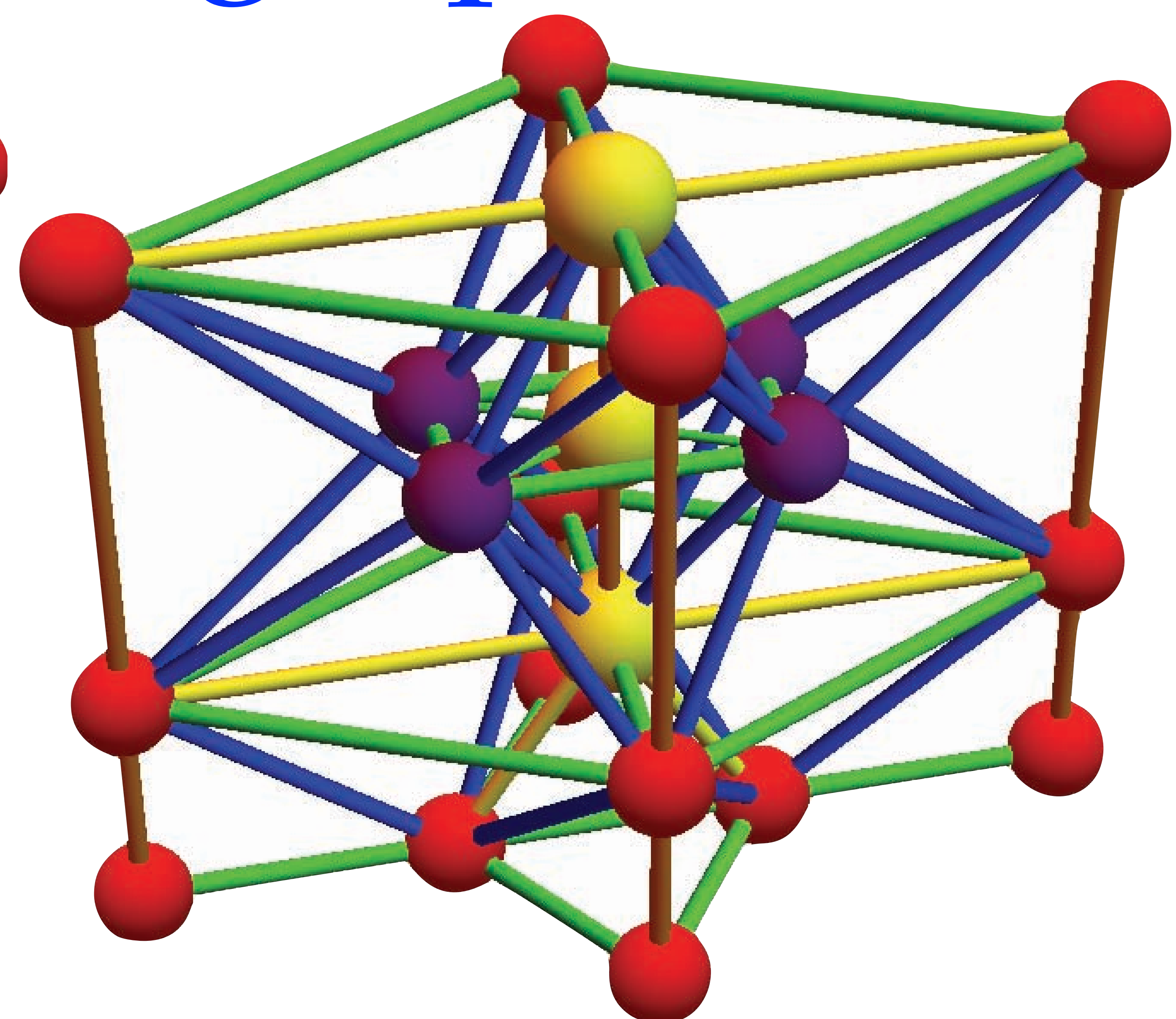
``Cobordism between spheres”

SELF-COBORDISM

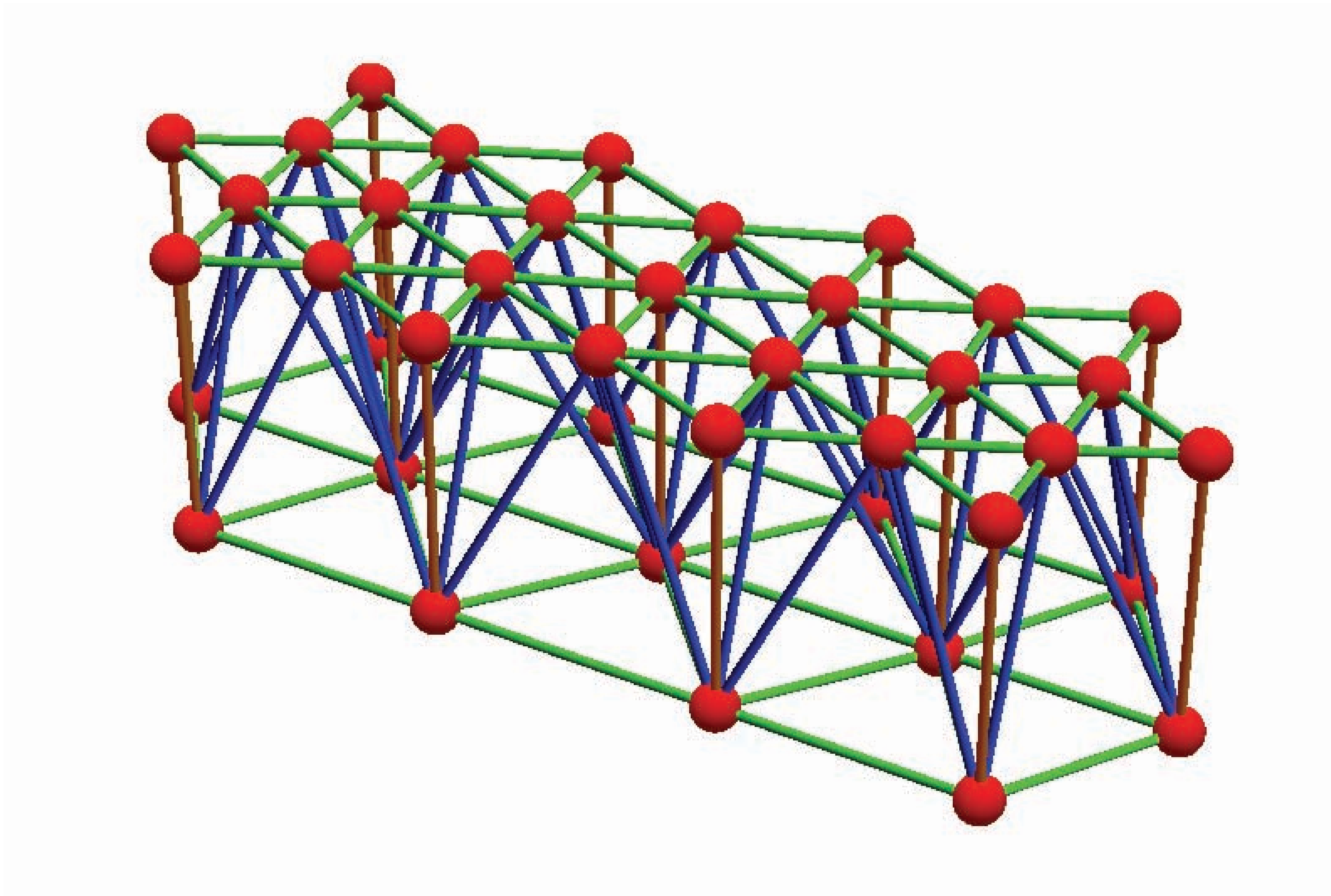


``Sandwich dual graph”



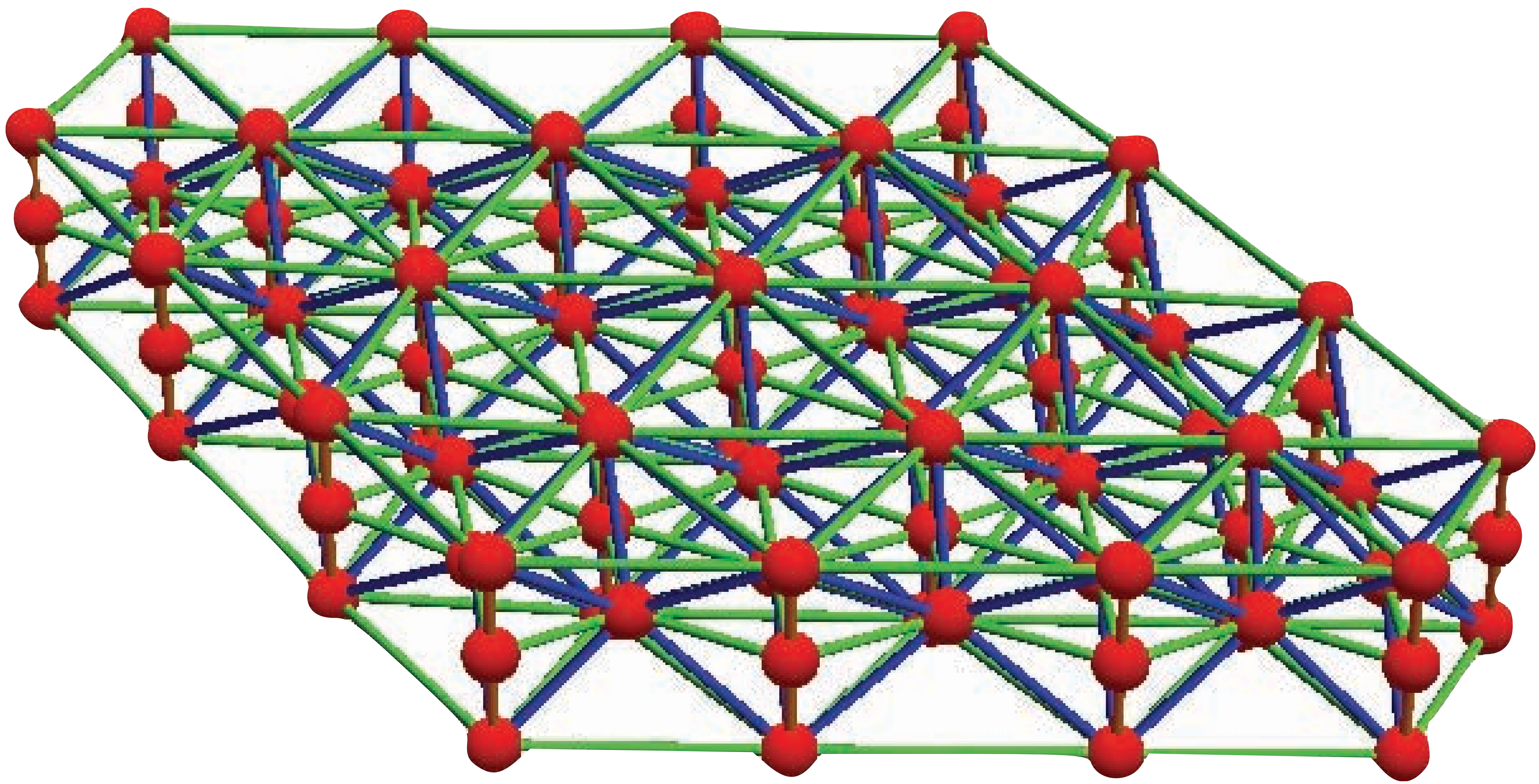


SELFCOBORDISM



``X=X in cobordism group"

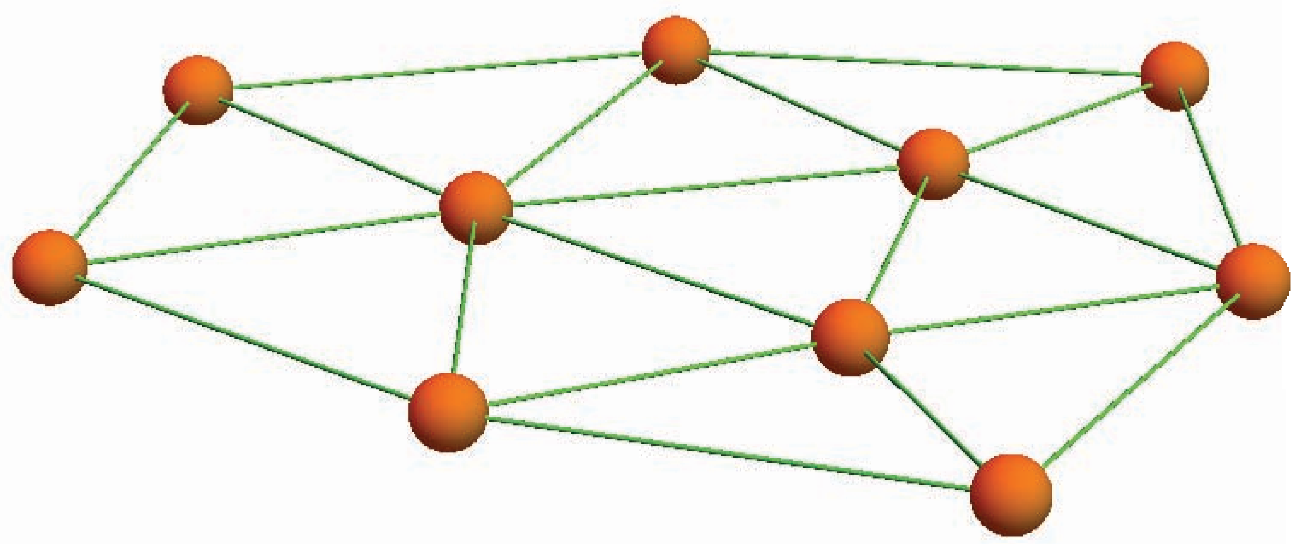
SELF-COBORDISM

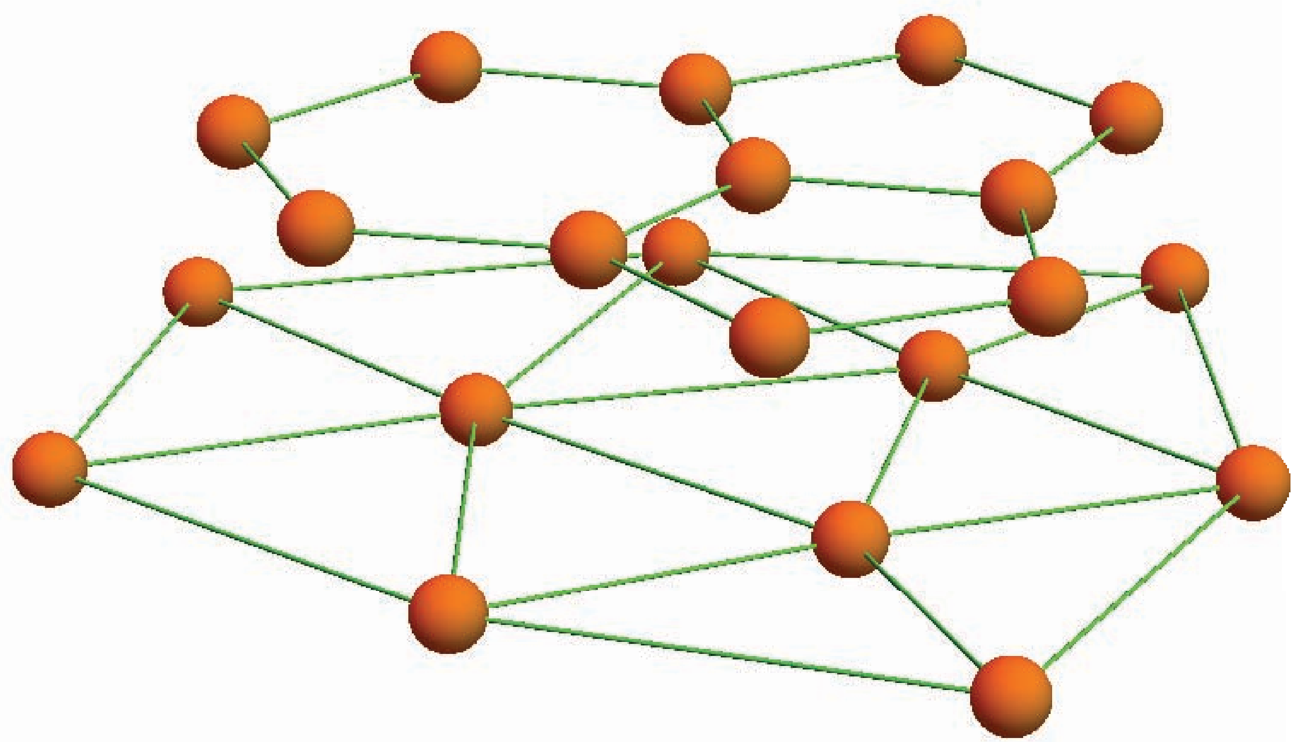


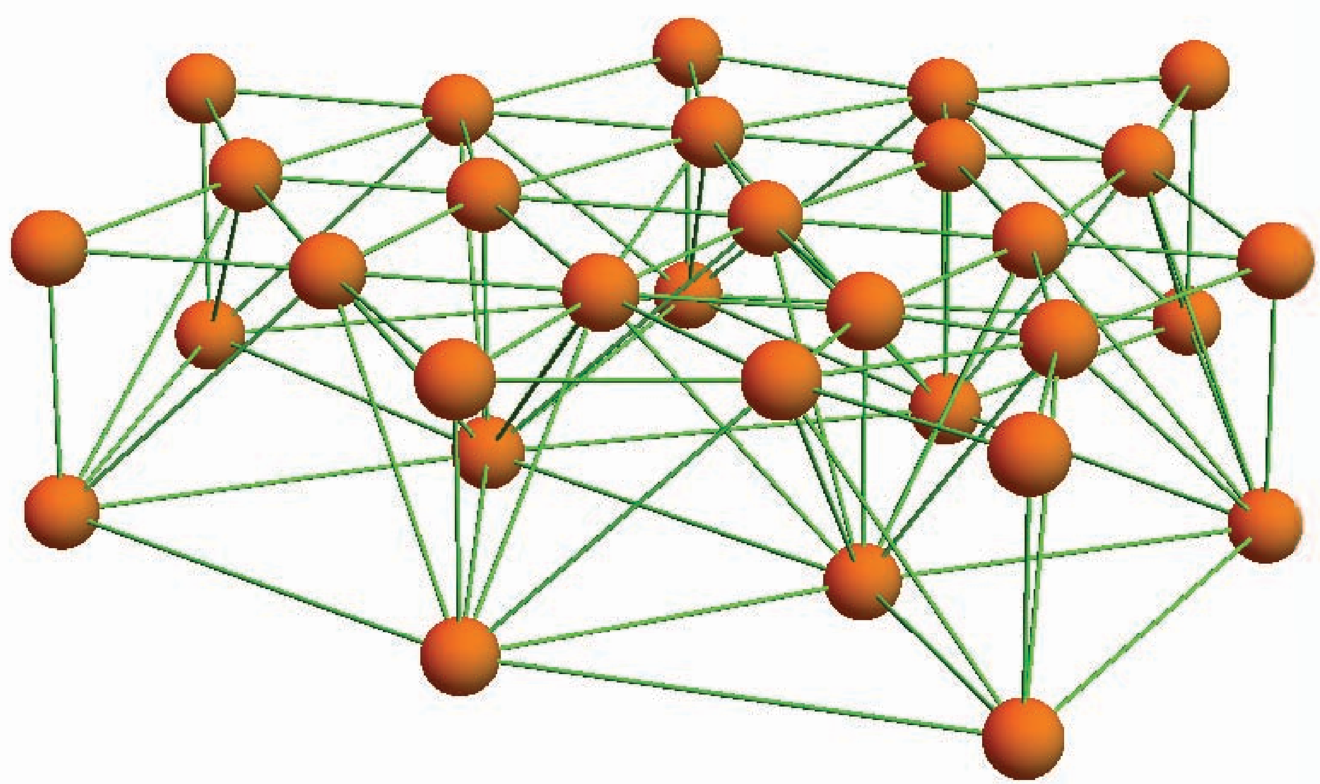
``crystal''

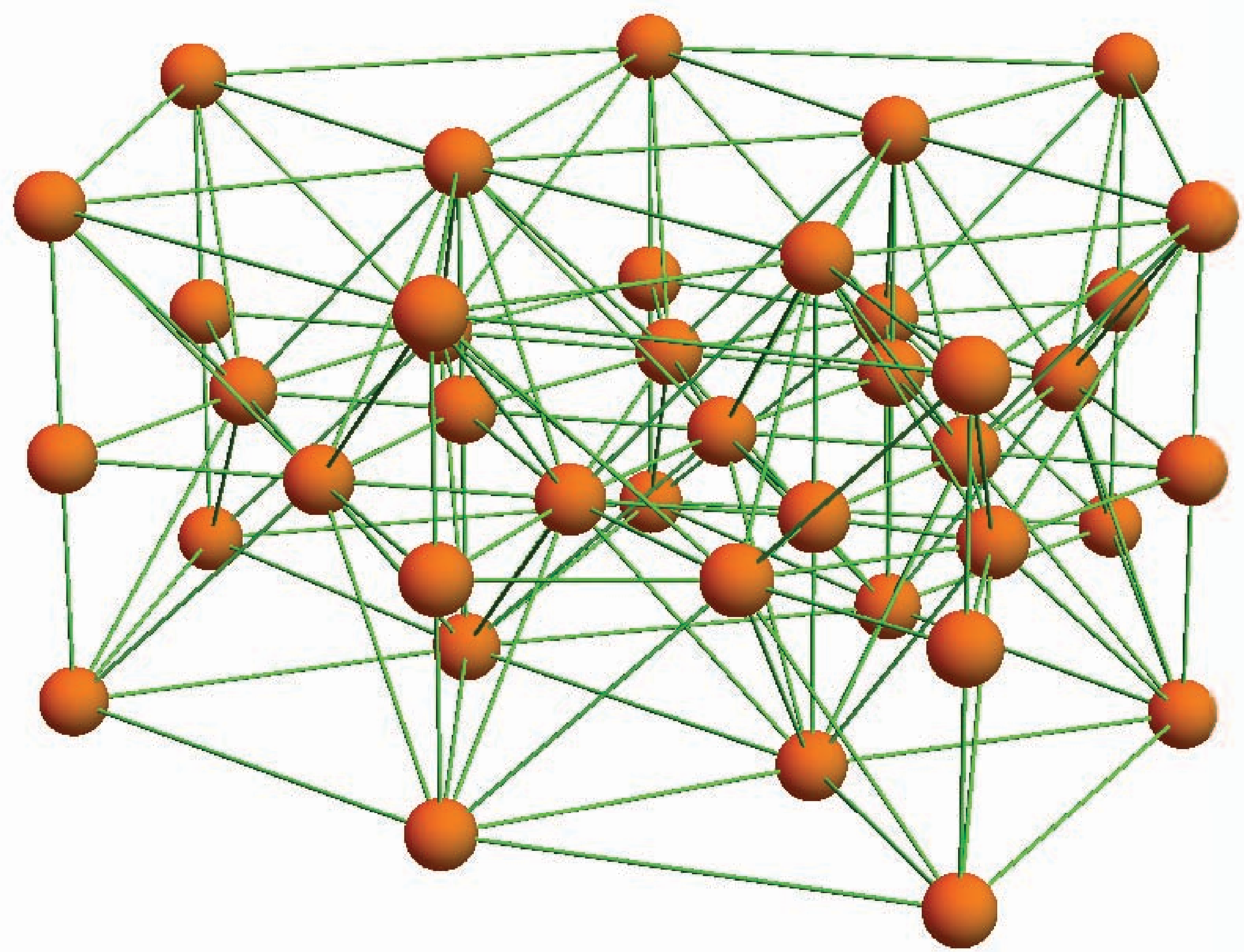
EXAMPLE COLORING

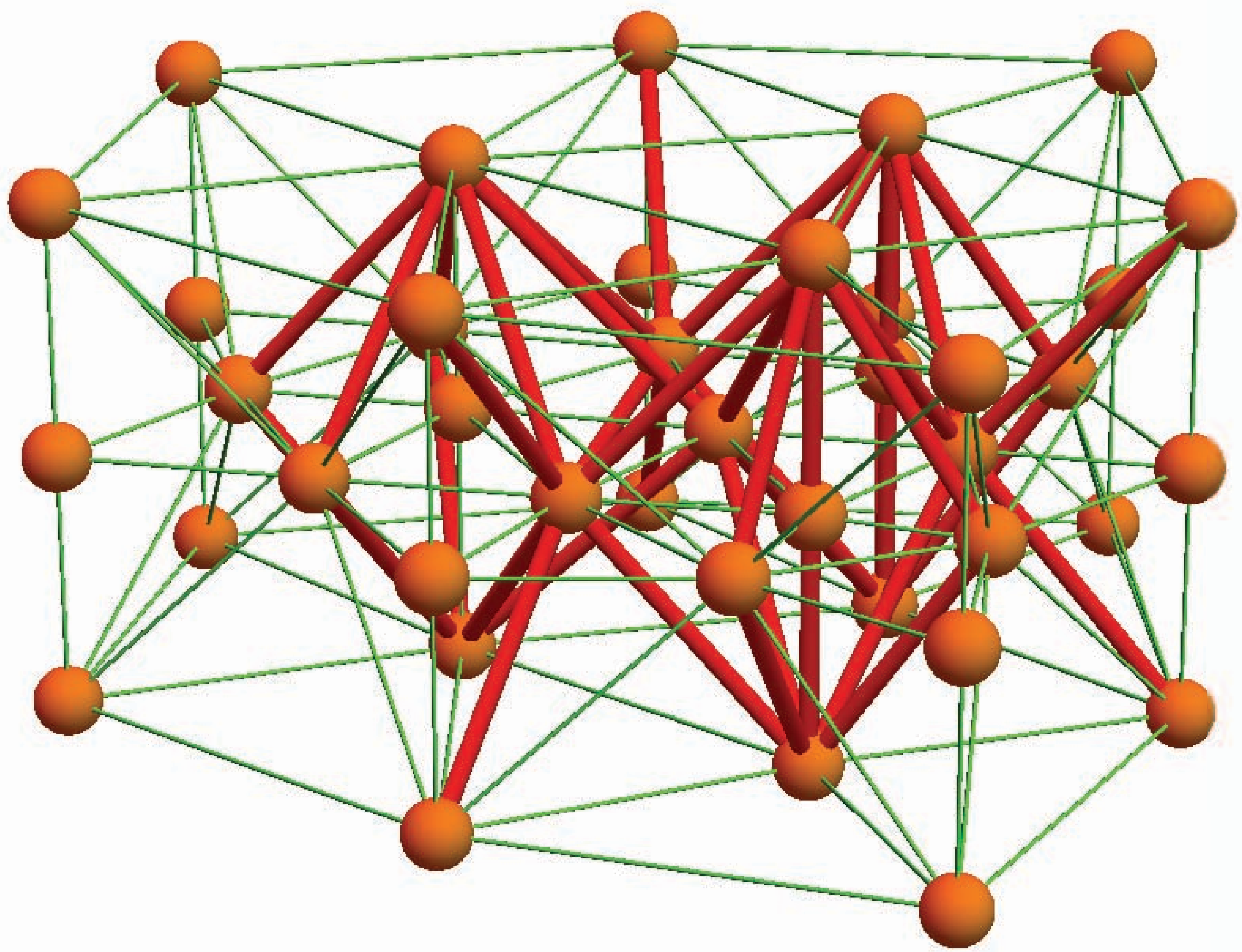
``yes it does”

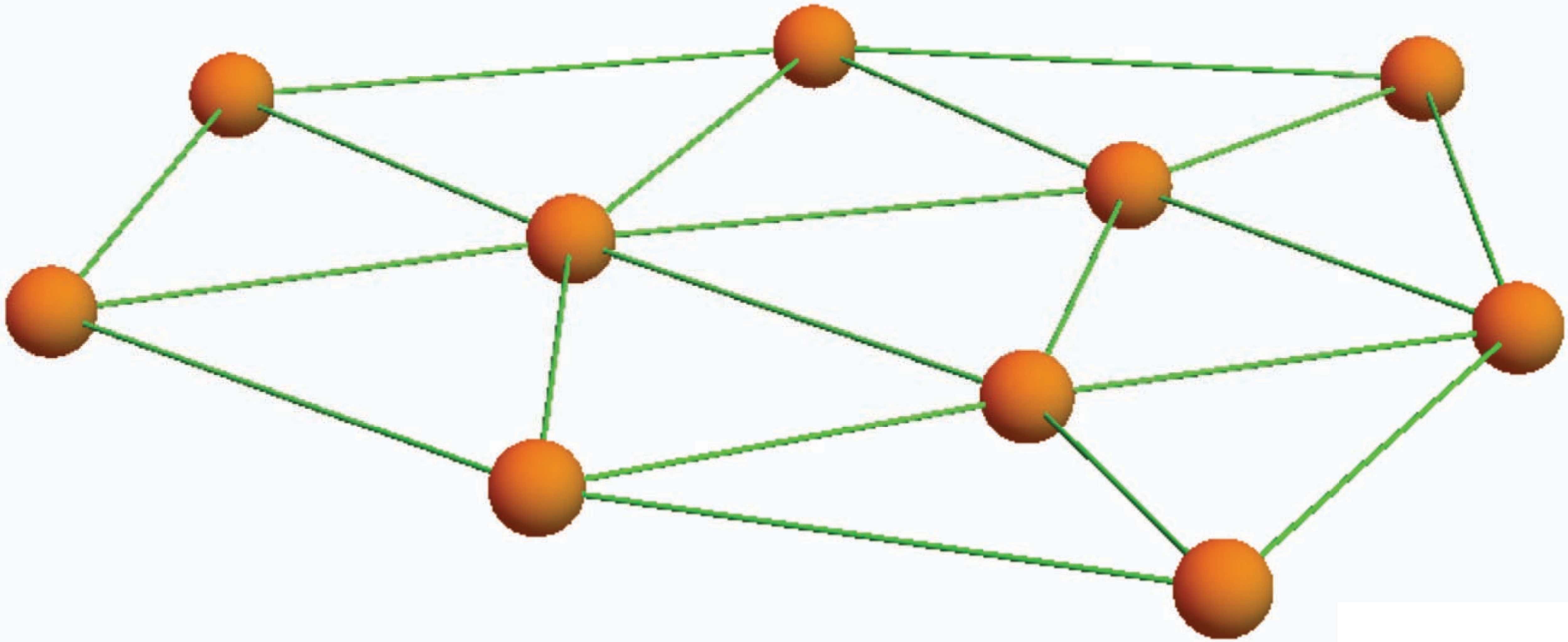


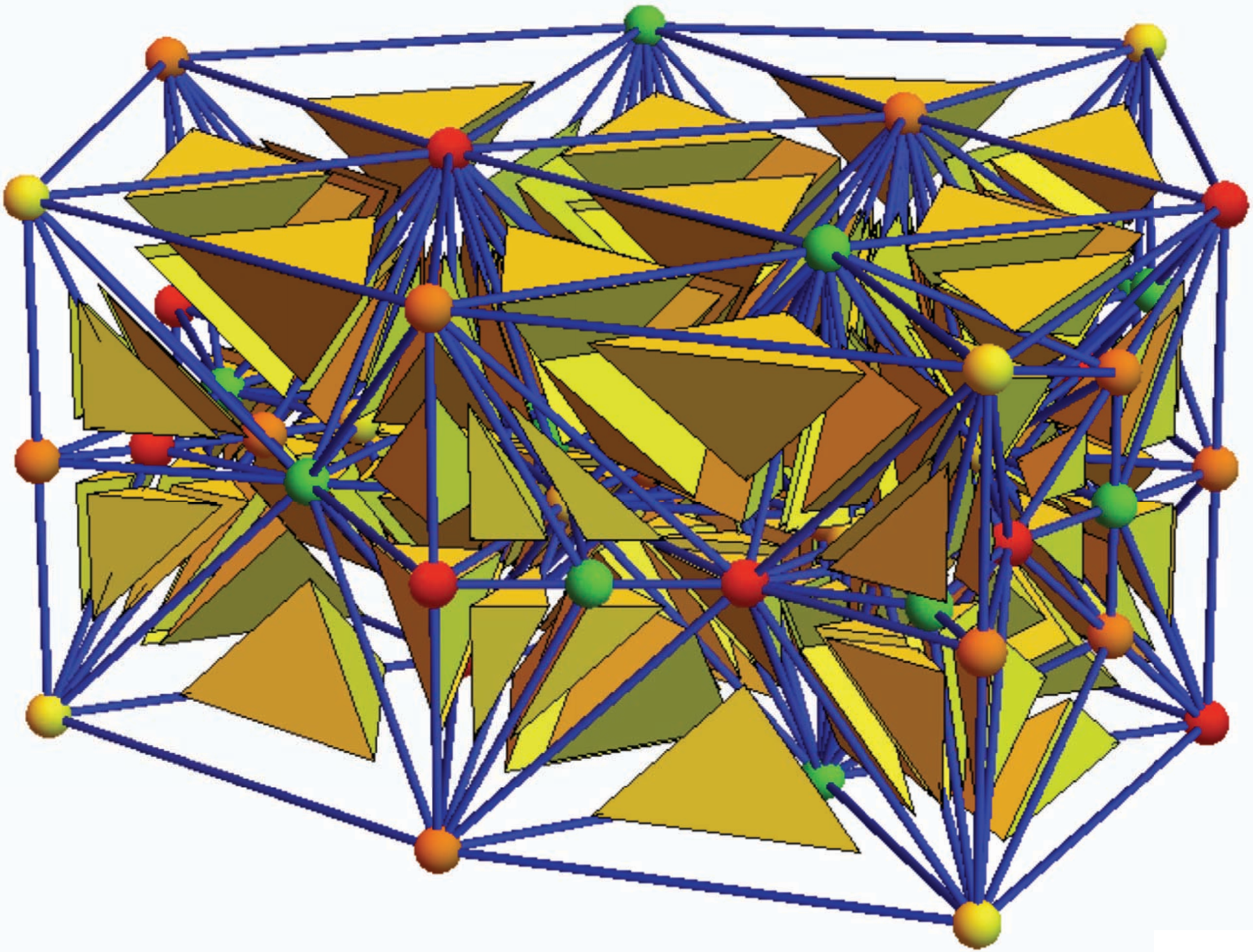






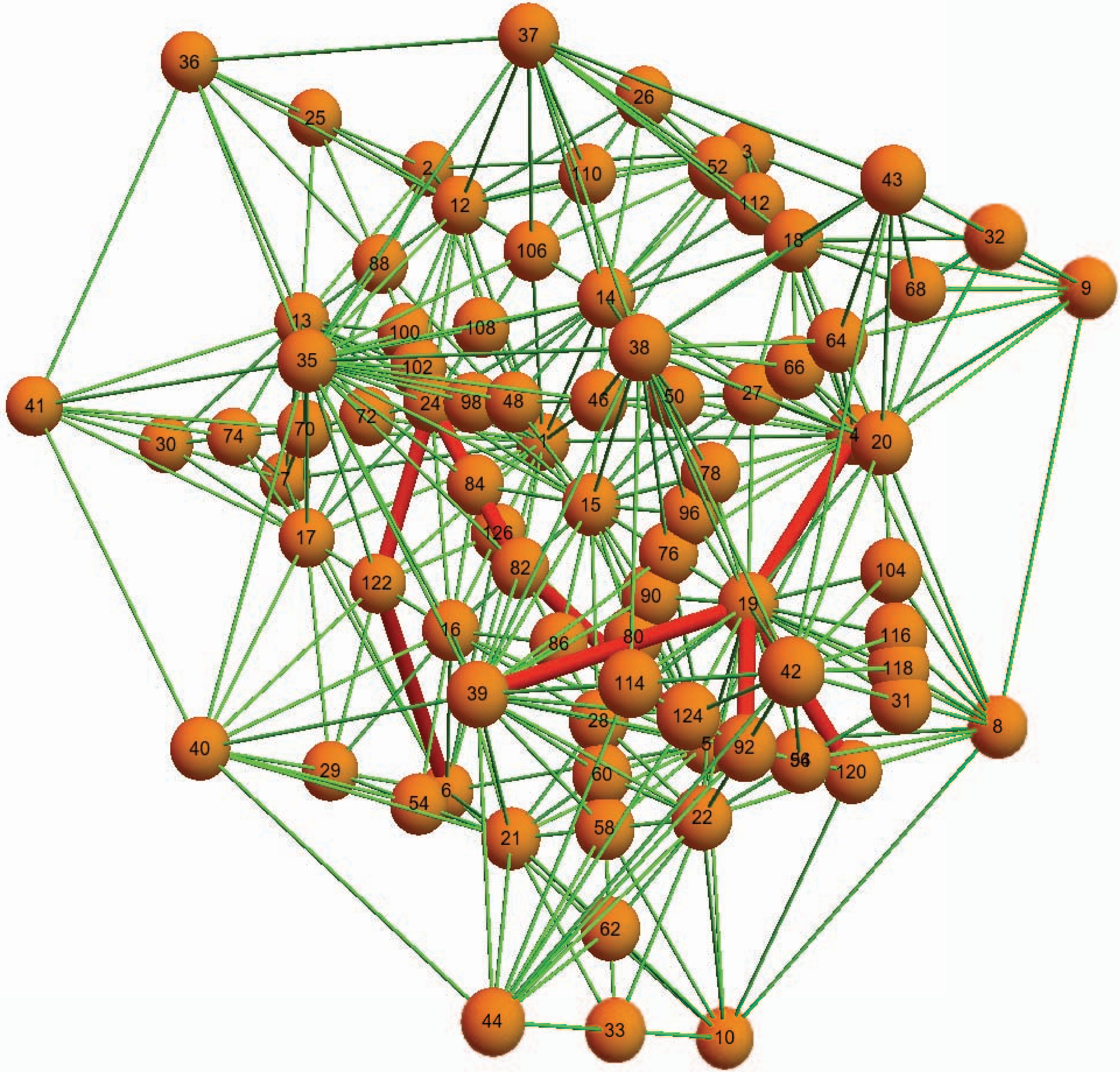






SIMULATED ANNEALING

``does it always work?”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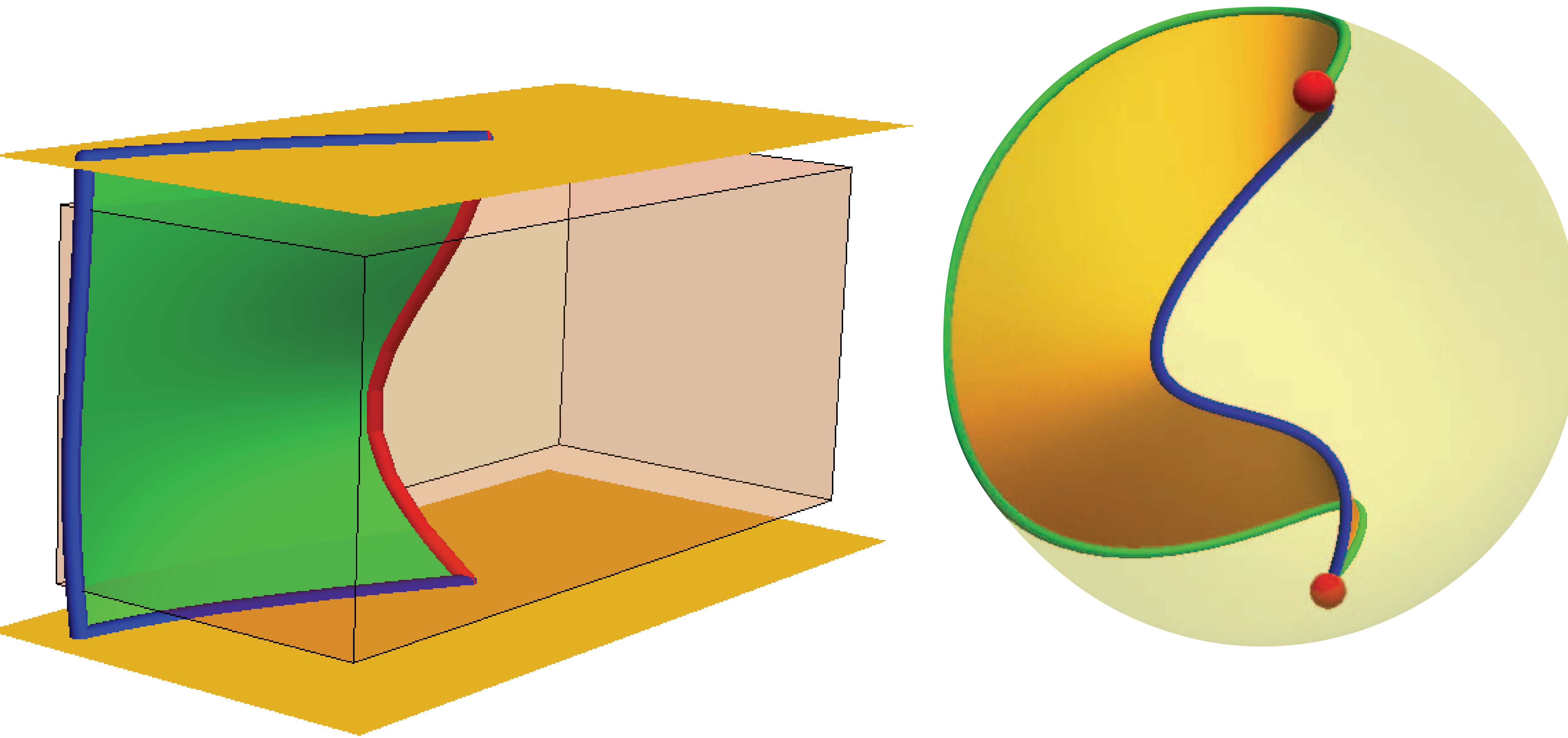
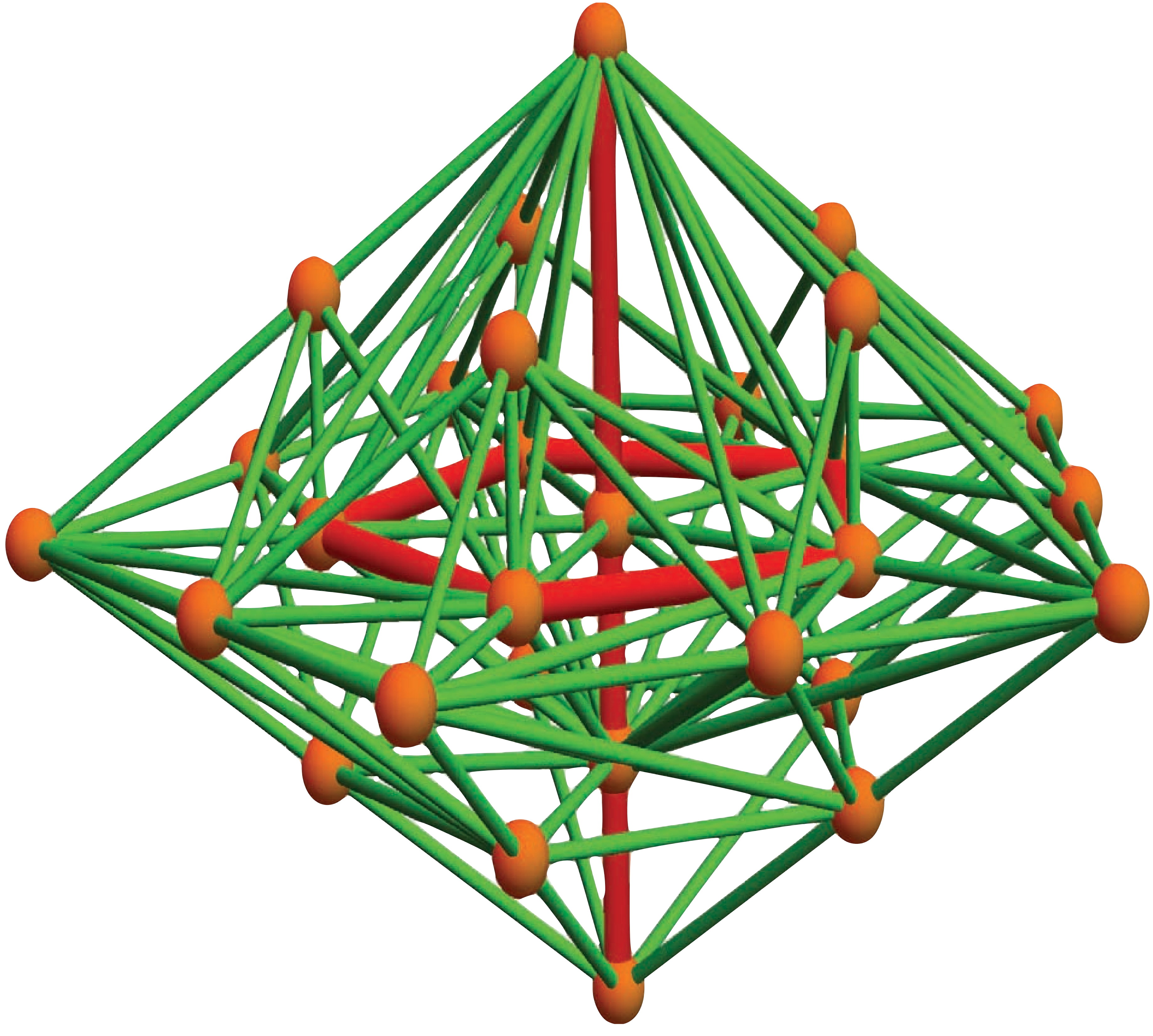
CUTTING STEP





BEYOND SPHERES

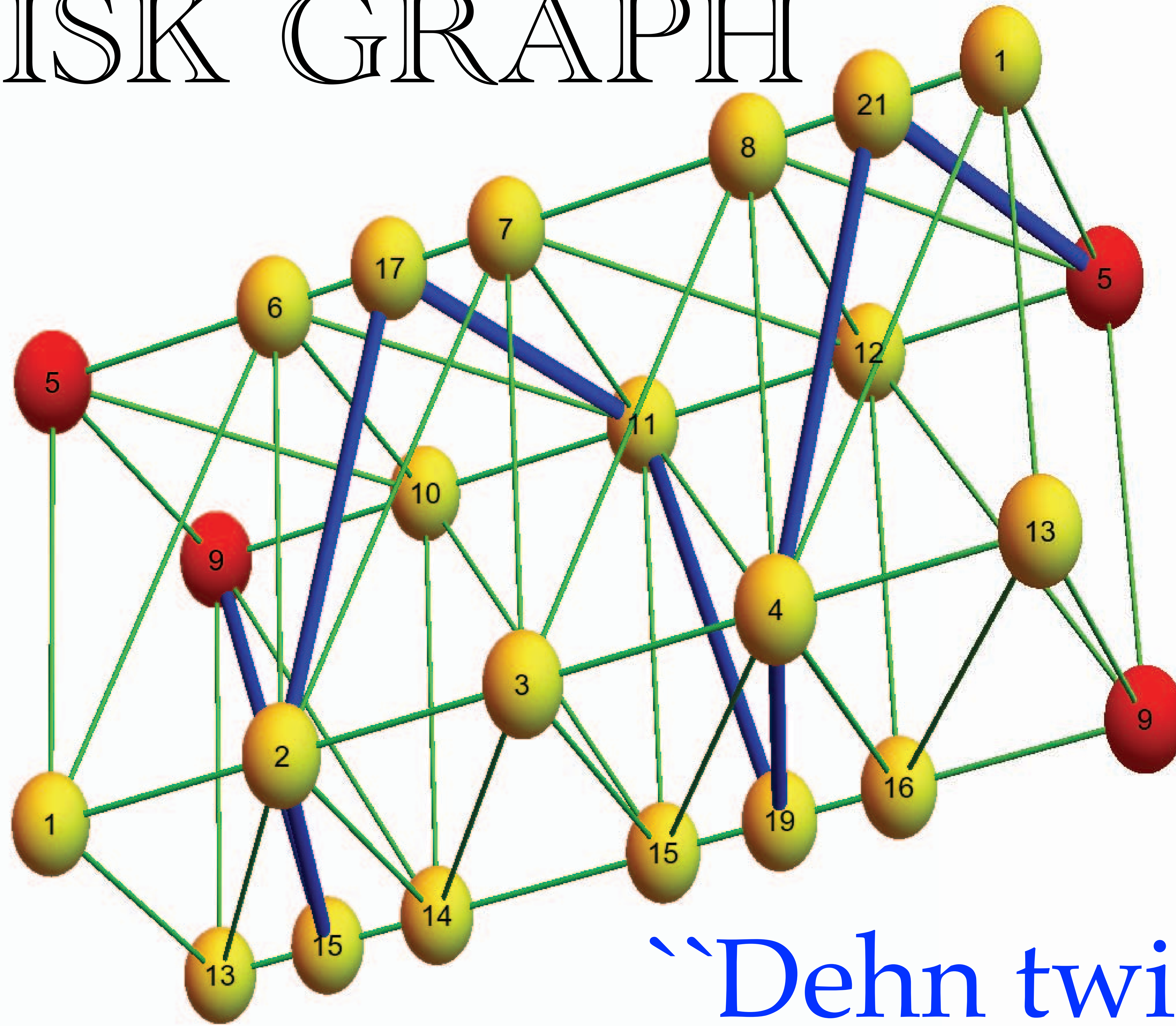
$\mathcal{G}_{2,g,o} \cap \mathcal{C}_3$ not empty

$\mathcal{G}_{2,g,o} \cap \mathcal{C}_4$ not empty

$\mathcal{G}_{2,g,o} \cap \mathcal{C}_5$ not empty

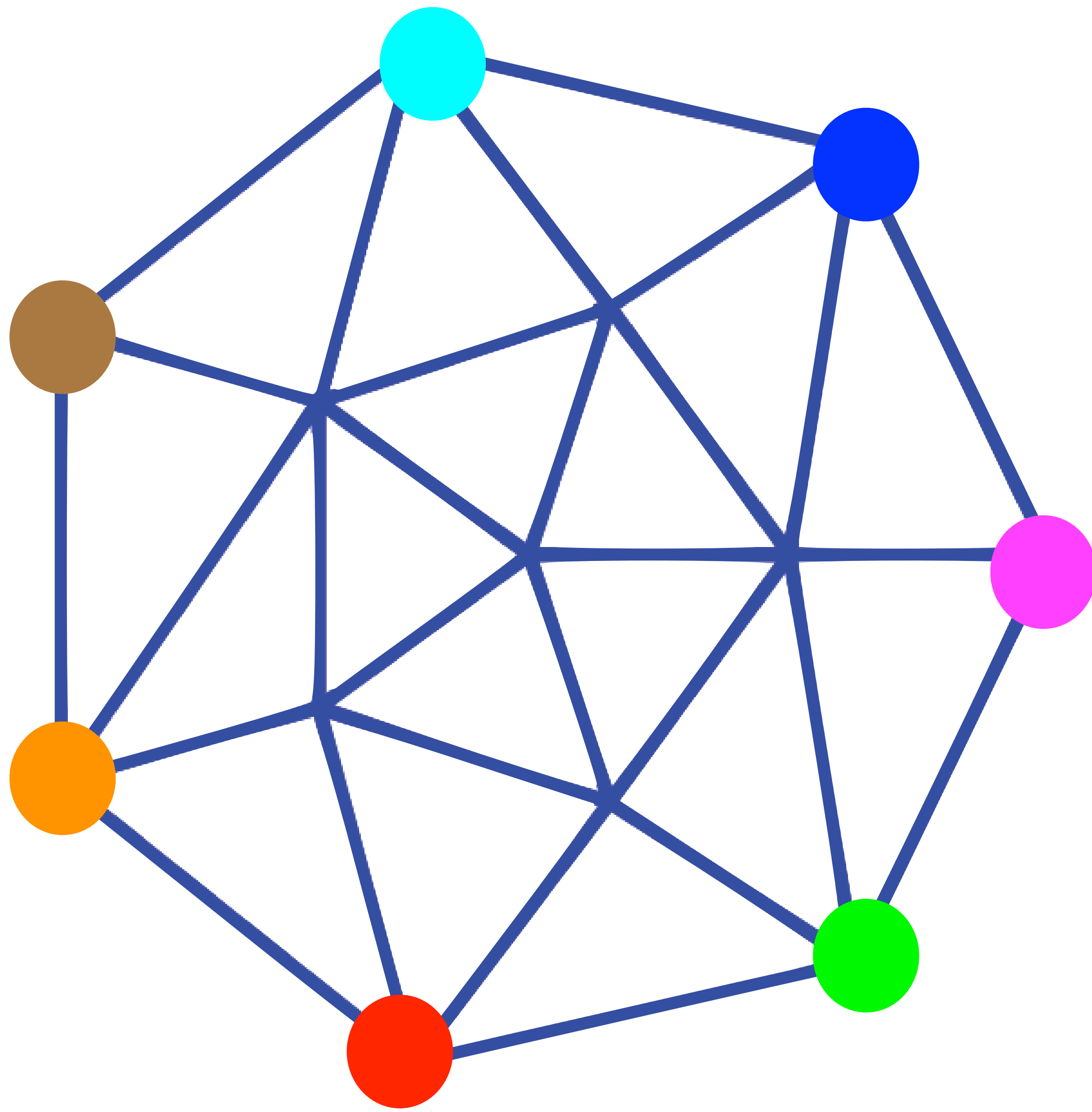
``for c=5: Fisk theory”

FISK GRAPH

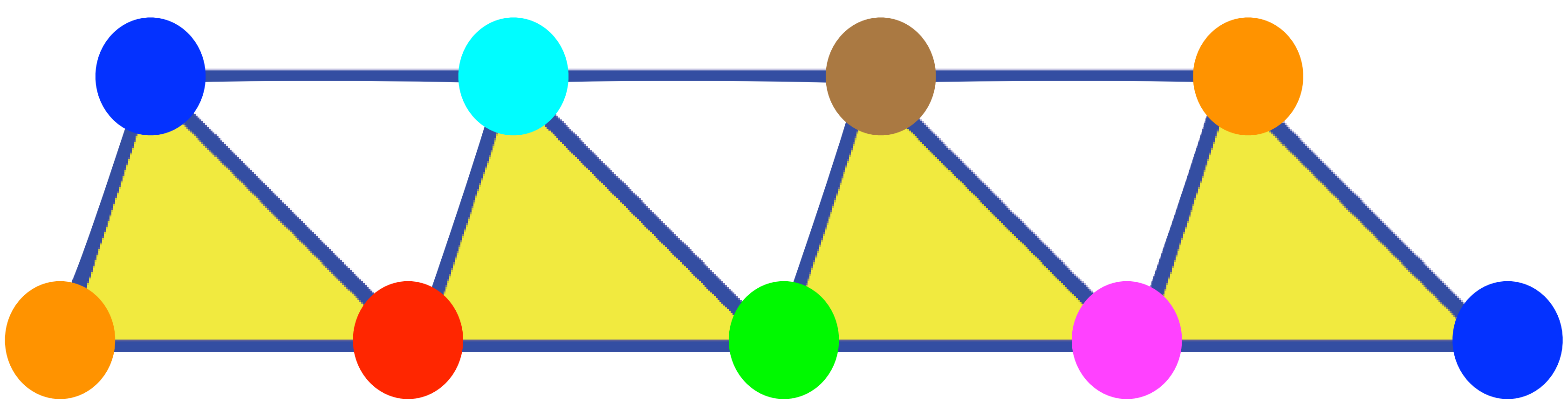


``Dehn twist''

JENNY'S GRAPH

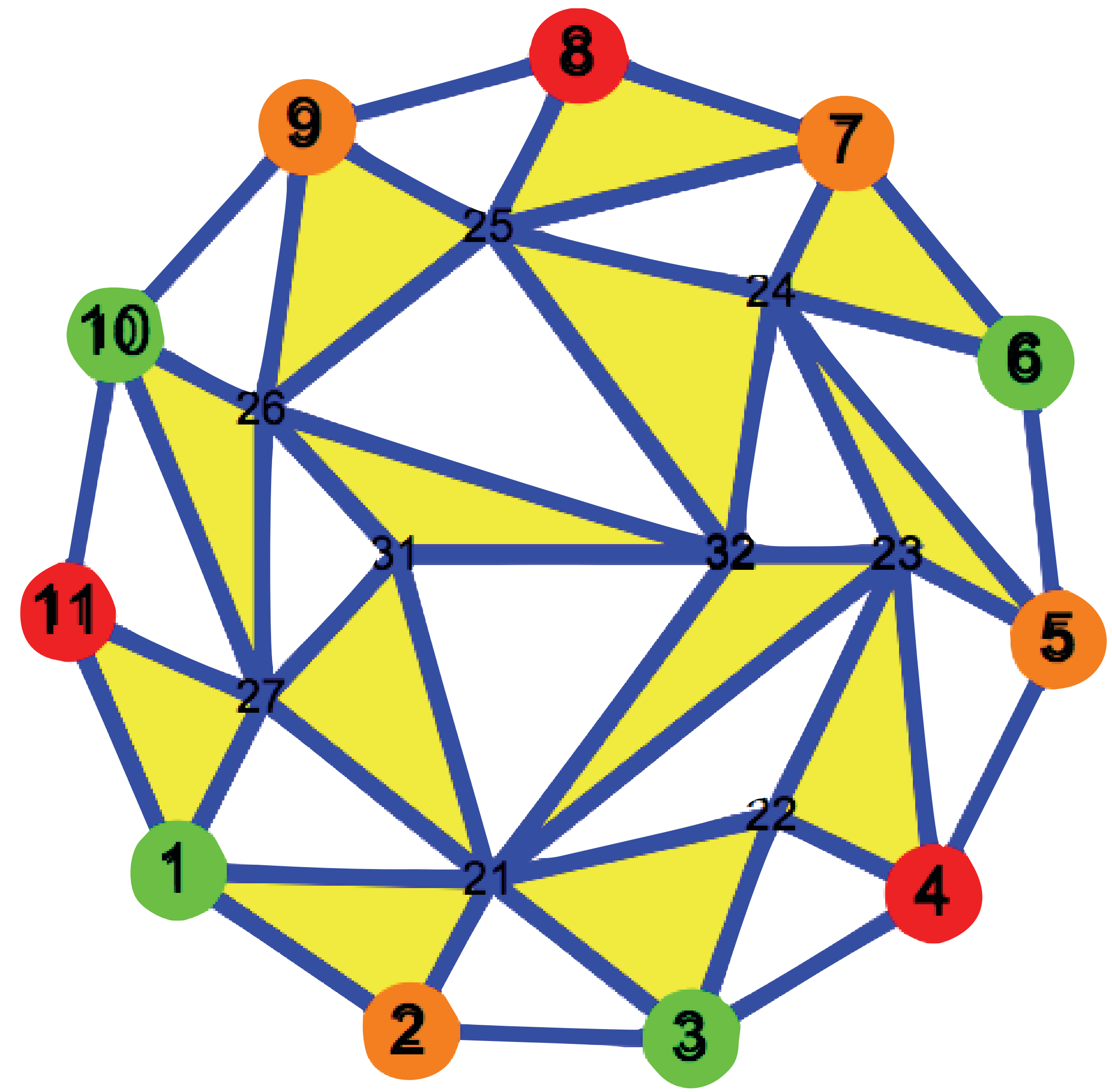


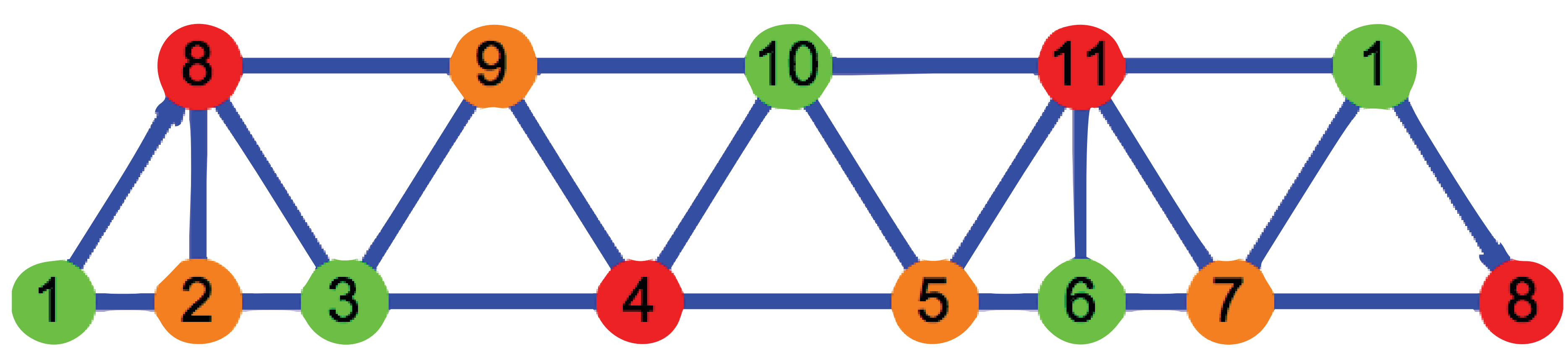
``a projective plane of chromatic number 5!”



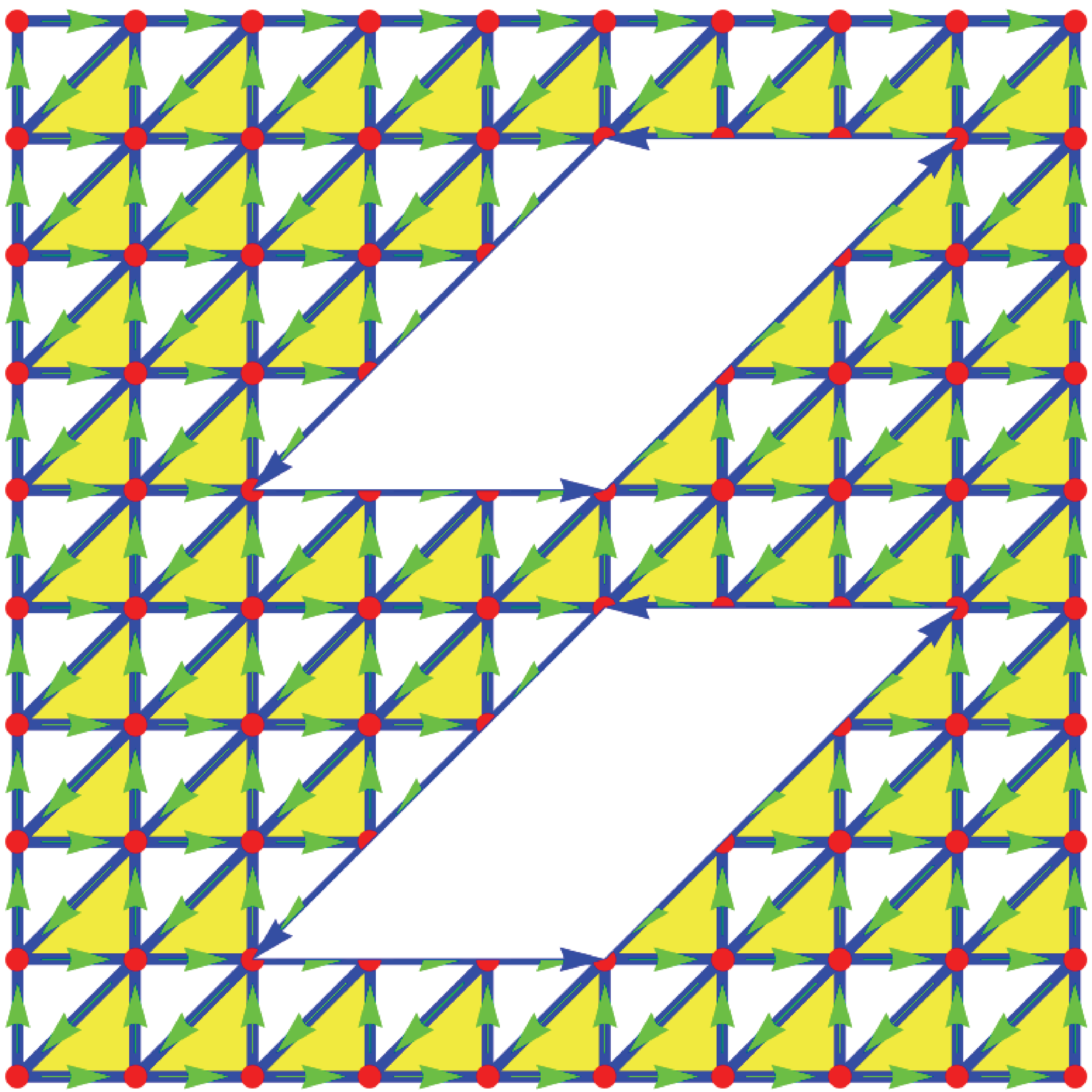
3COLORABLE

``A projective plane with minimal color"





HIGHER GENUS

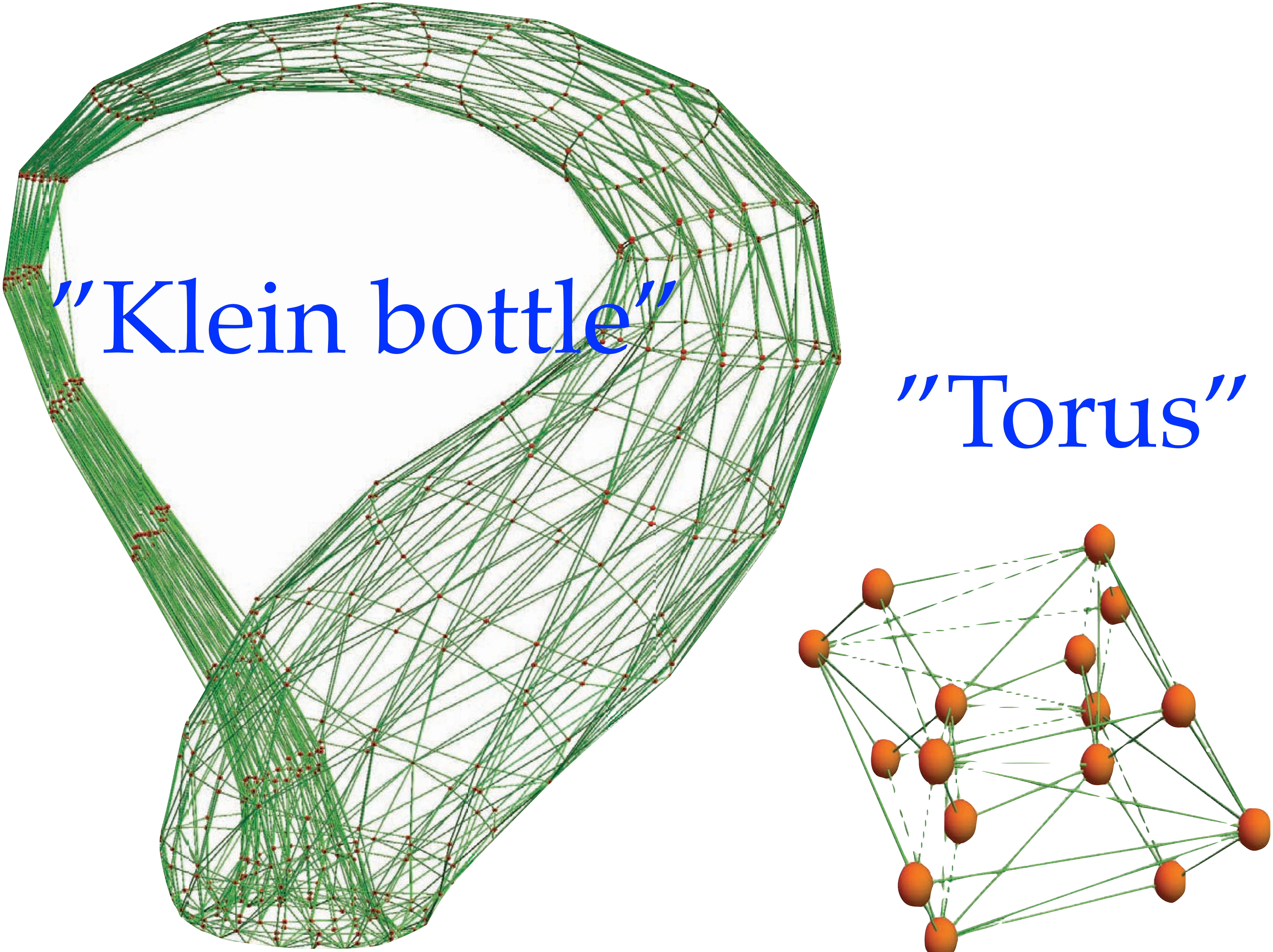


``glueing game”

5 COLOR CONJECTURE

$$\mathfrak{G}_2 \subset \mathfrak{C}_5$$

``motivated from Stromquist-Albertson type question for tori”


"Klein bottle"
"Torus"

D+2 COLOR CONJECTURE

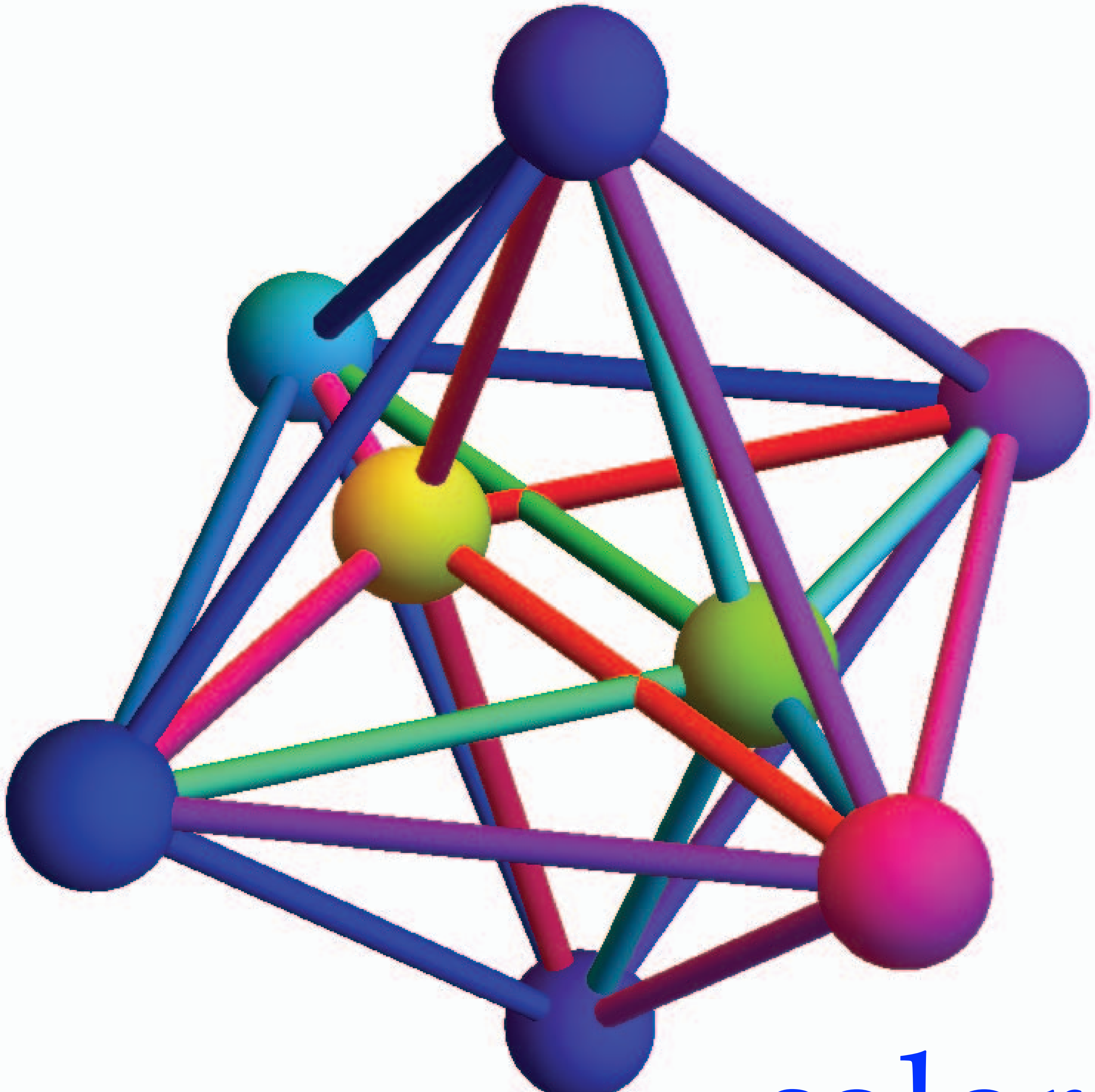
$$\mathcal{S}_d \subset \mathcal{C}_{d+2}$$

``higher dimensional analogue of 4 color problem”

WHY?

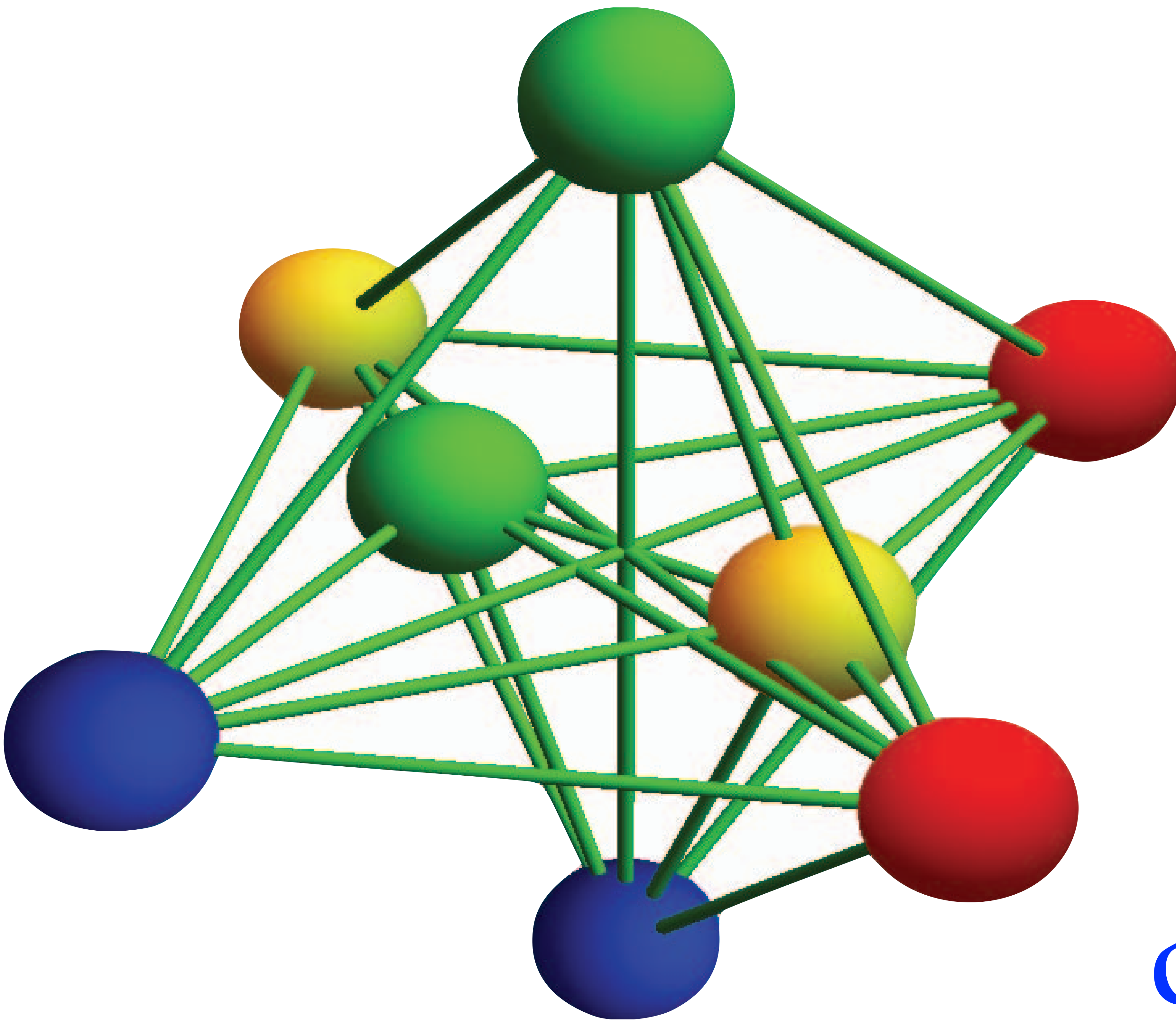
"Embed sphere G in d+1 dimensional minimally colorable sphere H."

16 CELL in $\mathcal{S}_3$



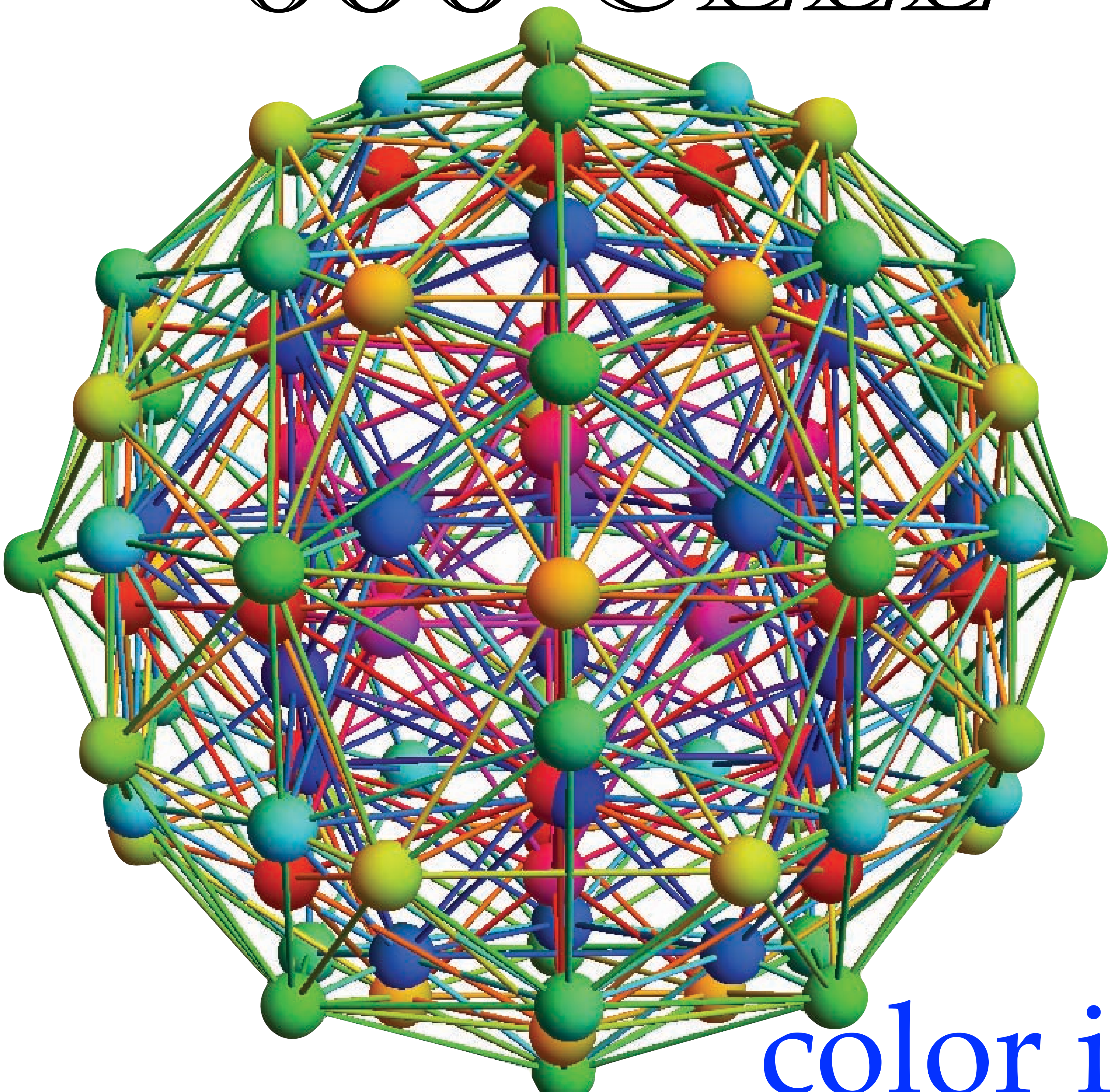
color is 4th dim

16 CELL in $\mathcal{S}_3$



colored

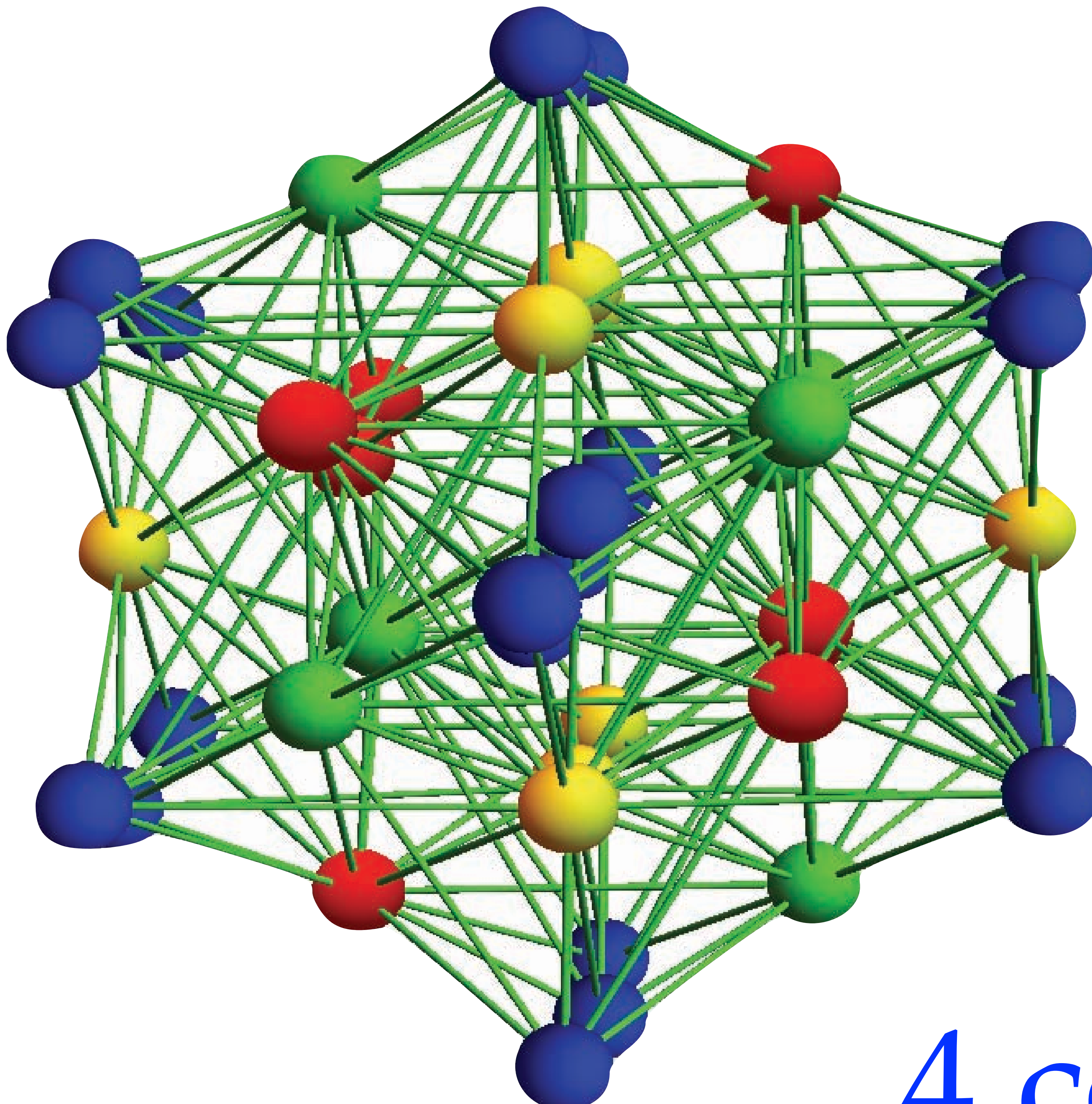
600 CELL in $\mathcal{S}_3$



color is 4th dim

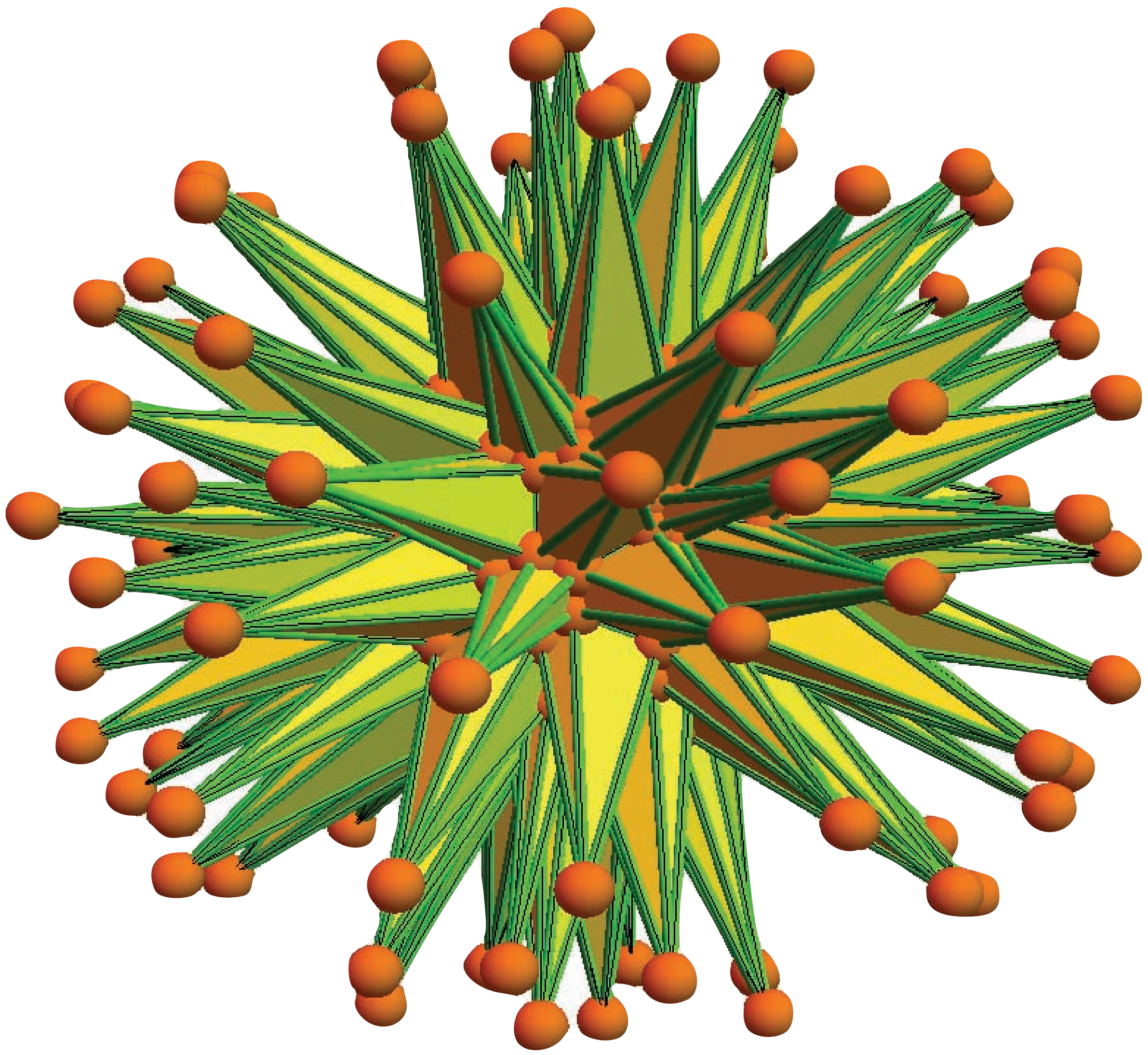
CAPPED CUBE

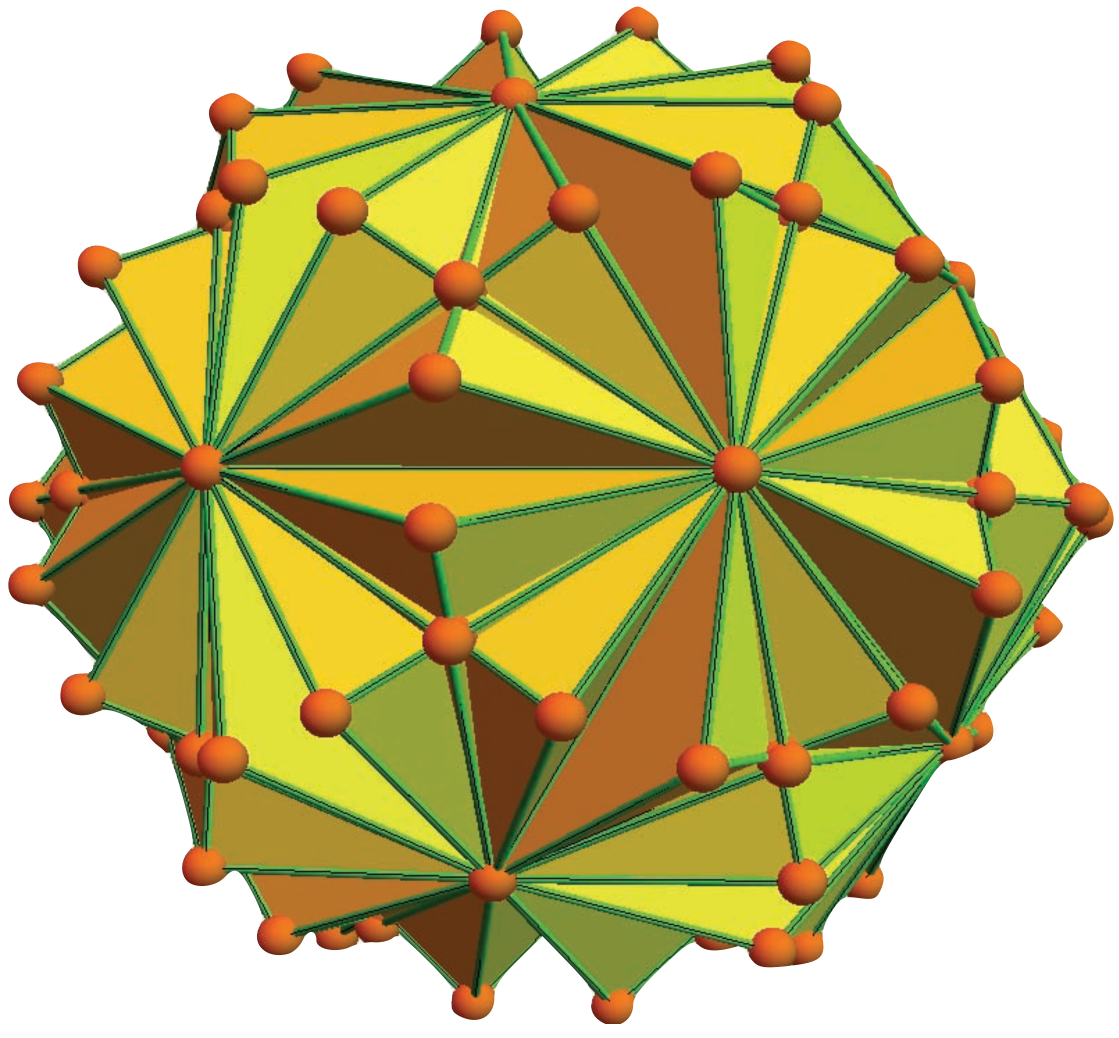
in $\mathcal{S}_3$



4 colored

THE END





details:

http://arxiv.org/abs/1412.6985