TOWARDS A TOPOLOGICAL PROOF OF THE 4 COLOR THEOREM I

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Abstract. Rough outline for a possible topological proof of the four color theorem.

Notation is from [1, 2]. The goal is

Theorem 1 (4 color theorem). $\mathcal{S}_2 \subset \mathcal{C}_4$

This is equivalent to the 4-color theorem [1]: spheres are equivalent to maximal planar 4-connected graphs [1]. The idea of the proof is to write $G \in \mathcal{S}_2$ as the boundary of a ball $B \in \mathcal{B}_3$ with Eulerian interior $\text{int}(B) = B \setminus G$, meaning that every interior vertex v has an Eulerian unit sphere $S(v)$. We aim therefore to prove that G is cobordant to an Eulerian sphere, using a 3-dimensional graph with Eulerian interior. To start the construction, we need to have an initial interior point x of B so that $S(x)$ is still part of the interior of B:

Lemma 2 (G is cobordant to a unit sphere). *G is the boundary of $B \in \mathcal{B}_3$ such that there exists $x \in \text{int}(B)$ with $B(x) \subset \text{int}(B)$.*

Proof. Start with a suspension o of the graph G which is already a graph in $\mathcal{B}_3$ which has G as boundary. Now edge refine every connection $e = (o, y)$ with $y \in G = \delta B$. Now the unit sphere $S(o)$ of o is in the interior of B. □

First we show that a 2-sphere can be refined to become Eulerian. For $G \in \mathcal{S}_2$, Eulerian means that every vertex has even degree. An edge refinement is the opposite of an edge-collapses and is a special type of homotopy which keeps $\mathcal{S}_2$ invariant.

Proposition 3 (Every 2-sphere is homotopic to an Eulerian one). *Every $G \in \mathcal{S}_2$ can be refined to become Eulerian.*

Proof. Lets call a vertex $x \in V(G)$ Eulerian, if the vertex degree is even. By the Euler handshaking lemma, the sum of the vertex degrees

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is twice the number of edges and so even, the number n of non-Eulerian vertices is therefore even. We proceed by induction with respect to n. For $n = 0$ we are done. In general, choose two non-Eulerian vertices and connect them with a path connecting dual points of edges. e can cut along the path to change the parity at the boundary only at two points. To show this, it is enough to show that if two vertices x, y are connected by an edge e. It can be done by cutting the edge e twice, then cut through the pair of newly generated edges from x to y. □

Notation for $G \in \mathcal{S}_2$. Given two discs $U, W \in \mathcal{B}_2$ which are subgraphs of G of non-empty interior $U \cup W = G$ and $U \cap W = C$ is homotopic to a circle then (U, W) is called a **two-disk cover**. Start with $G_0 = G$. The refinement steps G_k can be done so that the Eulerian part contains a nested sequence U_k of discs for $k \geq 1$ such that the end product G_n is Eulerian.

Let now G be the boundary of a ball B. By Lemma (2) B can be refined to contains a ball $B(x)$ of radius 1 in the interior. By Proposition (3), we refine $S(x)$ so that x is Eulerian. We could add more vertices in the interior of $S(x)$ so that we have a ball U for which $\text{int}(U)$ is not empty. We have now a cover (U, W) of B, where W is allowed to contain Eulerian vertices too but where every vertex in U is Eulerian.

Proposition 4. *Given a cover (U, W) of B we can recursively take a new x and edge refine $S(x)$ to make it Eulerian without modifying U. This adds x to U. All newly added vertices y satisfy $d(y, S) \leq d(x, S)$ and are Eulerian if $y \in U$.*

Proof. Take a point x near the boundary such that $U \cup \{x\}$ is still a ball. If $S(x)$ is Eulerian, we just add it to U. Otherwise, by Proposition(3) we can refine the sphere $S(x)$ such that x becomes Eulerian. This can be done without modifying edges in U but modifying edges in $S(x)$ only. □

While the digging steps adds more vertices outside U, the newly added vertices are either Eulerian or closer to the boundary.

References

[1] O. Knill. Coloring graphs using topology. http://arxiv.org/abs/1410.3173, 2014.
[2] O. Knill. Graphs with eulerian unit spheres. http://arxiv.org/abs/1501.03116, 2015.

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