TOWARDS A TOPOLOGICAL PROOF OF THE FOUR COLOR THEOREM X

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Abstract. Proof outline that every $G \in \mathcal{S}_2$ is in $\mathcal{C}_4$, where the coloring is done in a constructive way in $O(n)$ steps, where n are the number of vertices in V.

Lemma 1 G is the boundary of a $B \in \mathcal{B}_3$ of radius 2. The proof is an explicit construction.

Lemma 2 $G \in \mathcal{S}_2$ can be refined to become Eulerian. The proof is an explicit construction: the number of odd degree vertices is even by Gauss-Bonnet. Take a pair A, B and cut from A to B. This is possible as we can define a "dual sphere graph" which is connected. We do the cutting along geodesics in that graph and have implemented it by doing as few cuts as possible overall. This is pretty solid and refined in Lemma 4, where we add additional constraints. **Lemma 3** B is 4-colorable if and only if every interior edge has even degree. This is 3D version of a Kempe-Heawood result. It needs that B is simply connected. We have generalized this to arbitrary dimensions by introducing a degree also in higher dimensions. See http://arxiv.org/abs/1501.03116 We have tried half of the summer 2014 with Jenny to generalize such a result in the non-simply connected case and its subtle. Our hope had been to give an entirely cohomological characterization of such minimal colorings and not succeeded yet. In the simply connected case this Kempe-Heawood is all pretty trivial and explicit. Just color a maximal simplex, then its neighbor etc. The degree condition assures that we we can do that locally. The simply connectedness assures that there are no global constraints. **Lemma 4** $G \in \mathcal{S}_2$ with disc $V \subset G$ and triangle $T \subset G$ can be refined to become Eulerian without cutting edges in V and T. This is an explicit construction, now fully implemented in a deterministic way. **Lemma 5** Assume a ball D inside B is already Eulerian and D is not yet B, we can find a boundary point $y \in D$ so that $D \cup B(y)$ is Eulerian and a ball. This can be done without cutting edges in the

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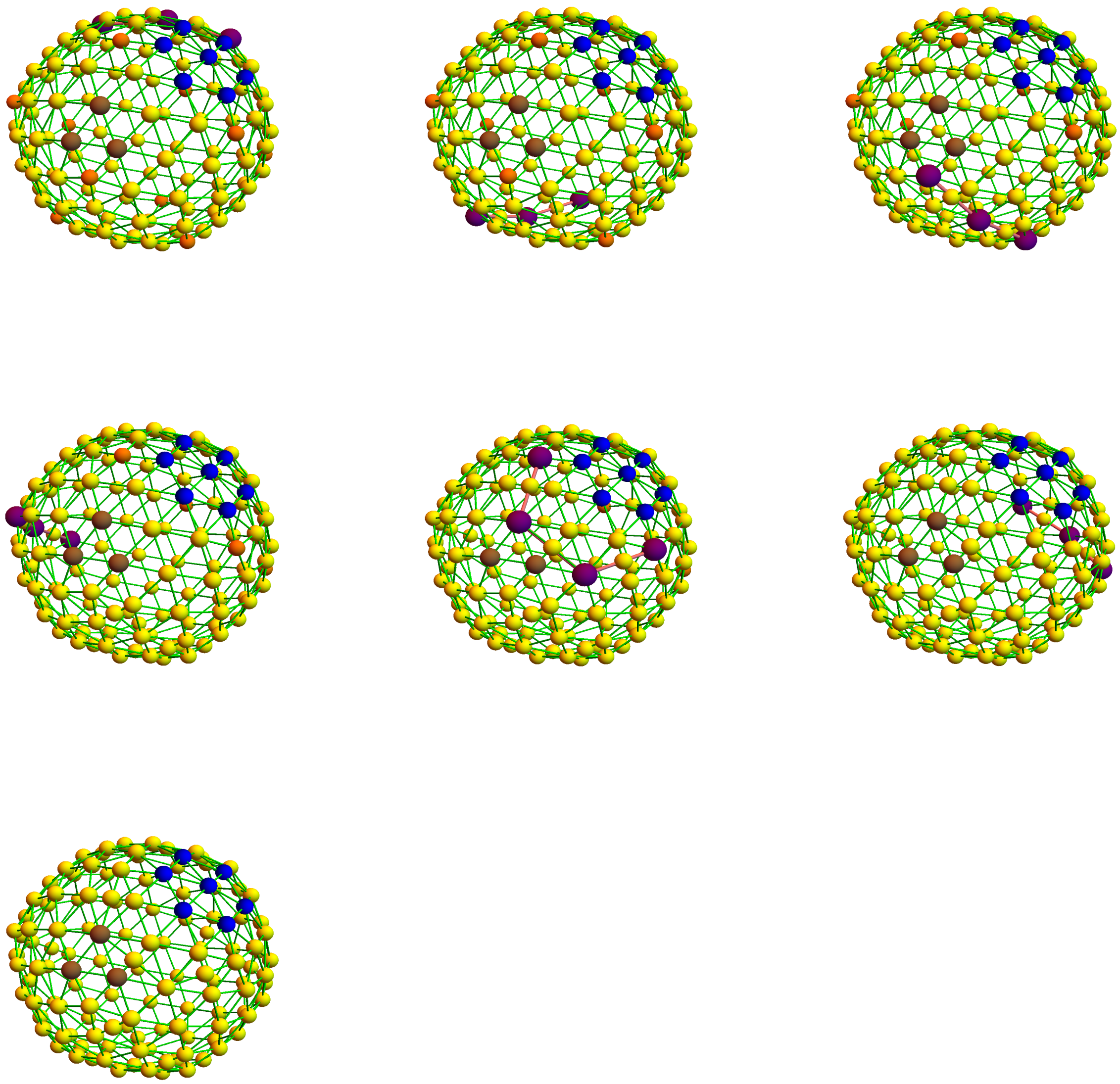
original graph, nor cutting interior edges in D. This is also an explicit construction for which we have all tools, but it is not yet implemented automatically in the computer. Theoretically, the extension to larger balls appears to give no problems. Yes, we are producing additional vertices when cutting, but in each step, we make definite progress in terms of the size of the region. **Lemma 6** The endgame lemma: if a 2D sphere has only odd degree vertices contained in a triangle, then it is Eulerian. By Gauss-Bonnet, we only have to look at the case with 2 neighboring odd degree vertices. This can not happen. So, once we have cut up everything and all balls $B(x)$ within B are Eulerian, then everything is done.

The picture so far strongly suggest, that this constructive coloring algorithm produces a coloring of G in $O(n)$ steps, where even C can be given explicitly if we restrict to graphs with a maximal degree. Theoretically, one might be able to prove an $O(n^2)$ bound relatively easily. We have to remember however that graph coloring is a treacherous, slippery topic in mathematics. We start believe everything is ok, if the computer has learned to do that all without external guidance.



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