# TOWARDS A TOPOLOGICAL PROOF OF THE FOUR COLOR THEOREM II

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ABSTRACT. We look at the process of making $G \in \mathcal{S}_2$ Eulerian.

Notation is from [1, 2] and previous notes. $G \in \mathcal{S}_2$, $U, W \in \mathcal{B}_2$ two-disc cover of $G$ so that $U \cap W$ is homotopic to a circle. A vertex in $G$ is **Eulerian** if its degree is even. We refer to $U$ as an **Eulerian disc**, as every vertex in $U$ is Eulerian. We do not require all vertices in $W \setminus U$ to be non-Eulerian.

**Proposition 1** (Every 2-sphere is homotopic to an Eulerian one)**.** *Every $G \in \mathcal{S}_2$ can be refined to become Eulerian. There is a sequence of graphs $G_0 = G, G_1, \dots, G_n$ where $G_n$ is Eulerian and every step renders one more vertex Eulerian. This refinement adds more Eulerian vertices. Each $G_n$ has a two-disk cover $(U_n, W_n)$, where $U_n$ consists of Eulerian vertices and $U_n \subset U_{n+1}$. The edge refinement does not modify edges already in $U_n$ and will terminate in finitely many steps.*

*Proof.* Given an edge $e$, we look at the dual sphere $\hat{e}$. When cutting $e$, the parity of the vertices in $\hat{e}$ changes. Now chose a vertex $v$ in distance 1 from $U$ (if there should be none, increase $U$ keeping it a disc). The best is to keep at any moment the set $U$ maximal in the sense that adding an other of its Eulerian vertices renders $U$ no more a disc. Now chose a sequence of dual spheres $\hat{e}_k$ pairwise intersecting in a vertex, so that the last entry is near $U$ again. If we should reach an Eulerian point, we stop earlier. Splitting the $e_k$ now switches the parity of the initial and end point. We have added two more vertices to be Eulerian. After continuing like that we will end up with $U$ filling up the entire sphere. □

**Proposition 2** (Every 2-sphere is homotopic to an Eulerian one)**.** *This process can be done also in a disc with boundary rather than a sphere and can be done without modifying the boundary.*

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**Remarks:**

- In some sense, the analogue construction in higher dimensions will prove the result. We will show in three dimensions that $G \in \mathcal{S}_2$ can be embedded into a 3-sphere in $\mathcal{S}_3$ for which $G$ is the equator. As we will only need to fill in half of the sphere, we will write $G$ as the boundary of a three dimensional ball $B$, then start with an Eulerian part $U$ and grow it until it reaches all of $int(G)$. The main proof will produce a sequence $B_n$ of balls with boundary $\delta B_n = G$ so that $B_0 = B$ and so that $B_n$ is Eulerian. In each step, there is an Eulerian set $U_n$ which grows. The refinement process will add new vertices but these newly added vertices are Eulerian.
- It will be necessary to do the completion also if part of the sphere has reached already the boundary.

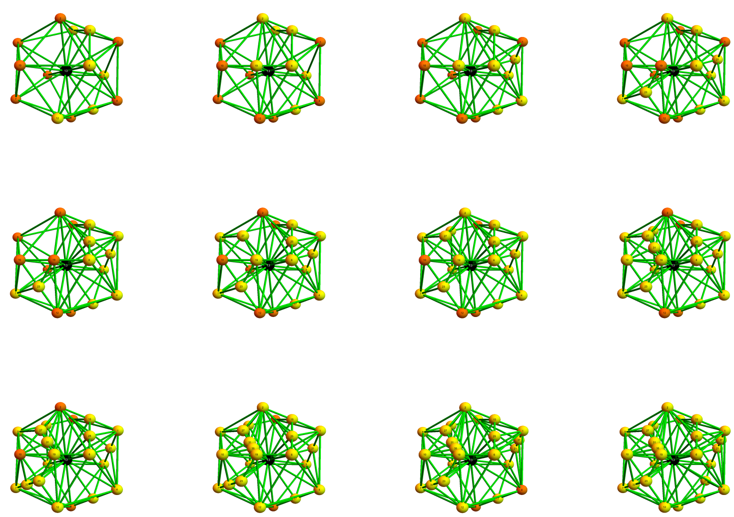

FIGURE 1. A preliminary automatic refinement process to make a given graph Eulerian. It does not produce the necessary sequence $U_n$ of disks.

## REFERENCES

[1] O. Knill. Coloring graphs using topology. http://arxiv.org/abs/1410.3173, 2014.
[2] O. Knill. Graphs with eulerian unit spheres. http://arxiv.org/abs/1501.03116, 2015.

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