

# ON THE DIMENSION AND EULER CHARACTERISTIC OF RANDOM GRAPHS

OLIVER KNILL

ABSTRACT. The inductive dimension  $\dim(G)$  of a finite undirected graph  $G$  is a rational number defined inductively as 1 plus the arithmetic mean of the dimensions of the unit spheres  $\dim(S(x))$  at vertices  $x$  primed by the requirement that the empty graph has dimension  $-1$ . We look at the distribution of the random variable  $\dim$  on the Erdős-Rényi probability space  $G(n, p)$ , where each of the  $n(n-1)/2$  edges appears independently with probability  $0 \leq p \leq 1$ . We show that the average dimension  $d_n(p) = E_{p,n}[\dim]$  is a computable polynomial of degree  $n(n-1)/2$  in  $p$ . The explicit formulas allow experimentally to explore limiting laws for the dimension of large graphs. In this context of random graph geometry, we mention explicit formulas for the expectation  $E_{p,n}[\chi]$  of the Euler characteristic  $\chi$ , considered as a random variable on  $G(n, p)$ . We look experimentally at the statistics of curvature  $K(v)$  and local dimension  $\dim(v) = 1 + \dim(S(v))$  which satisfy  $\chi(G) = \sum_{v \in V} K(v)$  and  $\dim(G) = \frac{1}{|V|} \sum_{v \in V} \dim(v)$ . We also look at the signature functions  $f(p) = E_p[\dim]$ ,  $g(p) = E_p[\chi]$  and matrix values functions  $A_{v,w}(p) = \text{Cov}_p[\dim(v), \dim(w)]$ ,  $B_{v,w}(p) = \text{Cov}[K(v), K(w)]$  on the probability space  $G(p)$  of all subgraphs of a host graph  $G = (V, E)$  with the same vertex set  $V$ , where each edge is turned on with probability  $p$ .

## 1. DIMENSION AND EULER CHARACTERISTIC OF GRAPHS

The inductive dimension for graphs  $G = (V, E)$  is formally close to the Menger-Uhryson dimension in topology. It was in [7] defined as

$$\dim(\emptyset) = -1, \dim(G) = 1 + \frac{1}{|V|} \sum_{v \in V} \dim(S(v)),$$

where  $S(v) = \{w \in V \mid (w, v) \in E\}$ ,  $\{e = (a, b) \in E \mid (v, a) \in E, (v, b) \in E\}$  denotes the unit sphere of a vertex  $v \in V$ . The inductive dimension is useful when studying the geometry of graphs. We can look at the local dimension  $\dim(v) = 1 + \dim(S(v))$  of a vertex which is a local property like "degree"  $\deg(v) = \text{ord}(S(v))$ , or "curvature"  $K(v)$  defined below. Dimension is a rational number defined for every finite graph; however it is in general not an integer.

The dimension is zero for graphs  $P_n$  of size 0, graphs which are completely disconnected. It is equal to  $n-1$  for complete graphs  $K_n$  of order  $n$ , where the size is  $\binom{n}{2} = n(n-1)/2$ . Platonic solids can have dimension 1 like for the cube and dodecahedron, it can be 2 like for the octahedron and icosahedron or be equal to

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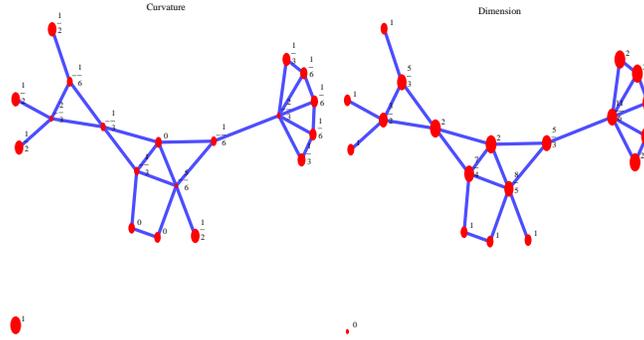


FIGURE 1. This figure shows a particular graph  $G$  with 20 vertices and Euler characteristic  $\chi(G) = 1$ . The left side shows the graph with curvatures

$$\{0, -1, -2, 6, 3, -5, -2, -4, 3, 0, -4, -1, 0, 2, 1, 1, 1, 2, 3, 3\}/6$$

which sum up to the Euler characteristic 1 by Gauss-Bonnet. The right figure shows the same graph with local dimensions

$$\{2, 5/3, 2, 0, 1, 8/5, 7/4, 3/2, 1, 1, 11/6, 5/3, 1, 2, 2, 2, 2, 2, 1, 1\}$$

which average to the dimension of the graph  $\dim(G) = 1801/1200$ .

3 like for the tetrahedron. The 600 cell with 120 vertices is an example of a three dimensional graph, where each unit sphere is a two dimensional icosahedron. Figure 3 illustrates two Archimedean solids for which fractional dimensions occur in familiar situations. All Platonic, Archimedean and Catalan solids are graph theoretical polyhedra: a finite truncation or kising process produces two-dimensional graphs where each unit sphere is a circular graph.

Computing the dimension of the unit spheres requires to find all unit spheres for vertices in the unit sphere  $S(x)$  of a vertex and so on. The local dimension  $\dim(x) = 1 + \dim(S(x))$  satisfies by definition

$$(1) \quad \dim(G) = \frac{1}{|V|} \sum_{x \in V} \dim(x).$$

There is an upper bound of the dimension in terms of the average degree  $\deg(G) = \frac{1}{|V|} \sum_V \deg(v)$  of a graph:  $\dim(G) \leq \deg(G) - 1$  with equality for complete graphs  $K_n$ . The case of trees shows that  $\dim(G) = 1$  is possible for arbitrary large  $|V|$  or  $\deg(G)$ .

An other natural quantity for graphs is the Euler characteristic

$$(2) \quad \chi(G) = \sum_{k=0}^{\infty} (-1)^k v_k,$$

where  $v_k$  is the number of  $K_{k+1}$  subgraphs in  $G$ . We noted in [8] that it can be expressed as the sum over all curvature

$$K(x) = \sum_{k=0}^{\infty} (-1)^k \frac{V_{k+1}(x)}{k+1},$$

where  $V_k(x)$  is the number of  $K_{k+1}$  subgraphs in the sphere  $S(x)$  at a vertex  $x$ . As in the continuum, the Gauss-Bonnet formula

$$(3) \quad \chi(G) = \sum_{x \in V} K(x)$$

relates a local quantity curvature with the global topological invariant  $\chi$ . For example, for a graph without triangles and especially for trees, the curvature is  $K(v) = 1 - \text{deg}(v)/2$ . For graphs without 4 cliques and especially two dimensional graphs  $K(v) = 1 - \text{deg}(v)/2 - \text{size}(S(v))/3$ . For geometric graphs, where each sphere  $S(v)$  is a cyclic graph,  $K(v) = 1 - \text{deg}(v)/6$ . For the standard Petersen graph  $P_{5,2}$ , the dimension is 1, the local dimension constant 1, the Euler characteristic  $-5$  and the curvature is constant  $-1/2$  at every vertex. The Petersen graph  $P_{9,3}$  has dimension  $4/3$  and Euler characteristic  $-6$ . There are 9 vertices with curvature  $-1/2$  and 9 with curvature  $-1/6$ . The sum of curvatures is  $-6$ .

The Gauss-Bonnet relation (3) is already useful for computing the Euler characteristic. For inhomogeneous large graphs especially, the Gauss-Bonnet-Chern formula simplifies in an elegant way the search for cliques in large graphs. An other application is the study of higher dimensional polytopes. It follows immediately for example that there is no 4-dimensional polytope - they are usually realized as a convex set in  $R^5$  - for which the graph theoretical unit sphere is a three dimensional 600 cell: the Euler characteristic is 2 in that dimension and the curvature would by regularity have to be  $2/|V|$ . But curvature of such a graph would be constant and since the 600 cell has 120 vertices, 720 edges, 1200 faces and 600 chambers, the curvature were

$$K = \frac{V_{-1}}{1} - \frac{V_0}{2} + \frac{V_1}{3} - \frac{V_2}{4} + \frac{V_3}{5} = \frac{1}{1} - \frac{120}{2} + \frac{720}{3} - \frac{1200}{4} + \frac{600}{5} = 1$$

which would force  $|V| = 2$  and obviously does not work. While such a result could certainly also be derived with tools developed by geometers like Schläfli or Coxeter, the just graph theoretical argument is more beautiful. It especially does not need any ambient space realization of the polytope.

We have studied the Gauss-Bonnet theme in a geometric setting for  $d$ -dimensional graphs for which unit spheres of a graph satisfy properties familiar to unit spheres in  $R^d$ . In that case, the results look more similar to differential geometry [8].

The curvature for three-dimensional graphs for example is zero everywhere and positive sectional curvature everywhere leads to definite bounds on the diameter of the graph. Also for higher dimensional graphs, similar than Bonnet-Schoenberg-Myers bounds assure in the continuum, positive curvature forces the graph to be of small diameter, allowing to compute the Euler characteristic in finitely many cases allowing in principle to answer Hopf type questions about the Euler characteristic of finite graphs with positive sectional curvature by checking finitely many cases.

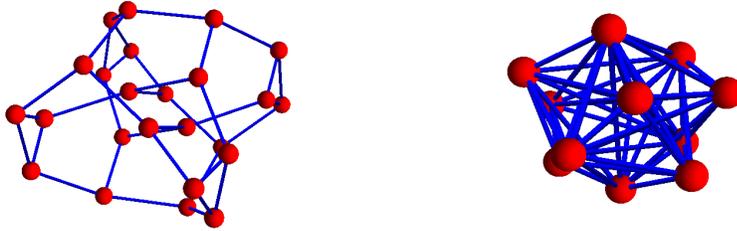


FIGURE 2. The Petersen graph  $P(12, 4)$  of dimension  $4/3$  and Euler characteristic  $-8$  can be seen to the left. The curvatures take 12 times the value  $-1/2$  and 12 times the value  $-1/6$ . To the right, we see the Turan graph  $T(13, 4)$  of dimension 3 and Euler characteristic  $-23$ . The curvatures take values  $-2$  nine times and  $-5/4$  four times.

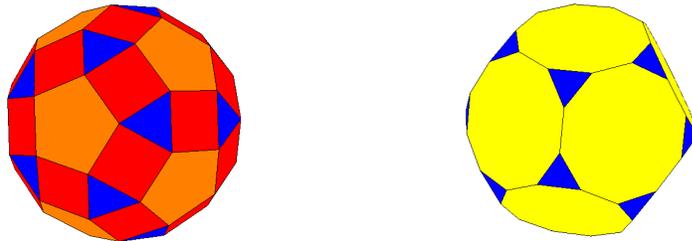


FIGURE 3. The dimension of the **small rhombicosidodecahedron** is  $3/2$ : each point has dimension  $3/2$  because each unit sphere is a graph with 4 vertices, where two points have dimension 1 and two points have dimension 0. The unit sphere has dimension  $1/2$ . The dimension of the **truncated dodecahedron** is  $5/3$  because every unit sphere has dimension  $2/3$ .

Such questions are often open in classical differential geometry but become a finite combinatorial problem in graph theory. Obviously and naturally, many question in differential geometry, whether variational, spectral or topological can be asked in pure graph theory without imposing any additional structure on the graph. Dimension, curvature and Euler characteristic definitely play an important role in such quests.

What are the connections between dimension and Euler characteristic and curvature? We don't know much yet, but there are indications of more connections:

all flat connected graphs we have seen for example are geometric with uniform constant dimension like cyclic graphs or toral graphs. An interesting question is to study graphs where Euler characteristic is extremal, where curvature (also Ricci or scalar analogues of curvature) is extremal or where curvature is constant. So far, the only connected graphs with constant curvature we know of are complete graphs  $K_n$  with curvature  $1/n$ , cyclic graphs  $C_n$  with curvature 0, discrete graphs  $P_n$  with curvature is 1, the octahedron  $O$  with curvature  $1/6$ , the icosahedron  $I$  with curvature  $1/12$ , higher dimensional cross polytopes with  $2n$  vertices and curvature  $1/(2n)$ , the 600 cell 120 vertices, where each unit sphere is an icosahedron and which is “flat”

$$K = \frac{V_{-1}}{1} - \frac{V_0}{2} + \frac{V_1}{3} - \frac{V_2}{4} = \frac{1}{1} - \frac{12}{2} + \frac{30}{3} - \frac{20}{4} = 0$$

like for any three dimensional geometric graph [8], twisted tori  $K_{n,m}$  with curvature 0 as well as higher dimensional regular tessellations of tori. We could not yet construct a graph with constant negative curvature even so they most likely do exist. We start to believe that geometric graphs - for which local dimension is constant like the ones just mentioned - are the only connected constant curvature graphs.

We look in this article at connections with random graph theory [1, 10], an area of mathematics which has become useful for the study of complex networks [2, 9, 11, 6]. The emergence of interest in web graphs, social networks, neural networks, complex proteins or nano technology makes it an active area of research.

## 2. RANDOM SUBGRAPHS OF THE COMPLETE GRAPH

We inquire in this section about the dimension of a typical graph in the probability space  $G(n, 1/2)$ , where each edge is turned on with probability  $1/2$ . To investigate this, we can look at all possible graphs on a fixed vertex set of cardinality  $n$  and find the dimension expectation by computing the dimension of each graph and adding this up. When we counted dimensions for small  $n$  by brute force, we noticed to our surprise that the sum of all dimensions of subgraphs is an integer. Our limit for brute force summation was  $n = 7$ , where we have already  $2^{21} = 2'097'152$  graphs. For the next entry  $n = 8$ , we would have had to check 128 times more graphs.

Note that as usual, we do not sum over all subgraphs of  $K_n$  but all subgraphs of  $K_n$  for which  $|V| = v_0 = n$ . While this makes no difference for dimension because isolated points have dimension 0, it will matter for Euler characteristic later on, because isolated points have Euler characteristic 1. For  $K_3$  for example, there are 3 graphs with dimension 1, one graph with dimension 2 and 3 graphs with dimension  $2/3$ . The sum of all dimensions is 7. For  $K_4$ , the sum of dimensions over all  $2^6 = 64$  subgraphs is 75: there are 22 subgraphs of dimension 1, there are 12 of dimensions  $5/3$  and  $3/4$  each, 6 of dimensions  $1/2$  and 2 each, 4 of dimension  $3/2$  and one of dimension 0 and 3 each. Also for  $K_5$ , dimension 1 appears most with 237 subgraphs followed with 120 graphs of dimension  $22/15$ . For  $K_6$  already, we have two integer champions: dimension 1 appears for 3'981 subgraphs and dimension 2 for 2'692 subgraphs. The total sum of dimension is 53'275.

**Theorem 2.1** (Average dimension on  $G(n, 1/2)$ ). *The average dimension on  $G(n, 1/2)$  satisfies the recursion*

$$d_{n+1} = 1 + \sum_{k=0}^n \frac{\binom{n}{k}}{2^n} d_k,$$

where  $d_0 = -1$  is the seed for the empty graph. The sum over all dimensions of all order  $n$  subgraphs of  $K_n$  is an integer.

*Proof.* Let  $g(n) = 2^{\binom{n}{2}}$  be the number of graphs on the vertex set  $V = \{1, \dots, n\}$ , and let  $f(n)$  the sum of the dimensions of all subgraphs of the complete graph with  $n$  vertices. We can find a recursion for  $f(n)$  by adding a  $(n+1)$ 'th vertex point  $x$  and then count the sum of the dimensions over all subgraphs of  $K_{n+1}$  vertices by partitioning this set of subgraphs up into the set  $Y_k$  which have  $k$  edges connecting the old graph to the new vertex. There are  $\binom{n}{k}$  possibilities to build such connections. In each of these cases, the unit sphere  $S(x)$  is a complete graph of  $k$  vertices. We get so the sum  $\sum_{k=0}^n (f_k + g(k)) \binom{n}{k}$  of dimensions of such graphs because we add 1 to each of the  $\binom{n}{k}$  cases. From this formula, we can see that the sum of dimensions is an integer. The dimension itself satisfies the recursion

$$d_{n+1} = 2^{\binom{n}{2} - \binom{n+1}{2}} \sum_{k=0}^n (d_k + 1) \binom{n}{k}.$$

With  $\binom{n}{2} - \binom{n+1}{2} = -n$  and using  $2^{-n} \sum_{k=0}^n \binom{n}{k} = 1$ , we get the formula.  $\square$

**Remarks.**

1. We see that  $d_{n+1}$  is 1 plus a Bernoulli average of the sequence  $\{d_k\}_{k=0}^n$ .
2. Theorem 2.1 will be generalized in Theorem 4.1 from  $p = 1/2$  to general  $p$ .

### 3. AVERAGE DIMENSION ON CLASSES OF GRAPHS

We can look at the average dimension on subclasses of graphs. For example, what is the average dimension on the set of all 2-dimensional graph with  $n$  vertices? This is not so easy to determine because we can not enumerate easily all two-dimensional graphs of order  $n$ . Only up to discrete homotopy operations, the situation for two dimensional finite graphs is the same as for two dimensional manifolds in that Euler characteristic and the orientation determines the equivalence class.

We first looked therefore at the one-dimensional case. Also here, summing up the dimensions of all subgraphs by brute force showed that the sum is always an integer and that the dimension is constant equal to  $3/4$ . This is true for a general one dimensional graph without boundary, graphs which are the disjoint union of circular graphs.

**Theorem 3.1** ( $3/4$  theorem). *For a one-dimensional graph  $G = (V, E)$  without boundary, the average dimension of all subgraphs of  $G$  is always equal to  $3/4$ . The sum of the dimensions of subgraphs  $H$  of  $G$  with the same vertex set  $V$  is an integer.*

*Proof.* A one-dimensional graph without boundary is a finite union of cyclic graphs. For two disjoint graphs  $G_1, G_2$  with union  $G = G_1 \cup G_2$ , we have  $\dim(G) = (\dim(G_1)|G_1| + \dim(G_2)|G_2|)/|G|$ . It is therefore enough to prove the statement for a connected circular graph of size  $n \geq 4$ . This can be done by induction. We

can add a vertex in the middle of one of the edges to get from  $C_n$  to  $C_{n+1}$ . The dimensions of the other vertices do not change. The new point has dimension 0 with probability  $1/4$  and 1 with probability  $3/4$ . Since the smallest one dimensional graph has 4 nodes, and  $2^{\binom{n}{2}}$  is already a multiple of 4, the sum dimensions of all subgraphs is an integers.  $\square$

**Remarks.**

1. We will generalize this result below and show that for a one dimensional circular graph, the expected dimension of a subgraph is  $p(2-p)$ . The  $3/4$  result is the special case when  $p = 1/2$ . The result is of course different for the triangle  $K_3 = C_3$ , which is two dimensional and for which the expected dimension is  $p(2-p+p^2)$ . We will call the function  $f(p) = p(2-p)$  the signature function. It is in this particular case the same for all one dimensional graphs without boundary.
2. Is the sum of dimensions of subgraphs of a graph with integer dimension and constant degree an integer? No. Already for an octahedron, a graph  $G$  of dimension 2 which has  $2^{12} = 4096$  different subgraphs, a brute force computation shows that the sum of dimensions is  $a = 15424/3$  and that the average dimension of a subgraph of  $G$  is  $a/2^{12} = 1.25521$ . The unit ball  $B_1(v)$  in the octahedron is the wheel graph  $W_4$  with four spikes in which the sum of the dimensions is 284 and the average dimension is  $284/256 = 1.10938$ .
3. We would like to know the sum of all dimensions of subgraphs for flat tori, finite graphs for which each unit disc is a wheel graph  $W_6$ . Such tori are determined by the lengths  $M, K$  of the smallest homotopically nontrivial one dimensional cycles as well as a "Dehn" twist parameter for the identification. Already in the smallest case  $M = K = 4$ , the graph has 48 edges and summing up over all  $2^{\binom{48}{2}} = 3.6 \cdot 10^{339}$  possible subgraphs is impossible. It becomes a percolation problem [5]. It would be interesting to know for example whether the dimension signature functions  $f_{K,M}(p) = E_p[\text{dim}]$  have a continuous limit for  $\min(K, M) \rightarrow \infty$ .
4. For the wheel graph  $W_6$ , the unit disc in the flat torus, the average dimension is  $(159368/35)/2^{12} = 1.1116\dots$  For a flat torus like  $T_{5,5}$  we can not get the average dimension exactly but measure it to be about 1.3. We observe in general however that the dimension depends smoothly on  $p$ .
5. Brute force computations are not in vain because they allow also to look at the distribution of the dimensions of graphs with  $n$  vertices. Since we do not yet have recursive formulas for the higher moments, it is not clear how this behaves in the limit  $n \rightarrow \infty$ .

The first sums of dimensions are  $s_1 = 0, s_2 = 1, s_3 = 7, s_4 = 75, s_5 = 1451, s_6 = 53275, s_7 = 3791451, s_8 = 528662939, s_9 = 145314459035, s_{10} = 79040486514843,$  and  $s_{11} = 85289166797880475$ . On a graph with 5 vertices for example, the sum of all dimensions over all the  $2^{10}$  subgraphs is 1451. We could not possibly compute  $s_{11}$  by brute force, since there are just too many graphs.

Favoring integer dimensions for concrete or random graphs is a resonance phenomena of number theoretical nature. Whether it is a case for "Guy's law of small numbers" disappearing in the limit  $n \rightarrow \infty$  remains to be seen.

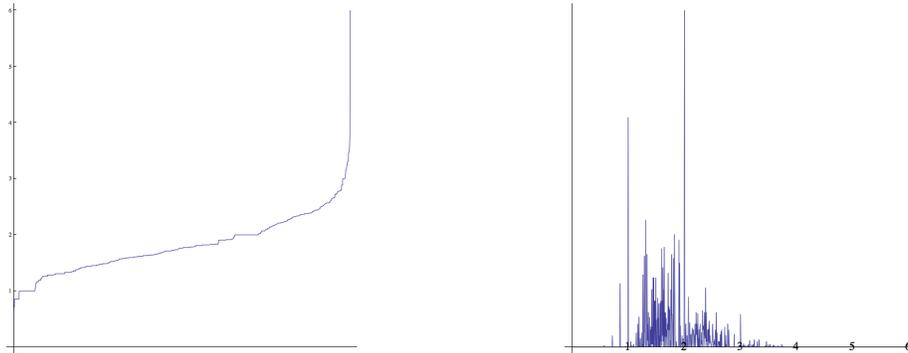


FIGURE 4. The distribution of the random variable  $\dim$  on the finite probability space  $G(7, 1/2)$  of all graphs with vertex cardinality 7. The left picture shows the cumulative distribution function, the right the density function. Graphs with integer dimension appear to have higher probability than graphs with fractional dimension. This is a phenomenon, we also see in concrete networks, like social graphs or computer networks. The function  $\dim$  takes 245 different dimension values on  $G(7, 1/2)$ . The dimension 2 appears for 146'583 graphs, the next frequent dimension is 1 appears in 99'900 graphs. Only in third rank is dimension  $55/42$  is a fraction. It appears for 55'440 graphs.

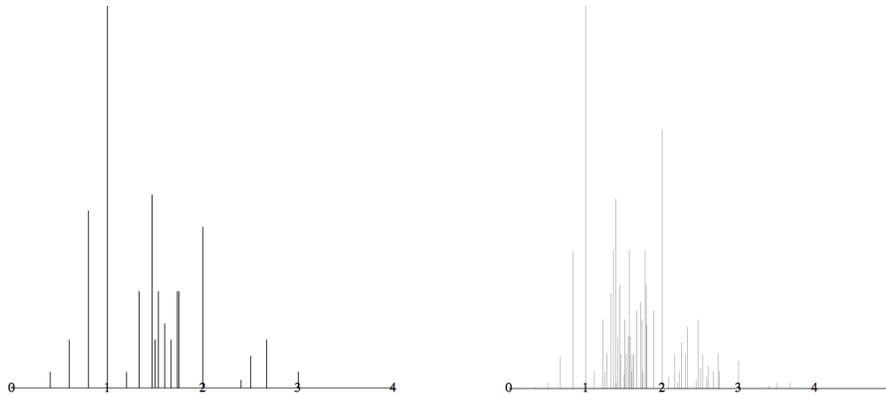


FIGURE 5. The dimension statistics on  $G(5, 1/2)$  and  $G(6, 1/2)$  already indicates a preference for integer dimension of subgraphs.

#### 4. THE DIMENSION OF A RANDOM P-PERCOLATING GRAPH

We generalize now the probability measure on the space of all graphs with  $n$  elements and switch each edge on with probability  $0 \leq p \leq 1$ . This is the classical Erdős-Rényi model [3]. With the probability measure  $P_p$  on  $X_n$ , the probability space is called  $G(n, p)$ . For this percolation problem on the complete graph, the mean degree is  $E[\text{deg}] = np$ .

The following result is a generalization of Theorem 2.1, in which we had  $p = 1/2$ . It computes  $d_n = E_{n,p}[\dim]$ .

**Theorem 4.1** (Average dimension on  $G(n, p)$ ). *The expected dimension  $E_p[\dim]$  on  $G(n, p)$  satisfies*

$$d_{n+1}(p) = 1 + \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} d_k(p),$$

where  $d_0 = -1$ . Each  $d_n$  is a polynomial in  $p$  of degree  $\binom{n}{2}$ .

*Proof.* The inductive derivation for  $p = 1/2$  generalizes: add a  $n+1$ 'th point  $P$  and partition the number of graphs into sets  $Y_k$ , where  $P$  connects to a  $k$ -dimensional graph within the old graph. The expected dimension of the new point is then

$$d(n+1) = \sum_{k=1}^n \binom{n}{k} p^k (1-p)^{n-k} (d(k) + 1)$$

and this is also the expected dimension of the entire graph. This can be written as

$$d(n+1) = \sum_{k=1}^n \binom{n}{k} p^k (1-p)^{n-k} d(k) + \sum_{k=1}^n \binom{n}{k} p^k (1-p)^{n-k} 1$$

which is

$$d(n+1) = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} d(k) + 1 - (1-p)^n d(0) - (1-p)^n$$

which is equivalent to the statement.  $\square$

Again, if we think of the vector  $d = (d_0, d_1, \dots, d_n)$  as a random variable on the finite set  $\{0, 1, \dots, n\}$  then  $d_{n+1}$  is 1 plus the expectation of this random variable with respect to the Bernoulli distribution on  $\{0, 1, \dots, n\}$  with parameters  $n$  and  $p$ .

Lets look at the first few steps: we start with  $-1$ , where the expectation is  $-1$  and add 1 to get 0. Now we have  $(-1, 0)$  and compute the expectation of this  $(-1)p^0(1-p)^1 + 0p^1(1-p)^1 = p - 1$ . Adding 1 gives  $p$  so that we have the probabilities  $(-1, 0, p)$ . Now compute the expectation again with  $(-1)p^0(1-p)^2 + 2p^1(1-p)^1 \cdot 0 + p^2(1-p)^0 p$ . Adding 1 gives the expected dimension of  $d_3(p) = 2p - p^2 + p^3$  on a graph with 3 vertices. We have now the vector  $(-1, 0, p, 2p - p^2 + p^3)$ . To compute the dimension expectation on a graph with 4 vertices, we compute the expectation  $(-1)p^0(1-p)^3 + 3p^1(1-p)^2 \cdot 0 + 3p^2(1-p)^1 \cdot p + 1p^3(1-p)^0 \cdot (2p - p^2 + p^3) = -1 + 3 \cdot p - 3 \cdot p^2 + 4 \cdot p^3 - p^4 - p^5 + p^6$  and add 1 to get the expected dimension  $d_4(p) = 3 \cdot p - 3 \cdot p^2 + 4 \cdot p^3 - p^4 - p^5 + p^6$  on a graph of 4 elements. Here are the first polynomials:

$$\begin{aligned} d_2(p) &= p \\ d_3(p) &= 2p - p^2 + p^3 \\ d_4(p) &= 3p - 3p^2 + 4p^3 - p^4 - p^5 + p^6 \\ d_5(p) &= 4p - 6p^2 + 10p^3 - 5p^4 - 3p^5 + 5p^6 - p^8 - p^9 + p^{10} \end{aligned}$$

The expected number of cliques  $K_k$  of the complete graph  $K_n$  is  $\binom{n}{k} p^{\binom{k}{2}}$ . By approximating the Binomial coefficients  $\binom{n}{k}$  with the Stirling formulas, Erdős-Rényi have shown (see Corollary 4 in [4]), that for  $p < n^{-2/(k-1)}$ , there are no  $K_k$  subgraphs in the limit  $n \rightarrow \infty$  which of course implies that then the dimension is

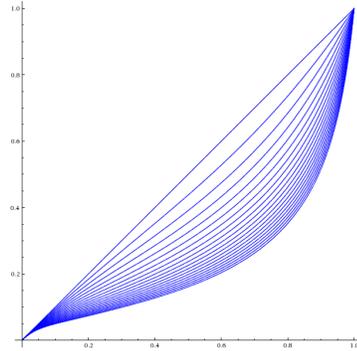


FIGURE 6. The expected dimension of a graph as a function of  $p$  is monotone and is shown here for  $n = 1, \dots, 20$ . We see the graphs of the functions  $d_n(p)/(n-1)$  for  $n = 1, 2, \dots, 20$ . A division by  $(n-1)$  produces a scaling so that  $d_n(1) = 1$ . We experimentally see that they satisfy a power law: there are functions constants  $a = a(p)$  and  $c = c(p)$  for which with the scaling law  $d_n(p) \sim c/n^a$  holds in the limit  $n \rightarrow \infty$ . Especially  $d_n(p)/(n-1)$  is monotone in  $n$  for every  $p$ . This needs to be explored more.

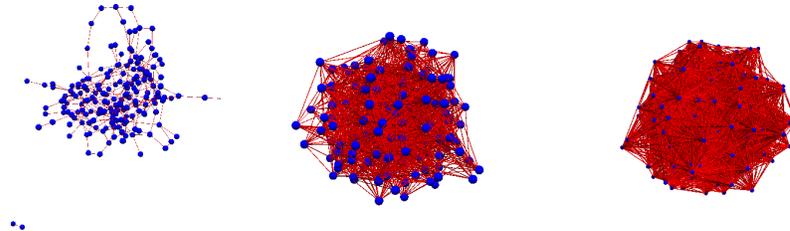


FIGURE 7. Three random graphs in which the percolation probability  $p$  parameter is chosen to have an expected dimension 1, or 2 or 3.

$\leq (k-1)$  almost surely. For  $p < n^{-2/2}$ , there are no triangles in the limit and  $\dim(G) \leq 1$  for  $p < n^{-2/3}$ , there are no tetrahedra in the limit and  $\dim(G) \leq 2$ .

Is there a threshold, so that for  $p < n^\alpha$  the expectation of dimension converges? We see experimentally for any  $p$  that  $E[\dim] \sim c(p)/n^{a(p)}$ , where  $c, a$  depend on  $p$ .

Finally lets look at a generalization of the 3/4 theorem:

**Theorem 4.2** (Expected dimension of a subgraph of a one dimensional graph). *The expected dimension  $E_p[\dim]$  on all subgraphs of any one-dimensional graph without boundary is  $p(2-p)$ .*

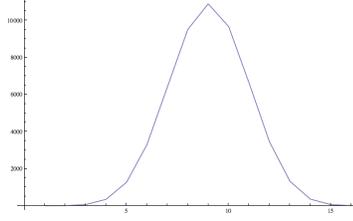


FIGURE 8. The sum  $s_n(k)$  of dimensions of size- $k$  and order- $n$  subgraphs of a complete graph  $K_n$  is an integer too. The sequence  $s_n(k)$  appears to converge to a Gaussian distribution. This is not surprising, when considering the de Moivre-Laplace limit theorem and that dimension is highly correlated with the size of the graph. We see in experiments, that this distribution limit is achieved also for subgraphs of other graphs like circular graphs.

*Proof.* Proceed by induction. Add an other vertex in the middle of a given edge. The expectation of the dimension of the remaining points does not change. The expectation of the dimension of the new point is  $1 \cdot p^2 + 2p(1-p) + 0(1-p)^2 = p(2-p)$  because the dimension of the point is one if both one of the two adjacent edges are present and 0 if none is present.  $\square$

**Remark.** More interesting is to introduce the variable  $q = p - 1$  and write the expectation as a sum

$$\sum_{k=0}^n a_{k,n} \binom{n}{k} q^{n-k} p^k ,$$

where  $a_{k,n}$  is the expectation of the dimension on the probability space  $G(p)$  of all graphs  $(V, E_k)$ , where  $E_k$  runs over all subsets of the vertices of the host graph  $G$ . For example, for the host graph  $C_8$  we have

$$E[\text{dim}] = 2q^7 p + 13q^6 p^2 + 36q^5 p^3 + 55q^4 p^4 + 50q^3 p^5 + 27q^2 p^6 + 8q^1 p^7 + p^8 = p(2-p)$$

from which we can deduce for example that  $27/28$  is the expected dimension if two links are missing in a circular graph  $C_8$  or  $13/28$  if two links are present and that  $2/8$  is the expected dimension if only one link is present. The polynomial has the form

$$2pq^{n-1} + (2n - 3)q^{n-2}p^2 + n^2q^{n-3}p^3 + \dots + nqp^{n-1} + p^n .$$

### 5. THE EULER CHARACTERISTIC OF A RANDOM GRAPH

Besides the degree average  $\text{deg}$  and dimension  $\text{dim}$ , an other natural random variable on the probability space  $G(n, p)$  is the Euler characteristic  $\chi$  defined in Equation (2).

For  $p = 1/2$ , we sum up the Euler characteristics over all order  $n$  sub graphs of the complete graph of order  $n$  and then average. The list of sums starts with

$$1, 3, 13, 95, 1201, 25279, \dots$$

leading to expected Euler characteristic values

$$1, 1.5, 1.625, 1.48438, 1.17285, 0.771454, 0.34842, -0.0399132 .$$

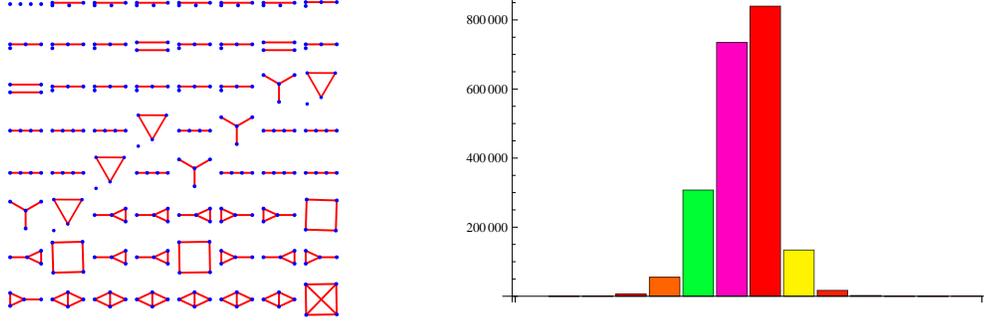


FIGURE 9. The total number of triangles  $K_3$  in all subgraphs of  $K_n$  is  $2^{\binom{n}{2}} \binom{n}{3} / 8$ . To the left, we see all the  $2^6 = 64$  subgraphs of  $K_4$ . There are 32 triangles in total so that  $E[v_3] = 1/2$ . To the right we see the statistics of the Euler characteristic on the probability space  $G(7, 1/2)$ . This is the last row of Table 1. The most frequent case is  $\chi = 1$  followed by  $\chi = 0$ .

Already for  $n = 5$ , there are some subgraphs with negative Euler characteristics. For  $n = 8$ , the expectation of  $\chi$  has become negative for the first time. It will do so again and again. These first numbers show by no means any trend: the expectation value of  $\chi$  will oscillate indefinitely between negative and positive regimes and grow in amplitude. This follows from the following explicit formula:

**Theorem 5.1** (Expectation of Euler characteristic). *The expectation value of the Euler characteristic on  $G(n, p)$  is*

$$E_{n,p}[\chi] = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} p^{\binom{k}{2}}.$$

*Proof.* We only need expectation of the random variable  $v_k$  on  $G(n, p)$ . But this is well known: (see [1]). We have

$$E[v_k] = \binom{n}{k+1} p^{\binom{k+1}{2}}.$$

This later formula is proven as follows: if  $S_k$  is the set of all  $k$ -cliques  $K_k$  in the graph. Now count the number  $n(s)$  of times that a  $k$ -clique  $s$  appears. Then  $E[v_k] 2^{\binom{n}{2}} = \sum_{s \in S_k} n(s) = \binom{n}{k} p^k$  because  $n(s)$  is constant on  $S_k$ . We especially have  $E[v_0] = n$ ,  $E[v_1] = \binom{n}{2} p$  is the expected value of the number of edges, and  $E[v_2] = \binom{n}{3} p^3$  is the expected number of triangles in the graph and  $E[v_3] = \binom{n}{4} p^6$  is the expected number of tetrahedra  $K_4$  in the graph.  $\square$

**Remarks.**

1. The Euler characteristic expectation  $E_{n,p}[\chi]$  as a function of  $n$  oscillates between different signs for  $n \rightarrow \infty$  because for each fixed  $k$ , the function  $n \rightarrow \binom{n}{k} / 2^{\binom{n}{2}}$  dominates in some range than is taken over by an other part of the sum.
2. We could look at values of  $p_n$  for which the expectation value of the dimension gives  $E_{n,p}[\dim] = 2$  and then look at the limit of the Euler characteristic.
3. The clustering coefficient which is  $3v_2/a_2$  where  $a_2$  are the number of pairs of

$\chi =$	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
n=1	0	0	0	0	0	0	1	0	0	0	0	0	0
n=2	0	0	0	0	0	0	1	1	0	0	0	0	0
n=3	0	0	0	0	0	0	4	3	1	0	0	0	0
n=4	0	0	0	0	3	35	19	6	1	0	0	0	0
n=5	0	0	10	162	571	215	55	10	1	0	0	0	0
n=6	10	105	1950	9315	16385	4082	780	125	15	1	0	0	0
n=7	35	420	6321	54985	307475	734670	839910	133693	17206	2170	245	21	1

TABLE 1. The Euler characteristic statistics of subgraphs of  $K_n$ .  
 We see that  $\chi = 1$  is the most frequent situation in all cases  $n = 1$  to  $n = 7$ .

adjacent edges is a random variable studied already. It would be interesting to see the relation of clustering coefficient with dimension.

**4.** The formula appears close to  $\sum_{k=1}^n (-1)^k \binom{n}{k} p^k$  which simplifies to  $1 - (1 - p)^n$  and which is monotone in  $p$ . But changing the  $p^k$  to  $p^{\binom{n}{k}}$  completely changes the function because it becomes a Taylor series in  $p$  which is sparse. We could take the sequence  $E_{n,p}[\chi] / \binom{n}{n/2}$  to keep the functions bounded, but this converges to 0 for  $p < 1$ . A natural question is whether for some  $p(n)$ , we can achieve that  $E_{n,p(n)}[\chi]$  converges to a fixed prescribed Euler characteristic. Since for every  $n$ , we have  $E_{n,0}[\chi] = n$  and  $E_{n,1}[\chi] = 1$  and because of continuity with respect to  $p$ , we definitely can find such sequences for prescribed  $\chi \geq 1$ .

**5.** Despite correlation between the random variables  $v_k$  and different expectation values, its not impossible that the distribution of the random variable  $X_n(G) = (\chi(G) - E[\chi(G)]) / \sigma(\chi(G))$  on  $G(n, p)$  could converge weakly in the limit  $n \rightarrow \infty$  if  $\sigma(\chi)$  is the standard deviation. We would like therefore to find moments of  $\chi$  on  $G(n, p)$ . This looks doable since we can compute moments and correlations of the random variables  $v_k$ . The later are dependent: the conditional expectation  $E[v_3 = 1 | v_2 = 0]$  for example is zero and so different from  $E[v_3 = 1] \cdot E[v_2 = 0] > 0$ .

**6.** When measuring correlations and variance, we have no analytic formulas yet and for  $n \geq 8$  we had to look at Monte Carlo experiments. Tests in cases where we have analytic knowledge and can compare the analytic and experimental results indicate that for  $n \leq 15$  and sample size 10'000 the error is less than one percent.

$$\begin{aligned}
 d_1(p) &= 1 \\
 d_2(p) &= 2 - p \\
 d_3(p) &= 3 - 3p + p^3 \\
 d_4(p) &= 4 - 6p + 4p^3 - p^6 \\
 d_5(p) &= 5 - 10p + 10p^3 - 5p^6 + p^{10}
 \end{aligned}$$

TABLE 2. Polynomials  $d_n(p)$  which give the average Euler characteristic  $E_p[\chi]$  on the probability space  $G(n, p)$ .

### 6. STATISTICAL SIGNATURES

For any graph  $G = (V, E)$  called "host graph" we can look at the dimension and Euler characteristic signature functions

$$f(p) = E_p[\dim(H)], \quad g(p) = E_p[\chi(H)]$$

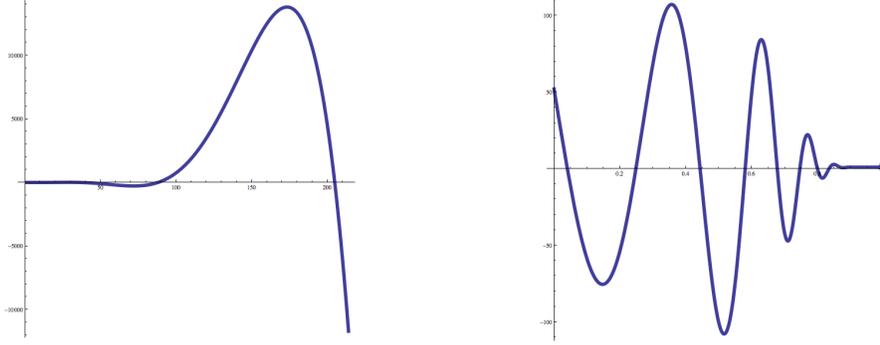


FIGURE 10. The average Euler characteristic as a function of  $n$  oscillates. The intervals, where the behavior is monotone gets larger and larger however, the amplitudes grow too. Also the  $p$  dependence is interesting. The right picture shows the average Euler characteristic as a function of  $p$  for a fixed order  $n = 52$ . The function is  $n - 1$  at  $p = 0$  and 1 at  $p = 1$ .

which give the expected dimension and Euler characteristic on the probability space  $G(p)$  of all subgraphs of  $G$  if every edge is turned on with probability  $p$ . These are polynomials in  $p$ . We have explicit recursive formulas in the case of  $K_n$ , in which case the coefficients of  $f$  are integers. We can explore it numerically for others. The signature functions are certainly the same for isomorphic graphs but they do not characterize the graph yet. Two one dimensional graphs which are unions of disjoint cyclic graphs, we can have identical signature functions  $f, g$  if their vertex cardinalities agree. The union  $C_5 \cup C_5$  and  $C_4 \cup C_6$  for example have the same signature functions  $f, g$ .

Since the global signature functions are not enough to characterize a graph, we can also look at the local dimension and Euler characteristic signature functions

$$h_v(p) = E_p[1 + \dim(H, S(v))], \quad k_v(p) = E_p[K(H, v)],$$

where  $H$  runs over all subgraphs and  $v$  over all vertices. Here  $K(H, v)$  is the curvature of the vertex  $v$  in the graph  $H$ . Of course, by definition of dimension and by the Gauss-Bonnet-Chern theorem for graphs, we have

$$\frac{1}{|V|} \sum_{v \in V} h_v(p) = f(p), \quad \sum_{v \in V} k_v(p) = g(p).$$

Since for one-dimensional graphs without boundary, the curvature is zero everywhere, also these do not form enough invariants for graphs.

We could also look at the dimension and curvature correlation matrices

$$A(p) = \text{Cov}[h_v(p), h_w(p)], \quad B(p) = \text{Cov}[k_v(p), k_w(p)]$$

which are  $n \times n$  matrix-valued functions on  $[0, 1]$  if  $|V| = n$  is the cardinality of the vertex set. We have

$$\frac{1}{|V|} \text{tr}(A(p)) = \text{Var}_p[\dim], \quad \text{tr}(B(p)) = \text{Var}_p[\chi].$$

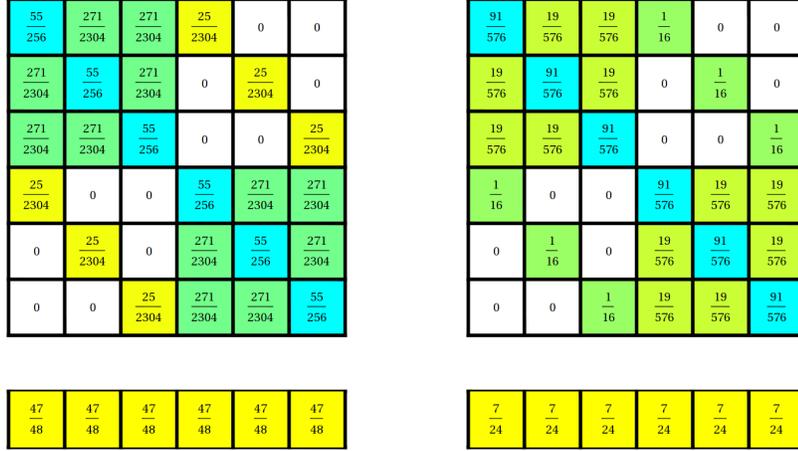


FIGURE 11. The dimension correlation matrix  $A(1/2)$  and the curvature correlation matrix  $B(1/2)$  for the Petersen graph  $G = P(3, 2)$  is shown. Below the dimension correlation matrix, we see the average local dimension. Below the curvature correlation matrix, we see the average curvature. The dimensions and correlations were computed in the probability space  $G(p)$  of all subgraphs of  $G$  which have the vertex set of  $G$ . The curvatures and dimensions at vertices which are not connected are uncorrelated.

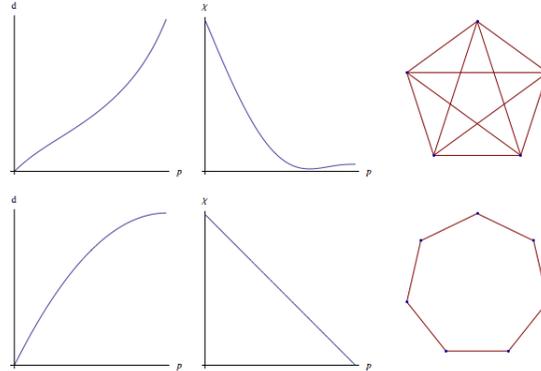


FIGURE 12. The dimension and Euler characteristic signature functions for the complete graph  $K_5$  are

$$f(p) = p(p^9 - p^8 - p^7 + 5p^5 - 3p^4 - 5p^3 + 10p^2 - 6p + 4)$$

$$g(p) = 5 - 10p + 10p^3 - 5p^6 + p^{10}.$$

The bottom figure shows the case when the host graph is the circular graph  $C_7$ , where

$$f(p) = p(2 - p), \quad g(p) = 7(1 - p)$$

The function  $f$  is the same for every one dimensional graph, and in general  $g(p) = |V|(1 - p)$ .

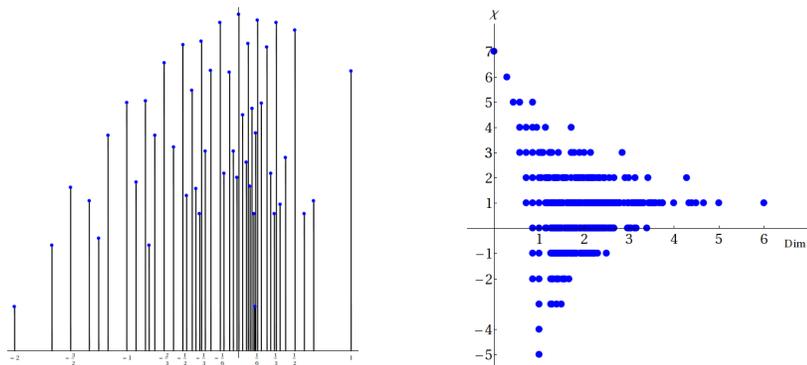


FIGURE 13. The left figure shows the logarithm of the number of vertices having a given curvature on the probability space  $\{V \times G(7, 1/2)\}$ , where  $V = \{1, 2, \dots, 7\}$  is the vertex set on the set  $G(7, 1/2)$  of graphs with 7 vertices. There are 47 different curvature values attained between  $-2$  and  $1$ . The right figure shows the pairs  $(\dim(G), \chi(G))$ , where  $G$  runs over all  $2^{\binom{7}{2}}$  graphs of 7 vertices. The Euler characteristic takes values between  $-5$  and  $7$ , the dimension of the graph takes values between  $0$  and  $6$ .

### Remarks.

1. In the computation to Figure 13 we observed empirically that most nodes are flat: 2784810 of the  $7 \cdot 2^{\binom{7}{2}}$  vertices have zero curvature. Graphs are flat for example if they are cyclic and one dimensional, which happens in  $6! = 720$  cases. The minimal curvature  $-2$  is attained 7 times for star trees with central degree 6.
2. Extreme cases of in the “dimension-Euler characteristic plane” are the complete graph with  $(\dim, \chi) = (6, 1)$ , the discrete graph with  $(\dim, \chi) = (0, 7)$  as well as the case  $(\dim, \chi) = (1, -5)$  with minimal Euler characteristic. An example is a graph with 12 edges, 7 vertices and no triangles obtained by taking an octahedron, remove all 4 vertices in the  $xy$ -plane then connect these 4 vertices with a newly added central point.

	1	2	3	4	5	6	7
0	$\frac{-1}{4}$	$\frac{-19}{64}$	$\frac{-981}{4096}$	$\frac{-138043}{1048576}$	$\frac{-4889125}{1073741824}$	$\frac{540429585637}{4398046511104}$	

TABLE 3. The correlations of  $\dim$  and  $\chi$  for until  $n = 7$  for  $p = 1/2$ . These are exact rational numbers because dimension is a rational number and Euler characteristic is an integer. The correlation has become positive the first time at  $n = 7$ . For  $n$  larger than 7 we only numerically computed it with Monte Carlo runs in Figure 14.

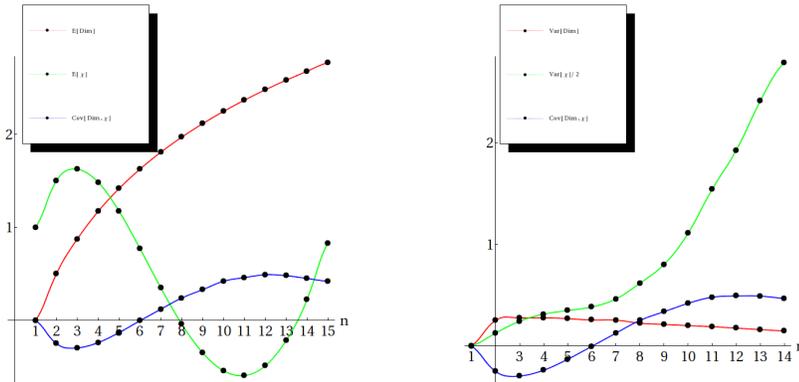


FIGURE 14. The left figure shows the expectation of the random variables  $\dim$  and  $\chi$  on  $G(n, 1/2)$  as well as  $\text{Cov}[\dim, \chi]$  for  $n = 1, \dots, 14$ . The right figure shows the variance  $\text{Var}[\dim]$ ,  $\text{Var}[\chi]$  as well as  $\text{Cov}[\dim, \chi]$  on  $G(n, 1/2)$  again for  $n = 1, \dots, 14$ . While the variance of the Euler characteristic grows rather rapidly on that interval, the variance of dimension decreases on that interval. For  $n = 20$ , we measure it to be about 0.1. We can explore experimentally however only a very small part of the probability space  $G(20, 1/2)$  consisting of  $2^{\binom{20}{2}} \sim 1.5 \cdot 10^{53}$  graphs.

n	1	2	3	4	5	6	7	8	9	10	11	12
min	1	1/2	0	-1/2	-1	-1.5	-2	-2.5	-3	-3.5	-4	-4.5
max	1	1	1	1	1	1	1	$\geq 1$	$\geq 1$	$\geq 4/3$	$\geq 3/2$	$\geq 5/3$

TABLE 4. The maximal and minimal curvature which can occur at points of graphs with  $n$  vertices. This was settled for  $n = 1, \dots, 7$  by checking over all graphs. The minimal curvatures are obtained at star shaped trees, where curvature satisfies  $K(v) = 1 - \text{deg}(v)/2$ . For larger  $n$ , we ran Monte Carlo experiments over 10'000 random graphs.

Having looked at the random variables  $\dim$  and  $\chi$  on  $G(n, p)$  it might be of interest to study the covariance

$$\text{Cov}[\dim, \chi] = E[\dim \cdot \chi] - E[\dim] \cdot E[\chi]$$

between them. The extremal cases of size 0, order  $n$  graphs of Euler characteristic  $n$  and dimension 0 or complete graphs with Euler characteristic 1 and dimension  $n - 1$  suggest some anti correlation between  $\dim$  and  $\chi$ ; but there is no reason, why there should be any correlation trend between dimension and Euler characteristic in the limit  $n \rightarrow \infty$ . Like Euler characteristic, it could oscillate.

While we have a feel for dimension of a large network as a measure of recursive "connectivity degree", Euler characteristic does not have interpretations except in geometric situations with definite constant dimensions. For example, for a two dimensional network, it measures the number of components minus the number

of "holes", components of the boundary  $\delta G$  the set of points for which the unit sphere is one dimensional but not a circle. While for geometric  $d$  dimensional graphs,  $\chi(G)$  has an interpretation in terms of Betti numbers, for general networks, both dimensions and curvatures vary from point to point and the meaning of Euler characteristic remains an enigma for complex networks.

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*E-mail address:* [knill@math.harvard.edu](mailto:knill@math.harvard.edu)

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA, 02138