

DYNAMICALLY GENERATED NETWORKS

OLIVER KNILL

ABSTRACT. Simple algebraic rules can produce complex networks with rich structures. These graphs are obtained when looking at a monoid operating on a ring. There are relations to dynamical systems theory and number theory. This document illustrates this class of networks introduced together with Montasser Ghachem in [11, 10]. Besides showing off pictures, we look at elementary results related to the Chinese remainder theorem, the Collatz problem, the Artin constant, Fermat primes and Pierpont primes.

1. INTRODUCTION

In September 2013, we stumbled upon networks generated by finitely many maps T_i on a ring R [11, 10]. The rule is that two different points x, y in R are connected if there is a map T_i from x to y . Some constructions of finite simple graphs can be seen below in this document. The idea is based on the old concept of **Cayley graphs** which visualizes finitely presented groups equipped with finitely many generators $T_i(x) = a_i x$ on groups. For a single transformation T on a ring R , one has a dynamical system: $T : R \rightarrow R$. The networks visualize the orbit structure of these systems if we think of the monoid generated by T as “time”. We disregard here however the digraph structure, self-loops and multiple connections and look at finite simple graphs only. As in complex dynamics, where the simplest polynomial maps already produce a rich variety of fractals, the discrete structures can be complex. The maps T_i on the ring do not have to be algebraic, they could be any permutation on a finite set. One can see these networks as graph homomorphic images of Cayley graphs generated by the transformations in the permutation groups of the vertex set. An other motivation comes from dynamical systems theory: since a computer always simulates a system on a finite set, it is interesting to see what the relation

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between this discrete and continuum is. The relation between the continuum and arithmetic systems have been investigated for example in [27, 22, 21]; a dynamical system like $T(x) = cx(1 - x)$ on the interval $[0, 1]$ is realized on the computer as a map on a finite set leading to finitely many cyclic attractors. For two or more maps, this can become a rather complex and geometric network.

The fact that simple polynomial maps produce arithmetic chaos is exploited by **pseudo random number generators**. There is a mild justification in that random variables like $x, T(x) = x^2 + c$ are asymptotically independent [17] on Z_n in the limit $n \rightarrow \infty$. While the pseudo random nature given by the quadratic map Z_n is unclear, the orbits are sufficiently random to be exploited or example in the ‘‘Pollard rho method’’ for integer factorization [23]. Maps like $x \rightarrow x^2$ modulo p or cellular automata maps were visualized in [32, 33] using state transition digraphs.

What is new? By allowing more transformations, we can get more structure and more variety. Our point of view is motivated heavily from the study of deterministic and random networks. The generalization from groups to monoids and by looking at the state space instead of the group itself, we break symmetries: while Cayley graphs look more like rigid crystals, the monoid graphs tend to produce rather organic structures resembling networks we see in social networks, computer networks, synapses, chemical or biological networks and especially in peer to peer networks. This happens already in the simplest cases: we can for example take quadratic maps $T(x) = x^2 + a, S(x) = x^2 + b$ or affine maps like $T(x) = 3x + 1$ and $S(x) = 5x + 3$ on Z_n . Whether we take affine, or nonlinear or arithmetic functions, the networks obtained in such a dynamical way can be intriguing:

a) The dynamical graphs show **visually interesting structures** which bring arithmetic relations to live. **b)** Their **statistical properties** of path length, global cluster and vertex degree are interesting. **c)** Some examples lead to **deterministic small world examples** with small diameter and large cluster. **d)** The graphs display **rich-club phenomena**, where high degree nodes are more interconnected. **e)** Many feature **garden of eden states**, unreachable configurations like transient trees. **f)** They can feature **attractors** like cycle sub graphs but also more complex structures. **g)** By definition, these graphs are **factors of Cayley graphs** on the permutation group of V . **h)** They are **universal** in the sense that any finite simple graph can be obtained like

that. **i)** In many cases, classes of networks produce natural probability spaces as we can parametrize maps. **j)** In certain cases, the graphs appear to be **triangularizations of manifolds** or varieties. **k)** In the arithmetic case, the topology like connectivity and dimension leads to Diophantine problems.

Some pictures can be seen at the end of the article. Here are two experimental observations:

A) ([11]) We measure that the mean length $\mu(G)$ and the global clustering coefficient $\nu(G)$ have the property that $\lambda(G) = -\mu(G)/\log(\nu(G))$ often has a compact limit set if the number of nodes go to infinity. When choosing random permutations and averaging, we see actual convergence in the limit $n \rightarrow \infty$. The two quantities $\mu(G)$ and $\nu(G)$ are essential to see small world phenomena as seen in [31]. A reasonable conjecture is that in the probability space of all pairs of random permutations f, g on Z_n , the random variable $\lambda(G(f, g))$ has an expectation which converges for $n \rightarrow \infty$. There is strong numerical evidence for that. In [11] we also showed how one can naturally construct large bipartite or multipartite graphs using dynamical constructions.

B) ([10]) We get deterministic examples of networks which feature all the statistical properties of Watts-Strogatz [31] in the sense that μ, ν and vertex distributions behave in the same way. The statistic is almost indistinguishable from W-S. These examples are of the form $T_i(x) = [x^{1+\epsilon_i} + i]$, where at least one ϵ_i is 0 and the others are equal to p , a permutation parameter. For $\epsilon_i = 0$ and k maps, we have the initial wiring setup of Watts-Strogatz for $p = 0$. While in Watts-Strogatz, the rewiring is done in a probabilistic way, this is taken care by increasing the nonlinearity. If the maps are nonlinear but close to linear, we see interesting geometric features appearing discussed in [10]. For example: if maps are close to linear maps, interesting topological structures can appear.

In this paper, we prove a couple of elementary which indicate how these graphs can relate to elementary number theory. To do so, we have questions from dynamical systems as a guide. The subject of networks has exploded in the last decade, as a look onto the library shows [14, 3, 13, 6, 20, 19, 28, 16, 29, 2, 15, 26, 20, 9, 8, 24]. It has been made accessible to a larger audience in books like [30, 4, 1, 25, 5].

The topics in the next sections are in an obvious way motivated from corresponding problems in complex dynamics, where the question of connectivity and dimension of the Julia sets is of interest. One can also look at the analogue of the Mandelbrot set, the set of parameters for which graphs are connected. If we have a class of dynamically generated graphs, we can ask how the clique size is distributed on the parameter space. As in dynamical system terminology, we look for the connectivity locus and the dimension of the object. The subject can lead to relatively simple but unexplored questions. It is suited for experimentation (as we have accessed it ourselves primarily) and is almost unexplored. We illustrate this by formulating some simple questions related to connectivity. Computer algebra code will be available on the project website and the Wolfram demonstration project.

2. AFFINE MAPS AND SMOOTH NUMBERS

We first look at graphs generated by a single affine map $T(x) = ax + b$ on the ring $Z_n = Z/(nZ)$. Given parameters (a, b) , for which n is the graph connected on Z_n ? Lets assume first $T(x) = ax$ on Z_n . The following result will be used later on:

Lemma 1. *The graph on Z_n generated by $T(x) = 2x$ is connected if and only if n is a power of 2.*

Proof. If n is a power of 2, then $T^k x = 2^k x$ is divisible by n if k is larger or equal than n . We see that every x is eventually attracted by 0 and that the graph is a tree. If $n = p2^m$ with p being relatively prime to 2, we can look at the orbit $T^k x = x2^k$ modulo p . If x is divisible by p , then $T^k x$ stays divisible by p and the graph is not connected. \square

If we look at Z_n^* , we get additionally some primes. Which ones? We see that 3, 5, 11, 13, 19, 29, 37, 53, 59, 61, 67, 83, 101, ... lead to connected graphs while 7, 17, 23, 31, 41, 43, 47, 71, 73, 79, 89, 97, 103, ... lead to disconnected ones.

Proposition 2 (Miniature I: Artin). *The graph on Z_n^* generated by $T(x) = 2x$ has one component if and only if n is a power of 2 or a prime p for which 2 is a primitive root.*

Proof. We look at the dynamics on $Z_n^* = Z_n \setminus \{0\}$, where n is a prime. If 2 is a primitive root modulo n , then by definition, the orbit $\{2^k \bmod n\}$ covers the multiplicative group Z_n^* and the graph is connected. If 2 is not a primitive root, then there exists x for which the discrete logarithm problem $2^k = x \bmod n$ has no solution. The graph is not connected. \square

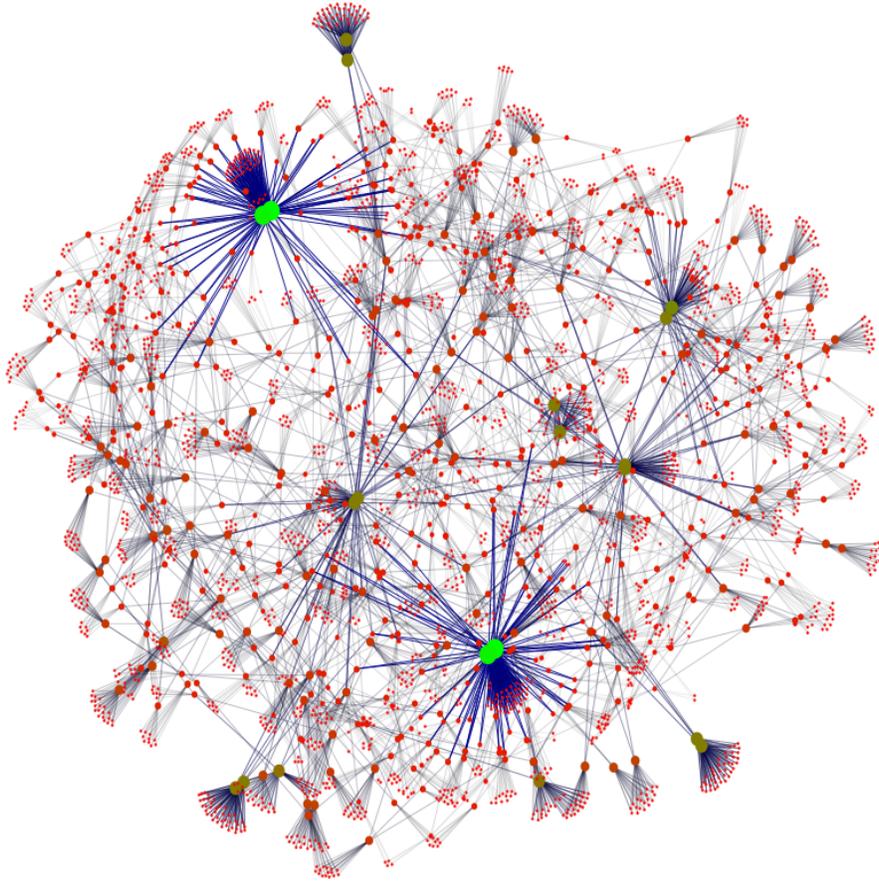


FIGURE 1. A graph is generated by $f(x) = x^2+1, g(x) = x^2+2$ on Z_{3000} . It has diameter 9, average vertex degree 3.99, characteristic path length $\mu = 5.8$, mean clustering $\nu = 0.00074$ and a length-cluster coefficient $\lambda = 0.806994$.

Remarks.

1) It is an open problem to determine the fraction of the set of primes for which 2 is a primitive root. Among the first $n = 10^6$ primes, 374023 have this property. A conjecture of Artin implies that this probability should converge to the **Artin constant** $\prod_{p \text{ prime}} (1 - 1/(p(p-1))) = 0.3739558\dots$

2) There are analogous results, when 2 is replaced with an other prime. What matters for prime n whether a is a primitive root in the field Z_n or not.

Definition 1. Given a finite set P of primes, we call the set of all products $\{\prod_{p_i \in P} p_i^{n_i} \mid n_i \geq 0\}$ the set of all **P -smooth numbers**. If P is the set of P -smooth numbers, let's call the set $P \cup 2P$ the set of **double P -smooth numbers**. If a is an integer, we call the set of numbers which have only prime factors from a to be **a -smooth**. Similarly, if a, b are integers, the set of numbers which have only prime factors from a, b are called **(a, b) -smooth**.

Double smooth numbers appear as the "connectivity locus" in the case $x \rightarrow 3x + 1$: the graph is connected for $n = 1, 2, 3, 6, 9, 18, 27, 54, 81, \dots$ which is the set of double $\{3\}$ -smooth numbers, numbers which are a power of 3 or twice a power of 3. We do not have a complete picture yet but state only:

Lemma 3. *The set of n which are connected for $T(x) = ax + b$ is a subset of all $\{a, a - 1\}$ smooth numbers.*

Proof. If q is a prime factor of n which does not divide $a - 1$, then the graph is not connected: there is a congruence class modulo q which is a fixed point of T . The reason is that $ax + b = x \pmod{q}$ has a solution $x = -b(a - 1)^{-1}$. For example, if $T(x) = 5x + 1$ and $n = 21$ which has a factor $q = 3$, then the congruence class 2 modulo 3 is a fixed point of T . Since this congruence class is invariant, the graph is not connected. If a divides n , then $n = ka$ and T^a produces a translation on Z_q and there is a chance that we have several graphs. If a is prime and does not divide n , \square

Here are some results, where we denote by (p_1, \dots, p_n) the $P = \{p_1, \dots, p_n\}$ smooth numbers and by $(2^*, p_1, \dots, p_n)$ the double P smooth numbers. We computed the table by constructing the graphs, seeing which are connected and matching it with P smooth sequences.

	$b = 0$	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$b = 5$	$b = 6$
$a = 2$	(2)	(2)					
$a = 3$	(3)	(2*, 3)	(3)				
$a = 4$	(2)	(2, 3)	(2, 3)	(2)			
$a = 5$	(5)	(2, 5)	(5)	(2, 5)	(5)		
$a = 6$	(2, 3)	(2, 3, 5)	(2, 3, 5)	(2, 3, 5)	(2, 3, 5)	(2, 3)	
$a = 7$	(7)	(2*, 3, 7)	(3, 7)	(2*, 7)	(3, 7)	(2*, 3, 7)	(7)
$a = 8$	(2)	(2, 7)	(2, 7)	(2, 7)	(2, 7)	(2, 7)	(2, 7)

We always get smoothness sequences or double smoothness sequences involving the prime factors of a and $a - 1$. For $T(x) = px$ with prime p , we have connectivity for $\{p\}$ -smooth numbers $\{p, p^2, p^3, \dots\}$. For $T(x) = px + 1$ with prime p we have connectivity for $\{p, (p - 1)\}$ smooth numbers if $p - 1$ is divisible by 4 and double $\{p, P(p - 1) \setminus \{2\}\}$ -smooth

numbers if $p - 1$ is divisible by 2.

We should also look at the case $a = 1$. Now $T^k(x) = x + kb$ and the graph is connected if and only if b has no common divisor with n . If $R = Z_n^m$ and $T(\vec{x}) = x + k\vec{b}$, then the graph is connected if all b_i have no common divisor with n . This is the Chinese remainder theorem.

3. QUADRATIC MAPS AND FERMAT PRIMES

An other simple example of a dynamically generated graph is obtained with $T(x) = x^2$ on Z_n . In [32], this is attributed to a suggestion of Stan Wagon. How many components does the graph have? We see that for $n = 2^k$ with $k \in N$, there are two components, the even and odd numbers.

Definition 2. A prime of the form $n = 2^{2^k} + 1$ is called a **Fermat prime**.

Proposition 4 (Miniature II: Fermat). *The graph on Z_n^* is connected if and only if $n = 2$ or if n is a Fermat prime $n = 2^{2^k} + 1$.*

Proof. If n is not prime $n = pq$, then the orbit of $x = p$ has the property that every point $T^k(x)$ is divisible by p modulo n and the graph is not connected. We can therefore assume that n is a prime. This allows find a primitive root a and write $Z_n^* = \{a^k\}$. If n is a Fermat prime, then the graph is a tree centered at 1. The elements 1 and the quadratic non residues have one neighbor, while the quadratic residues have 3 neighbors $x^2, \pm\sqrt{x}$. If n is not a Fermat prime, then $n - 1$ is not a power 2 and has therefore a factor q different from a power of 2. We look now on the dynamics of the system $x \rightarrow 2x$ modulo $q2^l$. We have seen in Lemma (1) that the graph is connected if and only if $q = 1$. In other words, the graph is connected if and only if n is a Fermat prime. \square

The only Fermat primes known are F_0, F_1, F_2, F_3, F_4 . It would be interesting to know the Euler characteristic of the graphs G_n generated by $T(x) = x^2$ on Z_n . The list of Euler characteristics starts with

$$1, 2, 2, 2, 2, 4, 3, 2, 3, 4, 2, 4, 3, 6, 4, 2, 2, 6, 3, 4, 6, 4, 2, \dots$$

The graphs do not need to be simply connected. An example is G_{59} , a case with 3 components and one large cycle. To get triangles, we have to solve $((x^2)^2)^2 = x^8 = x$ which is only possible if $x - 0$ or $x^7 - 1$ is a multiple of n . Indeed, for $n = 127$ we get the first such graph with 2 triangles. Since graphs generated by one map never has a tetrahedron,

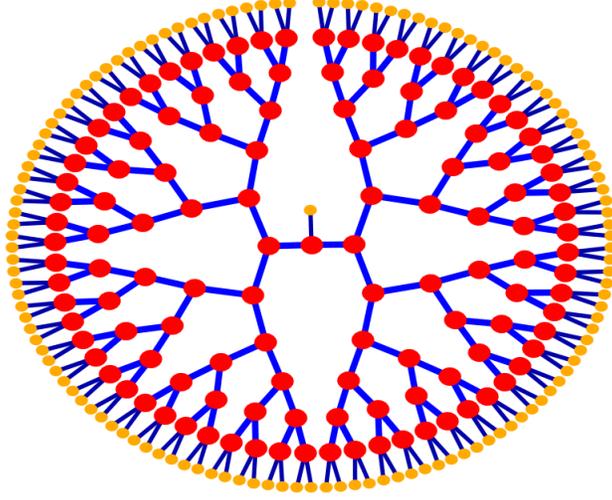


FIGURE 2. The Fermat graph for $F_3 = 2^{2^3} + 1 = 257$.

the Euler characteristic of a graph G_n is $v - e + f$, where v is the number of vertices, e the number of edges and f the number of triangles.

If we take $T(x) = x^3$, we see no nontrivial graphs with 1 or 2 components. Graphs with $n = 3^k$ have three components. For $T(x) = x^n$ with even n we have the same list of graphs with 2 components as in the case of $n = 2$. For $n = 5$, the list of integers on which $T(x) = x^5$ has a graph with 3 components starts with 3, 4, 11, 251, \dots

In the case $T(x) = 2^x$, the distribution of the number of components is smaller and we measure about $M/\log(M)$ graphs among all graphs with $n = 1, \dots, M$ which have one component. Now, we can ask for which n the graphs generated by $x \rightarrow 2^x$ are connected.

4. A CLASS OF COLLATZ TYPE NETWORKS

This example is inspired by the famous Collatz $3x + 1$ problem, a "prototypical example of an extremely simple to state, extremely hard to solve, problem" to cite [18]. It deals with two maps: a number n is divided by 2 if it is even and mapped to $3n + 1$ if it is odd. The problem is whether every starting point converges to 1 when applying this rule. We can look at networks generated by affine maps $T(x) = 2x$ and

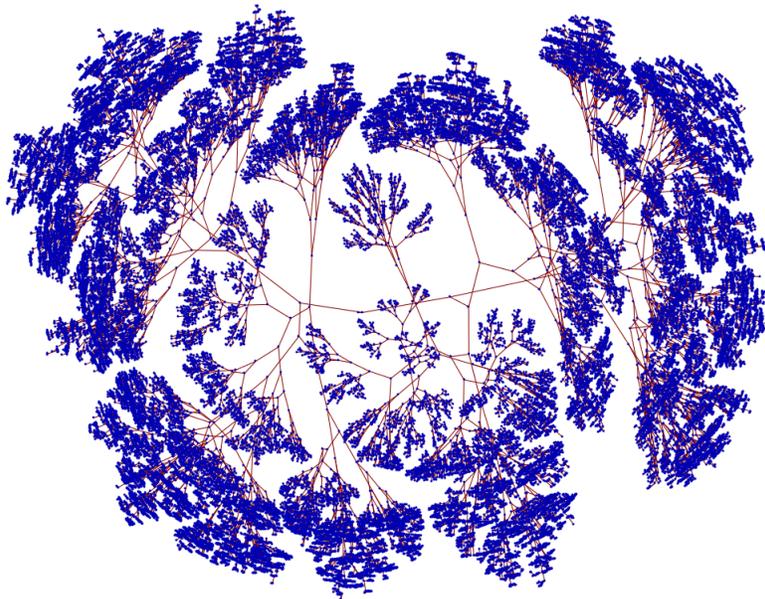


FIGURE 3. The Fermat graph for the largest known Fermat prime $F_4 = 2^{2^4} + 1 = 65537$.

$S(x) = 3x + 1$ on Z_n , disregarding the conditioning and ask about the structure of this network. Of course, we can not expect any trivial relations with the Collatz problem any more. The graphs look surprisingly random. It is the non-commutativity of the monoid generated by T, S as well as the conditional application of these two maps which makes the original Collatz problem difficult. When the graph is generated by $T(2x + 1) = 6x + 4, T(2x) = 2x, S(2x) = x/2, S(2x + 1) = x$, then this leads to the Collatz graph which is believed to be a tree. Note that the original Collatz problem is generated by one map. We look at the two affine maps unconditionally.

We see experimentally that all networks C_n on Z_n^* generated by the dynamical system $(Z_n, T(x) = 3x + 1, S(x) = 2x)$ are connected. This is not related to the Collatz problem, where connections are only done under conditions leading to a digraph called Collatz graph which if the Collatz conjecture is true, is a connected tree. The connectivity question is interesting for general affine maps. For $d \geq 2$ -dimensional

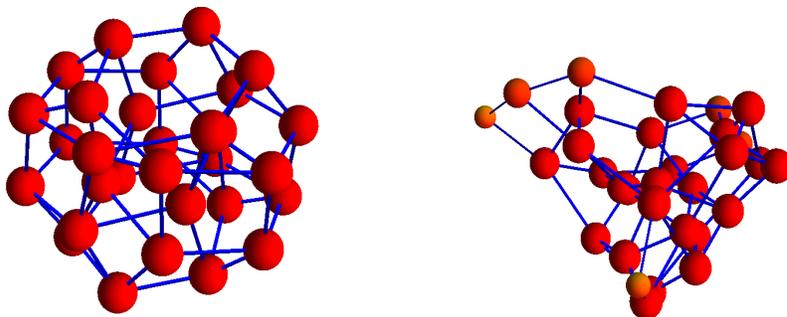


FIGURE 4. The first graph shows the graph generated by $T(x) = 2x, S(x) = 3x$ on Z_{31} . This is a Cayley graph on the multiplicative group. The second shows the Collatz graph generated by $T(x) = 2x, S(x) = 3x+1$ on Z_{31} . The symmetry is broken: the maps T, S do not commute any more.

graphs, non-connectivity (ignoring the isolated vertex 0) is more rare but still can happen. For the graph generated by $5x + 2, 3x + 1$ for example, we have for prime n about $2/3$ connected graphs and $1/3$ disconnected; the number of connected components looks exponentially distributed. For the graph generated by $5x + 1, 3x + 1$ we appear to have connectivity for all primes.

The connectivity question can be seen as a non-commutative Chinese remainder theorem problem because we want to solve $T^{n_1} S^{n_2} \dots T_{n_k} S_{n_k} x = y$. In the commutative case, this reduces to $T^n S^m x = y$. An example is $T(x) = ax$ and $S(x) = bx$ where the maps commute. It is enough to assure that for every x we can find u, v such that $a^u b^v = x \pmod{n}$. If either a or b is a primitive root of unity, then $b = a^l$ and we have the problem to solve $a^{u+lv} = x$, which is no problem already for $v = 0$.

Here is an amusing fact for Collatz networks:

Proposition 5 (Miniature III: Collatz). *All Collatz networks $C_n = (Z_n, 2x, 3x + 1)$ have exactly 4 triangles if n is prime and larger than 17.*

Proof. Since Z_p is a field if $n = p$ is prime, a linear equation modulo a prime can be solved in a unique way. Let $T(x) = 2x$ and $S(x) =$

$3x + 1$. Since $T^3(x) = 8x = x$ has only the solution $x = 0$ and $S^3(x) = 13 + 27x = x$ has a unique solution which is already a solution of $S(x) = x$, we see that every triangle must be formed with 2 maps T and one map S or two maps S and one map T . In each of the four possible cases $T^2(S(x))$ and $T(S(T(x)))$ and $S^2T(x)$ $STS(x)$ we have exactly one solution. Now for small primes, there are either less or more solutions. For $n = 13$ for example, the equation $S^3(x) = x$ is solved by any x . There are 6 triangles. The combinations of three maps from T, S, T^{-1}, S^{-1} produces $4^3 = 64$ possible polynomials. The largest coefficient which appears is 27. For $p > 27$ there are exactly 4 solutions. We check by hand that also for $p = 19, 23$, there are exactly 4 solutions. \square

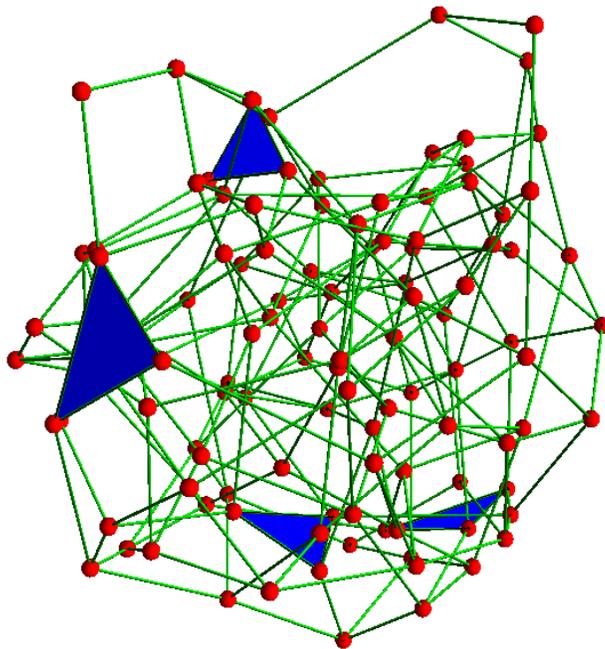


FIGURE 5. The Collatz graph C_{113} generated by $T(x) = 3x + 1, S(x) = 2x$ on Z_{113} with the four triangles.

Remarks.

1) This result is neither special nor universal. The dynamical graph generated by $2x + 1, 3x + 1$ also has 4 triangles for prime $n > 17$. Similarly, the graph generated by $5x + 1, 3x + 1$, but the one generated by $T(x) = 5x + 2, S(x) = 3x + 1$ has no triangles for prime $n > 37$. Why? For $n = 41$ for example, we have $T(S(T(20))) = 20$ but also $T(20) = 20$ and $S(20) = 20$. For prime n , the point $(n - 1)/2$ is a fixed point of both T and S so that $x, T(x), S(T(x))$ is not a triangle but a point. What about non-prime n ? It can be subtle for $n = pq$ already, where we can see different numbers of triangles.

2) The result seems to generalize to higher dimensions. We see 16 triangles for prime $p > 37$ and three transformations like $T(x) = 2x, S(x) = 3x + 1, U(x) = 5x + 2$. We do not see any tetraedra in that case for prime n .

The statistics of the Collatz networks on Z_n generated by $T(x) = 3x + 1, S(x) = 2x$ is typical among other dynamical networks. We see that the mean path length-mean cluster coefficient relation $\lambda(C_n) = -\mu(C_n)/\log(\nu(C_n))$ converges for $n \rightarrow \infty$. The dimension of the Collatz graphs converges to 1 for $n \rightarrow \infty$ as the number of triangles are rare. It would be nice to know the exact number of triangles in the general case (not only for primes). This might give hope that we might be able to compute the Euler characteristic of C_n exactly.

5. ARITHMETIC FUNCTIONS

The map $T(x) = \sigma(x) - x$ gives the sum of the proper divisors of x . Fixed points of T are called **perfect numbers**. While the structure of even perfect numbers is known it is an ancient problem whether there are odd perfect numbers. Periodic points of period 2 are **amicable numbers**. It is not known whether there are infinitely many. The dynamical system generated by T on \mathbb{Z} is the Dickson dynamical system [7]. We call the graphs on Z_n generated by T Dickson graphs. Their topological properties depend on number theoretical properties. We can now play with other graphs like graphs generated by $T(x)$ and $S(x) = x + 1$.

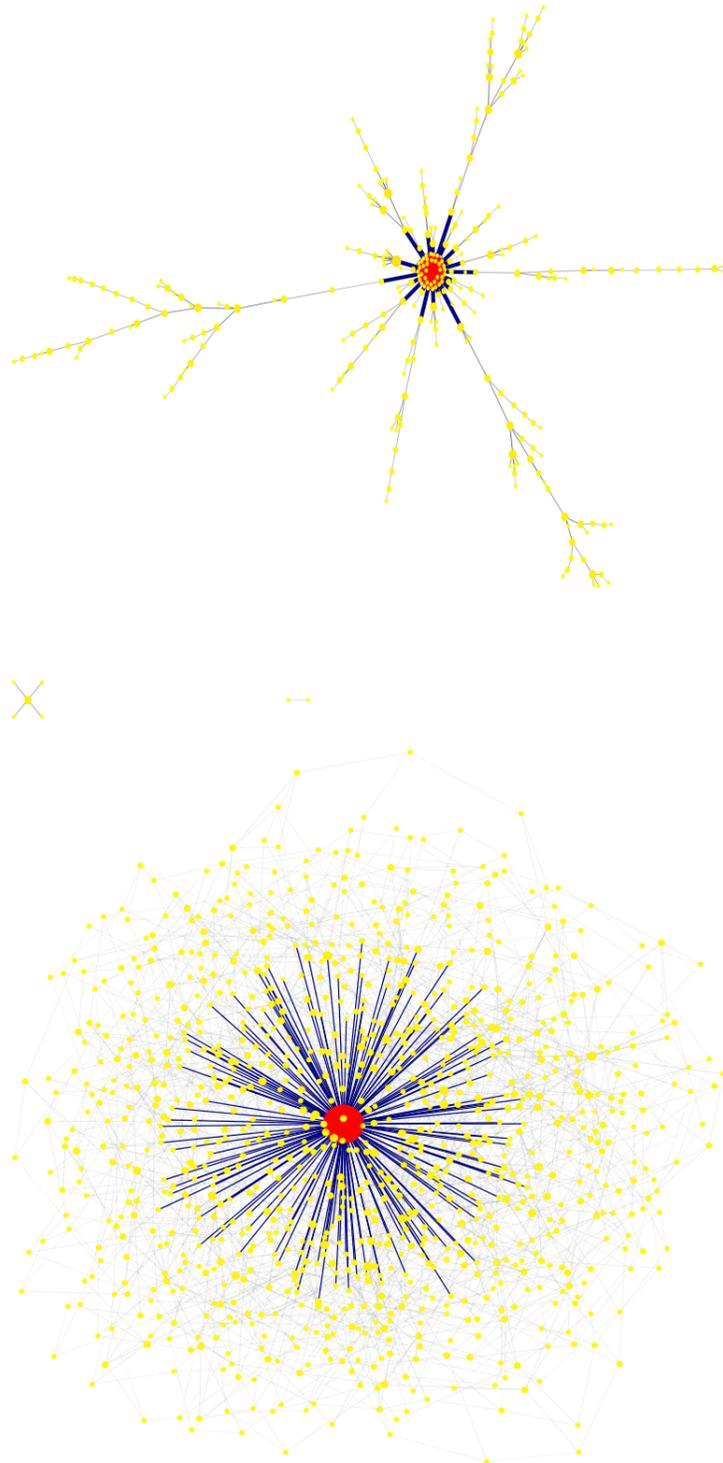


FIGURE 6. Dickson graphs: the first graph is generated by $T(x) = \sigma(x) - x$. The second is the graph generated by T and $S(x) = x + 1$.

As the Dickson dynamical system illustrates, questions involving the iteration of arithmetic functions can be difficult. The oldest known open problem in mathematics is related to it.

6. PIERPONT PRIMES

One of the simplest nonlinear cases with two dimensional time is $T(x) = x^2$ and $S(x) = x^3$ on Z_n . This is a case where the monoid is commutative. One can ask, for which n , the graph on $\{2, \dots, n-1\}$ generated by x^2, x^3 is connected. This can be answered because $T(x) = x^2$ and $S(x) = x^3$ commute. We have $T^n(S^m(x)) = x^{2^n 3^m}$.

Definition 3. A prime n is called a Pierpont prime if is of the form $2^t 3^s + 1$, where s, t are integers.

The list of Pierpont primes starts with

2, 3, 5, 7, 13, 17, 19, 37, 73, 97, 109, 163, 193, 257, 433, 487, 577,

It is unknown whether there are infinitely many. Gleason has shown that these primes play an important role with ruler, compass and angle trisection [12]. He was also the first to conjecture that there are infinitely many Pierpont primes. Here is an elementary proposition which illustrates the relation between arithmetic and graph theory:

Proposition 6 (Miniature IV: Pierpont). *The graph on Z_n^* generated $T(x) = x^2$ and $S(x) = x^3$ is connected if and only if n is a Pierpont prime.*

Proof. If $n = pq$, then any multiple of p will remain a multiple of p after applying T or S . This implies that the graph is not connected. We therefore need that n is prime. If n is prime, then the multiplicative group of Z_n is cyclic. A generator is a primitive root. We can now look at indices (discrete logarithms) and get maps $x \rightarrow 2x$ and $x \rightarrow 3x$ on the multiplicative group $\{1, \dots, n-1\}$. Either n is even and equal to 2 in which it is a Pierpont prime or n is odd and of the form $2^t k + 1$, where k is an integer. The proof is concluded with two things: (i) if $k = 3^s$, then the graph is connected. (ii) If k has a factor different from 2 and 3, then the graph is disconnected. Part (i) is clear because the maps $U(x) = 2x, V(x) = 3x$ on $Z_{n-1} = Z_{2^t 3^s}$ eventually lead to 0 modulo $2^t 3^s$, showing that the graph is connected. Part (ii) follows from Lemma (3). \square

This can be generalized. For example a graph generated by x^2, x^5 is connected if and only if the primes are of the form $2^t 5^u + 1$. The sequence of these primes is 2, 3, 5, 11, 17, 41, 101,

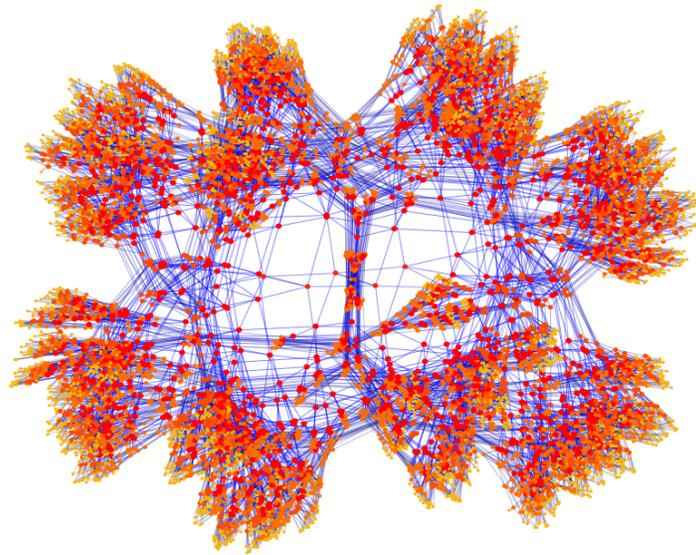
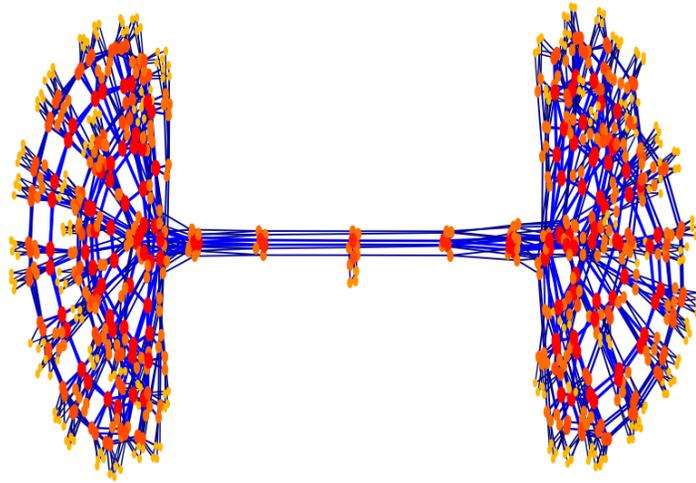


FIGURE 7. Pierpont graphs generated by x^2, x^3 are connected on Z_n^* if n is a Pierpont prime. We see three cases: $n = 769$ and $n = 10369$.

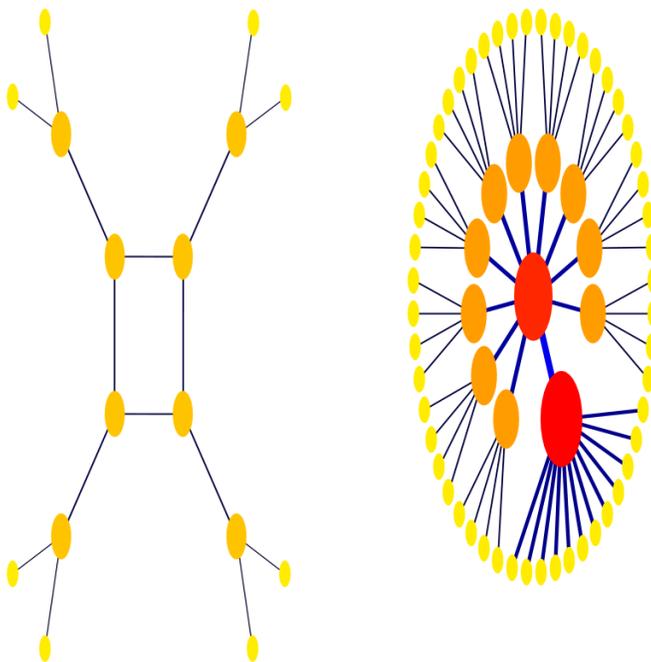


FIGURE 8. The graph given by of $T(x) = x^2$ on the ring R of upper triangular 2×2 matrices over Z_5 has two main components. There are other components associated to matrices with zero eigenvalues.

7. RINGS WITH MORE STRUCTURE

Lets take the non-commutative ring $R = M(2, Z_n)$. If $T(x) = x^2$, then since diagonal matrices are left invariant by T we can not have one component but have at least two components in the graph. We need a Fermat prime n to have two components.

Lets stay with $M(2, Z_n)$ and consider maps $T_i(x) = x^2 + c_i$. The networks generated like that look more complex but the statistics is not much different than in the commutative case.

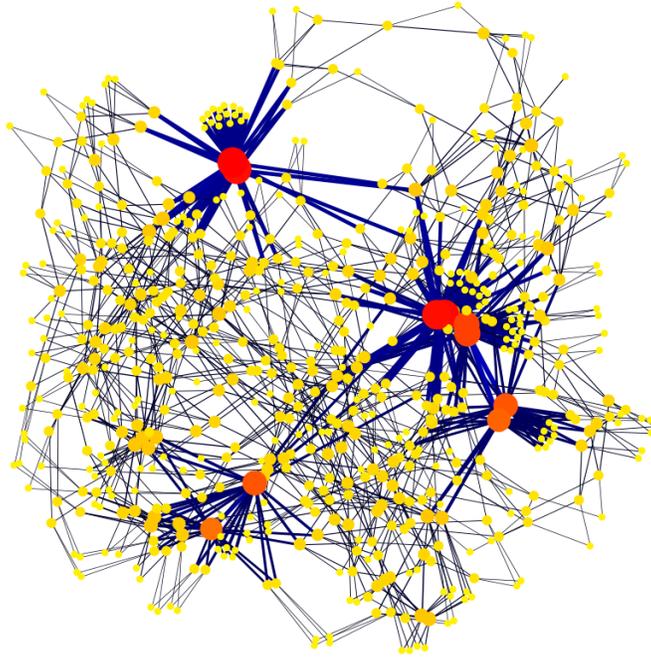


FIGURE 9. An example of graph using a noncommutative ring. We take the graph generated by the two quadratic transformations $x \rightarrow x^2 + A$ and $x \rightarrow x^2 + B$ on the matrix ring $M(2, Z_5)$.

We can also look at polynomial rings $R[x]/(p)$ where p is a polynomial.

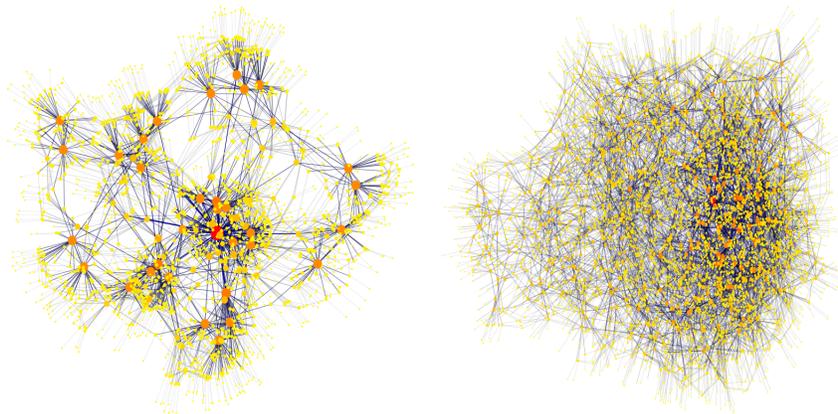


FIGURE 10. Graphs on $Z_n[x]/x^6$ ($n=4,5$) generated by three transformations $T_1(f) = f'$, $T_2(f) = f^2$ and $T_3(f) = f + x^4 + x^3 + x^2 + x + 1$.

More generally, one can look at small rings. For example, there are 11 rings of size 4.

Finally, we can look at transformations of the finite vector space Z_p^n . A natural choice are **elementary cellular automata**. With the Wolfram numbering for the 256 possible rules, we have for every n a parameter space with 2^{16} elements. We see experimentally that for some parameters, the connectivity fluctuates with n while for others, the connectivity stays. We could explore this however only for $n \leq 25$ because this produces already graphs with number of entries reaching the human population on earth.

8. FIVE MANDELBROT PROBLEMS

Here are five connectivity problems which are unsettled at the moment but look accessible.

1) Find necessary and sufficient conditions which assure that the graph generated by $T(x) = ax + b$ on Z_n is connected.

We have seen that n is a (double) P smooth number where P is a subset of the union of primes of a and $a - 1$.

2) Are all graphs on Z_n generated by $T(x) = 3x + 1, S(x) = 2x$ connected?

We have checked this at the moment only for all graphs up to $n = 200'000$ nodes.

3) Find necessary and sufficient conditions which assure that the graph generated by $T(x) = x^a$ and $S(x) = x^b$ on Z_n^* is connected.

We have seen that if $a = 2$ and n is a Fermat prime, then the graph is connected because it is already connected for one transformation. An example is $n = 257, a = 5, b = 56$.

4) Which 2×2 matrices A in $R = M(2, Z_n)$ have the property that $T(x) = x^2 + A$ produces a connected graph?

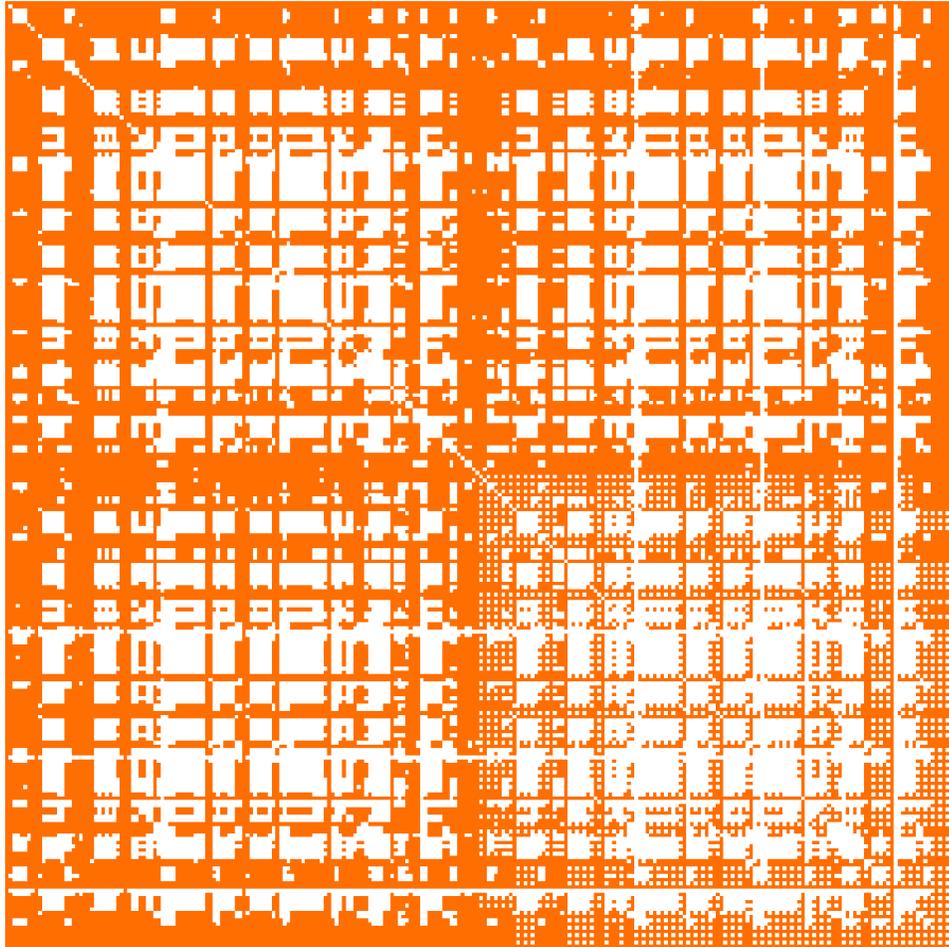


FIGURE 11. The CA Mandelbrot set M for pairs of elementary CA on Z_2^9 . Colored are the parameters (a, b) with $0 \leq a, b \leq 255$ for which one has a connected graph.

An example of a connected graph is obtained for $n = 5$ with $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

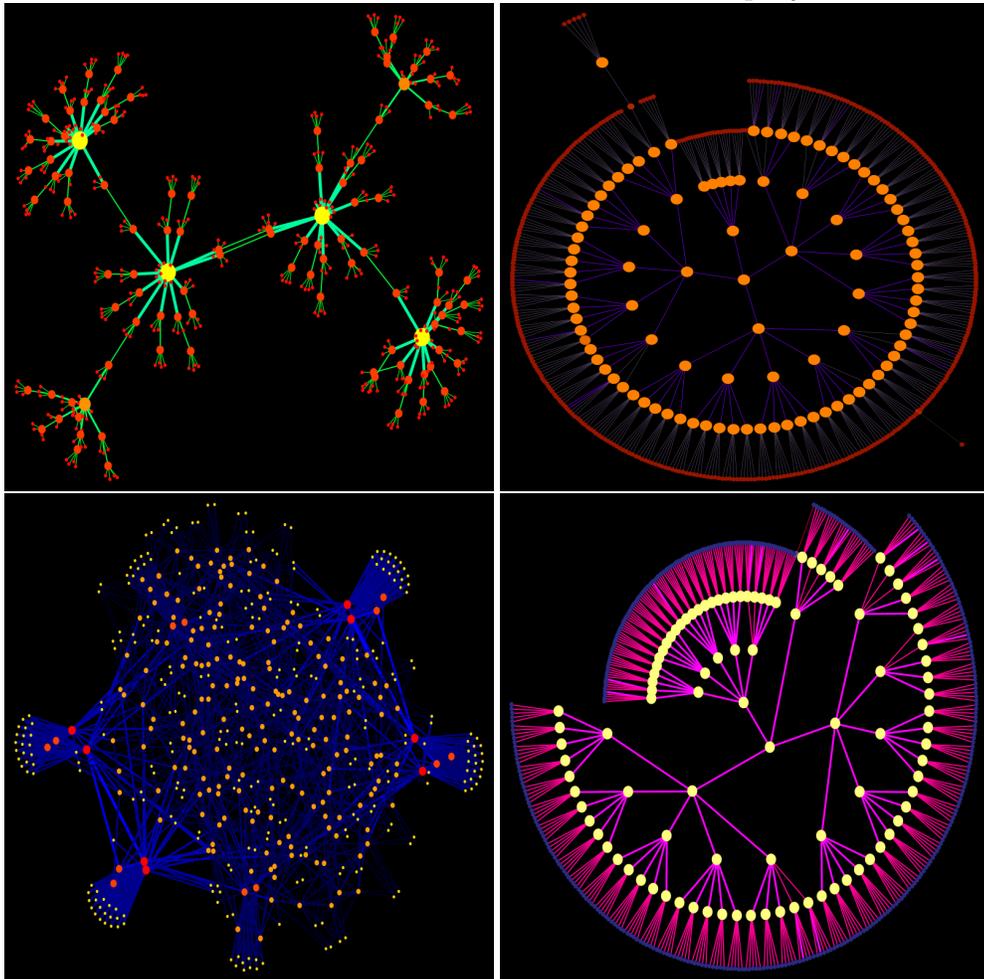
5) For which n, a, b do the elementary cellular automata T_a, T_b acting on the field Z_2^n produce connected graphs?

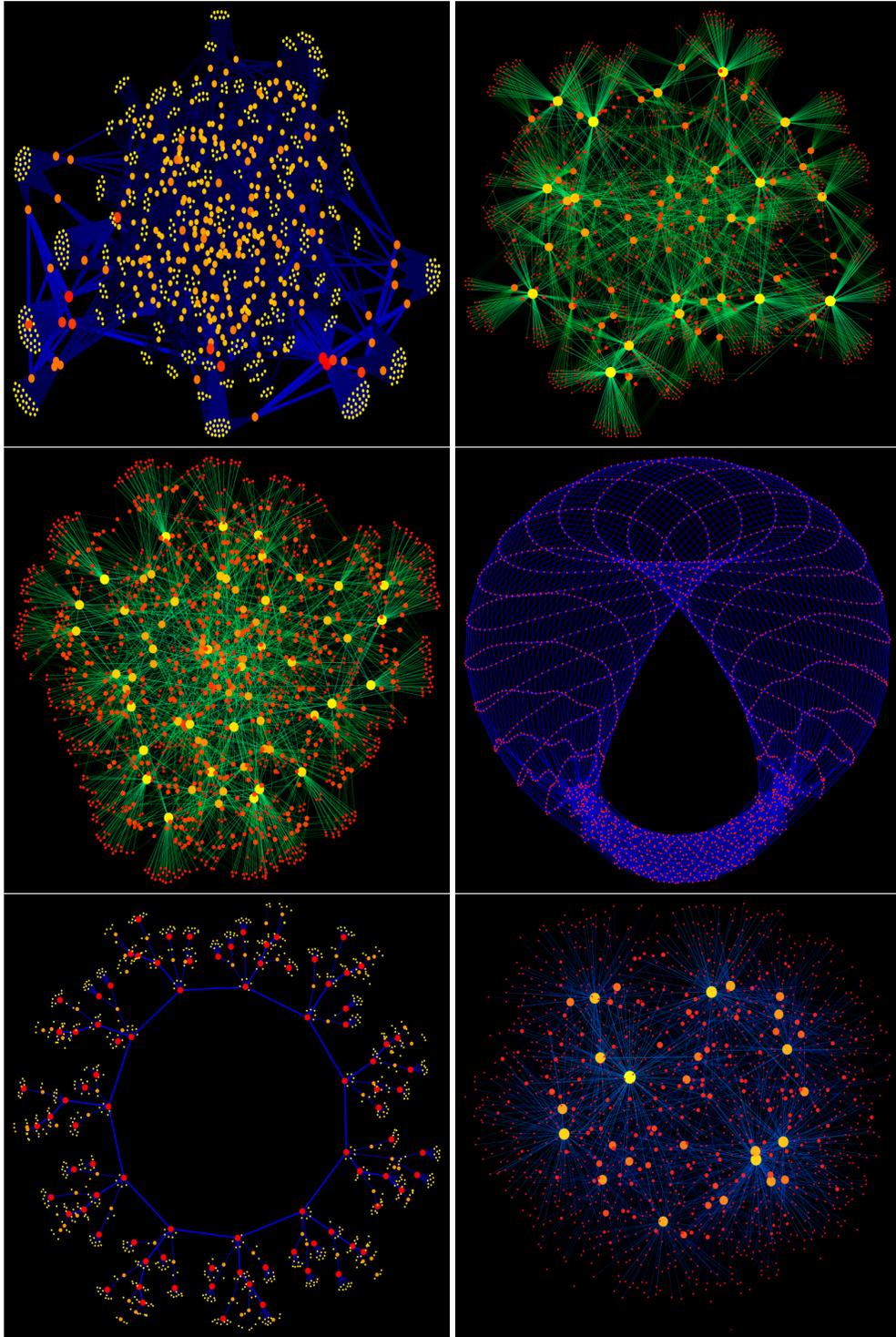
For every n there are 2^{16} possible graphs. One can ask whether we have for fixed a, b convergence of the number of connectivity components when $n \rightarrow \infty$. This seems to be the case for $a = 101, b = 110$. For other pairs like $a = 3, b = 4$ we see fluctuations in n which seem to

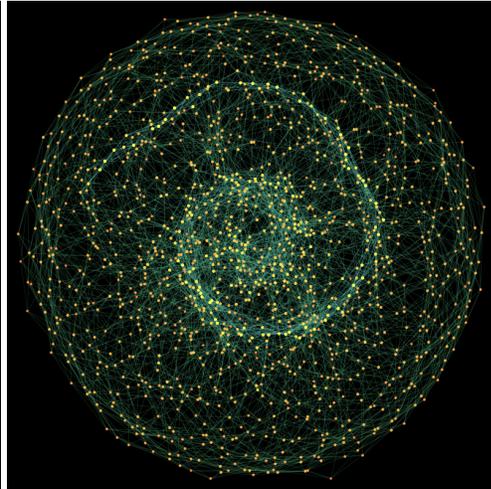
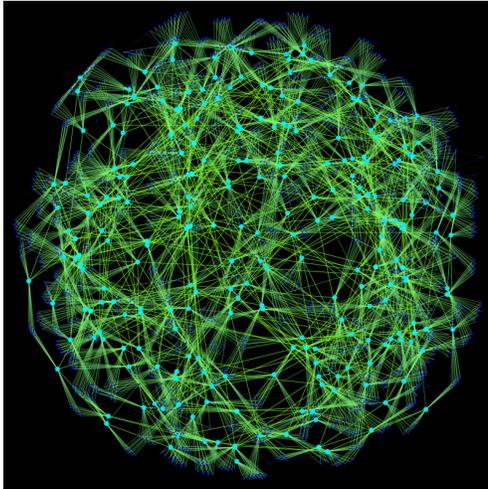
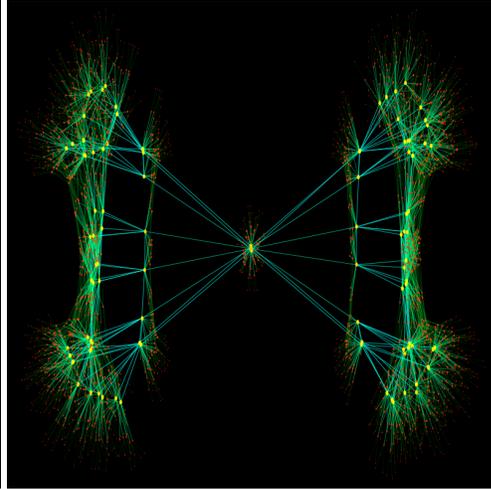
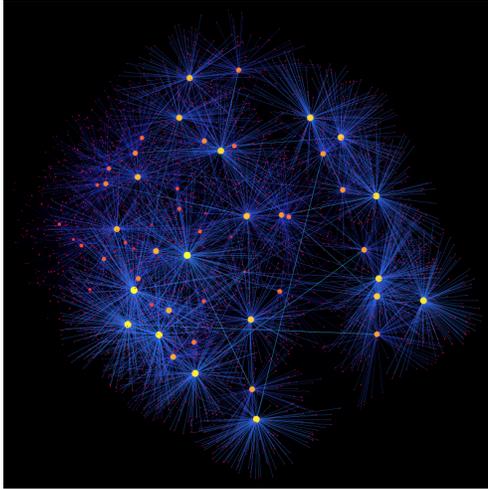
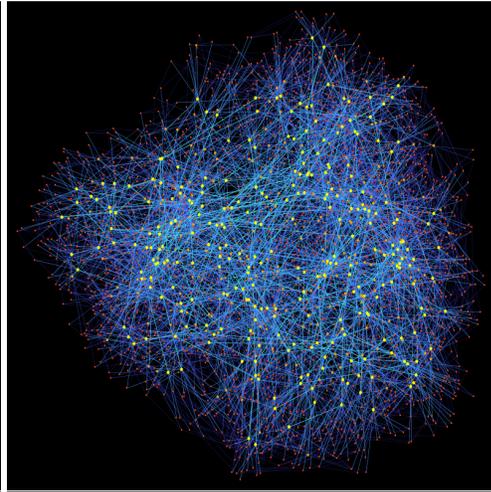
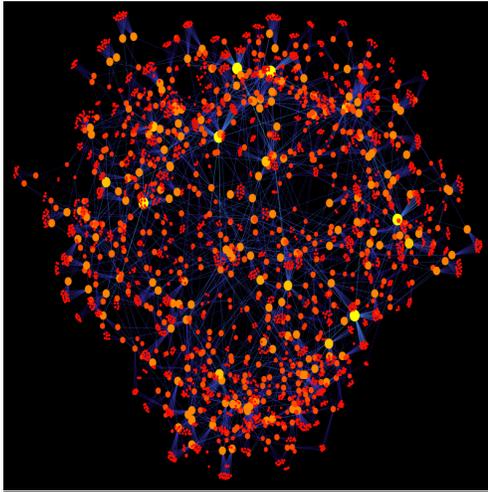
depend on the prime factorization of n at least for the n in which we can construct the graphs like $n \leq 25$.

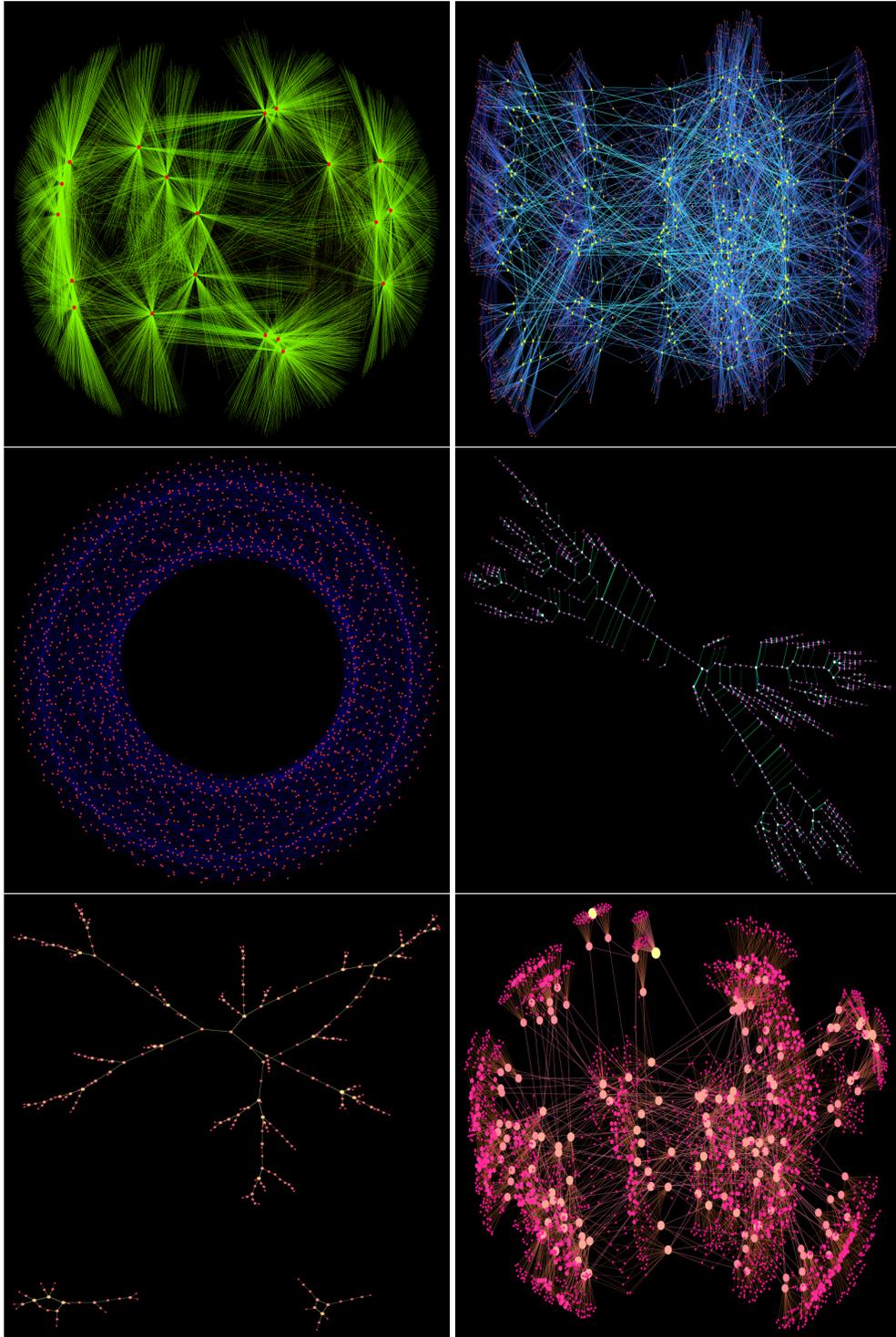
9. PICTURES

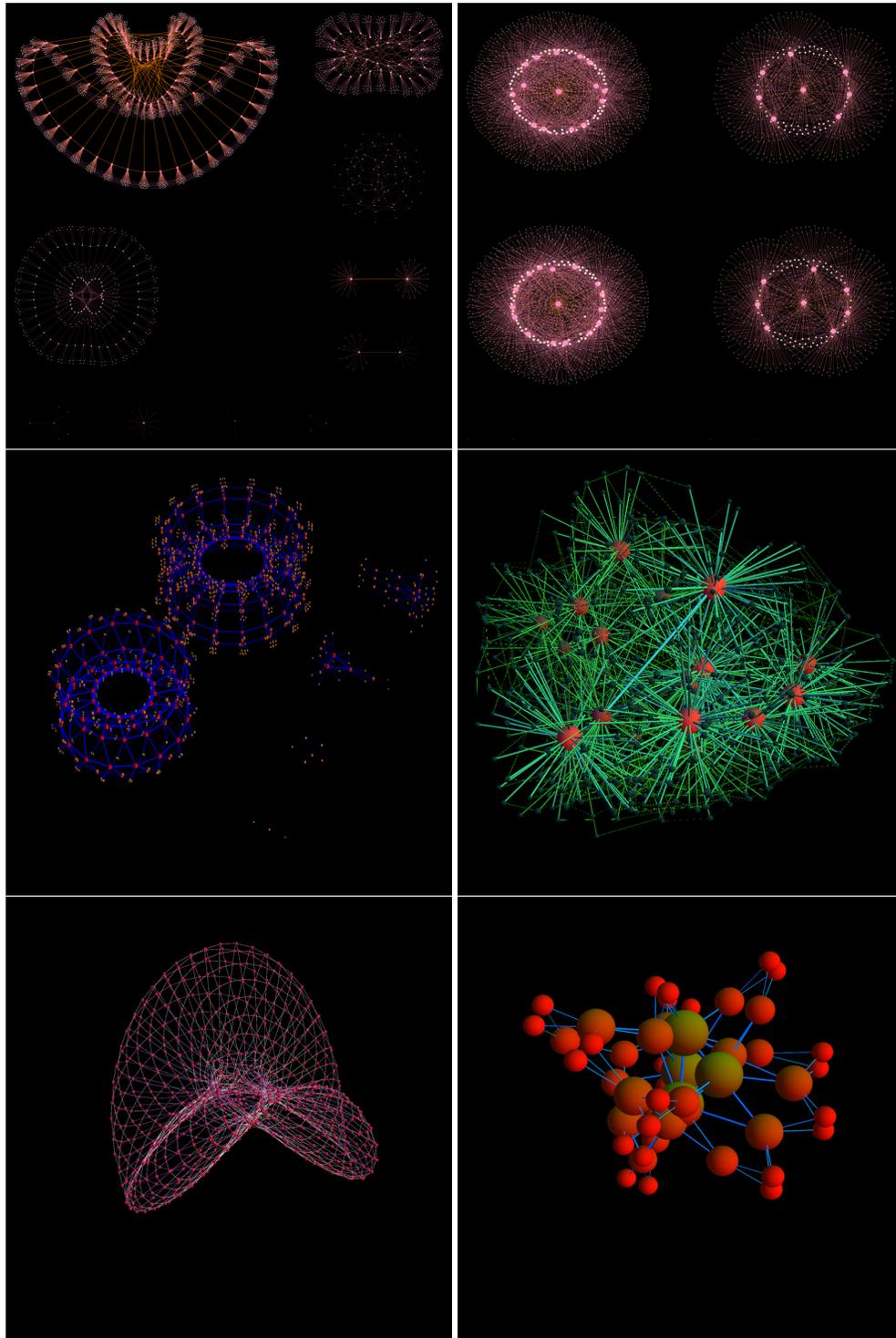
The following pictures have been produced with code we will submit to a demonstration project allowing readers to draw the graphs on their own. In the mean time the code is also available on the project website.

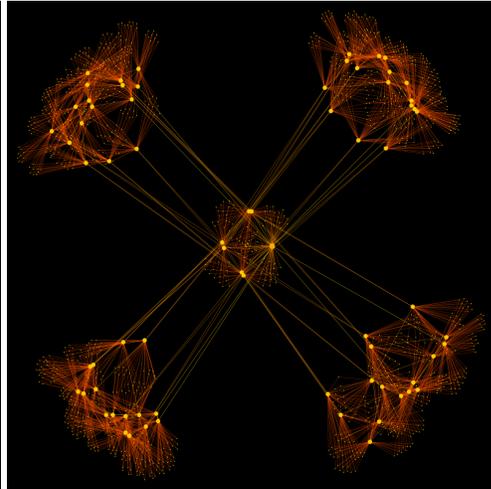
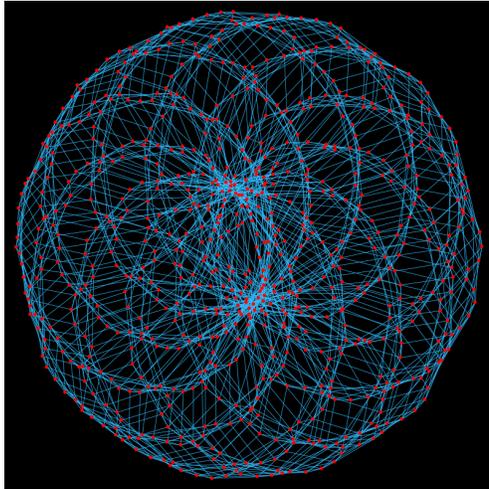
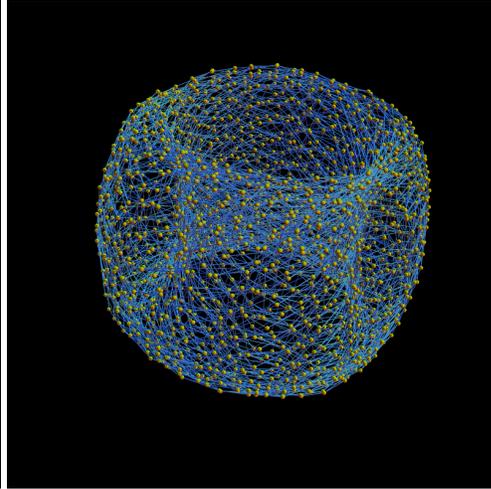
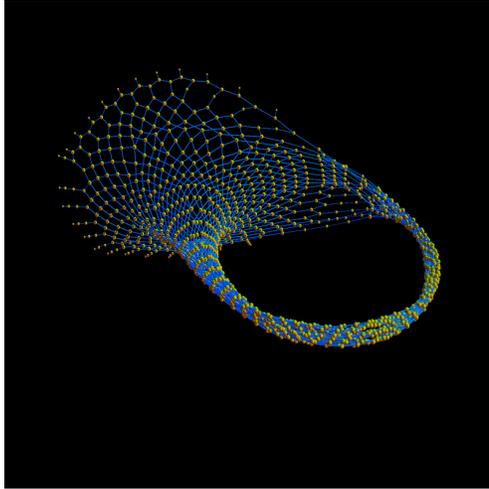
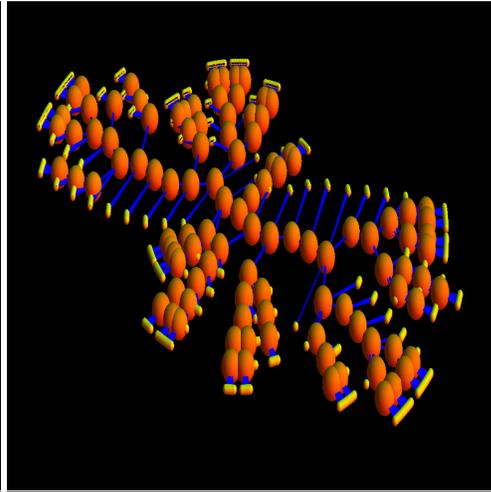
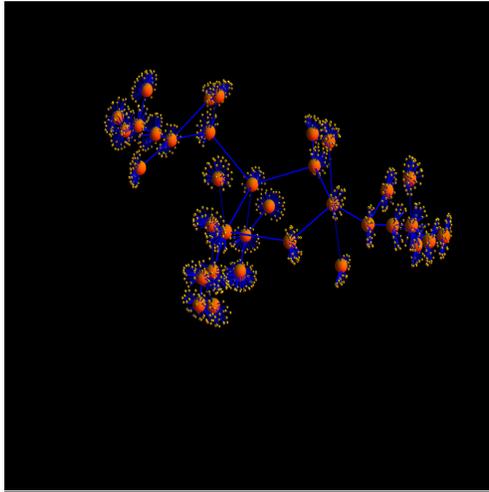


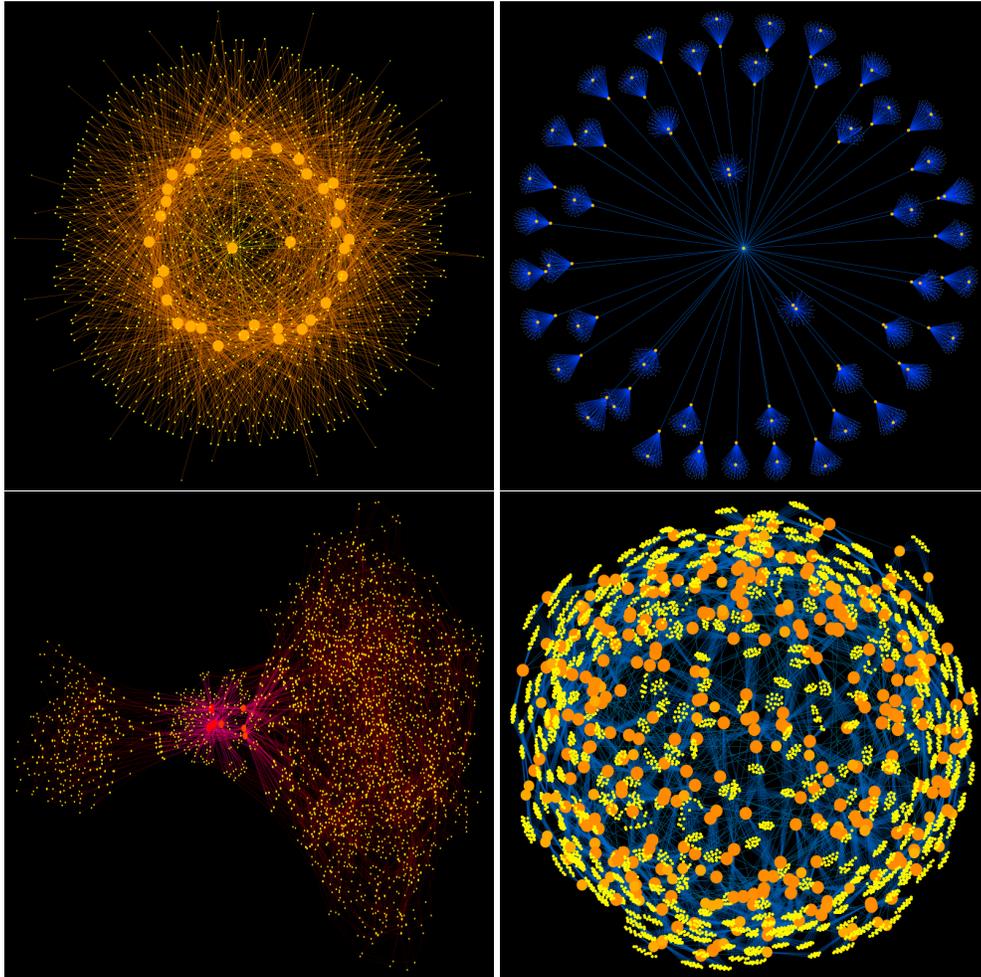












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DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA, 02138,
HARVARD UNIVERSITY