POSITIVE CURVATURE AND BOSONS

OLIVER KNILL

Abstract. Compact positive curvature Riemannian manifolds M with symmetry group G allow Conner-Kobayashi reductions M to N, where N is the fixed point set of the symmetry G. The set N is a union of smaller-dimensional totally geodesic positive curvature manifolds each with even co-dimension. By Berger, N is not empty. By Lefschetz, M and N have the same Euler characteristic. By Frankel, the sum of dimension of any two components in N is smaller than the dimension of M. Reverting the process N to M allows to build up positive curvature manifolds from smaller ones using **division algebras** and the geodesic flow. From dimension 6 to 24, only four exceptional manifolds have appeared so far, some of them being flag manifolds and related to the special unitary group in three dimensions. We can now draw a periodic system of elements of the known **even-dimensional positive curvature manifolds** and observe that the list of even-dimensional known positive curvature manifolds has an affinity with the list of known **force carriers in physics**. Positive mass of the boson matches up with the existence of of two linearly independent harmonic k-forms on the manifold. This motivates to compute more quantities of the exceptional positive curvature manifolds like the lowest non-zero eigenvalues of the **Hodge Laplacian** L=d d*+d* d or properties of the pairs (u,v) of harmonic 2,4 or 8 forms in the positive mass case.

1.1. Similarly as **representation theory** lets **Lie groups** act on linear spaces, one can see some Lie groups realized as symmetry groups acting as isometries on **compact even-dimensional positive curvature manifolds**. So far, in even dimensions, only spheres $\mathbb{S}^{2d}$, projective spaces of division algebras $\mathbb{RP}^{2d}, \mathbb{CP}^d, \mathbb{HP}^d, \mathbb{OP}^2$ as well as the four special manifolds W^6, E^6, W^{12}, W^{24} are known to admit positive curvature. All of them also admit a continuum isometry group.

1.2. The exceptional manifolds are the **Wallach manifolds** W^6, W^{12}, W^{24} of complete flags in three-dimensional [1] vector spaces $\mathbb{C}^3, \mathbb{H}^3, \mathbb{O}^3$ over the **division algebras** $\mathbb{C}, \mathbb{H}, \mathbb{O}$ and the **Eschenburg manifold** E^6 which is a twisted version of W^6 with same cohomology but different cohomology ring [46]. The manifold E^6 is a **bi-quotient**, where $(z_1, z_2) \in \mathbb{T}^2$ acts on $SU(3)$ as $g \to z_1 g z_2^{-1}$. These exceptional $2d$-manifolds differ from the **projective spaces** of division algebras in that some of their **inner Betti numbers** $b_k(M)$ are larger than 1.

1.3. An affinity to **particle physics** appears: the **force carrier bosons** W^+, W^-, Z^0, H (**vector gauge bosons** and the **scalar Higgs boson** H) which all have **positive mass** can be lined up with the exceptional positive curvature manifolds W^6, E^6, W^{12}, W^{24} with "heavier" **cohomology**, the projective spaces match with the rest: complex projective spaces with **photons**, quaternionic projective spaces with **gluons** and the **Moufang-Cayley plane** $\mathbb{OP}^2$ with the **graviton**, a particle which is not expected to be discoverable with current technology [38].

1.4. It appears that one can look at compact even-dimensional positive curvature manifolds M constructively, starting with the 0-manifold $1 = \mathbb{RP}^0$ and build them with successive extensions $N = N_1 \cup \cdots \cup N_k \to M$ using a few principles: **1. Conner-Kobayashi:** extensions must come from a Lie group symmetry. [8, 25], **2. Frobenius-Hurwitz:** extensions come from unit

Date: 06/28/2020, updated 7/7/2020.

[1] These are 2*3=6,4*3=12,8*3=24 dimensional real vector spaces.

spheres $Z_2, U(1), SU(2), S^7$ of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ [15, 22, 28] (Frobenius (1877) assumes an associativ division algebra and gives $\mathbb{R}, \mathbb{C}, \mathbb{H}$, Hurwitz (1923 posthumous) assumes normed division algebra and gives all 4), **3. Moufang-Cayley:** only one single S^7 extension is possible, the Moufang-Cayley plane is the last. [32, 40], **4. Frankel-Synge:** the fundamental group is $\mathbb{Z}_2$ or 0 and there is a dimension inequality from variational principles. [14, 42], **5. Euler-Lefschetz:** Euler characteristic is invariant under extensions. [26, 35]. A guiding principle is therefore to place the manifolds in the **dimension-Euler characteristic** plane which then has the structure of a **periodic system of elements**.

1.5. Writing $M_{\chi(M)}$, where $\chi(M)$ is the Euler characteristic, the list is $\mathbb{RP}^{2d}_1, \mathbb{SP}^{2d}_{d+1}, \mathbb{CP}^d_{d+1}, \mathbb{HP}^d_{d+1}$ for all d and $\mathbb{OP}^d_{d+1}$ for $d = 1, 2$, the Wallach manifolds [43] $\mathbb{W}^6_6, \mathbb{W}^{12}_6, \mathbb{W}^{24}_6$ (Euler characteristic 6 has been pointed out in [44] already), and the Eschenburg manifold [12] $\mathbb{E}^6_6$. See [46] for an overview or [41] for symmetry groups, which are surprisingly large. Both the W^{24} or the Moufang-Cayley plane $\mathbb{OP}^2$ have the symmetry of F_4 an exceptional simple Lie group [3].

1.6. The sequence of $\mathbb{CP}^d$ with $U(1)$ extensions correspond to **photons**, the $\mathbb{HP}^d$ with $SU(2)$ extensions correspond to **gluons**. While $\mathbb{O}^1 = \mathbb{S}^8$, the Moufang-Cayley plane $\mathbb{O}^2$ would correspond to the **graviton** G, a spin 2 particle. The $\mathbb{W}^6, \mathbb{E}^6, \mathbb{W}^{12}$ are constructed from $SU(3)$ and pair with the **vector bosons** W^+, W^-, Z^0. The $\mathbb{W}^{24}$ then remains to be paired with the **H Boson** [7]. The link of division algebras with fundamental physics had been suggested in 1934 by Jordan, von Neumann and Wigner [34, 3]. An obvious mismatch is that particle physics links $SU(3)$ with the strong and $SU(2)$ with the weak force. [2] On the other hand, all $\mathbb{CP}^d = SU(d+1)/SU(d)$ have a $SU(d+1)/U(1)$ symmetry and all $\mathbb{HP}^d = Sp(d+1)/(Sp(d) \times SU(2)) \rtimes \mathbb{Z}_2$ (semi-direct product) have $Sp(d+1)/\mathbb{Z}_2$ symmetry, expressed in the **quaternionic unitary group** $Sp(d)$ symmetry [41]. Having the **Cayley flag** W^{24} (the set of complete flags in $\mathbb{O}^3$) produces a natural link to $\mathbb{OP}^2$.

1.7. We do not know whether the currently known **"periodic system of elements"** for positive curvature manifolds is complete. The **standard model of particle physics** also provides a system of fundamental forces and their bosons. Also there, we do not know whether it is complete. In this note, we suggest a remarkable affinity between these two seemingly unrelated classification problems. This association forces to think about the nature of the fundamental forces in physics. This is not a new idea. Atiyah in a talk[2] speculated about the possibility of associating octonions with gravity. See also [19, 10, 4] and [9, 3, 16, 39].

1.8. There are areas in mathematics, where one has complete classifications of objects like for **finite simple groups**, or **Lie groups**, for **regular polyhedra** or for **division algebras**. Similarly, in physics, one has a complete classification of say crystal structures related to wallpaper groups, a classification of eigenvectors of the Hydrogen operator leading to the periodic system of elements in chemistry. There are also areas in mathematics, where a classification of objects is missing, like compact positive curvature manifolds, four manifolds or a classification of knots.

1.9. Symmetries are important in any classification. They are related to **conserved quantities** by Noether's theorem and to **variational principles** like for example in the isoperimetric inequalities. Classification and cross field associations always come with risks. One can also get "lost in math" [21]. Kepler's **Harmonices Mundi** of 1618 linked polyhedra with planets in the solar system, William Thomson (Baron Kelvin) imagined in 1870 **a vortex theory** modeling atoms using knots in the aether. More examples are in [30]. In the representation

[2] There is no consideration of dynamics with Fermions yet however.

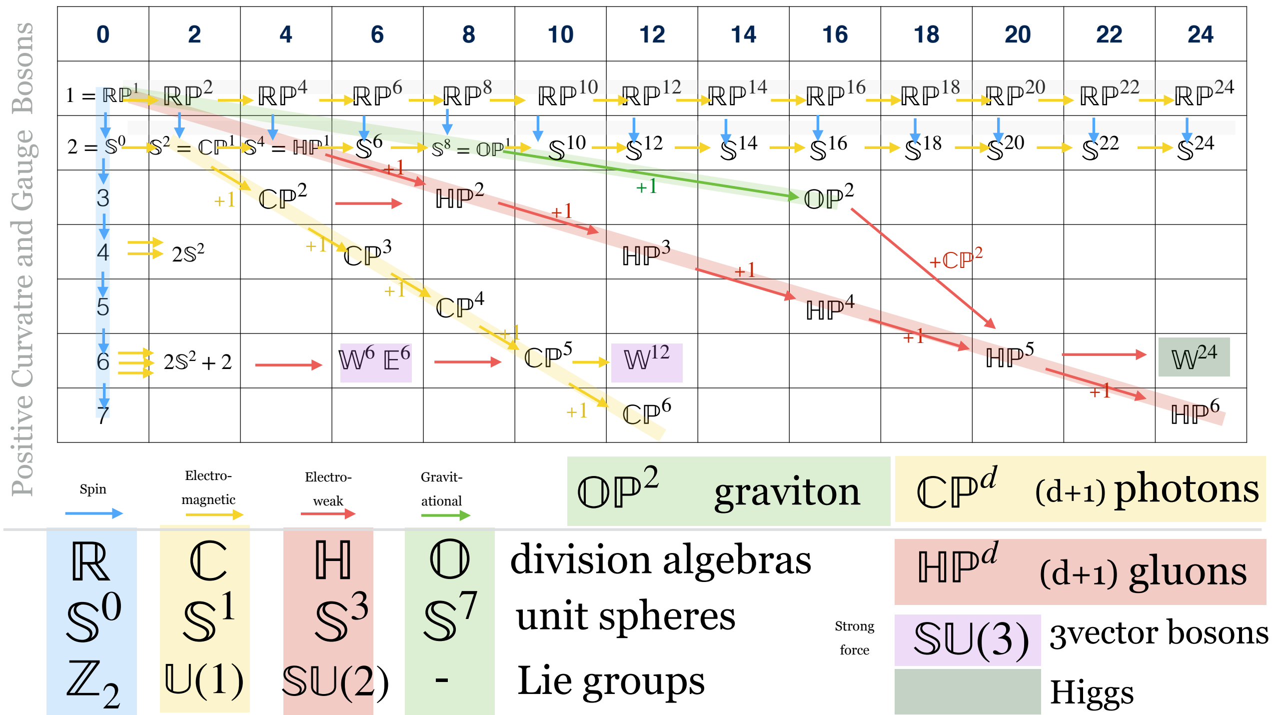

FIGURE 1. Periodic system of positive curvature manifolds.

theory of Lie groups, affinities with particle physics has worked out. See [11] for the history. Some pictures have not worked out like [17], seeing Baryons and mesons in the Rubik cube, or [23] for recognizing **Lepton and Hadron structures** in equivalence classes of primes over division algebras $\mathbb{C}, \mathbb{H}$.

1.10. The case of **real normed division algebras** $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ is even more remarkable than the finite list of **Platonic solids**. According to the **Hurwitz theorem**, the real complex numbers, quaternions and the octonions are the only real normed division algebras. The associated projective spaces are equivalence classes of lines in these algebras. The non-associativity of the octonions alters the projective construction. In the associative cases, there are infinite sequences $\mathbb{RP}^{2d}, \mathbb{CP}^d, \mathbb{HP}^d$ of even-dimensional projective spaces, coming from extensions $U(1), SU(2)$. The Moufang-Cayley plane $\mathbb{OP}^2$ however is the last octonionic one, the reason being non-associativity [27]. One can see Z_2-extension as a passage going from projective spaces to spheres. But this is only needed for $\mathbb{RP}^{2d}$, as $\mathbb{CP}^d, \mathbb{HP}^d, \mathbb{OP}^d$ are orientable and do not require a double cover. Universal covers can often linked to **spin** like in the cover $SU(2) \to SO(3)$.

1.11. All projective spaces over division algebras are compact manifolds all allowing for positive curvature metrics with symmetry. Similarly as in the classification of regular polytopes, there are cases with infinite families like the 1-dimensional polyhedra, regularity in the larger dimensional case as in dimension 4 and higher only the higher-dimensional tetrahedra (simplices), cubes (hypercubes) and octahedra (cross polytopes) exist, in dimension 3 and 4, a richer structure of two or three **exceptional polyhedra** exist. In the case of polyhedra, the structure is determined by combinatorial constraints like the **Gauss-Bonnet theorem**.

1.12. The known even-dimensional positive curvature manifolds are four infinite sequences: **spheres** and **projective spaces** $\mathbb{RP}^{2d}, \mathbb{SP}^{2d}, \mathbb{CP}^d, \mathbb{HP}^d$ then $\mathbb{O}^d$ for $d = 1, 2$ over the four division algebras. The "rare elements" are the **Wallach manifolds** $\mathbb{W}^6, \mathbb{W}^{12}, \mathbb{W}^{24}$ and the **Eschenburg manifold** $\mathbb{E}^6$. No other example has been found. It is also remarkable that all of these examples admit metric with symmetries. One can constrain the situation more and ask at least one component N to have co-dimension 2. This has been completely classified by Grove-Searle [18]. The co-dimension-4 case with $SU(2)$ symmetry adds $\mathbb{HP}^d$. An example of a co-dimension 6 case is $N = W^6$ for a $\mathbb{T}^1$ action on $M = W^{12}$. [3].

1.13. Why should positive curvature manifolds and physics be linked at all? Lie groups have to have a home somewhere. Representation theory lets them work on linear spaces. If we want to have them act on a compact space, then positive curvature is appealing as there is a **reduction theory** similarly as in representation theory. One can build up positive curvature manifolds M from smaller dimensional ones N which are fixed point sets of a group G action. It is nice that this reduction is always non-trivial because of **Berger's theorem** [5] assuring that the fixed point set is non-empty in even-dimensions. This is not the case for negative curvature, where by the **Cartan-Hadamard theorem**, there is no such theory. Still, both positive or negative curvature case could be relevant as the relativistic constant curvature models **de Sitter** and **anti-de Sitter** illustrates.

1.14. Many more questions remain beside the mathematical question to complete the list of positive curvature manifolds with or without symmetry. In order to find out whether the affinity between positive curvature manifolds and force carriers in physics is just a structural accident, one should try to link the **mass** $91.19\text{GeV}/c^2$ of the Z^0 boson [37] with properties of W^{12} or the mass $80.39\text{GeV}/c^2$ of $W^{\pm}$ bosons with the Wallach or Eschenburg manifold W^6, E^6 and the mass $125.35\text{GeV}/c^2$ of the Higgs boson with properties of W^{24}. Beside dimension and Euler characteristic, **volume**, **diameter** and Betti vectors, properties of harmonic forms or **spectra**, especially ground state energies offer themselves.

1.15. We have $b(W^6) = b(E^6) = (1,0,2,0,2,0,1)$ (with different cohomology ring) [46] or $b(W^{12}) = (1,0,0,0,2,0,0,0,2,0,0,0,1)$,
as well as $b(W^{24}) = (1,0,0,0,0,0,0,0,2,0,0,0,0,0,0,0,2,0,0,0,0,0,0,0,1)$. [4]. The $b_0 = b_{2d} = 1$ are not of interest in connected and orientable cases. The interior Betti numbers are associated with harmonic forms by **Hodge theory**. They can be represented by varieties or even manifolds with properties like volume or intersection numbers.

1.16. The picture appears that the geodesic flow on W^6 and E^6 are very similar, just introducing different oriented return maps, obtained by starting the geodesic flow on one sphere $\mathbb{S}^2$ of the symmetry and ending on the second. It is the only case where we see two positive dimensional components N_1, N_2 in N. By Frankel [14], $2+2 = \dim(N_1)+\dim(N_2) \leq \dim(M)-2 = 4$. This Frankel inequality is hard to meet with two positive dimension connected components in higher dimensions as it forces extensions $N \to M$ to go beyond the known list of division algebras. All even-dimensional positive curvature manifolds appear to admit metrics with **integrable geodesic flow**. This can be understood by looking at the return properties of the flow outside N and noticing that for every $x \in N$, every $y \in N$ is **conjugate** as the G-action produces families of geodesics.

[3]Personal communication Burkhard Wilking

[4]Personal communication Wolfgang Ziller

1.17. If M admits an isometry group G, by Hodge, the harmonic forms representing cohomology classes are G-invariant. They can live also on the fixed point set N. For the flag manifold $M = W^6$ admitting a $G = U(1)$ action, where $N = \mathbb{S}^2 + \mathbb{S}^2 + \mathbb{S}^0$, the two harmonic cohomology classes in the sector of differential 2-forms live on two copies of $\mathbb{S}^2$. They are represented by **algebraic cycles** which here are manifolds even. The flag manifolds, complete flags in $\mathbb{C}^3, \mathbb{H}^3$ and $\mathbb{O}^3$ have remarkable properties in Riemannian geometry and are under active investigaation (i.e.[13, 6])

1.18. How do we get from W^6 to W^{12}? We can think of passing over $\mathbb{CP}^5$ by first making an $G = SU(2)$ extension, then do a $G = U(1)$-extension. But the later is not possible because of Grove-Searle theorem [18] which forces for co-dimension 2 extensions $N \to M$ that N is a sphere or projective space. What happens is that the $U(1)$-symmetry on W^{12} directly gives the fixed point set $N = W^6$ or E^6. Also seeing $N = \mathbb{CP}^2 + \mathbb{CP}^2$ as a $G = U(1)$ fixed point set of $M = W^6$ is not possible by Frankel but despite that $N = \mathbb{S}^2 + \mathbb{RP}^0 \to M = \mathbb{CP}^2 + \mathbb{CP}^2$ is. But $N = \mathbb{S}^2 + \mathbb{S}^2 + \mathbb{S}^0$ can be a $SU(2)$ fixed point set of $M = W^6$. The extension fills $M - N$ by 4-spheres $\mathbb{S}^4$.

1.19. For W^6, E^6 we have two harmonic 2-forms and two harmonic 4-form, for W^{12} we have two harmonic 4 and two harmonic 8-forms and for W^{24} we have two harmonic 8 and two harmonic 16-forms. Their **cup products** could produce interesting numbers, when integrated over M. This still needs to be computed. Compare with the harmonic 2-forms of $\mathbb{T}^2$ with Betti vector $b(\mathbb{T}^2) = (1, 2, 1)$. The lengths of the two cycles depends on the metric but they always intersect in 2 points.

1.20. The association with mass appears because the gauge bosons, photon, gluon or graviton all have zero mass. The cohomology of projective spaces is different. For $\mathbb{RP}^{2d}$ it is always the Betti vector $b = (1, 0, 0, \dots)$. For $\mathbb{CP}^d$ it is $(1, 0, 1, 0 \dots, 0, 1, 0, 1)$ for $\mathbb{HP}^d$ it is $(1, 0, 0, 0, 1, 0, 0, \dots, 1, 0, 0, 0, 1)$ and for $\mathbb{OP}^1$ it is $(1, 0, 0, 0, 0, 0, 0, 0, 1)$ and for $\mathbb{OP}^2$ it is $(1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1)$. The concrete question therefore is whether M a Betti number 2 or more can be linked to **transport properties** and so with mass.

1.21. This could happen by some kind of interaction between the different cohomology classes of the same kind. The picture is not unfamilar when looking at **ground states** and especially **harmonic forms** have always been important in physics. Here, these forms appear as 2 and 4 forms u_2, v_2, u_4, v_6 in 6 manifolds W^6, E^6 and can define, an intersection forms $dV = u_2 \wedge v_6$. In the 12-manifold W^6 we can look at pairs $dV = u_4 \wedge v_8$ and in W^{24} at pairs $dV = u_8 \wedge v_{16}$. It would be good to be able to compare the numbers $\int_M dV$ with the masses of the particles.

1.22. For $M = W^6$ for example, one has two different homotopically nontrivial 2-spheres embedded in M and one can ask for the minimal volume or notions of distance which are topologically robust. This motivates to compute more quantitative numbers coming out of even-dimensional positive curvature manifolds. The **volumes of cup products of cohomology classes** could be computed using the heat flow e^{-Lt} using the Hodge Laplacian $L = (d + d^*)^2$ which keeps every space Λ_k of k-forms invariant. It would not surprise that the **spectra** of the L_k have a relation with the mass ratios. One would be able to "hear the mass".

1.23. Speaking about Laplacians, one always can look at the **spectrum of the Laplacians** L_k on k-forms. Of special interest are the **ground states**, the smallest non-zero eigenvalues in each k-sector. In the case of trivial cohomology, one can look at the smallest eigenvalue. By McKean-Singer [31], there is a symmetry between non-zero eigenvalues of even differential forms and non-zero eigenvalues of odd-dimensional differential forms. This fact could help to compute the ground states. It would be of extreme importance to know the lowest eigenvalues. Speaking about spectral invariants, also the **Zeta regularized determinants** are of interest. See e.g [33, 29].

1.24. An other source of rich interest are the **automorphism groups** of the manifolds. That is especially interesting for $\mathbb{OP}^2$ and $\mathbb{W}^{24}$, where the symmetry group is F_4 one of the exceptional Lie groups G_2, F_4, E_5, E_7, E_8. This Lie group F_4 has dimension 26. So, when looking for any numbers which make up the mass, one should also look at properties of these Lie groups. Rather than doing numerology, it would be better to see whether there are physical reasons why these manifolds at all should appear in transport.

1.25. One has not observed a single example of an even-dimensional compact positive curvature manifold, where not some metric with a symmetry exists. A continuum symmetry forces geodesics to come in families away from the fixed point set N (a picture familiar in string theory, where particles are loops, but where also the variational problem changes from arc length to area [1]). The presence of a symmetry in a positive curvature manifold M always has a non-empty fixed point set N by a theorem of Berger [5] which can be seen geometrically: a geodesic segment in M away from a fixed point set N longer than the diameter of M, because of the focusing nature of the geodesic flow acting on families of geodesics governed by the **Jacobi fields**.

1.26. It is intriguing that with a G-symmetry, $G = U(1), SU(2)$ any pair of **fat geodesics** $G\gamma_1, G\gamma_2$ intersecting outside N must outside N intersect in sets Gx which are topologically robust meaning that if we change some initial point or velocity of a geodesic, this number does not change. The symmetry renders the intersection points to be either circles $U(1)$ or 3-spheres $SU(2)$. One can study the interaction between such tubes. If two such sets $G\gamma_1, G\gamma_2$ intersect, they intersect in a sphere Gx. The number of such intersection points for geodesics starting and ending in N is an integer **intersection number** that is topologically stable.

1.27. The structure of **odd-dimensional positive curvature manifolds** with symmetry is a bit more complicated [45]. We first have the **space forms** $\mathbb{S}^{2d+1}_m = \mathbb{S}^{2d+1}/\mathbb{Z}_m$ which include projective spaces $\mathbb{RP}^{2d+1} = \mathbb{S}^{2d+1}/\mathbb{Z}_2$. The analog of W^6 are the **Aloff-Wallach spaces** $W^7_{p,q}$, which generalize the **Berger space** B^7. The analogue of E^6 are the Eschenburg spaces $E_{k,l}$. The analog of W^{12} are the **Bazaikin spaces** B^{13}_q which generalize the **Berger space** B^{13}. There appears no analog of W^{24}. Since the Euler characteristic is zero for all odd-dimensional manifolds, one needs to organize therm using the fundamental group or look at Lefschetz numbers for some symmetry.

1.28. Whether also affinities of Fermions with $(2d+1)$-manifolds of positive curvature can be drown is not at all clear. Unlike in even dimensions, where all known positive curvature manifolds have also symmetry, in odd dimensions, there are also positive curvature space forms without continuum symmetry. In odd dimensions, we also can have discrete symmetries G which have maps $T : M \to M$ as generators. In that case, one can look at the **Lefschetz number** $\chi(M,T)$ which is the super trace of the linear map induced on cohomology. It is equal to $\chi(N)$ in general and $\chi(M)$ if T is the identity. For $M = \mathbb{S}^{2d+1}$ and a reflection $T : x \to -x$ for example, $\chi(M,T) = 2$ and this is $\chi(N) = 2$, the manifold $N = \mathbb{S}^{2d}$ fixed by T.

1.29. Is it possible to match up the Leptons "electrons" and "neutrini" or the six quarks **up**, **down**, **strange**, **charm**, **top**,**bottom** with positive curvature $(2d+1)$-manifolds, where the later are possibly equipped with some discrete symmetry $T : M \to M$? Inspired from the Harmonices Mundi analogy given by Kepler, one can note that **space form** symmetries

[45] can be **"tetrahedral"**, **"octahedral-cube"** like or **"icosahedral-dodecahedron"** like. A most naive analogy would try to match symmetry types with **three quark and lepton generations**. Matching the three possible platonic solid types in dimension larger than 4 corresponding to 3 generations of fundamental particles is in the Kepler spirit.

1.30. Using a discrete isometric symmetry $T : M \to M$ one can again place the objects in a **dimension-Lefschetz number plane** as Euler characteristic $\chi(M) = \chi(M, Id)$ is always zero in odd-dimensions if Poincaré duality holds, but the **Lefschetz number** $\chi(M, T)$ is more informative. The non-orientable $\mathbb{RP}^{2d+1}$ with $\chi(\mathbb{RP}^n) = 1$ independent of dimension suggests a match-up of real projective spaces with **neutrini**. And the **dynamical spheres** $\mathbb{S}^{2d+1}, T(x) = -x$ associate with **electron-positron** pairs. That would take care of the Leptons. The **quarks** should then have to be matched up with suitable pairs (M, T) the **Aloff-Wallach-Berger** and **Bezaikin-Berger spaces** M equipped with some discrete symmetry T. That no 25-manifold analog of W^{24} has been detected in math is compatible with the absence of a Fermion like analog of the Higgs boson.

1.31. The discrete symmetry also allows to associate some mass to electrons and neutrons if one puts some symmetry on it. Euler characteristic can in nature be tied with energy [24]. Now, by taking space forms, the Euler characteristic can become negative. One might associate odd cohomology $b_{2k+1} > 1$ with negative mass like anti-particles. For $2d$-manifolds, the statement $b_{2k+1} = 0$ for all k is stronger than the **Hopf conjectures** [20]. Taking quotients changes Betti numbers according to spectral sequences: for $SU(3)$ for example one has $b(SU(3)) = (1, 0, 0, 1, 0, 1, 0, 0, 1)$ with $\chi(SU(2)) = 0$ while for $W^6 = SU(3)/\mathbb{T}^2$ one has $b(W^6) = (1, 0, 2, 0, 2, 0, 1)$ with $\chi(W^6) = 6$.

1.32. The concept of "Mass" is still not well understood. There are mechanisms for particles to acquire mass via the Higgs field, but the **neutrino mass** is not believed to come from a Higgs mechanism. Indeed, positive neutrino mass is placed beyond the standard model. The cohomological interpretation with associating mass with higher Betti numbers would allow to get mass by factoring out discrete symmetries or by using fibrations. The change of flavor of neutrini would have to mean a **symmetry change**. Discrete symmetry changes can be explained already by **harmonic analysis**. Waves are superposition of an even and an odd wave for example. A function on a sphere is a superposition of **spherical harmonics** which can have different symmetries. For $2d$-manifolds of positive curvature, there is no interesting discrete symmetry beside $\mathbb{Z}_2$ which allows for **charge** but not for **flavors**: gauge bosons do not come in generations. This changes in odd dimensions: already on the circle, Fourier series shows different type of symmetries of the eigenfunctions e^{inx}. The richer varieties of space forms in odd dimensions correlate with a richer variety of Fermions.

1.33. The analogy drawn urges to compute the **numerical values** coming from even and odd-dimensional positive curvature manifolds. We do not even know all the Betti numbers $b_k(M)$ of all odd-dimensional M and Lefschetz numbers $\chi(M, T)$ for discrete transformations. We have not seen any properties of the non-trivial cohomology classes, especially in the Wallach space W^6, W^{12}, W^{24} case. We miss **quantitative numbers** like **ground state energies** of the Hodge Laplacian of positive curvature manifolds for all k-forms as well as volumes of cup products of cohomology classes. Computing these numbers and seeing them not match up with masses is a way to find out to **falsify the analogy** [36]. We can compute a basis of the cohomology group for any simplicial complexes with a few lines of code, but for simplicial complexes which come from 6, 12 or 24- dimensional manifolds, we can not even store the incidence matrices corresponding to the exterior derivatives. It needs new ideas to compute those.

References

[1] S. Albeverio, J. Jost, S. Paycha, and S. Scarlatti. *A mathematical introduction to string theory*, volume 225 of *Lecture Note Series*. London Mathematical Society, 1997.

[2] M. Atiyah. From algebraic geometry to physics - a personal perspective. Talk given at Simons Center in Stony Brook, 2010.

[3] J.C. Baez. The octonions. *Bull. Amer. Math. Soc. (N.S.)*, 39(2):145–205, 2002.

[4] J.C. Baez. Division algebras and quantum theory. *Found. Phys.*, 42(7):819–855, 2012.

[5] M. Berger. Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 15:179–246, 1961.

[6] R.G. Bettiol and R.A.E. Mendes. Flag manifolds with strongly positive curvature. *Mathematische Zeitschrift*, 280:1031–1046, 2015.

[7] B. Duplantier C. Bachas and V. Rivasseau. *The H Boson*, volume 72 of *Progress in mathematical physics*. Birkhäuser, 2017.

[8] P.E. Conner. On the action of the circle group. *Mich. Math. J.*, 4:241–247, 1957.

[9] J.H. Conway and D.A. Smith. *On Quaternions and Octonions*. A.K. Peters, 2003.

[10] G.M. Dixon. *Division algebras: octonions, quaternions, complex numbers and the algebraic design of physics*, volume 290 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1994.

[11] H.G. Dosch. *Beyond the Nanoworld, Quarks, Leptons and Gauge Bosons*. A.K. Peters, Ltd, 2008.

[12] J.-H. Eschenburg. New examples of manifolds with strictly positive curvature. *Invent. Math*, 66:469–480, 1982.

[13] J.-H. Eschenburg. Cohomology of biquotient. *Manuscripta Math.*, 75:151–166, 1992.

[14] T. Frankel. Manifolds with positive curvature. *Pacific J. Math.*, 1:165–174, 1961.

[15] G. Frobenius. Über lineare Substitutionen und bilineare Formen. *J. Reine Angew. Math.*, 84:1–63, 1878.

[16] G. Furey. *Standard model physics from an algebra*. University of Waterloo thesis, 2015. https://arxiv.org/abs/1611.09182.

[17] S.W. Golomb. Rubik's cube and quarks: Twists on the eight corner cells of rubik's cube provide a model for many aspects of quark behavior. *American Scientist*, 70:257–259, 1982.

[18] K. Grove and K. Searle. Positively curved manifolds with maximal symmetry-rank. *J. of Pure and Applied Algebra.*, 91:137–142, 1994.

[19] F. Gürsey and C-H. Tze. *On the role of division, Jordan and related algebras in particle physics*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[20] H. Hopf. Differentialgeometrie und Topologische Gestalt. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 41:209–228, 1932.

[21] S. Hossenfelder. *Lost in Math*. Basic Books, 2018.

[22] A. Hurwitz. Über die Komposition der quadratischen Formen. *Math. Ann.*, 88(1-2):1–25, 1922.

[23] O. Knill. On particles and primes. https://arxiv.org/abs/1608.07175, 2016.

[24] O. Knill. The energy of a simplicial complex. *Linear Algebra and its Applications*, 600:96–129, 2020.

[25] S. Kobayashi. Fixed points of isometries. *Nagoya Math. J.*, 13:63–68, 1958.

[26] S. Kobayashi. *Transformation groups in Differential Geometry*. Springer, 1972.

[27] M. Lackmann. The octonionic projective plane. https://arxiv.org/abs/1909.07047, 2019.

[28] T.Y. Lam. *A first course in noncommutative rings*. Springer Verlag, 1991.

[29] M.L. Lapidus and M.van Frankenhuijsen. *Fractal Geometry, Complex Dimensions and Zeta Functions*. Springer, 2006.

[30] M. Livio. *Brilliant Blunders: From Darwin to Einstein - Colossal Mistakes by Great Scientists That Changed Our Understanding of Life and the Universe*. Simon and Schuster, 2013.

[31] H.P. McKean and I.M. Singer. Curvature and the eigenvalues of the Laplacian. *J. Differential Geometry*, 1(1):43–69, 1967.

[32] R. Moufang. Alternativkörper und der Satz vom vollständigen Vierseit (D_9). *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität*, 9:207–222, 1933.

[33] B. Osgood, R.Phillips, and P.Sarnak. Extremals of determinants of Laplacians. *J. Funct. Anal.*, 79:148–211, 1988.

[34] J.v. Neumann P. Jordan and E. Wigner. On an algebraic generalization of the quantum mechanical formalism. *Annals of Mathematics, Second Series*, 35:29–64, 1934.

[35] P. Petersen. *Riemannian Geometry*. Springer Verlag, third and second 2006 edition, 2016.

[36] K. Popper. Falsifizierbarkeit, zwei Bedeutungen. In H. Seiffert and G. Radnitzky, editors, *Handlexikon der Wissenschaftstheorie*. Deutscher Taschenbuch Verlag, 1989.

[37] C. Verzegnassi R. Tenchini. *The Physics of the Z and W Bosons*. World Scientific, 2008.

[38] T. Rothman and S. Boughn. Can gravitons be detected? *Foundations of Physics*, 36:1801–1825, 2006.

[39] P. Rowlands and S. Rowlands. Are octonions necessary to the standard model? *J. of Physics, Conf. Series*, 1251, 2019.

[40] H. Salzmann, D. Betten, T. Grundhoefer, H. Haehl, R. Loewen, and M. Stroppel. *Compact Projective Planes: With an Introduction to Octonion Geometry*, volume 21 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter, 1995.

[41] K. Shankar. *Isometry groups of homogeneous spaces with positive sectional curvature*. 1999. Dissertation University of Maryland.

[42] J.L. Synge. On the connectivity of spaces of positive curvature. *Quarterly Journal of Mathematics*, 7:316–320, 1936.

[43] N.R. Wallach. Compact homogeneous Riemannian manifolds with strictly positive curvature. *Annals of Mathematics, Second Series*, 96:277–295, 1972.

[44] N.R. Wallach. Three new examples of compact manifolds admitting riemannian structures of positive curvature. *Bulletin of the AMS*, 78:55–56, 1972.

[45] J.A. Wolf. *Spaces of Constant Curvature*. AMS Chelsea Publishing, sixth edition, 2011.

[46] W. Ziller. Riemannian manifolds with positive sectional curvature. In *Geometry of Manifolds with Non-negative Sectional Curvature*. Springer, 2014. Lecture given in Guanajuato of 2010.

Department of Mathematics, Harvard University, Cambridge, MA, 02138