ON THE COHOMOLOGY OF MEASURABLE SETS

OLIVER KNILL

Abstract. If T is an ergodic automorphism of a Lebesgue probability space $(X, \mathcal{A}, m)$, the set of coboundries $\mathcal{B} = dB = T(B) + B$ with symmetric difference $+$ form a subgroup of the set of cocycles $\mathcal{A}$. Using tools from descriptive set theory, Greg Hjorth showed in 1995 that the first cohomology group $\mathcal{H} = \mathcal{A}/\mathcal{B}$ is uncountable. This can surprise, given that in the case of a finite ergodic probability space, one has $\mathcal{H} = \mathbb{Z}_2$. Hjorth's proof used descriptive set theory in the complete metric space $(\mathcal{A}, d(A, B) = m(A + B))$, leading to the statement that $\mathcal{B}$ is meager in $\mathcal{A}$. We use a spectral genericity result of Barry Simon to establish the same. It leads to the statement noted first by Karl Petersen in 1973 that for a generic A, the induced system T_A is weakly mixing, which is slightly stronger than a result of Nate Friedman and Donald Ornstein about density of weakly mixing in the space of all induced systems T_A coming from an ergodic automorphism T.

1. Cohomology of measurable sets

1.1. This is a remark on a spectral and cohomological aspect in classical ergodic theory [15, 18, 9, 11, 43, 36, 10, 35, 21]. A **measure preserving invertible** map T of a standard non-atomic **Lebesgue probability space** $(X, \mathcal{A}, m)$ defines an automorphism of the σ-algebra $\mathcal{A}$ which preserves measure $m(T(A)) = m(A)$ for $A \in \mathcal{A}$. The map T is called **ergodic** if $0 = \emptyset$ and X are the only fixed points of T. The automorphism T defines a group translation on the additive Abelian and Boolean group $(\mathcal{A}, +)$. The set of **coboundaries** $\mathcal{B} = \{dB = B(T) + B = B(T) - B\}$ form a subgroup of $\mathcal{A}$. If one thinks of $\mathcal{A}$ as **cocycles**, then $\mathcal{H} = \mathcal{A}/\mathcal{B}$, the set of cocycles modulo coboundaries is the **first cohomology group** of the dynamical system with coefficient group $\mathbb{Z}_2$. This coefficient group has first been considered in [26] in the context of representation theory. Both $\mathcal{A}$ and $\mathcal{B}$ share the cardinality of the continuum so that the cardinality of $\mathcal{H}$ is not obvious. With the metric $d(A, B) = m(A + B)$, the set $\mathcal{A}$ becomes a **Polish space**, a separable, metrizable and complete topological group. The standard assumption is always that $d(A, B) = 0$ is equivalent to $A = B$. Sets A, B which are the same up to a set of measure zero are identified.

1.2. We assume that the probability space is **non-atomic** because if T induces a permutation of a finite set X, then $\mathcal{H}^1 = \mathcal{Z}_2^k$, where k is the number of ergodic components. Ergodicity is assumed because one can reduce the problem in the non-ergodic case by ergodic decomposition [45, 11]. The assumption to have a Lebesgue space is important but not really much of a restriction: any Borel σ-algebra of a complete separable metric space (X, d) and so virtually any probability space that appears in applications is a Lebesgue space. Without the Lebesgue assumption, the cohomology can be trivial, as was first pointed out in [1], answering a question of Paul Halmos.

1.3. The construction of concrete non-coboundaries is in general not easy. Under some circumstances,it is possible to give non-coboundaries: for example, if T^2 is ergodic, then the full space X itself is not a coboundary. This is important for example because the **set complement operation** $A \to A + X$ preserves coboundaries if and only if X is a coboundary.

Date: 7/23/2023.

1.4. If we have ergodicity for T, T^2, then X is not a coboundary and any A which tiles the space using transformations commuting with T can not be a coboundary. Therefore, if X is a compact Abelian group with Haar measure m and Y is a finite subgroup, then X/Y defines a fundamental domain A which satisfies $X = \{y + A, y \in Y\}$. If T is an ergodic translation on X such that T^2 is ergodic, then A can not be a coboundary. In the case $X = \mathbb{T}^d$ with irrational rotation for example, there is a countable set of explicit non-coboundaries. For $T(x) = x + \alpha$ on $\mathbb{T}^d$, every non-empty union of rational rectangles $\prod_{j=1}^{d}[k_j/n, (k_j + 1)/n]$ can not be a coboundary.

1.5. It is an interesting question to inquire about the structure of the group $\mathcal{H}$. Are there some interesting invariants which one can distill from $\mathcal{H}$ that allow to distinguish dynamical systems? This is not so clear. $\mathcal{A}/\mathcal{B}$ is a topological group again it is not complete. As it is Boolean and $A + A = 0$, it only contains finite subgroups of order $\mathbb{Z}_2^k$. We asked in [29] about the cardinality of $\mathcal{H}$ and whether this could depend on the dynamical system T. Hjorth [1] showed me the proof of the following result:

Theorem 1. *For any ergodic automorphism T on a Lebesgue space, $\mathcal{H}$ is uncountable.*

1.6. As a corollary, one knows in more generality that $\mathcal{H}^1(T, U)$ is uncountable, if U is a compact topological group with normal subgroup $\mathbb{Z}_2$. One can either take the proof and generalize it or then write an element as $B = \pm k$ with $k \in K = U/\mathbb{Z}_2$. If A is a $\mathbb{Z}_2$ cocycle and $A = B(T)B^{-1}$ then B can be chosen to be $\mathbb{Z}_2$-valued. It follows that $\mathcal{H}^1(T, U)$ is uncountable.

1.7. The proof of Hjorth already used a Baire category argument. Here is a new proof using a result developed in the context of mathematical physics:

Proof. Because of the Lebesgue assumption, one can assume that $X = [0, 1]$ and $m = dx$ is the Lebesgue measure. We can associate to very $A \in \mathcal{A}$ of positive measure the induced probability measure $\mu(A) = 1_A/m(A)$. Its cumulative distribution function $F_{\mu A}(x) = \mu(A)[0, x]$ allows to rewrite the dynamics of the induced map T_A on $[0, 1]$. One has now a unitary Koopman operator $U_A : f \to f(T)$ on $L^2([0, 1], dx)$ for every A [33]. The map $A \to U_A$ is continuous if one takes the strong operator topology on the space of unitary operators. By a result of Friedman-Ornstein the set $\{A \mid T_A \text{ is weakly mixing }\}$ (even mixing) is dense in the topological group $\mathcal{A}$. But this means it is dense in the corresponding set $\mathcal{U} = \{U_{T_A}, A \in \mathcal{A}\} \subset L^2([0, 1], dx)$ of unitary operators, which is again a compact topological space in the induced metric. But since this metric is stronger than the strong operator topology the set $\mathcal{U}$ is also closed in the strong operator topology. By the Simon Wonderland theorem [49], the set of operators with no point spectrum on the orthogonal complement of constant functions must be residual. □

1.8. For the spectral theory of unitary operators, see also [7], about genericity, see [38]. As a comparison, here is Hjorth's proof (as transcribed in [30], any misunderstanding of course would be my mistake):

Proof. T defines a continuous map on the Polish group $\mathcal{A}$. Coboundaries are a continuous image of the automorphism $f : A \to A(T)A^{-1}$ which has he kernel $\{-1, 1\}$. So, coboundaries are the injective image of the automorphism $\mathcal{A}/\ker(f)$. By the theorem of Lusin-Souslin (Theorem 15.1 in [25]), it is a Borel set and so has the Baire property. The set $\mathcal{C} = im(f)$ is not open and therefore meager by a theorem of Pettis (Theorem 9.9 in [25]). The equivalence relation on $\mathcal{A}$ given by $A \sim B$ is $AC = BC$ is meager [13]. A Theorem of Mycielski (Theorem 19.1 [25]) implies that there are uncountably many equivalence classes. □

[1] Gred Hjorth (1963-2011) was an instructor at Caltech 1994-1995

1.9. The following corollary of the proof is a remark of Karl Petersen in 1973 [42]. The density of weakly mixing has been a result of Friedman-Ornstein [15]. We can upgrade density to residual:

Corollary 1. *The set of A for which T_A is weakly mixing is residual.*

Proof. Lemma 3.2 in [28] notes that T_A^2 is ergodic if and only if A is not a coboundary. Having no pure point spectrum on the ortho-complement of the constant functions, is equivalent to weak mixing. The set of sets A for which the induced map T_A is weakly mixing, is residual. So, for a generic measurable set A of positive measure, the induced map T_A is ergodic. Because weakly mixing is residual and weakly mixing implies ergodic, this especially means also that T_A is again ergodic for generic A if T was ergodic. $\square$

1.10. Let us also reprove also one of the first genericity results [3]:

Corollary 2 (Anzai)**.** *The set of A for which $T \times A$ is weakly mixing is residual.*

Proof. Anatoly Stepin [54] showed that $T \times A$ is ergodic on $X \times \mathbb{Z}_2$ if and only if A is not a coboundary. $\square$

2. Remarks

2.1. William Veech asked whether if T is an irrational rotation of the circle, the set of $A \in \mathcal{A}$ for which T_A has pure point spectrum is dense (see [42] p. 229 and [8]). If that were the case, then by the Wonderland theorem, we would have a generic set of A such that T_A has **purely singular continuous spectrum**. We mentioned this Veech question in an earlier version [27] to [31], a document which focused on genericity alone while [31] also contained quantitative mixing scales. The question of Veech can be asked for any T with pure point spectrum. We expect in this case that for a generic A the system T_A has purely singular continuous spectrum. The interest of Veech in 1969 was in the context of Sturmean sequences. If $A = [0, \beta]$ one gets a new dynamical system T_A from the irrational rotation $T(x) = x + \alpha$ mod 1.

2.2. The definition of an induced system was given by Kakutani in 1943 [50, 42]. It is a measurable analog of a Poincaré return map. The induced system T_A can upgrade ergodic to weakly mixing and so tends to make the dynamics more chaotic. If the **Kolmogorov-Sinai entropy** $h(T)$ (introduced in 1958 by Kolmogorov [46] and refined by Sinai later) of T is positive, then the entropy $h(T_A) = h(T)/m(A)$ of T_A is larger **formula of Abramov**. The case of Bernoulli shifts shows that T_A can be mixing. A theorem of Sinai assures that that every T_A has a Bernoulli shift as a factor. By a result of Rohlin, there is then some absolutely continuous spectrum of infinite multiplicity. Already Rohlin knew however that the set of transformations with pure absolutely continuous spectrum is meager both in the strong and weak topology of measure preserving trasformations.

2.3. To illustrate the difficulty of cocycles and coboundaries, here is a challenge: assume we look at the Arnold cat map $T(x, y) = (2x + y, x + y)$ on $X = \mathbb{T}^2$ which is measure preserving and hyperbolic. Because also T^2 is ergodic, the entire torus X is not a coboundary. Is $Y = [0, 1/2] \times [0, 1/2]$ a coboundary or not? More generally, which of the sets $Y_{a,b} = [0, a] \times [0, b]$ are coboundaries? A measurable set Z such that $T(Z) + Z$ is $Y_{a,b}$ is expected to be very complicated.

2.4. If T is a permutation of a finite set X, a subset A can serve as a jump switch to toggle the two branches. If T on X has one cycle then the doubled system either has one or two cycles. If A has an even number of elements, then we have two branches. If A is odd, we go round and reach after n steps the other side meaning that we cover all of the two branches.

2.5. Anatoly Stepin [54] first showed that A is not a coboundary if and only if the cocycle action $T \times A$ is ergodic on $X \times \mathbb{Z}_2$. We used this to show that switching stable and unstable directions of a skew flow does not necessarily kill Lyapunov exponents. We proved in [28] that if we had an oracle that determines the Lyapunov exponents of $SL(2,\mathbb{R})$ cocycles, then we could decide whether a set is a coboundary or not. The problem of deciding whether $Sl(2,\mathbb{R})$ cocycle Lyapunov exponents are positive or not is at least as hard as the coboundary problem.

2.6. A *conjecture of Kirillov* asks whether two automorphisms T, S with the same coboundaries are isomorphic. In the finite periodic case, Kirillov's question is negative because conjugation requires that the orbits of the cycles have the same length. We are not aware that the Kirillov question has been answered. The question can even be asked in subclasses. If two irrational rotations $x \to x+\alpha, x \to x+\beta$ have the same coboundaries. Is $\alpha = \beta$?

2.7. Related to the cohomology problem is a more general question in group theory which goes back to Halmos. Let G be an arbitrary Abelian group and T a group automorphism. One has then the **Halmos cohomology group** $H(G,T) = G/dG$, where $dG = \{g(T)g^{-1} \mid g \in \mathcal{G}\}$. One can then define $H^k(G,T) = d^{k-1}G/d^kG$ for any $k \in \mathbb{Z}$. I wondered once whether $H^k(G,T) = H^{-k}(G,T)$ but has been answered negatively by an example of Ashbacher.

2.8. The thesis of Jerome Depauw [12] looked at the second cohomology groups [2]. Such higher cohomology groups were defined independently also in [29], as well as in work of Katok. The idea is simple: replace the usual partial derivatives $\partial_j f$ of a function of several variables with the dynamical derivative $f(T_j) - f$. Let us only look at the two-dimensional case $d = 2$, where S, T be two commuting ergodic measure preserving transformations of the probability space. The standard assumption is that the action is **free** meaning that the set of fixed points of S^nT^m are of measure zero if $(n,m) \neq (0,0)$ Cocycles are vector fields in $\mathcal{A}$ for which the curl is zero. This means we have pairs (P,Q) in the σ-algebra such that the curl, the line integral along a basic plaquette in $\mathbb{Z}^2$ is zero. This means $Q_x - P_y = P + Q(S) + P(T) + Q = 0$. Coboundaries $\mathcal{B}$ are sets of the form $(P,Q) = [C(S)+C, C(T)+C]$. The curl of the gradient is zero. The first cohomology group are the equivalence classes of all 2-forms modulo all 1-forms which are coboundaries. This can be generalized to cohomology groups $\mathcal{H}^p(T,U)$ with other coefficient groups $U = \mathbb{Z}_2$. There is no interesting higher cohomology by a theory of Feldman and Moore [14]. In the 2-dimensional case, this has also been treated by Depauw [12]. For $d \geq 2$, the group $H^d(T,U)$ is trivial if T is a free and U is an Abelian Polish group.

2.9. There are no cohomological constraints in higher dimensions. The De Rham type cohomology is equivalent to a simplicial type group cohomology of Eilenberg-McLane and the cohomology of ergodic equivalence relations, as developed Feldman and Moore [14]. Douglas Lind showed in 1978 that for a general (not necessarily Abelian) Polish group U and any measurable function $F : X \to U$, the equation $F(x) = P(T(x))P(x)Q(S(x))Q(x)$ can be solved with functions P, Q [34]. The proof uses a two-dimensional version of the **Halmos-Rohlin lemma**, stating that one can approximate the $\mathbb{Z}^2$ action by periodic actions up to arbitrary small measure. To summarize: for non-Abelian Polish U, the group $H^2(T,U)$ is trivial if T is a free $\mathbb{Z}^2$ action. This has interesting consequences in mathematical physics. Assume for example we have a lattice $\mathbb{Z}^2$ and we want to generate a **field** $F_{n,m}(x) = F(S^n x, T^n x)$ on the plaquette at position (n,m) given by a two-dimensional stochastic process defined by two commuting transformations S, T. No decorrelation is assumed. The fields can be almost periodic for example like $F_{n,m}(x,y) = f(x + n\alpha + m\beta)$. Then, there is a "lattice gauge field" (P,Q) such that the curl agrees with the vector field, meaning $F(x) = P(x)Q(Sx)P(Tx)^{-1}Q^{-1}(x)$. The

[2] Jack Feldman informed me about de Pauw's work in 1994

curl is the product of the cocycle values around a plaquette. In physics, one sees (P, Q) as a **lattice gauge field**, a discrete in general non-abelian 1-form. The Feldman-Moore picture of ergodic equivalence relations even shows that one can realize any magnetic field process on a Penrose lattice as the curl of a process defined on edges of the tiling.

2.10. The spectral theory of unitary operators $f \to f(T)$ coming from dynamical systems were first investigated by Koopman and von Neumann [9]. The history of cohomology in ergodic theory is long. The cohomological equation $f(T) - f = g$ appears at the end of the statement of Hilbert's 5th problem [19]. It is an example of an analytic functional equation with only continuous but non-differentiable solution. Von Neumann's 1932 paper [55] (p.641) mentions cohomology of dynamical systems in the context of spectra of ergodic flows. The term "cohomology of dynamical systems" was coined by Kirillov in [26] in 1967. [3] Other pioneers are Katok and Stepin [23, 24, 53]. Alexandre Kirillov also already looked at the cohomology of sets $H^1(T, \mathbb{Z}_2)$. Also the 1972 paper of [2] uses cohomology of dynamical systems, but unaware of [26]. [3] used the name "equivalent" for cohomologuous. [17] introduced "generalized eigenvalues". [1] continued the work aiming to find invariants for dynamical systems. The cohomology of dynamical systems as a special case of group cohomology was worked out in [14] in the context context of ergodic equivalence relations. See also Chapter 2 of [48].

2.11. The history of what happens generically and especially with respect to mixing is interesting. Steven Smale looked in 1967 at the general question of what happens typically for (a not necessarily conservative) diffeomorphisms of a compact manifold [52, 51]. Having a dense set of hyperbolic periodic orbits or the existence of homoclinic points, or the existence of a dense set of periodic elliptic orbits is generic [37]. In [34] is something about the history of mixing. Eberhard Hopf first defined various notions of mixing including weak mixing. A weakly mixing transformation that is not mixing was first constructed by Michiku Kakutani [50] and von Neumann [55, 42], but this was not published. Rafael Chacon [5] first showed that a change of speed of any ergodic flow could lead to weakly mixing flows. The number of classes of dynamical systems with with generic weak mixing has exploded. A generic set of measure preserving transformations of a probability space are weakly mixing by a theorem of Katok and Stepin [23]. For a new proof see [31].

2.12. The work [3] was one of the earliest works on **cocycle dynamics** and includes examples, where $T \times A$ on $X \times \mathbb{Z}_2$ has purely singular continuous spectrum. A generic volume preserving homeomorphism of a compact metric space is weakly mixing by a result of Oxtoby-Ulam [39], enhanced by Halmos and Rohlin pushing ergodicity to mixing. A generic invariant measure of a shift is weakly mixing [31] (enhancing [40]). A generic C^1 diffeomorphism is either weakly mixing with zero Lyapunov exponent or then weakly mixing and partially hyperbolic. A generic symplectomorphism has this property too by the last theorem of Mané [44] which is now a theorem [4]. A generic measure preserving differential equation on the 2-torus is weakly mixing [20]. A generic permutation of a Hamiltonian system with a $(k \geq 2)$-dimensional KAM torus produces a weakly mixing invariant torus [32]. A generic polygonal billiard is weakly mixing [6] and [16]. Remarkable is [47] showing that a Baire generic vertical-horizontal billiard is weakly mixing along a set of directions of full measure. In [30], we asked whether a generic convex body in a square is weakly mixing. This is indeed the case: for strictly convex Sinai scatter bodies, one has no pure point spectrum by Pesin theory [22], for rational polygonal scatterers one has pure point spectrum. Rational polygonal bodies are dense in all convex bodies. One believes that a generic right angle triangle produces a weakly mixing, non-mixing dynamics.

[3] We owe this reference to J.P. Conze, who informed us about this around 1993.

Unproven is whether in the smooth case, there is a generic set of diffeomorphisms which have a weakly mixing invariant component of positive measure. The vague attractor of Kolmogrov (VAK) picture suggests that there should be ergodic mixing components of positive measure mixed with quasi-periodic motion. For a more recent list of problems [41].

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Department of Mathematics, Harvard University, Cambridge, MA, 02138