THE COLORFUL RING OF PARTITIONS

OLIVER KNILL

Abstract. We visualize the identity $p(n) = \sum_{k=1}^{n} \sigma(k)p(n-k)/n$ for the partition function $p(n)$ involving the divisor function σ, add comments on the history of visualizations of numbers, illustrate how different mathematical fields play together when proving $\lim_{n\to\infty} p(n)^{1/n} = 1$ and introduce the modular ring of partitions.

1. Introduction

1.1. As far as we know, the earliest mathematicians worked with the help of **manipulatives** in the form of sticks, pebbles and ropes. Finger or toe cardinalities might have initiated the **decimal** or **base 20** systems, divisibility considerations the hexadecimal number system. Marks were carved into bones, clay, stone or bark [29, 19, 5]. Numbers were then written on Clay tablets or Papyrus or encoded in knots [45]. While historical discoveries are correlated to pedagogical time lines [14], history is always an effective ingredient for teaching [30]. Manipulatives are not only relevant historically or for teaching, they help to build intuition, even in a time of smartphones, tablets, virtual reality and artificial intelligence. The use of physical manipulative has flared up in the 3D printing age [43, 56]. Dubbed as the **forth industrial evolution** [52], it was recently over shadowed by the AI revolution [64]. Manipulatives might reappear more in virtual reality set-ups, once interfaces become fully VARK: **visual**, **auditory**, **read related** or **kinesthetic** [15].

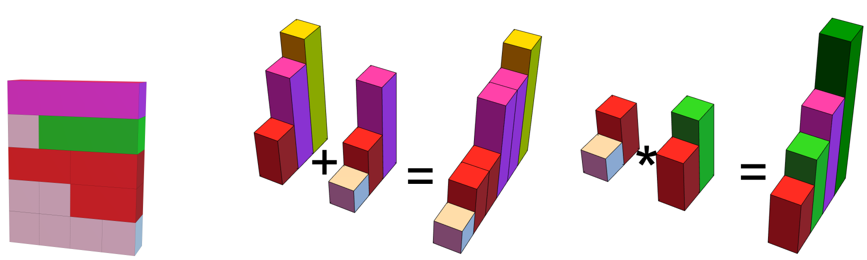

Figure 1. Partitions like the $p(4) = 5$ partitions of 5 generalize numbers. There is addition and multiplication in the ring $(\mathcal{P}, +, *, 0, 1)$. Note that $1 * 1 = 1$ but $1 + 1 = (1, 1) \neq (2)$. While $(2, 2) = (2) + (2)$, the partition (4) is not the sum of smaller partitions; it is an **additive partition prime**. Every $n = (n_1, \dots, n_k)$ with either $|n| = \sum_{j=1}^{k} n_j$ or k a rational prime is a **multiplicative partition prime** but there are more: $(1, 2, 3, 4)$ is a multiplicative partition prime, even so $|n| = 1 + 2 + 3 + 4$, and $k = 4$ are both not prime.

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2. Partitions as numbers

2.1. To illustrate how manipulatives can lead to **innovation**, we look at partitions as "generalized numbers" enlarging "natural numbers" that initiate the build-up $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$. The generalization is geometric: we can "add "finite simple graphs using the join operation [65]. After completing the monoid $(\mathcal{G}, \oplus)$ to a group, it leads with the large multiplication [53] to the **Sabidussi ring** $(\mathbb{G}, \oplus, \otimes, 0, 1)$ in which the empty graph is the zero element and the 1-point graph 1 is the unit. This ring is isomorphic to the **Shannon ring** $(\mathcal{Z}, +, *, 0, 1)$, in which $+$ is the **disjoint union** and $*$ is the **Shannon multiplication** [57]. See [21, 58, 60, 36, 37, 38].

2.2. The upshot is that there is a build-up $\mathcal{N} \subset \mathcal{Z} \subset \mathcal{Q} \subset \mathcal{R} \subset \mathcal{C} \subset \mathcal{H} \subset \mathcal{O}$ also for partitions $\mathcal{P} \sim \mathcal{Z}$. We do not have division algebras because the list of real normed division algebras is limited to to the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ by **Hurwitz theorem** [23] and $\mathcal{R}$ contains $\mathbb{R}$ so that also $\mathcal{C}, \mathcal{H}, \mathcal{O}$ would be division algebras which is impossible. However, the **partition complex numbers** $\mathcal{C}$ carry a **Banach algebra structure**. This can be done for any set of networks as generators. The simplest is to start with one single network which then completes it to a Banach algebra, isomorphic to the **Wiener algebra** $A(\mathbb{T})$. See [39].

2.3. Partitions are relevant in the context of the **discrete Sard theorem** [41]: if we take a discrete d-manifold G and a $(k+1)$-partite graph $P = K_{n_0,\dots,n_k}$ and a function $f : V(G) \to V(P)$ such that all facets F in P are reached, then the **level surface** $\{x \in G, f(x)$ contains at least an element in $F\}$, is either empty or then a $(d-k)$-manifold. Always. This is a remarkable result given that in the continuum, level sets can be singular like $x^2 + y^2 - z^2 = c$ for $c = 0$.

2.4. **The ring of partitions** $\mathcal{P}$ is a subring of the Sabidussi ring $\mathcal{S}$. Its elements are multipartite graphs, elements generated by zero dimensional subgraphs. The integers $\mathcal{Z}$ are the subring of the Sabidussi ring generated by complete graphs. In the partition ring, the concept of a number like "5" is richer. It contains 7 flavors of 5, the 7 partitions of 5. The addition is the concatenation. This comes up naturally in the Cuisenaire picture [9]. The multiplication can be visualized geometrically also.

Theorem 1. *The ring of partitions $\mathcal{P}$ is an extension of the ring $\mathbb{Z}$ of integers and is contained in the Sabidussi ring of graphs $\mathcal{G}$ which is isomorphic to the Shannon ring of graphs $\mathcal{Z}$. A partition n that adds up to a rational prime $p = |n|$ or contains a rational prime $k(n) = p$ number of elements is necessarily a multiplicative partition prime.*

Proof. $\mathcal{Z}$ is generated by complete graphs. $\mathcal{P}$ is generated by 0-dimensional graphs. Note that $K_1 \oplus K_1 \cdots \oplus K_1 = K_n$ is a complete graph so that $\mathcal{Z} \subset \mathcal{P}$. $\mathcal{G}$ is generated by all graphs. The duality in the form of **graph complement** represents $\mathcal{Z}$ in the Shannon ring as generated by zero dimensional graphs and $\mathcal{P}$ generated by complete graphs and $\mathcal{G}$ generated by all graphs. To see the prime statement, note that both $n = (n_1, \dots, n_k) \to |n| = \sum_{j=1}^{k(p)} n_j$ and $n \to k(n)$ are **ring homomorphisms** from $\mathcal{P}$ to $\mathbb{Z}$. The homomorphism statements follow from $|n+m| = |n| + |m|, k(n+m) = k(n) + k(m), |n * m| = |n||m|, k(n * m) = |n||m|$. ☐

3. Additive number theory

3.1. We illustrate the power of colored sticks, we use a recursion theorem which was obtained in [31]. It was found independently with the help of Cuisenaire material. It turns out that the mathematical theorem result was known since more than 100 years [49, 1] and probably was known to Euler already. Let $\sigma(n)$ denote the **divisor σ-function** which gives the sum of the proper divisors of an integer n. The **partition number** $p(n)$ gives the number of different ways to write n as a sum of smaller positive integers, where the order of the summands does not matter.

Theorem 2 (1918)**.** $p(n) = \sum_{i=1}^{n} \sigma(i)p(n-i)/n$

Proof. The arrangement of all partitions is a rectangle $P(n)$ of area $np(n)$. Let $f \downarrow P(n)$ denote how many times the stick f appears in the pile. One can now take from every row containing an f one piece away to see

$$f \downarrow P(n) = f \downarrow P(n-f)\}| + p(n-f) \ .$$

Repeating this gives

$$f \downarrow P(n) = f \downarrow P(n-f) + f \downarrow P(n-2f) + P(n-2f) \ .$$

Therefore

$$f \downarrow P(n) = \sum_{i=1} p(n-if)$$

if we assume $p(k) = 0$ for negative k and $p(0) = 1$. Now, we sum up the area as a sum over all the sticks:

$$\begin{array}{rcl} p(n)n & = & [p(n-1) + p(n-2) + p(n-3) + p(n-4) + ...] * 1 \\ & + & [p(n-2) + p(n-4) + p(n-6) + p(n-8) + ...] * 2 \\ & + & [p(n-3) + p(n-6) + p(n-9) + p(n-12) + ...] * 3 \\ & + & [p(n-4) + p(n-8) + p(n-12) + p(n-16) + ...] * 4 \\ & + & [p(n-n)] * n \ . \end{array}$$

The theorem follows by noticing that $p(n-k)$ appears $\sigma(k)$ times in that array. ☐

3.2. The theorem can be seen also as a fixed point result $p = T(p)$ for a **convolution average**

$$\sigma * g(n) := \frac{1}{n} \sum_{k=1}^{n} \sigma(k) g(n-k)$$

with the divisor function σ. As convolution is commutative, we also have $g*\sigma(n) = \frac{1}{n}\sum_{k=1}^{n} \sigma(n-k)g(k)$.

3.3. The book title of the ATM resources [6] shows similar arrangements for partitions of numbers if the order matters. See also [31] (page 133). [6] states *"After Gattegno Curriculum Graph 1974"*. The idea of illustrating identities as such must have appeared first in [18] page 5, interlocking staircases illustrating $10 = 1 + 9 = 2 + 8 + \cdots = 9 + 1 = 10$. We could not find so far any **partition illustrations** earlier than [31]. We could also not spot it in Gattegno's original books [17], nor in [9].

3.4. Computer algebra systems like Mathematica have the integer partition algorithm already built in, but Euler's formula for the generating function

$$\sum_{k=0}^{\infty} p(k)x^k = \prod_{j=1} \frac{1}{1-x^j}$$

[13] quickly allows to compute it from scratch. The third part of the following code uses the implementation of the theorem. The built in Mathematica routine probably uses the Euler Pentagonal number theorem.

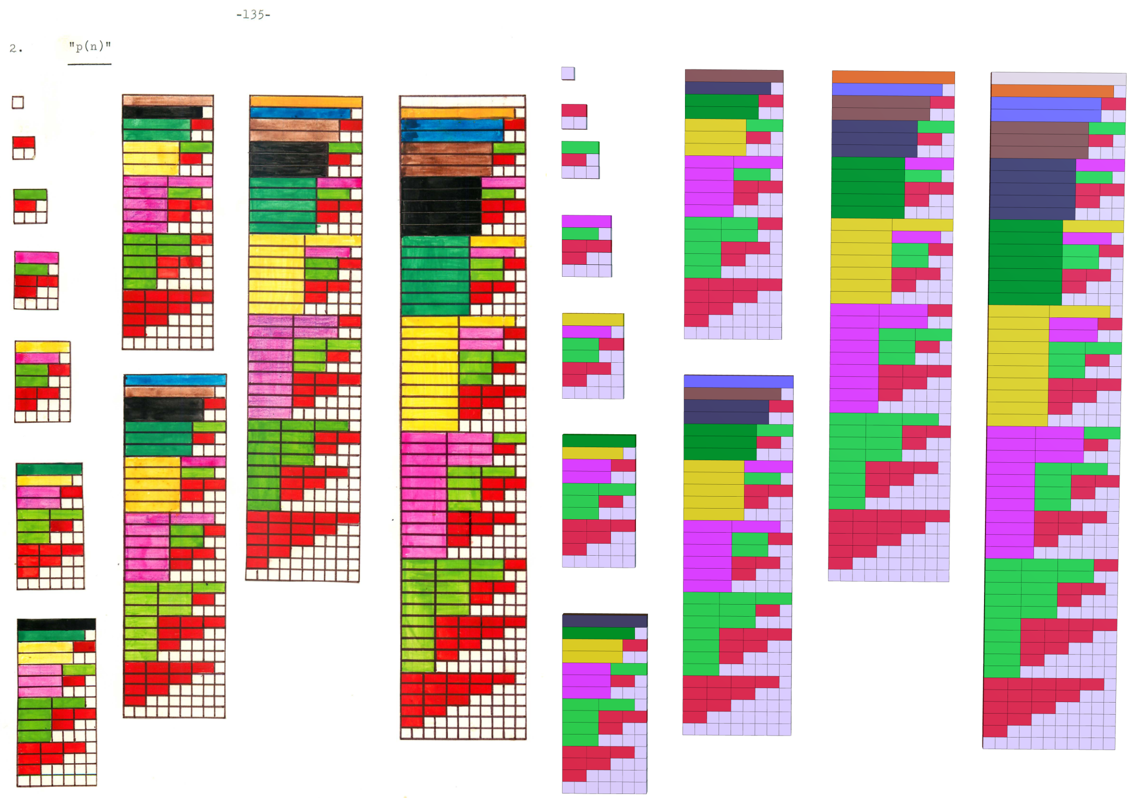

FIGURE 2. We see to the left first 10 partitions (see [31] on page 135, hand drawn and pencil colorized. See [32] for a summary). We have redrawn them in Mathematica in 2024.

```
Table[PartitionsP[n],{n,0,20}]
CoefficientList[Series[Product[1/(1-x^j),{j,20}],{x,0,20}],x]
F[0]:=1;F[n_]:=Sum[DivisorSigma[1,k]*PartitionsP[n-k]/n,{k,n}];
Table[F[n],{n,0,20}]
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3.5. As Euler noticed, the partition function and the divisor function satisfy the same recursion. But the initial and boundary conditions are different:

$$\sigma(n) = \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \sigma(n-12) + \sigma(n-15) + \cdots$$

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) + \cdots .$$

This is amazing because $\sigma(n)$ uses **multiplicative properties** of n and $p(n)$ uses **additive structures**. The initial conditions are different. Both assume $p(n) = \sigma(n) = 0$ if $n < 0$. The partition function starts with $(p(1), p(2)) = (1, 2)$ and takes $p(0) = 1$. The divisor function starts with $(\sigma(1), \sigma(2)) = (1, 3)$ and uses $\sigma(n-n) = n$ in the recursion, when computing $\sigma(n)$.

FIGURE 3. The author presenting for a youtube recording on January 27th, 2024. It illustrates the recursion formula visually, in an example $6*11 = 7*1 + 5(2+1) + 3(3+1) + 2(4+2+1) + 1(5+1) + 1(6+3+2+1)$.

4. Dynamics

4.1. How fast does the partition function $p(n)$ grow? The growth rates is known to be exponential. The **Hardy-Ramanujan asymptotic formula** shows that $p(n) \sim \exp(\pi\sqrt{2n/3})/(4n\sqrt{3})$. We repeat here a dynamical systems proof [33] of the following theorem:

Theorem 3. $\limsup_n p(n)^{1/n} = 1$

Proof. **Euler's pentagonal number theorem** tells that

$$Q(z) = \prod_{k=1}^{\infty} (1-z^k) = \sum_n a_n z^n ,$$

where a_n are the **pentagonal numbers**. We also have

$$P(z) = \sum_{n=1}^{\infty} p(n) z^n = \prod_{k=1}^{\infty} (1-z^k)^{-1} = \frac{1}{Q(z)} .$$

Hadamard's lacunary theorem [26] about power series $\sum_k c_k z^{n_k}$ shows that if $n_k + 1/n_k \geq r > 1$, then $r = 1$ is a natural boundary. This means that boundedness at one point $|z| = r$ would imply that the radius of convergence is larger than r. Define $g(\alpha) = (1 - re^{2\pi i \alpha})^{-1}$ so that $f(\alpha) = \log(P(z)) = -\sum_{k=1}^{\infty} g(k\alpha)$. For $r = |z| < 1$, and $z = r\exp(2\pi i \alpha)$ with Diophantine α, the **theorem of Gottschalk-Hedlund** [28] assures that f is a **coboundary**, meaning $f(x) = F(x+\alpha) - F(x)$, implying that the sum is bounded. Having so proven that the Taylor series has no singularity on $|z| < 1$, the partition function $p(n)$ satisfies $\limsup_n |p(n)|^{1/n} \geq 1$. The other inequality $\limsup_n |p(n)|^{1/n} \leq 1$ implies together with $p(n) \geq 1$ that $\sum_{n=1}^{\infty} p(n) z^n \geq \sum_{n=1}^{\infty} z^n$ so that the radius of convergence of the former is larger or equal than the radius of convergence of the later, which is 1. ∎

4.2. The proof illustrates how number theory, complex analysis, and dynamical systems theory as well as Diophantine notions can come together. Such connections also appeared when studying Dirichlet series with almost periodic coefficients [42].

5. Psychology

5.1. How do humans learn, teach and do research? How do they gain intuition about mathematical objects? Psychological aspects of teaching have been looked at in research [20] or education [48]. Hadamard stresses that visual thinking is key for intuition. Norton suggests that a number is defined through **our own actions** and mentions in the chapter "What are numbers" cites historical definitions of numbers like Euclid's notion *"A number is a multitude composed of units"* which appearing in book VII of his collected work. The development of numbers is also linked to language and notation [46]. A more topological approach to numbers are quipu (=khipu), talking knots [45, 2].

5.2. In [40], the question was raised whether one can challenge even the beginning of counting. Kronecker is reported to have told the famous **"God made the integers; all else is the work of man"** [22, 4]. But are the integers really the most natural way for counting?

5.3. In [40] we argued that the **infinite dihedral group** $Z_2 * Z_2$ generated by two involutions is more "natural" than $\mathbb{Z}$. This was neither opinion nor a philosophical question; it emerged from a definition: a group is defined to be "natural" if there exists a metric space X such that there is only one way up to isomorphism to put a group structure on X with the constraint that all group operations are isometries. The infinite dihedral group is natural. The integers are not. There are also more down-to-earth reasons why the dihedral group is natural. Maybe the most important one is that there is no **Grothendieck group completion** from the monoid $(\mathbb{N}, +, 0)$ to the group $(\mathbb{Z},, 0)$ required. From the natural numbers, a young student first has to be convinced about **negative numbers** using analogies like "money debt" or "negative temperatures". In the dihedral case $1 = ab$ and $-1 = ba$. A completely different approach sees numbers as games [8]. Donald Knuth visualized the surreal numbers in the novel "Surreal numbers" (or "Insel der Zahlen" which I read in German and which is a much better "selling" book title. It should also have been "Island of Numbers" in English.) [44].

5.4. While commutative structures have been challenged since the emergence of quantum mechanics and under the umbrella of non-commutative mathematics [7], the commutativity axiom for the additive structure of the integers has never been put up to debate. But as argued in [40], there are reasons to consider a group generated by two involutions more fundamental than a group by a single generator 1. The dihedral approach very much is in the philosophy of reflection geometry [3, 24] where reflections are used to generate the non-Abelian Euclidean group $\mathbb{R}^n \rtimes O(n)$ consisting of translations and orthogonal transformations. Reflections at affine planes are all involutions. **Dihedral counting** enters geometry if we look at the group generated by the exterior derivative d and its dual d^*. We have $d^2 = 0$ and $(d^*)^2 = 0$.

5.5. Also notation and implementation is important. Good notation pays off in the long term as it allows to think more effectively. Leibniz's notations and nomenclature in calculus for example was much persuasive than Newton's notation. In my personal experience, using intuitive and elegant notation is one of the most important factors for clarity in teaching undergraduate mathematics. In the case of manipulatives, the analog of notation is how to use **material**, **shape** or **color** and **labels** implemented in texture. Katherine Stern used colored cubes around a similar time than Cuisenaire. But the colors did not carry intuitive meaning. Other manipulatives had writings on them, making them less elegant.

5.6. In the case of Cuisenaire sticks, the colors have been carefully chosen. Cuisenaire thought a great deal about the psychology of numbers. He tried to use colors which match division properties. Of course this can not be pulled through in detail but the choice $red = 2 \to crimson = 4 \to brown = 8$ or $yellow = 5 \to orange = 10$ or $green = 3 \to darkgreen = 6 \to blue = 9$ and then black=7 shows taste. Each of the 4 smallest **primes** $2, 3, 5, 7$ in the range of 1 to 10 used. Multiples then are using similar color components.

- Each of the primes 2,3,5,7 starts a new color family.
- The even numbers 2,4,6,8 have a red-purple components,
- The multiples of three, 3,6,9, have a green-bluish side,
- The multiples of 5, given by 5 and 10 have a yellow-orange part.

5.7. A young mathematics student playing with these sticks can gain so additional understanding of numbers. There is a geometric link in the form of length. There are also color impressions. Also music exposure can provide **auditory link** to numbers. The do-re-mi-fa-so-la-ti-do already introduces a modular arithmetic, in this case Z_8, the smallest three dimensional vector space. In music, there are other dimensions like amplitude or modulation. Music implements number systems like the modular arithmetic on scales. But it is much richer.

6. History

6.1. We take the opportunity to make a list of manipulative material used by educators. We see that the first attempts to build manipulatives were done at the end of the 18th century by educators like Friedrich Froebel (1782-1852). Froebel is known primarily as a philosopher from the book [16]. He believed in the fundamental principle that there must be an inner connection between the pupil's mind and the objects which are studied. Froebel was a student of the Swiss pedagogue and education reformer Johann Heinrich Pestalozzi (1746-1827) who is considered the first "modern educator". He is known for the picture that training children should involve **"heart, mind and body"** and that nature was the best way to teach.

6.2. As for the development for mathematical toolboxes or manipulative, see [25]. Jobbe-Duval gives as a reference about the first appearance (1947) of the Cuisenaire rods an article [61]. Wikipedia dates the first appearance of the sticks to 1945. We cite from this Wikipedia entry *"In 1945, following many years of research and experimentation, Cuisenaire created a game consisting of coloured cardboard strips of various lengths that he used to teach mathematics to young children."* We can see simple tasks like counting trees or estimating the number of trees in a forest or to watch for Fibonacci patterns in pine cones or sunflowers or see fractions in the harmonies of music as examples of such teaching methodologies.

Author	Year	Method
Friedrich Froebel	(1782-1852)	wooden building blocks
Ernst Tillich	(1780-1807)	Rechenkasten
Catherine Stern	(1894-1973)	blocks
Maria Montessori	(1870-1952)	barres numeriques rouges es blues
Georges Cuisenaire	(1891-1975)	wooden block in 1951
Mina Audemars	(1883-1971)	66 blocks
Louise Lafandel	(1872-1971)	66 blocks
Setton Pollock's	(1910-1983)	Color Factor system 1960ies
Artur Kern	(1902-1988)	Rechenkasten
Paulette Calcia	(1904-1985)	Buchettes'dor
Maria-Antonia Canals	(1930-)	Reglets
Adam Langer	(1836-1919)	Rechenkasten
Remi Brissiaud	(1949-)	Noumes

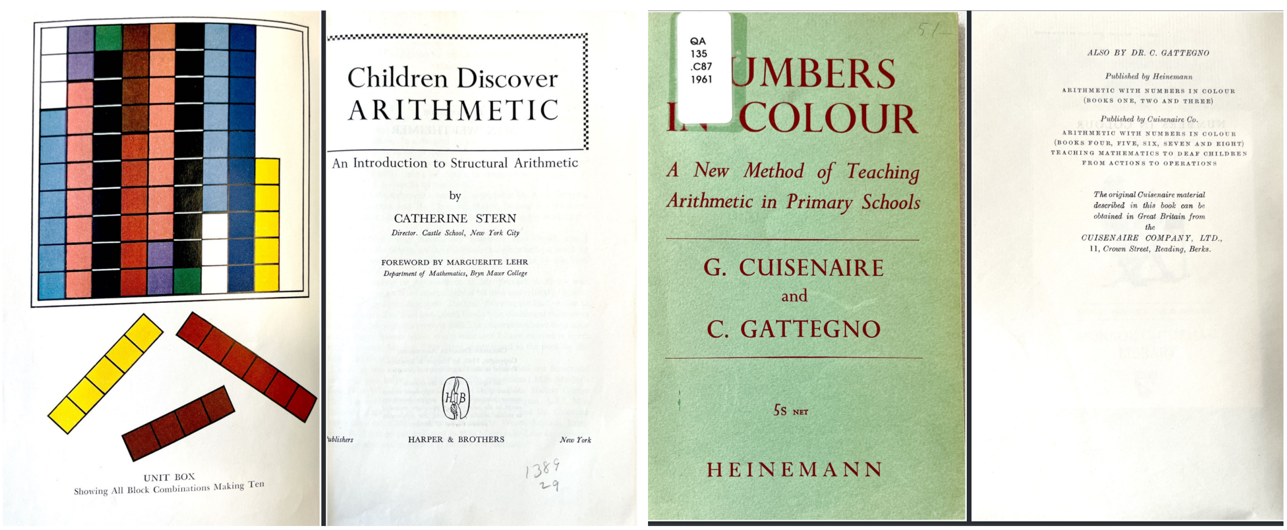

FIGURE 4. The book cover of Stern [62] from 1949 and the book cover of Cuisenaire-Gattegno [9] from 1954.

6.3. The first colored Cuisenaire were known since 1947 and were published in 1951. It appears as if the approach of Catherine Stern was independent from Cuisenaire's and that Stern was influenced by Montessori. The "Stern Math" approach is also still used today. On the website of the organization is a quote of Einstein saying *"I believe that her (Catherine Stern's) idea is sound would be of real value in the teaching of the elements of arithmetic"*.

6.4. Research in partitions was in full swing at the time of Ramanujan who would mention partitions already in his first letter to Hardy in January 1913.

FIGURE 5. We see the two pioneers Catherine Stern and George Cuisenaire. The my-heritage colorized photo shows her in 1950 with her grandson Fred. According to [63], Catherine Stern was influenced by Maria Montesori's ideas and presented arithmetic manipulatives in 1934 during a conference of the Swiss Kindergarden Association. She immigrated to the US in 1938. The photo of George Cuisenaire is from [10], also colorized with my-heritage.

FIGURE 6. Hardy (played in the movie by Jeremy Irons) explains the asymptotic Hardy-Ramanujan formula for the partition function in the 2015 film: "The man who knew infinity".

6.5. Partitions are mentioned in the book about Ramanujan [27] (pages 246-250). The movie adaptation of the book makes partitions to a small drama in the form of a competition. Hardy explains the problem in the movie similarly as in the book with a small example $p(4)$. One can the see in a scene a manuscript showing the Hardy-Ramanujan formula $p(n) \sim R(n) = 1/(4n\sqrt{3}) \exp(\pi\sqrt{2n/3})$ meaning $p(n)/R(n) \to 1$.

7. Creativity

7.1. Partitions appear when defining discrete sub-manifolds of given manifolds [41]. While working on this emerged the picture that the addition of two partitions $n = (n_1 \dots, n_k)$, $m = (m_1, \dots, m_l)$, in the form of concatenation $n+m = (n_1, \dots, n_k, m_1, \dots, m_l)$ and the product by foiling out: $n*m = (n_1m_1, n_2m_1, \dots, n_km_1, \dots, n_1m_k \dots, n_km_l)$. For example $(3,4)+(1,1,2) = (3,4,1,1,2)$ and $(3,4)*(1,1,2) = (3,3,6,4,4,8)$. It leads to the idea to associate to a partition $n = (n_1, \dots, n_k)$ the k-partite graph $K_{n_1,\dots,n_k}$. The partition $5 = K_{1,1,1,1,1}$ for example is K_5 and $6 = K_{3,3}$ is known as the utility graph. The addition is the Zykov join and the multiplication is the Sabidussi multiplication. This ring structure on multi-partite graph is dual to the Shannon ring structure, where addition is disjoint union and multiplication is Shannon multiplication.

7.2. Let me add a personal remark about education, which is very much related to my early exposure to Cuisenaire sticks. In the last 40 years, one can observe a **standardization tendency** in K-12 education. The reason is a reflex to **minimize risk** from the administration side and to make sure that basic goals are achieved. But chaining of teachers to fixed curricula indicates an increasing lack of trust for individual initiative. It is a vicious circle because such a system discourages creativity, it also is also less likely attracts creative minds into the profession. It leads to a de-valuation of public schools, the emergence of after-school programs or private schools, which are not chained to such administrative handcuffs and where teachers can fully show their potential.

7.3. In the primary school up to high school of 50 years ago, our teachers could get us out into the open, mix art or sport with mathematics without administrative oversight. I myself enjoyed orienteering competitions in larger forest with professional orienteering maps and compass learning about direction, level curves, or excursions to collect plants in order to build our own botany book, or collect rock samples, climb on the cold volcanoes in the southern Germany, or visit the Danube sinkhole. In high school chemistry, we were for once just given a powder and access to the generous school chemistry lab. The task was to use whatever tool we could find and gets hand on to figure out what the powder was. There were no worksheets, just expert guidance and supervision by teacher and an expert lab assistant helping to keep us safe. I still remember the excitement after finding out that the powder given to us was "Aspirin".

7.4. The freedom which these teachers had appears more exceptional today. It has become the privilege of private institutions. The times of the "Black Mountain college" [55, 11], where rival methodologies were explored and experimental forms were practiced, are long over. (The college is featured in the 2013 novel "The longest ride" by Nickolas Sparks, and also made it into a movie in 2015, where the free teaching style at the college is very well visible.)

7.5. By the way, the mathematician Max Dehn, who was a student of Hilbert, had taught from 1945-1952 at the Black Mountain college and is buried there [59]. The Wikipedia article about him (which follows [59]) states *He enjoyed the forested mountains found in Black Mountain, and would often hold class in the woods, giving lectures during hikes. His lectures frequently drifted off topic on tangents about philosophy, the arts, and nature and their connection to mathematics. He and his wife took part in community meetings and often ate in the dining room. They also regularly had long breakfasts with Buckminster Fuller and his wife.* As as source, [50] is given.

7.6. The following quotation given in [59] about math education is remarkable: *All instruction, including mathematics, should be an end in itself, not, in the first place, means. Therefore in mathematics one should be so simple and elementary as possible so that the structure of mathematical thinking becomes clear. If this is achieved then the student will have less difficulty to see mathematical structures in physical phenomena or even in phenomena of biology and social life.* And from [50]: *As Dehn was famous for his way of picturing abstract concepts, many of his great mathematical ideas can be intuitively understood through pictures.*

7.7. Also in higher education, the administrative cages have been rolled out more and more. I could still develop calculus courses independently without oversight 10 years ago. Of course, there were course goals and coordination between different courses but the details, the homework or lesson plans were not prescribed. Today, we live in a time, with much more constraints. It appears to be safer like that but a teaching profession that lacks initiative and originality will be replaced by technology much faster than anticipated.

FIGURE 7. This photo is credited to the photography critic **Nancy Newhall** (1908-1974) is titled "Max Dehn at the Black Mountain College" is dated around 1947. The gelatin silver negative had been gifted by **Christi Newhall** to the Eastman Museum, where one can find a high resolution version. Nancy Newhall visited together with her husband Beaumont Newhall the Black Mountain College for three summers in the years 1946-1949. The Newhalls photographed during that time the campus and its people.

8. More personal remarks

8.1. I myself got introduced to numbers using the Cuisenaire material [9]. This had happened on the personal initiative by my first grade teacher **Edwin Ilg** [12] who worked at a time, when teachers could shape their curricula. We must maybe stress that this happened at an "ordinary" public school in Uhwiesen, a village near Schaffhausen in Switzerland. I still own the same Cuisenaire sticks from first grade.

8.2. The independent research done in high-school had led to the theorem mentioned above and others like the Pentagonal number theorem (which I however could not prove then). The topic of partitions came up during the winter 2023/2024 in the context of geometry, when looking again at the **discrete Sard theorem**. That theorem has classically has been developed by Morse and Sard [47, 54]. It is pretty remarkable that one can do a similar result within finite mathematics: first define carefully what a d-manifold is : it is a finite simple graph for which every unit sphere is a $(d-1)i$-sphere, where a d-sphere is a d-manifold that becomes contractible if a vertex is taken away. In 2015, I noticed that one can define hyper-surfaces in such a manifold by looking at all simplices on which the function changes sign [34]. The only requirement is that the function f is locally injective (a coloring). During Fall 2023 this was then generalized to higher dimensions [41]

8.3. Again related to teaching, when leading a day long workshop in 2017 on the "Adventures in algebra", [35], I wanted to make the point of "being creative" and spent the winter 2016/2017 preparing for that January 2017 event in McAllen in the Rio Grande Valley. During that exploration time, I wondered whether there is a multiplication which fits the join operation in graph theory. A longer search revealed the Sabidussi multiplication (as I later would find out). It is frequent in research that one rediscovers something that has already been found. But as with the high school research in number theory, it does not diminish the joy of finding it. It is also fun to find something that has already been known. It gives "street cred" to talk about "creativity". One can only preach something if one actually lives it. Walk the talk.

8.4. During my adventures in algebra, I had not yet known about the Sabidussi ring and the Shannon ring and was primarily focussed also in the case of "additive primes" in the Zykov

FIGURE 8. An undercover fraction lab, operative in the winter 2016/2017.

monoid. Taking graph complements showed there that the additive primes in the Zykov monoid are the graphs for which the graph complement is connected. The question to characterize multiplicative primes in the Shannon ring is interesting too. I took up the topic in this document again now in the fall of 2024 while teaching a tutorial Math 99R at Harvard on "Visualization of Mathematics" and thought about how to characterize multiplicative primes in the ring $\mathcal{P}$ of partitions. (Characterizing additive primes in $\mathcal{P}$ is a nice exercise left for the reader.)

9. Rings of Partitions

9.1. To represent partitions uniquely in the ring of partitions, we always sort them. Apropos primes, we end this exposition with an open question. How can we characterize primes in the ring of partitions? We have seen that $n = (n_1, \dots, n_k)$ is prime if either $n = \sum_j n_j$ is prime or if k is prime. But there are more primes like $p = (3, 4, 5, 2)$ in which case both $\sum_j n_j = 14$ and $k = 4$ are not prime, but where p can not be factored into smaller components. The only possible factors would be of the form $(a, b), (c, d)$ and since $3, 5$ are prime, it would force either $(a, b) = (1, 1)$ or $(c, d) = (1, 1)$ or $(a, b) = (1, b)$ and $(c, d) = (1, d)$ which all do not work.

9.2. This motivates to investigate the growth rate of **partition primes**, similarly as we are interested in the growth rate of **rational primes**. Unexplored is whether one can use partitions for cryptology. Given a rational integer $N \in \mathbb{N}$, is there a finite commutative ring to work with? We have experimented with some set-up. One could try to use it for schemes like **Diffie-Hellman key exchange** or **RSA public key systems** in such a **modular ring of partition**. In any finite Abelian group, there are cryptological algorithms to factor integers could be used [51]. A simple example is the Pollard ρ algorithm. It only requires to build in fast ways to form a^n if $a \in \mathcal{P}$ and $n \in \mathbb{Z}$ and to build the greatest common divisor $\gcd(n, m)$ of two elements n, m in $\mathcal{P}$.

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Department of Mathematics, Harvard University, Cambridge, MA, 02138