THE AVERAGE SIMPLEX CARDINALITY OF A FINITE ABSTRACT SIMPLICIAL COMPLEX

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ABSTRACT. We study the average simplex cardinality $\mathrm{Dim}^+(G) = \sum_{x \in G} |x|/(|G|+1)$ of a finite abstract simplicial complex G. It also defines an average dimension $\mathrm{Dim}(G) = \mathrm{Dim}^+(G) - 1$. The functional Dim^+ is a homomorphism from the monoid of simplicial complexes $\mathcal{G}$ to $\mathbb{Q}$: the formula $\mathrm{Dim}^+(G \oplus H) = \mathrm{Dim}^+(G) + \mathrm{Dim}^+(H)$ holds for the join $\oplus$ similarly as for the augmented inductive dimension $\dim^+(G) = \dim(G) + 1$ where dim is the inductive dimension (as recently shown by Betre and Salinger [1]). In terms of the generating function $f(t) = 1 + v_0 t + v_1 t^2 + \cdots + v_d t^{d+1}$ defined by the f-vector $(v_0, v_1, \ldots, v_d)$ of G for which $f(-1)$ is genus $1 - \chi(G)$ and $f(1) = \mathrm{card}(G) = |G| + 1$ is the augmented number of simplices, the average cardinality is the logarithmic derivative $\mathrm{Dim}^+(f) = f'(1)/f(1)$ of f at 1. So, $e^{\dim^+(G)}$, $e^{\mathrm{Dim}^+(G)}$, $e^{\max(G)}$, $\mathrm{genus}(G) = 1 - \chi(G) = f(-1)$, or $\mathrm{card}^+(G) = f(1)$ are all multiplicative characters from $\mathcal{G}$ to $\mathbb{R}$. After introducing the average cardinality and establishing its compatibility with arithmetic, we prove two results: 1) the inequality $\dim^+(G)/2 \leq \mathrm{Dim}^+(G)$ with equality for complete complexes. 2) In the Barycentric limit, $C_d = \lim_{n \to \infty} \mathrm{Dim}^+(G_n)$ is the same for any initial complex G_0 of maximal dimension d and the constant C_d is explicitly given in terms of the Perron-Frobenius eigenfunction of the universal Barycentric refinement operator $f_{G_{n+1}} = A f_{G_n}$ and is for positive d always a rational number in the open interval $((d+1)/2, d+1)$.

1. DIMENSIONS

1.1. A finite set G of non-empty sets closed under the operation of taking finite non-empty subsets is called an **abstract finite simplicial complex**. The sets x in G are also called **simplices** or (if the complex is the Whitney complex of a graph, the sets are also called **cliques** and the Whitney complex is known as clique complex). The **dimension** of a set $x \in G$ is $|x| - 1$, one less than the cardinality $|x|$ of the simplex x.

1.2. One sometimes also looks at the **augmented complex** $G^+ = G \cup \{\emptyset\}$ which is a set of sets closed under the operation of taking arbitrary subsets. The **maximal cardinality** $\max(G)$ is by one larger than the **maximal dimension** of the complex. A finite abstract simplicial complex G has the **maximal dimension** $\max_{x \in G} \dim(x)$, and the **inductive dimension** $1 + (1/|G|) \sum_{x \in G} \dim(S(x))$, where $S(x) = \{y \in G \mid y \subset x \text{ or } x \subset y\}$ is the **unit sphere** of $x \in G$. These dimensions are defined for arbitrary sets of sets not only for simplicial complexes.

1.3. The inductive dimension of G is a rational number less or equal than the maximal dimension of G. It behaves nicely with respect to the Cartesian product because of the inequality $\dim(G \times H) \geq \dim(G) + \dim(H)$ for the Cartesian product [9] (the Barycentric refinement of the Cartesian set product $G \times H$), which holds similarly also for the Hausdorff dimension does [3] (formula 7.2) for Borel sets in Euclidean space.

1.4. Recently, in [1] it was proven for the inductive dimension dim that the functional $\dim^+(G) = \dim(G) + 1$ is additive:

$$\dim^+(G \oplus H) = \dim^+(G) + \dim^+(H) .$$

This is a bit surprising as it is an identity, where one has rational numbers on both sides.
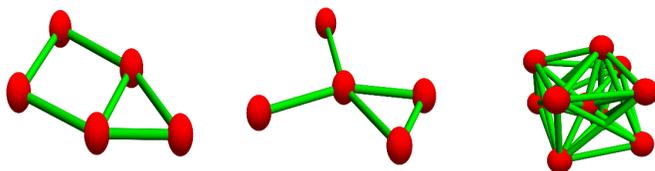

FIGURE 1. An illustration of the compatibility equality telling that the augmented dimension of $G \oplus H$ is the sum of the augmented dimensions of G and H. The first graph is the "house graph", the second the "rabbit graph". The average simplex cardinalities are $\mathrm{Dim}^+(G) = 20/13$, $\mathrm{Dim}^+(H) = 3/2$, $\mathrm{Dim}^+(G \oplus H) = 79/26 = 20/13 + 3/2$. The augmented inductive dimensions are $\dim^+(G) = 37/15$, $\dim^+(H) = 5/2$ and $\dim^+(G \oplus H) = 149/30 = 37/15 + 5/2$. The augmented maximal dimensions are $\max(G) = 3$, $\max(H) = 3$ and $\max(G \oplus H) = 6 = 3 + 3$.

1.5. We introduce and study here the **average dimension**

$$\mathrm{Dim}(G) = \frac{1}{|G| + 1} \sum_{x \in G} \dim(x)$$

and its augmentation

$$\mathrm{Dim}^+(G) = \frac{1}{|G| + 1} \sum_{x \in G} |x| ,$$

which we call the **average simplex cardinality** of simply the **average cardinality** of the simplicial complex G. It measures the expected size $|x|$ of a set $x \in G$. For a triangle $K_3^+ = \{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \emptyset\}$ for example, the average simplex cardinality is $(1 \cdot 3 + 3 \cdot 2 + 3 \cdot 1 + 1 \cdot 0)/8 = 12/8 = 3/2$.

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1.6. The use of $|G| + 1$ instead of $|G|$ in the denominator makes the arithmetic nicer. One can justify it also by looking at the **augmented complex** G^+ which includes the empty set $\emptyset$ in G and assign it to have dimension $|\emptyset| - 1 = (-1)$ which has cardinality or augmented dimension 0 so that it has no weight when taking the expectation but which adds 1 to the cardinality of the sets in G. In other words $|G| + 1 = |G^+|$ can explain the augmentation.

1.7. The presence of the empty set $\emptyset$ is also reflected in the use of the **simplex generating function**

$$f_G(t) = 1 + v_0 t + \cdots + v_d t^{d+1} ,$$

where v_k counts the number of k-dimensional simplices in G. The constant term in f can then be interpreted as $1v_{-1}$, where $v_{-1} = 1$ counts the number of (-1)-dimensional sets (the empty set $\emptyset$). We prefer to stick here however to the more traditional definition in topology and call "simplices" the non-empty sets in G and see the **empty set** as a (-1) dimensional **simplicial complex with no elements** and **not as a (-1)-dimensional set in a simplicial complex**.

1.8. The **join** of two simplicial complexes G, H is defined as the complex

$$G \oplus H = G \cup H \cup \{x = y \cup z \,|\, y \in G, z \in H\} .$$

Its graph is the **Zykov join** of the graphs $\Gamma(G)$ and $\Gamma(H)$ [16], which is the graph obtained by taking the disjoint union of the vertex sets of G and H and connecting any vertex in G with any vertex in H. The Zykov join is defined for all graphs, they don't need to come from a simplicial complex.

1.9. The **join monoid** $(\mathcal{G}, \oplus)$ has the class of spheres as a sub-monoid for which the augmented dimension $\mathrm{Dim}^+(G)$ is additive; this is all completely equivalent to standard topology, where the same identities hold. Also as in the continuum, the join with a 1-point graph is called a **cone extension** and the join with the 2-point graph (the 0-sphere) is the **suspension**.

1.10. From this point of view, also the **genus** $f(-1) = 1 - \chi(G)$ (which is sometimes called the **augmented Euler characteristic**) is a valid alternative to the Euler characteristic $\chi(G)$ as it is compatible with the arithmetic: it multiplies with the join operation.

1.11. Still, as Euler characteristic and simplicial complexes without empty set are far more entrenched outside of combinatorial topology, we prefer to stick to**simplicial complex** and **Euler characteristic**, as custom in topology. We will see in a moment that the average cardinality is the logarithmic derivative of f at 1. This immediately implies compatibility of $\mathrm{Dim}^+(G)$ with the join operation $\oplus$, facts which are trivial for the maximal dimension $\max(G)$ and which was recently shown to hold for the augmented inductive dimension $\dim^+$ by Betre and Salinger [1].

1.12. The average dimension Dim is most likely of **less topological importance** than the inductive dimension but instead of **more statistical interest**. Both the maximal dimension max as well as the inductive dimension dim are more intuitive than the average dimension Dim, as they align with dimension notions we are familiar with in the continuum. Inductive dimension is an integer for discrete manifolds for example. The average dimension however gives a richer statistical description of a network as tells about the distribution of the various simplices.

1.13. The fact that discrete manifolds have integer dimensions is a bit more general even: a simplicial complex G can inductively be declared to be a **discrete d-variety** if every unit sphere $S(x)$ is a discrete $(d-1)$-variety. The class d-**varieties** is more basic than d-manifolds as no homotopy is involved in its definition. For **discrete d-manifolds**, we insist that the unit spheres $S(x)$ are all $(d-1)$-spheres, which are defined to be $(d-1)$-manifolds which when punctured become contractible. The figure 8-graph is an example of a 1-variety which is not a discrete 1-manifold, because one of its vertices has a unit sphere which is not a 0-sphere.

1.14. The inductive dimension of "discrete variety" is booted up with the assumption that the **empty complex** 0 is a discrete variety. It is obvious from the definition that both maximal dimension as well as inductive dimensions of a variety are integers and that they agree with the maximal dimension. But this is not true for the average cardinality in general. The octahedron graph and icosahedron graph both have inductive dimension 2. The average cardinality of the octahedron graph is 2, the average cardinality of the icosahedron graph is $44/21 = 2.095...$ The Barycentric refinement of the octahedron graph has average cardinality $314/147 = 2.136...$ while the Barycentric refinement of the icosahedron graph has average cardinality $782/363 = 2.15....$ In dimension 2, the limiting average cardinality is $(1,3,2) \cdot (1,2,3)/(1+3+2) = 16/6 = 2.1666...$ as $(1,3,2)$ is the dominant eigenvector of the Barycentric operator $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{bmatrix}$ in dimension 2.

1.15. Examples: for the complete graph K_n, the cyclic graph C_n and the path graphs P_n and the complete bipartite graphs $K_{n,n}$ or the point graphs $E_n = 1 + 1 \cdots + 1$ with n vertices and no edges:

$\text{Dim}^+(E_n) = n/(n+1)$	$\text{dim}^+(E_n) = 1.$
$\text{Dim}^+(K_n) = n/2$	$\text{dim}^+(K_n) = n.$
$\text{Dim}^+(C_n) = 3n/(2n+1)$	$\text{dim}^+(C_n) = 2.$
$\text{Dim}^+(P_n) = (3n-2)/(2n-1)$	$\text{dim}^+(P_n) = 2.$
$\text{Dim}^+(K_{3,3}) = 3/2$	$\text{dim}^+(K_{3,3}) = 2.$
$\text{Dim}^+(K_{n,n}) = 2 - 2/(1+n)$	$\text{dim}^+(K_{n,n}) = 2.$

1.16. For a one-dimensional simplicial complex with Euler characteristic $\chi = |V| - |E| = n - m$, we have $\text{Dim}^+(G) = (|V| + 2|E|)/(|V| + |E| + 1) = (3n - 2\chi)/(2n - \chi + 1)$.

We see from this in particular that in the Barycentric limit of a one-dimensional complex we get the average simplex cardinality value $3/2$ in dimension $d = 1$. This can be expressed with the dominant eigenvector of the Barycentric refinement operator, the eigenvector to the maximal eigenvalue $d!$. The algebraically confirmation will be done in full generality; but in this particular case of dimension $d = 1$, the Barycentric refinement operator $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ has the **Perron-Frobenius eigenvector** $(1, 1)$ and $(1, 1) \cdots (1, 2)/(1 + 1) = 3/2$.

1.17. Remarkable for the utility graph $K_{3,3}$ is that **its average simplex dimension does not change under Barycentric refinements**. It remains $3/2$. This is why we can not claim $\mathrm{Dim}^+(G_1) > \mathrm{Dim}^+(G)$ in general. It also is not the case for the zero-dimensional case $n = 1 + 1 + \cdots + 1$ but there are no two or higher dimensional simplicial complexes G for which $\mathrm{Dim}^+(G_1) = \mathrm{Dim}^+(G)$. In dimension $d = 2$, where the limiting Barycentric limit dimension is $23/8 = 2.875$, we would need a complex of which the vertex, edge and triangle cardinalities v_0, v_1, v_2 would have to satisfy $(v_0 + 2v_1 + 3v_2)/(1 + v_0 + v + 1 + v_2) = 23/8$ which implies $v_2 = 23 + 15v_0 + 7v_1$ which is not possible as $v_2 \leq v_0$. In one-dimension we just need a graph with $v_1 = 3 + v_0$ to assure $(v_0 + 2v_1)/(1 + v_0 + v_1) = 3/2$. There are many one-dimensional simplicial complexes beside the utility graph for which this happens.

2. DIMENSION EXPECTATION

2.1. If G is a finite abstract simplicial complex of maximal dimension d, its *f*-**vector of** G is $(v_0, v_1, \cdots, v_d)$, where v_k is the number of k-dimensional simplices in G. If $f(t) = 1 + v_0 t + v_1 t^2 + \cdots + v_d t^{d+1}$ is the **simplex generating function**, then the **average simplex cardinality** is defined as

$$\mathrm{Dim}^+(G) = \sum_{x \in G} \frac{\dim(x) + 1}{|G| + 1} = \frac{f'(1)}{f(1)} \ .$$

2.2. We are also interested in the logarithm $g(t) = \log(f(t))$ because $g_G(-1) = \log(1 - \chi(G))$ is the logarithm of the genus which is additive and because $g'_G(1) = \mathrm{Dim}^+(G)$ is the average simplex cardinality. The rational function

$$g'(t) = \frac{f'(t)}{f(t)}$$

is therefore of interest:

Lemma 1. $g' = f'/f : \mathcal{G} \to \mathbb{Q}(t)$ *is an additive homomorphism:*

$$g_{G \oplus H}(t) = g_G(t) + g_H(t)$$

For any $a > 0$, the functional $G \to f_G(a)/f'_G(a)$ is an additive homomorphism from the join-monoid of simplicial complexes to $\mathbb{Q}$.

Proof. If follows immediately from $f_{G \oplus H} = f_G f_H$ which implies that any derivative of $\log(f_G)$ is additive.

The second statement follows from the fact that if f is a polynomial with positive entries, then all roots of f and f' are negative. $\qquad\square$

2.3. It follows that we can relate the value of g on a **unit ball** $B(x)$ with the value of the **unit sphere** $S(x)$ as $B(x) = S(x) \oplus K_1$.

Corollary 1 (Unit sphere and Unit ball dimension). $g_{B(x)}(t) = g_{S(x)}(t) + 1/(1+t)$. *Especially*
$$\text{Dim}^+(B(x)) = \text{Dim}^+(S(x)) + 1/2 .$$

Proof. $f_{B(x)}(t) = f_{S(x)}(t)(1+t)$ and $g_{B(x)}(1) = g_{S(x)}(1) + 1/2$. $\qquad\square$

2.4. We can evaluate the rational function at some points to get additive homomorphisms. For example $G \to f'_G(-1)/f_G(-1)$ is an additive homomorphism. It is sort of a **super cardinality average** as the degree $f(-1) = 1 - \chi(G)$ is sort of a super cardinality of the complex and $f'(-1)$ counts up the simplices with sign. But $f'(-1)/f(-1)$ is only defined for complexes G which do not have Euler characteristic $\chi(G) = 1$. Evaluating $f'(t)/f(t)$ for positive t however is never a problem as all roots of f are negative (as this holds for any polynomial with non-negative entries).

3. EXAMPLES

3.1.

Proposition 1.
$$\text{Dim}^+(K_n) = \frac{n}{2} = \frac{\dim^+(K_n)}{2} .$$
If the equality $\text{Dim}^+(G) = \dim^+(G)/2$ *holds, then* G *is* K_n.

Proof. For complete graphs K_n, we have $f'(t)/f(t) = n/(1+t)$ because $f_G(x) = (1+t)^n$ as $K_n = K_1 \oplus K_1 \cdots \oplus K_1$ is the join of n one-point graphs K_1. We therefore have $f'(1)/f(1) = n/2$. The inequality follows from $\dim^+(K_n) = n$. $\qquad\square$

3.2. The **cross polytopes** S^d are d-spheres which can be written as an iterated suspension $2 \oplus 2 \cdots \oplus 2$, where $G = 2$ is the 0-sphere, which is the zero-dimensional complex with 2 vertices. For $d = 1$, the cross-polytop is C_4 and for $d = 2$, it is the octahedron. For the 0-sphere 2, we have $f(t) = 1 + 2t$ so that $f'(1)/f(1) = 2/3$. Therefore, the average dimension is $f'(t)/f(t) = (d+1)2/(1+2t)$. This gives for the 0-sphere 2/3. For 1-sphere 4/3.

3.3. It follows quite directly from the **Dehn-Sommerville relations** that for any even dimensional sphere the function f always has a root at $x = -1/2$ while the other roots pair up to -1 (we will explore this a bit more elsewhere). For every odd-dimensional sphere in particular, the poles of f pair up to -1. For us, the Dehn-Sommerville topic is both a subject of excitement as well as of disappointments as we have met manifestations of it being under the impression to "discover" it: a first case was the fact that the eigenvectors of the transpose A^T of the Barycentric refinement operator A_d define functionals which lead to quantities which must be zero for manifolds [11]. Also the occurrence of $-1/2$ as roots of the simplex generating functions of even dimensional spheres first led to excitement until realizing that it is an other manifestation of Dehn-Sommerville.

3.4. To see a factor $1+2x$ in the simplex generating function f_G for even-dimensional sphere G is not so obvious because the roots in general change with homotopy deformations. The root at $x = -1/2$ also appears for other discrete manifolds with Euler characteristic 2 like the disjoint union G of two projective planes. The root $-1/2$ happens also for homology spheres (a complex which is not a sphere but which has the same homology and especially the same cohomology than the standard 3-sphere). There are homology spheres however, where the roots can become complex. And of course, because f has positive entries, all real roots of all derivatives of f are negative and in the sphere case are contained in the open interval $(-1, 0)$.

4. Lower bound from inductive dimension.

4.1. We now compare the two additive functionals $\dim^+$ and Dim^+. We consider it the main result of this paper as it is not totally obvious.

Theorem 1. *For all simplicial complexes G, the inequality*

$$\frac{\dim^+(G)}{2} \leq \mathrm{Dim}^+(G)$$

holds.

4.2. Examples.
1) For complexes with maximal dimension 0, the number n of 0-dimensional sets determines the complex. The left hand side $\dim^+(G)/2$ is then always 1/2. The right hand side is $n/(n+1)$. They are equal for $n = 1$ only.
2) For complexes with maximal dimension 1, there are two numbers n, m which determine the complex and $\dim^+(G)/2$ is in the interval $(1/2, 1]$. Now, because $m \geq 1$ (we would with $m = 0$ have a zero dimensional complex), we have $\mathrm{Dim}^+(G) = (n+2m)/(n+m+1) \geq 1$. There is only one possibility that $(n+2m)/(n+m+1) = 1$ and that is $m = 1$. If $m = 1$ and n is larger than 2, then $\dim^+(G)/2 < 1$. So, the inequality is strict if G is not the complete one-dimensional complex.
3) For circular graphs C_n in particular we have $\dim^+(C_n) = 2$ and $\mathrm{Dim}^+(C_n) = 3n/(2n+1)$.

4.3. If f is a functional on simplicial complexes, denote by $\mathrm{E}_G[f]$ the **expectation** of f when evaluated on all unit spheres $S(x)$ of G:

$$\mathrm{E}_G[f] = \sum_{x \in G} f(S(x))/|G| \ .$$

Lemma 2. *a)* $\mathrm{Dim}^+(G) \geq \frac{1}{2} + \mathrm{E}_G[\mathrm{Dim}^+]$.
b) If equality holds in a), then $G = K_n$.

Proof. a) Every simplex X in $S(x)$ corresponds to a simplex $X \cup x$ in G and its average count is 1/2 larger. This is reflected in the formula $\mathrm{Dim}^+(B(x)) = 1/2 + \mathrm{Dim}^+(S(x))$. Now we can use an elementary averaging (conditional expectation) result for finite probability spaces Ω: if A_n is a covering of Ω and X is a non-negative random variable, then $\mathrm{E}_G[E[X|A_j]] \geq \mathrm{E}_G[X]$. This becomes even strict if we average by the cardinality $+1$ rather than the cardinality. We can apply this

now to the situation where the unit balls $B(x)$ cover G.

b) Equality holds in the later case if and only if $j \to E_G[X|A_j]$ is constant which means that G is a complete complex. If $A_j = S(x_j)$ is not a complete graph at some point, we have an inequality. $\square$

4.4. Let's now turn to the proof of the theorem (1) which states that $\dim^+(G)/2 \le \text{Dim}^+(G)$.

Proof. We prove this by induction with respect to the maximal dimension d. Assume things work in maximal dimension $(d-1)$. This means $\dim^+(S(x))/2 \le \text{Dim}^+(S(x))$ or

$$\dim^+(S(x)) \le 2\text{Dim}^+(S(x)) .$$

Use the inductive definition of dimension on the left and the definition of expectation on the right, we get

$$\dim^+(G) - 1 \le 2E_G[\text{Dim}^+] .$$

From the lemma, which tells that

$$E_G[\text{Dim}^+] \le \text{Dim}^+(G) - \frac{1}{2}$$

we conclude

$$\dim^+(G) - 1 \le 2\text{Dim}^+(G) - 1 .$$

After adding 1 on both sides and dividing by 2 we end up with the claim of the theorem. $\square$

4.5. The inequality is sharp for complete complexes K_n. It is strict for discrete unions of complete complexes and gets even bigger if complete complexes are joined along lower dimensional complexes.

4.6. We also tried first to see what happens if we look at the complex as a CW complex and add a cell. Both the inductive dimension and average dimension are still defined then. What happens is that we change the f-vector at one point but it is not clear what happens with $f'(1)/f(1)$ after such a choice. There is no question however that the inequality holds for structures which are more general than simplicial complexes. Discrete CW complexes are examples.

5. ARITHMETIC COMPATIBILITY

5.1. A recent result of Betre and Salinger [1] tells that

$$\dim^+(G \oplus H) = \dim^+(G) + \dim^+(H) .$$

This result for inductive dimension is not so obvious. It is much easier to prove the same formula for the average simplex cardinality. There is no relation however and the following result does not imply the Betre-Salinger equality:

Proposition 2. $\text{Dim}^+(G \oplus H) = \text{Dim}^+(G) + \text{Dim}^+(H)$.

Proof. As $\text{Dim}^+(G) = f'_G(1)/f_G(1)$, this immediately follows from the Leibniz product rule $(fg)'/(fg) = f'/f + g'/g$, which holds for all t. $\square$

5.2. This formula brings in arithmetic compatibility similarly as the **genus functional**

$$\text{gen}(G) = 1 - \chi(G) \,,$$

where χ is the Euler characteristic. The identity

$$\text{gen}(G \oplus H) = \text{gen}(G)\text{gen}(H)$$

was useful for example when looking at the hyperbolic decomposition of a unit sphere with respect to the counting function $|x|$ on G which is Morse: we have $S(x) = S^-(x) \cup S^+(x)$, where $S^-(x)$ is the sphere consisting of all simplices contained in x and $S^+(x)$ is the **star** of x, the set of simplices containing x. Unlike $S^-(x)$ which is a simplicial complex, the star is in general not a simplicial complex, but its Barycentric refinement is.

5.3. Because $\text{gen}(S^-(x)) = \omega(x) = (-1)^{|x|-1}$ we got the formula $\text{gen}(S(x)) = 1 - \chi(S(x)) = \omega(x)\text{gen}(S^+(x))$. Because $\text{gen}(S^+(x))$ is a **Poincaré-Hopf index** of the function $g(x) = -|x|$, we had $\sum_x \omega(x)(1 - \chi(S(x))) = \chi(G)$ which is a McKean-Singer type formula, the reason for the terminology being that the super trace of $g = L^{-1}$ is $\chi(G)$, where L is the connection Laplacian of G, the reason being that the diagonal entries of $g(x,x)$ is the genus $1 - \chi(S(x))$ of the unit sphere $S(x)$. The Euler characteristic is by definition also the super trace $\sum_x \omega(x)L(x,x)$ of L. The McKean-formula for the Hodge Laplacian is $\text{str}(e^{-tH}) = \chi(G)$ which holds not only in differential geometry but works for simplicial complexes.

5.4. A consequence of the Betre-Salinger identity is that one can naturally extend the inductive dimension $\dim^+$ to the Zykov group defined by doing the **Grothendieck completion** of the monoid $\oplus$. After that completion, there are complexes of positive sign as well as of negative sign, completely analogue as the negative numbers were introduced in arithmetic. By extending the functional additively, we see that a negative complex $-G$ of a standard simplicial complex has a negative dimension. We summarize:

Corollary 2. *Inductive, average and maximal dimensions all extend to an additive group homomorphism from the Zykov group of simplicial complexes and gives a result in the additive group $\mathbb{Q}$. The zero element, the empty complex 0, gets mapped to 0.*

5.5. Extending the genus $1 - \chi(G)$ to the Zykov ring looks trickier at first because $-G$ is $1/\text{gen}(G)$ which is infinite if the genus of G is zero. The simple remedy is to look at the generating function $f_G(t) = 1 + v_0 t + v_1 t^2 + \cdots$ with f-vector $(v_0, v_1, \dots)$ of G. which is multiplicative

$$f_{G \oplus H}(t) = f_G(t)f_H(t) \,.$$

It is therefore possible to extend f_G to the Zykov group, where it becomes a rational function. Extending the rational function to the Zykov group is not a problem, extending particular values can be, as the values can become infinity at poles of the rational function.

6. Barycentric refinement

6.1. The **Barycentric refinement** G_1 of G is a new complex which can be seen as the set of vertex sets of complete subgraphs of the graph with vertex set G and edge set (x, y) for which either $x \subset y$ or $y \subset x$. One can also define it without graphs as the **order complex** of G. The sets of G_1 are all subsets A of 2^G which have the property that all elements of A are pairwise contained in each other.

6.2. The Barycentric refinement process has some nice limiting properties. (See [12] for an overview.) An example is that the spectra of the Kirchhoff Laplacian $L(G)$ have a **density of states** which converge to a limiting measure which only depends on the maximal dimension of the complex [10]. An other example is that the zeta function of the connection Laplacian converges in the one-dimensional case to a nice limiting zeta function. Unlike for the Kirchhoff Laplacians, where the limiting zeta function is difficult [14], the connection Laplacian case is easier [13] for zeta and because the connection Laplacians are unimodular.

6.3. Depending on the maximal dimension d there is a $(d + 1) \times (d + 1)$ matrix

$$A_{ij} = \text{Stirling}(j, i)i!$$

such that $f_{G_1} = A f_G$. While this induces a linear map on $f'_G(1)$, it only induces an affine map on $f_G(1)$. It also induces a transformation on the probability vector $f/|f|_1$. Here is the matrix A in the case $d = 10$:

$$A_{10} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 & 30 & 62 & 126 & 254 & 510 & 1022 & 2046 \\ 0 & 0 & 6 & 36 & 150 & 540 & 1806 & 5796 & 18150 & 55980 & 171006 \\ 0 & 0 & 0 & 24 & 240 & 1560 & 8400 & 40824 & 186480 & 818520 & 3498000 \\ 0 & 0 & 0 & 0 & 120 & 1800 & 16800 & 126000 & 834120 & 5103000 & 29607600 \\ 0 & 0 & 0 & 0 & 0 & 720 & 15120 & 191520 & 1905120 & 16435440 & 129230640 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5040 & 141120 & 2328480 & 29635200 & 322494480 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 40320 & 1451520 & 30240000 & 479001600 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 362880 & 16329600 & 419126400 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3628800 & 199584000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 39916800 \end{bmatrix}.$$

6.4. The eigenvalues are the diagonal entries $\lambda_k = k!$. The eigenfunctions v_k are the **Perron-Frobenius eigenvectors** of A_k. They all have non-negative components. The matrix is only non-negative but has the property that given a vector with positive entries (as must happen for f-vectors of simplicial complexes), the map $\overline{f} \to \overline{Af} = \overline{Af}$ on directions $\overline{f}/|f|$ is a contraction. There exists therefore a unique fixed point of $\overline{A}$. This is the Perron-Frobenius eigenvector of A and is the stable probability distribution $f/|f|_1$ in dimension d. Since the limiting mean of this probability distribution is the limiting average simplex cardinality, we have

Theorem 2. *Given a simplicial complex G of maximal dimension d. The successive Barycentric refinements G_n have a unique limit $\lim_{n \to \infty} \text{Dim}^+(G_n)$ which is $f_d \cdot$*

$(1, 2, \cdots, d+1)/|f_d|_1$, *where f_d is the unique Perron-Frobenius probability eigenvector of A_d.*

6.5. To compare with inductive dimension $\dim^+$: if the maximal dimension of G is d, then $c_d = \lim_{n \to \infty} \dim^+(G_n) = d + 1$ because the largest dimensional simplices replicate faster than lower dimensional ones. On the other hand, for the average simplex dimension Dim^+, the constant C_d is always in the interval $[(d+1)/2, (d+1))$. If d is positive then

$$C_d \in ((d+1)/2, d+1) .$$
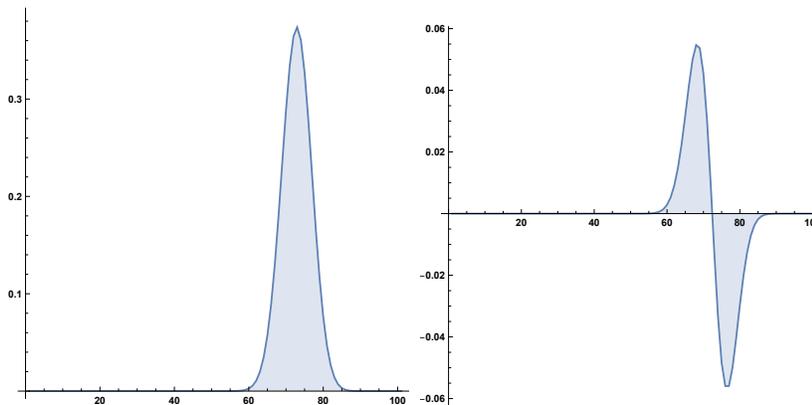
FIGURE 2. The Perron-Frobenius eigenvector f of A_d in the case $d = 100$. This vector defines the limiting distribution of the simplex cardinality for the limiting Barycentric refinement in dimension $d = 100$. In the Barycentric limit, most simplices have about dimension 75. To the right we show the discrete derivative $f'(k) = f(k+1) - f(k)$. The mean of f is 72.828.... This is the expectation $\mathrm{Dim}^+(G_n)$ in the limit $n \to \infty$. The actual limit is in dimension $d = 100$ a rational number p/q, with 4423 digit integers p, q.

7. REMARKS

7.1. Various notions of dimension exist for discrete geometries. For finite abstract simplicial complexes one has the **maximal dimension** $\max(G) = \max_{x \in G} |x| - 1$ which counts the dimension of the largest simplex $x \in G$. The **inductive dimension** [8, 6] motivated from the corresponding notion for topological spaces [5] starts with the assumption that the empty complex has dimension -1 and then defines the dimension of of a set of non-empty sets as 1 plus the average dimension of the unit spheres $S(x)$ in G, where $S(x)$ is the pre-simplicial complex given by the set of sets in G which either contain x or are contained in x.

7.2. For graphs $\Gamma = (V, E)$, the inductive dimension is one plus the average dimension of unit spheres $S(v)$ where the average is taken over all $v \in V$. The inductive dimension of the Whitney complex of Γ, (the vertex sets of complete subgraphs of G). The inductive dimension of that complex G is larger or equal

than the dimension of Γ. The Barycentric refinement of a complex, the order complex of G is always the Whitney complex of a graph, but there are complexes like $C_3 = \{\{1,2\}, \{2,3\}, \{3,1\}\}$ which are not Whitney complexes.

FIGURE 3. The icosahedron complex G with $f_G(t) = 1 + 12t + 30t^2 + 20t^2$ and its first two refinements G_1, G_2. The inductive dimension of G_k is $\dim(G_k) = 2$ so that $\dim^+(G_k)/2 = 3/2$. The average simplex cardinality are $\mathrm{Dim}^+(G_0) = (1*12+2*30+3*20)/(1+12+30+20) = 44/21 = 2.09..$, $\mathrm{Dim}^+(G_1) = 782/363 = 2.15..$ and $\mathrm{Dim}^+(G_2) = 4682/2163 = 2.16...$ The limiting dimension $\lim_{n\to\infty} \mathrm{Dim}^+(G_n)$ is $(1,3,2) \cdot (1,2,3)/(1+3+2) = 13/6 = 2.1666...$

7.3. Every simplicial complex G also defines a graph, in which $V = G$ are the vertices and two simplices are connected by an edge if one is contained in the other. The **embedding dimension** of a graph [15] is the minimal d such that G can embedded in $\mathbb{R}^d$ so that every edge length is 1. The **metric dimension** [4] is the minimal cardinality of a subset C such that all other vertices are uniquely determined by the distances to vertices in C. Also motivated by corresponding notions put forward by Lebesgue and Čech are notions of **covering dimension** which looks at the minimal order of an open cover, the maximal cardinality of open sets which a point can contain. In the discrete, this depends on the type of coverings chosen as open sets.

7.4. Both the maximal dimension as well as the inductive dimension are intuitive as they agree with classical topological dimensions in the case of discrete manifolds or varieties and both have arithmetic compatibility with the join operation. The inductive dimension captures subtleties of general networks as it allows for fractional dimension like **Hausdorff dimension** and also shares an inequality $\dim(X \times Y) \geq \dim(X) + \dim(Y)$ [9] which parallels the same inequality for Borel sets and Hausdorff dimension in the continuum [3]. (Of course, the notions in the discrete and the continuum are completely different but the analogy is remarkable).

7.5. We study here a dimension Dim which is of a more statistical nature as it counts the average dimension of the building blocks of a simplicial complex G. Unlike the inductive dim or maximal dimensions max which are rather "plain" for discrete manifolds or varieties (or more generally for homogeneous =pure simplicial complexes,

the averaging dimension adds additional information for manifolds. It also shares the arithmetic compatibility property $\mathrm{Dim}(G \oplus H) = (\mathrm{Dim}^+(G) + \mathrm{Dim}(H) - 1$ with the same property for maximal dimension $\max(G \oplus H) = \mathrm{maxdim}(G) + \mathrm{maxdim}(H) - 1$ and for inductive dimension $\dim(G \oplus H) = \dim(G) + \dim(H) - 1$. It is this -1 discrepancy which makes the use of the augmented versions $\mathrm{Dim}^+, \dim^+, \max$ more attractive.

8. QUESTIONS

8.1. (A) The answer to the question

Question 1. *Is* $\mathrm{Dim}^+(G_1) \geq \mathrm{Dim}^+(G)$ *always true?*

has been "yes" in all cases seen so far. It is a linear algebra problem: we have to show that f-vectors satisfying the **Kruskal-Katona** conditions have the property

$$\frac{(Af \cdot v)}{(1 + |Af|_1)} \geq \frac{(f \cdot v)}{(1 + |f|_1)} ,$$

where v is the Perron-Frobenius eigenvector of the **Barycentric refinement matrix**

$$A_{ij} = i!\mathrm{Stirling}(j, i) .$$

The inequality for vectors f does not hold for all vectors f but it appears to be true however for f-vectors which appear for simplicial complexes.

8.2. Example: for the first complete graphs $G = K_n$, we get $\mathrm{Dim}^+(G_1) - \mathrm{Dim}^+(G) = \delta_k$ with

$$\delta = (0, 1/6, 5/13, 91/150, 448/541, ...)$$

8.3. (B) One can ask whether

Question 2. $\mathcal{G}_{p/q} = \{G \mid f'_G(1)/f_G(1) = p/q\}$ *is finite on the class of varieties* G

(d-varieties as defined recursively as complexes for which all unit spheres $S(x)$ are $(d-1)$-varieties with the induction assumption that the empty complex is a -1 dimensional variety.)

8.4. The set $\mathcal{G}_{p/q} = \{G \mid \mathrm{Dim}^+(G) = p/q\}$ is infinite for arbitrary large one dimensional complexes with $v_0 = n, v_1 = m$ for example satisfying $n + 2m/(n+m+1) = 4/3$ by picking a number m of edges, then choosing $n = 2m - 4$. But there are only finitely many examples which are varieties. By adding isospectral examples, one get arbitrary large, arbitrary high dimensional complexes with the same $\mathrm{Dim}^+(G) = c$.

8.5. (C) The following question could have a simple answer, we just don't know yet. It is a pure linear algebra question. The Perron-Frobenius eigenvector $v_d = (v_0, \ldots, v_d)$ of the matrix A_d defines a probability distribution f_d on $[0, 1]$ by defining the real function $f_d(x) = v_d([d * x])$ normalized so that it is a probability density function. The sequence f_d converges in the weak-* sense as measures to a Dirac measures 1_{x_0} with $x_0 \sim 0.72....$

Question 3. *Can we locate x_0?*

Figure (6.5) shows the case $d = 100$, where we see the Perron-Frobenius eigenvector to the eigenvalue $\lambda = 100!$.

8.6. (D) The average inductive dimension $\mathrm{E}_p[\dim]$ on the Erdös-Renyi space $G(n, p)$ satisfies

$$d_{n+1}(p) = 1 + \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} d_k(p) ,$$

where $d_0 = -1$. Each d_n is a polynomial in p of degree $\binom{n}{2}$. We have for example:

$$d_1(p) = 0, d_2(p) = p, d_3(p) = p(2 - p + p^2), d_4(p) = p(3 - 3p + 4p^2 - p^3 - p^4 + p^5) .$$

Question 4. *What is the average simplex cardinality on $G(n, p)$?*

8.7. By counting through all graphs, we got the following values for $p = 1/2$. For $n = 1$ it is $1/2$, for $n = 2$ it is $5/6$ for $n = 3$, it is $35/32$ for $n = 4$ it is $6593/5040$ for $n = 5$ it is $18890551/12673024$.

8.8. We would have to be able to compute $\mathrm{E}_p[\log(f(t))]$ as a function of p and t and then differentiate at $t = 1$ to get the average simplex cardinality. While we know the expectation of the simplex generating function $f(t)$ itself, we do not have formulas for moments $\mathrm{E}_p[f(t)^k]$.

8.9. (D) Among all graphs with n vertices, which one maximizes

$$\delta(G) = \mathrm{Dim}^+(G) - \dim^+(G)/2 ?$$

We know it is the cyclic graph $C_4 = K_{2,2} = 2 \oplus 2$ in the case $n = 4$ where it the difference is $1/3$ and the bipartite graph $K_{3,3} = 3 \oplus 3$ (utility graph) in the case $n = 6$, where the difference is $1/2$. As $f_{K(n,n)}(t) = 1 + 2nt + n^2t^2$ we have $\mathrm{Dim}^+(K_{n,n}) = (2n + 2n^2)/(1 + 2n + n^2) = 2n/(1 + n)$ and $\delta(K_{n,n}) = (n-1)/(n+1)$. The bipartite graphs $K_{n,n}$ can not be maximal in general as the maximum goes to infinity for $n \to \infty$ and $\delta(K_{n,n}) < 1$.

Question 5. *For which graphs with n vertices is $\delta(G)$ maximal?*

8.10. (E) Having looked at the average cardinality $m(G) = \mathrm{Dim}^+(G)$, one can wonder about the **variance** $\mathrm{Var}^+(G) = \sum_{x \in G}(|x| - m(G))^2/|G^+|$ and **higher moments** $\sum_{x \in G}(|x| - m(G))^k/|G^+|$ and their behavior in the Barycentric limit or its typical value on Erdös-Rényi spaces.

Question 6. *Are there a limiting laws for the moments of the simplex cardinality?*

8.11. For complete complexes K_n, we computed the variance $v_n = \mathrm{Var}^+(K_n)$ and got the values $v_1 = 1/8, v_2 = 1/4, v_3 = 15/32, v_4 = 3/4, v_5 = 135/128$. For example, for $G = K_1 = \{\{1\}\}$, we have $\mathrm{Dim}^+(G) = 1/2$ and $\mathrm{Var}^+(G) = (1 - 1/2)^2/2 = 1/8$. The sequence v_n grows asymptotically linearly as the variance of the simplex cardinality grows exponentially under Barycentric refinements. It must do so as there is a limiting distribution given by the Perron-Frobenius eigenfunction of the Barycentric refinement operator shown in Figure (6.5) in the case $d = 100$.

9. ILLUSTRATIONS
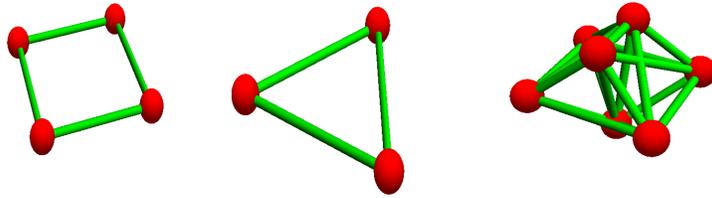
FIGURE 4. The cyclic graph C_4 has average simplex cardinality $(4 + 4*2)/9 = 4/3$. The complete graph K_3 has average simplex cardinality $(3 + 3*2 + 1*3)/8 = 3/2$. The join $C_4 \oplus K_3$ is 4-dimensional with f-vector $(7, 19, 25, 16, 4)$ and average simplex cardinality $(7 + 19*2 + 25*3 + 16*4 + 4*5)/(1 + 7 + 19 + 25 + 16 + 4) = 17/6$. The example illustrates the compatibility with join: $4/3 + 3/2 = 17/6$.
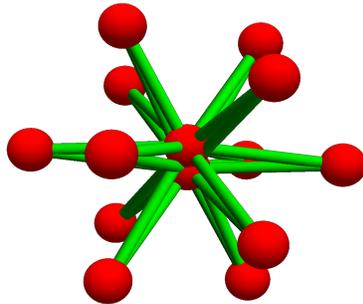

FIGURE 5. The join of two point graphs 12 and 2 gives the bipartite graph $K_{12,2}$. We have $\dim^+(12) = \dim^+(2) = 1$ and $\dim^+(K_{12,2}) = 2$. Also $\mathrm{Dim}^+(12) = 12/13, \mathrm{Dim}^+(2) = 2/3$ and $\mathrm{Dim}^+(K_{12,2}) = 62/39$. This illustrates the additivity of $\dim^+$ and Dim^+.
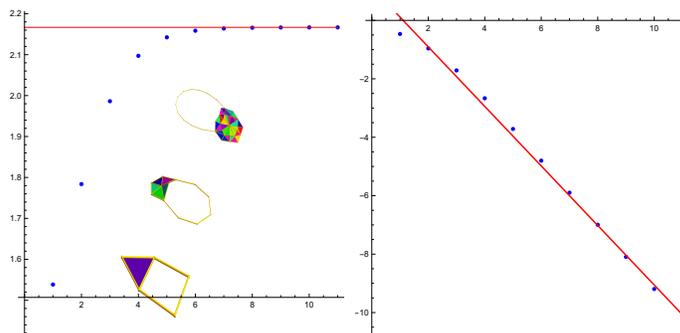
FIGURE 6. We see the average simplex density for the house graph and the first Barycentric refinements. The upper bound is $c = 13/6 = (1, 3, 2) \cdot (1, 2, 3)/(1 + 3 + 2)$ determined by the Perron-Frobenius eigenvector $(1, 3, 2)$ of the Barycentric refinement operator A. The right picture shows $\log(\text{Dim}^+(G_n) - c)$. It is a monotone sequence if $\text{Dim}^+(G_1) \geq \text{Dim}^+(G)$ holds. Verifying this inequality would be a linear algebra problem. Does the Kruskal-Katona constraint for f-vectors imply that $(Af) \cdot (1, \cdots, d)/((Af) \cdot (1, \ldots, 1))$ is larger or equal than $f \cdot (1, \cdots, d)/(f \cdot (1, \cdots, 1))$?
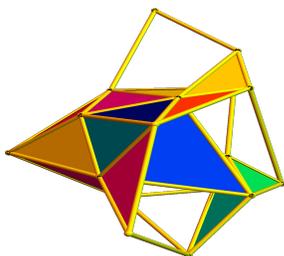

FIGURE 7. A graph with f-vector $(15, 36, 16, 1)$. Its inductive dimension is $8587/4725 = 1.817...$. Its average simplex cardinality is $139/69 = 2.01$. The augmented volume is $f(1) = 1 + 15 + 36 + 16 + 1 = 69$. The derivative f' of the simplex generating function is $f'(t) = 15 + 72t + 48t^2 + 4t^3$ which satisfies $f'(1) = 139$. The inequality $1.4087... = (1 + \dim(G))/2 \leq \text{Dim}^+(G) = 2.01...$ is satisfied. The Barycentric refinement G_1 satisfies $\dim(G_1) = 131002727/65345280 = 2.0048...$. It is an old theorem which assures $\dim(G_1) \geq \dim(G)$. We have $\text{Dim}^+(G_1) = 84/37 = 2.270...$ which is larger than $\text{Dim}^+(G) = 139/69 = 2.014...$. It is one of the open problems to prove this.
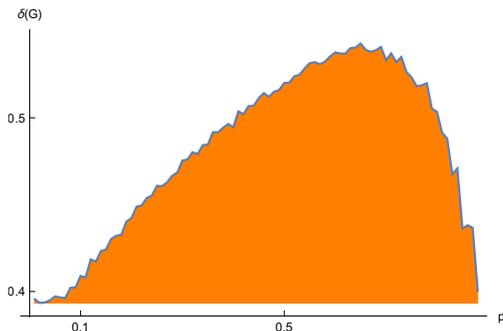
FIGURE 8. We see $\delta(G) = \text{Dim}^+(G) - \dim^+(G)/2$ which according to the theorem is positive. Every point is an average over 1000 Erdös-Renyi graphs [2] with 10 vertices and edge probability p. The difference $\delta(G)$ is quite significant but for $p = 1$, where we have the complete graph, it drops down to 0. For $p = 0$, where $\text{Dim}^+(G) = n/(n+1)$ and $\dim^+(G) = 1$ the difference is $n/(1+n) - 1/2$ which is in the case $n = 10$ equal to $9/22$.
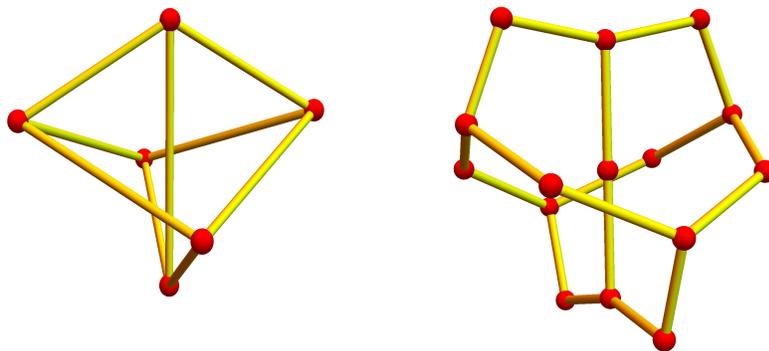

FIGURE 9. Among all connected graphs with 6 vertices, the **utility graph** $K_{3,3}$ is the one with maximal $\delta(G) = \text{Dim}^+(G) - \dim^+(G)/2$. It is a 3-regular graph of diameter 2 which can not be embedded of the complete bipartite graph $K_{3,3} = 3 \oplus 3$ in $\mathbb{R}^2$. The picture shows a natural embedding in $\mathbb{R}^3$. In the case $n = 2$, the extreme case was $C_4 = 2 \oplus 2$. To the right, we see the Barycentric refinement, which has the same $\delta(G)$. The case $\delta(G_1) = \delta(G)$ can not hold any more in higher dimensions.

9.1. The figures were generated in Mathematica. We give some code in the next section. As usual, the code can be copy-pasted from the ArXiv posting of this paper. The Wolfram language also serves as good pseudo code which could with ease be translated into any other programming language.

10. Code

10.1. The following Mathematica code implements the computation of the augmented inductive dimension and the augmented average simplex dimension and computes it for some random simplicial complex. It then compares the two numbers $\dim^+(G)/2$ and $\mathrm{Dim}^+(G)$ which appear in the inequality.

```
Generate[A_]:=Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]
R[n_,m_]:=Module[{A={},X=Range[n],k},Do[k:=1+Random[Integer,n-1];
  A=Append[A,Union[RandomChoice[X,k]]],{m}];Generate[A]];
Fvector[G_]:=Delete[BinCounts[Map[Length,G]-1],1];
Vol[G_]:=Total[Fvector[G]]+1;
f[G_]:=Module[{fv=Fvector[G]},1+Sum[ff[[k]]*t^k,{k,Length[ff]}]];
Dim[G_]:=Range[Length[Fvector[G]]].Fvector[G]/Vol[G];
Var[G_]:=Module[{m=Dim[G],v=Fvector[G],V=Vol[G]},
   v.Table[(k-m)^2,{k,Length[v]}]/V];
sphere[H_,x_]:=Module[{U={},n=Length[H]},Do[
   If[Sort[H[[k]]]!=Sort[x] && (SubsetQ[x,H[[k]]]
        || SubsetQ[H[[k]],x]),U=Append[U,H[[k]]]],
   {k,Length[H]}]; Table[Complement[U[[k]],{x}],{k,Length[U]}]];
dim[H_,x_]:=Module[{S},If[Length[H]==0,S={},S=sphere[H,x]];
   If[Max[Map[Length,H]]<=1,0,If[S=={},0,dim[S]]]];
dim[H_]:=Module[{n=Length[H],U},
   If[n==0,U={},U=Table[sphere[H,H[[k]]],{k,n}]];
   If[Length[U]==0,-1,Sum[1+dim[U[[k]]],{k,n}]/n]];
JoinComplex[G_,H_]:=Module[{HH=H+Max[G]+1,K}, K=Union[G,HH];
   Do[Do[K=Append[K,Union[G[[k]],HH[[l]]]],
   {k,Length[G]}],{l,Length[HH]}];K];

Do[G=R[6,6];Print[N[{(dim[G]+1)/2,Dim[G]}]],{n,5}]
CompleteComplex[n_]:=Generate[{Range[n]}];
Do[G=CompleteComplex[n];Print[{(dim[G]+1)/2,Dim[G]}],{n,2,5}]
CycleComplex[n_]:=Table[{k,Mod[k,n]+1},{k,n}];
Do[G=CycleComplex[n];Print[{(dim[G]+1)/2,Dim[G]}],{n,3,6}]

House =Generate[{{2,3,5},{1,4},{1,2},{3,4}}];
Rabbit=Generate[{{1,2,3},{3,4},{3,5}}];
RabbitHouse=JoinComplex[Rabbit,House];
{Dim[House],Dim[Rabbit],Dim[RabbitHouse]}
{dim[House]+1,dim[Rabbit]+1,dim[RabbitHouse]+1}  (* Patience! *)
```

10.2. Note that in the just done computation of the inductive dimensions of the house graph and rabbit graphs, we compute directly the dimensions of the Barycentric refinements (the Whitney complexes of the graphs) because we average over the simplices. The just given code gives for the average cardinalities $\mathrm{Dim}^+(\mathrm{House}) = 20/13$, $\mathrm{Dim}^+(\mathrm{Rabbit}) = 3/2$ and $\mathrm{Dim}^+(\mathrm{RabbitHouse}) = 79/26$ and for the inductive dimensions $\dim^+(\mathrm{House}) = 61/24$, $\dim^+(\mathrm{Rabbit}) = 13/5$, and $\dim^+(\mathrm{RabbitHouse}) =$

617/120. In Figure (1.4), we have noted the dimensions of the graphs themselves. The code for that is open source and given for example in [7].

10.3. We compute now the constants C_d, then the limiting value $C_d/d = 0.72...$ which for $d = 500$ is 0.722733. The limiting value is a real number which we can not yet identify. Is $\lim_{d\to\infty} C_d/d$ rational or irrational for example? It it algebraic or transcendental?

```
A[n_]:=Table[StirlingS2[j,i]*i!,{i,n+1},{j,n+1}];
EV=Eigenvectors;
c[k_]  :=Module[{v=First[EV[    A[k]]]},v.Range[k+1]/Total[v]];
cN[k_]:=Module[{v=First[EV[1.0*A[k]]]},v.Range[k+1]/Total[v]];

ListOfConstants        = Table[c[d],{d,0,10}]
LimitingValue          = cN[500]/500
DimensionIntervals     = Table[N[{(d+1)/2,cN[d],d+1}],{d,0,10}]
```

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