GAUSS-BONNET FOR FORM CURVATURES

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ABSTRACT. We look at curvatures that are supported on k-dimensional parts of a simplicial complex G. These curvature all satisfy the Gauss-Bonnet theorem, provided that the k-dimensional simplices cover G. Each of these curvatures can be written as an expectation of Poincare-Hopf indices. Linear or non-linear wave dynamics with discrete or continuous time allow to deform these curvatures while keeping the Gauss-Bonnet property.

1. INTRODUCTION

1.1. We start our story with a familiar special case of a convex polyhedron G. If G has $|V|$ vertices, $|E|$ edges and $|F|$ triangular faces, the **Euler polyhedron formula** $\chi(G) = |V| - |E| + |F| = 2$ holds. Discovered empirically first by Descartes [1], then proven by Euler in more generality what we call today **planar graphs** [11]. The polyhedral case needed time to stabilize. [34] made it a case study in the epitemology of sciences. The book of [40] illustrates the drama. Various combinatorial notions of curvature have been suggested. First maybe by Victor Eberhard (shown in Figure 6.3) [9]. The curvature $K(v) = 1 - d(v)/6$ with vertex degree $d(v)$ was used already in graph coloring contexts [5]. If all faces are triangular and no boundary is present, then the **Gauss-Bonnet formula** $\chi(G) = \sum_{v \in V} K(v)$ hold. To prove Gauss-Bonnet think about $\omega(x) = (-1)^{\dim(x)}$ at the "energy" [1] of $x \in V \cup E \cup F$. Euler characteristic $\chi(G) = \sum_{x \in G} \omega(x)$ is the total energy. Move the energy from every x equally to all vertices in x: every edges gives $-1/2$ to its vertices, every triangle gives $1/3$ to its vertices. 1 ends up at the vertex. Then $K(v) = 1 - d(v)/2 + d(v)/6 = 1 - d(v)/6$. More references are [14, 39, 15, 38, 10].

1.2. Combinatorial curvatures exist for any simplicial complex and especially all complexes that come from graphs, where the k-simplices are complete sub-graphs K_{k+1}. The **Levitt curvature** [?] on vertices $v \in V = \{x \in G, |x| = 1\}$ is

$$K(v) = \sum_{x \in G, v \in x} \frac{\omega(x)}{|x|}$$

with $\omega(x) = (-1)^{\dim(x)}$. It is a vertex curvature which works on a general **finite abstract simplicial complex** with Euler characteristic $\chi(G) = \sum_{x \in G} \omega(x)$. Gauss Bonnet means $\chi(G) = \sum_{v \in V} K(v)$. [2] As already noted in [19], the proof is a simple conservation principle in that the energy $\omega(x)$ on x is distributed to the $|x| = k + 1$ vertices of a k-dimensional simplex x. We generalize this slightly here and sow that the energies $\omega(x)$ can also be distributed to k-dimensional parts $G_k = \{x \in G, |x| = k + 1\}$ of G, producing curvatures K_k. In [19] experiments showed $K_0(v) = 0$ for odd-dimensional manifolds. It was proven using index

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[1][31] shows in full generality that $\chi(G)$ is always the sum of potential energies $g(x, y)$, where g is the inverse of the connection Laplacian, which is always unimodular. In physics, Green functions give potential values.

[2]Rediscovered in [19] while investigating curvatures for higher dimensional manifolds. We found the reference [?] in 2015, while working on multi-linear valuations [27].

expectation [22]. Curvatures can be seen as index expectation of **Poincaré-Hopf indices** i_f, where f are functions on facets. If $i_f = -i_{-f}$, the sum is zero if the probability measure invariant under the involution $f \to -f$.
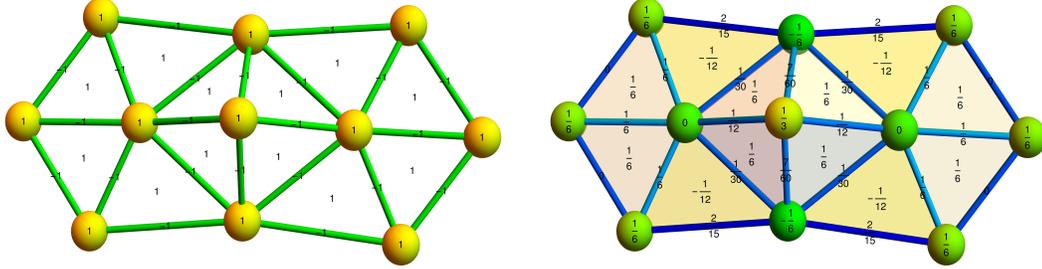

FIGURE 1. We see a two-dimensional region with 11 vertices V, 22 edges E and 12 faces F, where $V - E + F = 1$. The original energies 1 on $V \cup F$ and -1 on E can be distributed to vertices, edges and faces to get curvatures $K(v)$, $K(e)$ and $K(f)$ each satisfying Gauss-Bonnet $\chi(G) = \sum_{v \in V} K(v) = \sum_{e \in E} K(e) = \sum_{f \in F} K(f)$.

1.3. If the energies $\omega(x)$ of a 2 dimensional surface without boundary are moved to edges $e = (a, b)$, we end up with $K(e) = 1/d(a) + 1/d(b) - 1 + 2/3 = 1/d(a) + 1/d(b) - 1/3$. If the energies are moved to the triangles $f = (a, b, c)$, we end up with $K(f) = 1/d(a) + 1/d(b) + 1/d(c) - 3/2 + 1 = 1/d(a) + 1/d(b) + 1/d(c) - 1/2$. For 2-dimensional manifolds with boundary, the edge curvature at the rim is $K(v) = 1/d(a) + 1/d(b) - 2/3$ and the face curvature is at the boundary $K(f) = 1/d(a) + 1/d(b) + 1/d(c) - 2/3$. Beside the familiar Gauss Bonnet $\chi(G) = \sum_{v \in V} K(v)$ for vertex curvatures, there are now new Gauss-Bonnet statements for edge or face curvatures $\chi(G) = \sum_{e \in E} K(e)$ and $\chi(G) = \sum_{f \in F} K(f)$.

1.4. To summarize: in the special case of 2 dimensional manifolds, (graphs for which every unit sphere $S(v)$ is a circular graph with $d(v) \geq 4$ elements), we have:

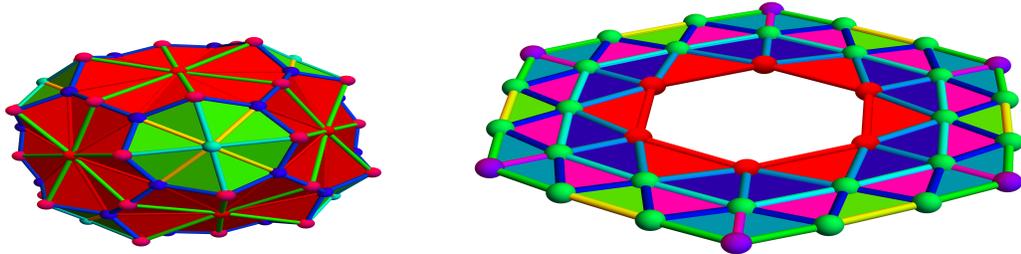|	in the interior	at the boundary
vertex	$1 - \frac{d(v)}{6}$	$\frac{2}{3} - \frac{d(v)}{6}$
edge	$\frac{1}{d(a)} + \frac{1}{d(b)} - \frac{1}{3}$	$\frac{1}{d(a)} + \frac{1}{d(b)} - \frac{2}{3}$
face	$\frac{1}{d(a)} + \frac{1}{d(b)} + \frac{1}{d(c)} - \frac{1}{2}$	$\frac{1}{d(a)} + \frac{1}{d(b)} + \frac{1}{d(c)} - \frac{2}{3}$

FIGURE 2. The manifold without boundary is the Barycentric refinement of a stellated cube. To the right, we see an example of a manifold with boundary that is flat in the interior. Also the edge curvature of interior edges is zero.

2. The general case

2.1. We assume from now on that G is a **finite abstract simplicial complex** meaning a finite set of non-empty sets closed under the operation of taking non-empty subsets. In order to introduce curvatures that are k-forms, meaning functions on $G_k = \{x \in G, \dim(x) = |x|-1 = k\}$, the k-dimensional simplices, we need to assume that the finite abstract simplicial complex $G = \bigcup_{k=0}^q G_k$ has the property that $\bigcup_{x \in G_k} x = V$ with $V = \bigcup_{x \in G} x$. We say in that case that $\bigcup_{x \in G_k} x$ **covers** G. A **pure q-dimensional complex** for example, is a complex $\bigcup_{k=0}^q G_k$ for which the **facets** G_q cover G meaning that every $x \in G$ is a subset of some facet. For $G = \{\{1\}, \{2\}, \{3\}, \{1,2\}\}$ for example, G_0 covers G but G_1 does not cover G. The statement G_1 **covers** G is equivalent to G **has no isolated vertices**.

2.2. We define then the **k-form-curvature** for $0 \le k \le q$ as

$$K_k(x) = \sum_{y \subset x} \frac{\omega(y)}{d_k(y)} + \sum_{j>k}(-1)^j \frac{d_j(x)}{\binom{j+1}{k+1}}$$

with $d_k(x) = |\{y \in G_k, y \subset x\}$. We get so functions $K_k : G_k \to \mathbb{R}$ which can be called a k-form, [3] as it is a function on G_k, or elements in $l^2(G_k)$, if we want to use an inner product on such functions. [4]

2.3. In the case $k = 0$, the first part $\sum_{y \subset x} \frac{\omega(y)}{d_k(y)}$ is zero. We get then the familiar **Levitt curvature**, where $d_j(x)$ counts the number of j dimensional simplices attached to the vertex x.

$$K_0(x) = \sum_j (-1)^j \frac{d_j(x)}{j+1} .$$

An other special case is the **facet curvature** for $k = q$. It assumes that the simplical complex is covered by the highest q-dimensional facets. One calls such a simplicial complex **pure** or **homogeneous** of maximal dimension q. Under this assumption, the second part of the form curvature is zero and we end up with

$$K_q(x) = \sum_{y \subset x} \frac{\omega(y)}{d_q(y)} ,$$

where the **facet degree** $d_q(y)$ of a simplex y counts the number of q dimensional simplices which contain y.

2.4. We have for every $k \ge 0$ the Gauss-Bonnet formula:

Theorem 1 (Gauss-Bonnet). *If $\bigcup_{x \in G_k} x$ covers G, then $\sum_{x \in G_k} K_k(x) = \chi(G)$.*

Proof. The proof is almost tautological if one thinks in terms of energies $\omega(y)$ attached to the simplices $y \in G$. Since G_k covers G, the distribution of energies does take care of all energies. There are two type of transactions. Either $y \subset x$, then the energy $\omega(y)$ is split into $d_k(y)$ simplices of dimension k. Or then the later case $x \subset y$, where we can count how many times this happens. We have $\binom{j+1}{k+1}$ simplices of dimension k in a simplex of dimension j. $\square$

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[3] As defined the curvature is a function on x, where x is seen as a set. It can be extended to ordered sets by setting $K(\pi(x)) = \text{sign}(\pi)K(x)$ so that it becomes a discrete differential form. This does not affect the integral $\sum_{x \in G_k} K_k(x)$ of the differential form. This is the point of view of Whitney who both worked in the continuum and discrete.

[4] $K_k(v)$, the curvature has nothing to do with the complete graph K_k.

2.5. Examples: We have already seen the cases for 2-manifolds, where $\chi(G) = |V| - |E| + |F| = |G_0| - |G_1| + |G_2|$, it becomes $K_0(a) = 1 - d(a)/6$, $K((a, b)) = 1/d(a) + 1/d(b) - 1/3$ and $K((a, b, c)) = 1/d(a) + 1/d(b) + 1/d(c) - 1/2$. For 3-manifolds, it simplifies to $K_0(a) = 1 - d(a)/2 + d_2(a)/3 - d_3(a)/4$, $K_1((a, b)) = 1/d_1(a) + 1/d_1(b) - 1 + d_2(a, b)/12$ and is $K_2((a, b, c)) = 1/d_2(a) + 1/d_2(b) + 1/d_2(c) - 1/d_2(a, b) - 1/d_2(b, c) - 1/d_2(a, c) + 1/2$ is $K_3((a, b, c, d)) = \sum_{y \subset \{a,b,c,d\}} \omega(y)/d_3(y)$. We will come back to this in the 3-manifold section.

2.6. If G has n elements, there is a $n \times n$ matrix $D = d + d^*$ which encodes the exterior derivatives of the complex. [5] The square $L = \oplus_{k=1}^{q} L_k$ is a block diagonal matrix and the kernel of $L_k : l^2(G_k) \to l^2(G_k)$ is the space of **harmonic k-forms**. The numbers $b_k = \dim(\ker(L_k)$ are called the **Betti numbers**. The **Euler characteristic** of G is $\sum_{x \in G} \omega(x)$ with $\omega(x) = (-1)^{\dim(x)}$. If we define the **super trace** for an operator on $l^2(G)$ as $\text{str}(A) = \sum_k A_{kk} \omega_k$, then $\text{str}(1) = \chi(G)$. The McKean-Singer symmetry [37, 23] tells $\text{str}(L^k) = 0$ for $k \geq 0$ so that $\text{str}(e^{-tL}) = \chi(G)$. Since for $t \to \infty$ this gives $\sum_{k=0}^{d}(-1)^k b_k = \chi(G)$, the Euler-Poincaré formula follows.

3. 3-MANIFOLDS

3.1. A finite simple graph is a **3-manifold** if every unit sphere $S(v)$ is a **2-sphere**, meaning a connected 2-dimensional manifold of Euler characteristic 2. A **2-manifold** is a finite simple graph for which every unit sphere is a cyclic graph of length 4 or more. 2-spheres can also be characterized as 2-manifolds G which after removing a vertex become contractible, a recursive notion which means that there is a vertex v such that both $G - v$ and $S(v)$ are contractible. [6]. 3-manifolds can be given as graphs. Libraries like [35] give them by listing the facets G_3, the highest dimensional simplices. The simplicial complex G generated by them then defines the graph with vertices $V = G_0$ and edges $E = G_1$.

3.2. In dimension 3, the curvature $K(v) = K_0(v)$ is given by

$$K(v) = 1 - \frac{V(v)}{2} + \frac{E(v)}{3} - \frac{F(v)}{4} \ ,$$

where $V(v), E(v), F(v)$ are the number of vertices, edges and faces in the unit sphere $S(v)$ of a vertex v. From the Euler relation $V - E + F = 2$ and the **Dehn-Sommerville relation** $2E = 3F$, (2 tetrahedra share exactly one triangle), we easily get $K(v) = 0$. In [19], we used the Dehn-Sommerville relations to get simplified relations in higher dimensions like $K(v) = 1 - E(v)/6 + F(v)/10$ for 4-manifolds and $K(v) = -E(v)/6 + F(v)/4 - C(v)/6$ for 5-manifolds, where $C(v)$ is the number of 3-dimensional **chambers** in the unit sphere $S(v)$. We had at that time not been aware of the second Dehn-Sommerville relation for 5-manifolds, which is equivalent to $K(v) = 0$ for 5-manifolds.

3.3. We looked in [19] already at various examples like the 16-cell $K_{2,2,2,2}$ which is the suspension of the octahedron $K_{2,2,2}$. The 600-cell is a 3-sphere for which all unit spheres are icosahedra. Or then an octahedral tesselation of the 3-torus. All the 5-manifold examples we have seen then displayed $K(v) = -E(v)/6 + F(v)/4 - C(v)/6 = 0$. The solution is given most elegantly in [30]: the curvature is a Dehn-Sommerville invariant for the unit sphere. Dehn-Sommerville graphs are graphs for which the Euler characteristic matches the sphere case and for which all unit spheres are Dehn-Sommerville. It is a very natural class of graphs which contains d-spheres but does not refer to contractibility.

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[5]We called it the Dirac operator [24] of the graph at a time, when most graph theorists would see "graphs" as **one-dimensional simplicial complexes**.

[6]See [32] for a recent more expository document

3.4. The edge curvature of a 3-manifold is for an edge $x = (a, b)$ given by

$$K_1(x) = \frac{1}{d(a)} + \frac{1}{d(b)} + \sum_{j=1,2,3} (-1)^j \frac{d_j(x)}{\binom{j+1}{2}}$$

which is $K_1(x) = \frac{1}{d(a)} + \frac{1}{d(b)} - 1 + \frac{d_2(x)}{3} - \frac{d_3(x)}{6}$, where $d_2(x) = |S(a) \cap S(b)|$ is the number of triangles containing (a, b) and $d_3(x) = |S(a) \cap S(b)|$ is the number of chambers (=tetrahedra) containing (a, b). So, for a 3-manifold, we have

Proposition 1 (Edge curvature for a 3-manifold). *For a 3-manifold, we have the edge curvature*

$$K_1((a, b)) = \frac{1}{d(a)} + \frac{1}{d(b)} + \frac{d(a, b)}{6} - 1$$

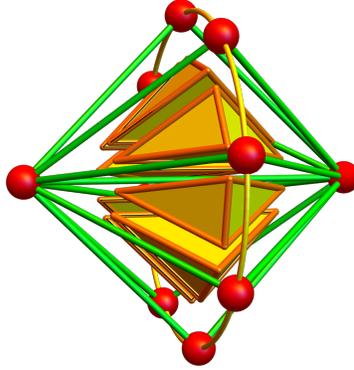*where $d(a), d(b)$ are vertex degrees and $d(a, b)$ is the edge degree.*

FIGURE 3. The edge degree of an edge $x = (a, b)$ in a 3-manifold is the number of tetrahedra hinging at (a, b). We have now vertex degrees $d(a) = |S(a)|$ and $d(b) = |S(b)|$ and an edge degree $d(a, b) = |S(a) \cap S(b)|$.

3.5. Examples:
a) For the smallest 3-manifold $K_{2,2,2,2}$, which is also known as the **16-cell**, with f-vector $(8, 24, 32, 16)$, we have $d(a) = d(b) = 6$ and $d(a, b) = 4$ for all edges (a, b). The edge curvature is $1/6 + 1/6 + 4/6 - 1$ which is constant 0.
b) For the 600-cell, also one of the Platonic 4-polytopes (3-spheres embedded in 4 dimensions), the f-vector is $(120, 720, 1200, 600)$. Every unit sphere $S(v)$ is an icosahedron with $|S(v)| = 12$ so that $d(v)$ is constant 12. The edge degree of one of the 720 edges is $|S(a) \cap S(b)| = 5$. Now $\frac{1}{12} + \frac{1}{12} + \frac{5}{6} - 1 = 0$.

3.6. The **face curvature** of a face $x = (a, b, c)$ in a 3-manifold is given by

$$K_2(x) = \frac{1}{d_2(a)} + \frac{1}{d_2(b)} + \frac{1}{d_2(c)} - \frac{1}{d_2(a,b)} - \frac{1}{d_2(b,c)} - \frac{1}{d_2(a,c)} + 1 - 2/4$$

where $d_2(a)$ is the number of triangles which contain a and $d_2(a, b)$ the number of triangles which contain (a, b). We called this the edge degree $d(a, b)$ before. The last part is because there are exactly two tetrahedra attached to each face and each tetrahedron shares with 4 faces.

Proposition 2 (Face curvature for a 3-manifold). *For a 3-manifold, we have*

$$K_2((a, b, c)) = \frac{1}{d_2(a)} + \frac{1}{d_2(b)} + \frac{1}{d_2(c)} - \frac{1}{d_2(a,b)} - \frac{1}{d_2(b,c)} - \frac{1}{d_2(a,c)} + \frac{1}{2},$$

where $d_2(a)$ counts the number of triangles containing a and $d_2(a, b) = d(a, b)$ is the edge degree.

3.7. If we go back to the two Platonic examples.
a) For the 16 cell we have exactly 12 triangles that contain a vertex (as the unit sphere is an octahedron with 12 edges). We have exactly 4 triangles that contain an edge (a, b). This is the edge degree $|S(a) \cap S(b)|$. The face curvature is now $K_2((a, b, c)) = 1/12 + 1/12 + 1/12 - 1/4 - 1/4 - 1/4 + 1/2 = 0$. The face curvature is constant 0.
b) For the 600 cell, where the unit sphere is an icosahedron we have $d_2(a) = 30$ as there are 30 edges in an icosahedron. We also have $d_2(a, b) = 5$. Therefore $K_2((a, b, c)) = 1/30 + 1/30 + 1/30 - 1/5 - 1/5 - 1/5 + 1/2 = 0$.

3.8. Lets look at the facet curvature

$$K_3(x) = \sum_{y \subset x} \frac{\omega(y)}{d_q(y)} \ ,$$

of a 3-manifold. We could also call it the **chamber curvature** of a chamber $x = (a, b, c, d)$. We can split it up into

$$\sum_{y \in \{(a),(b),(c),(d)\}} \frac{1}{d_3(y)} - \sum_{y \in \{(a,b),(a,c),(a,d),(b,c),(b,d),(c,d)\}} \frac{1}{d_3(y)} + \sum_{y \in \{(b,c,d),(a,c,d),(a,b,d),(b,c,d)\}} \frac{1}{d_3(y)} - 1$$

where $d_3(y)$ is the number of tetrahedra shared with the simplex y. For a vertex v, we have $d_3(v) = F(v)$, the number of faces in $S(v)$. For an edge $e = (a, b)$ we have $d_3((a, b)) = d(a, b)$ the edge degree and for a face $f = (a, b, c)$ we have $d_3(a, b, c) = 2$. This can be written as follows:

Proposition 3 (Facet curvature for a 3-manifold). *For a 3-manifold, we have for a chamber* $x = (a, b, c, d)$

$$K_3(x) = \frac{1}{F(a)} + \frac{1}{F(b)} + \frac{1}{F(c)} + \frac{1}{F(d)} - \frac{1}{d(a,b)} - \frac{1}{d(a,c)} - \frac{1}{d(a,d)} - \frac{1}{d(b,c)} - \frac{1}{d(b,d)} - \frac{1}{d(c,d)} + 1 \ ,$$

where $d_3(a) = F(a)$ *is the face degree of a vertex and* $d(a, b)$ *is the degree of an edge.*

3.9. Again, let us look at the Platonic examples.
a) For the 16 cell we have $F(v) = 8$ for all vertices v and $|S(a) \cap S(b)| = 4$ for all edges $e = (a, b)$. Then, there are Therefore $K_3((a, b, c, d)) = 4/8 - 6/4 + 1 = 0$. The face curvature is also constant 0.
b) For the 600 cell, where the unit sphere is an icosahedron we have $F(a) = 20$ as there are 20 triangles in an icosahedron. We also have $d_3(a, b) = d(a, b) = 5$. Therefore $K_3((a, b, c, d)) = 4/20 - 6/5 + 1 = 0$.

3.10. We should note that the other four Platonic 3-spheres (regular 4-polytopes) are not triangulations of a 3-sphere. The unit spheres $S(v)$ of a Platonic d-manifold have to be Platonic (d-1) manifold and there are only 2 Platonic 2-spheres (assuming to be a 2-manifold), the octahedron $K_{2,2,2}$ and the icosahedron. In all other Platonic cases, the unit spheres are not spheres. Gauss-Bonnet quickly shows that the only Platonic d-spheres for $d > 3$ are the cross polytopes $K_{2,2,\dots,2}$. Gauss-Bonnet prevents that a 600 cell is the unit sphere of all vertices of a 4-sphere: the curvature of a 4-manifold is $K(v) = 1 - E(v)/6 + F(v)/10$. Because The f-vector of the 600 cell is $(120, 720, 1200, 60)$, the curvature would have to be constant $K(v) = 1 - 720/6 + 1200/10 = 1$. The Euler characteristic of a 4-sphere is by Euler's Gem formula equal to 2. There is obviously no 4-manifold with 2 vertices. See [29] for an exposition.
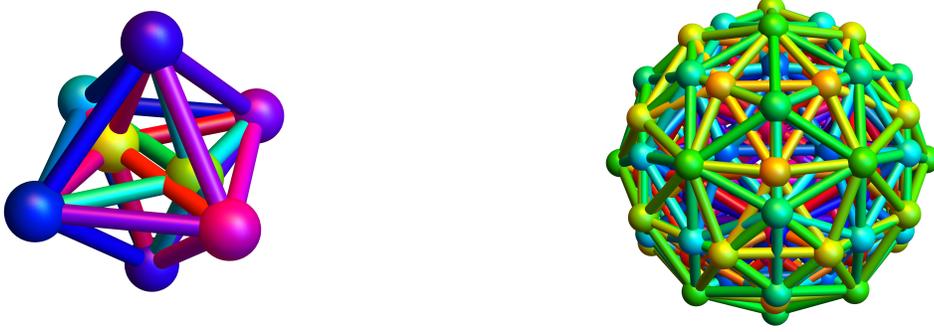
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FIGURE 4. The 16 cell $K_{2,2,2,2}$ and the 600 cells are the only Platonic 3-spheres. All the curvatures $K_k(x)$ are constant zero everywhere.

4. DEFORMATION OF THE CURVATURE

4.1. The curvature produces a dynamics on k-forms. We are using the **wave operator** $w_t = \cos(Dt)$ which solves the wave equation $w'' = -Lw$. [7] Define $\omega_t(x) = (-1)^{\dim(x)}\cos(Dt)(x, x)$ and the time-dependent curvature

$$K_{k,t}(x) = \sum_{y \leq x} \frac{\omega_t(y)}{\deg_k(y)} + \sum_{y \geq x} \frac{\omega_t(y)}{\binom{\dim(y)+1}{k+1}} .$$

This function is implemented in the Mathematica code below. If $t = 0$ it is the form curvature $K_k(x)$. For every t and every k, we still have Gauss Bonnet:

Corollary 1 (Gauss-Bonnet for time dependent curvatures). *If G_k covers G, then $\chi(G) = \sum_{x \in G_k} K_{k,t}(x)$ for all t.*

Proof. For any function f, the functional calculus defines $f(D)$ as D is symmetric. The wave operator $\cos(Dt)$ is a symmetric $n \times n$ matrix. The McKean-Singer symmetry assuring that $\mathrm{str}(L^k) = 0$ for all $k > 0$ implies that the super trace of $\cos(Dt)$ is equal to $\chi(G)$ and so that it is independent of t. We can now use the diagonal values $\omega_t(x)$ instead of $\omega(x) = \omega_0(t)$. $\square$

4.2. Note that the curvature functions $K_{k,t}$ on G_k do not satisfy the wave equation by themselves. It is that the underlying energies values given by the diagonal elements of the wave operator that move according to the wave equation. The curvature $K(x)$ on k-forms then aggregates all energies of simplices close to x. There is no choice of initial condition for this evolution because $K_{k,0}(x) = K_k(x)$ is the curvature on k-forms, introduced before.

4.3. Non-linear wave versions come from **integrable Lax pair systems** $D' = [g(D)^+ - g(D)^-, D]$ [26, 25] , leading to a unitary $Q'_t = (g(D)^+ - g(D)^-)Q_t$ satisfying $D_t = Q_t^* D_0 Q_t$. As noted in [33], the **non-linear iso-spectral deformation** $D' = [g(D)^+ - g(D)^-, D]$ is equivalent to the QR deformation like $e^{-tg(D_0)} = Q_t R_t$ and get a new Dirac matrix $D_t = Q_t^* D Q_t$ which has the form $D_t = d_t + d_t^* = Q_t^* dQ_t + Q_t^* d^* Q_t$ which is now block tri-diagonal $D_t = c_t + c_t^* + m_t$. The deformation can move spontaneously also in the complex by modifying to

$$(1) \qquad\qquad D' = [g(D)^+ - g(D)^- + icg(D)^0, D]$$

with some nonzero constant c.

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[7]More generally, we could take $w = e^{iDt}$ or e^{-iDt} and so get complex-valued curvatures.
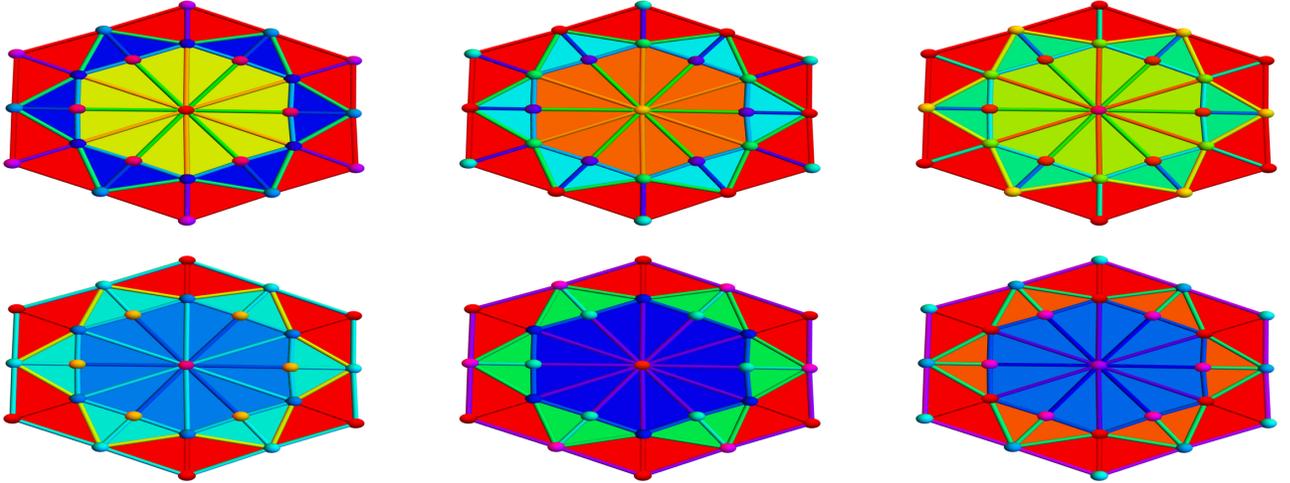
FIGURE 5. A wave deformation in the case of second Barycentric refinement of K_3. The wave is evaluated at times $k/2$ for $k = 1, \ldots, 6$. The colors encode the vertex, edge and wave curvatures.

4.4. If we want a linear and discrete time evolution, we could look at $T(u, v) = (c2Du - v, u)$, where c is so small that $||cD|| < 1$ rather than the pair $u' = \pm icDu$ or wave equation [18]. There is then still a unitary evolution $U_t = \exp(it \arccos(cD))$, and the super trace of U_t is still constant. The advantage of such a discrete time evolution is that it has **finite propagation speed**, meaning causality. Causality can also be obtained in the nonlinear Lax deformation mentioned above. We just need to make sure that $e^{-tg(D)}$ is local which is achieved for $g(D) = -\log(1 + cD)$ which is defined if $||cD|| < 1$. In that case $e^{-tg(D)} = (1 + cD)^t$. The orthogonal matrix Q_t then does not relate points in distance larger than t.

4.5. By the way, we introduced the discrete time evolution in [18] for numerical purposes, like to measure the nature of spectra of discrete Schrödinger operators. The growth rate of the spectral measures determines the spectral type. If we can numerically compute the growth rate with an evolution that has finite propagation speed, we know that any boundary effects do not play a role as long as the wave has not reached the boundary. When using rational or algebraic numbers, the evolution is **exact**. If the target space (usually $\mathbb{R}$ is replaced by a ring element, we can study arithmetic versions. With a finite ring, we get cellular automata [42]. Also in the present geometric frame work, we could replace real or complex valued functions with functions taking values in other rings, like also finite rings. An obvious generalization is to expand the non-linear evolution to quaternions. We would just have to replace the i in the differential equation (1) with a unit quaternion.

5. POINCARÉ-HOPF

5.1. A scalar function $g : G_k \to \mathbb{R}$ is called **locally injective** if $g(x) \neq g(y)$ if either $x \cap y$ is not empty or if $x \cup y$ are in a common simplex. Such a function defines a **total order** on the vertices of each simplex x of G_k. It allows to every simplex $x \in G$ assign a specific $y \in G_k$ with $y \subset x$ or $x \subset y$ on which g is the maximum. More generally, we just assign a function $F : G \to G_k$ such that $F(y) \subset y$ or $y \subset F(y)$. This gives the k-dimensional **Poincaré-Hopf index**

$$i_{g,k,t}(x) = 1 - \chi_t(S_g^+) - \chi_t(S_g^-) ,$$

8

where $S_g^+ = \{y, y \subset x, g(y) < g(x)\}$ and $S_g^- = \{y, x \subset y, g(y) < g(x)\}$.

5.2. There is now a more general Poincaré-Hopf formula, where the integer indices are located on k-dimensional simplices. It again holds only if the k-dimensional part of space covers it meaning $\bigcup_{x \in G_k} x = \bigcup_{x \in G} x = V$.

Theorem 2 (Poincaré-Hopf). *If $\bigcup_{x \in G_k} x$ covers G, then $\chi(G) = \sum_{x \in G_k} i_k(x)$.*

Proof. Every simplex $y \in G$ will be pointed to a specific $x \in G_k$. But this needs that the k-simplices cover G. Moving the energies accordingly from G to G_k does not change the total energy which is the Euler characteristic. $\qquad\square$

6. INDEX EXPECTATION

6.1. For $k = 0$, where K_0 is the Levitt curvature on vertices, the curvature is constant 0 if G is an odd-dimensional manifold. Conjectured in [19], it was verified using index expectation in [22, 21] and later seen it as a Dehn-Sommerville relation [28], establishing so zero curvature also for odd-dimensional Dehn-Sommerville manifolds [28], simplicial complexes that generalize d-spheres but satisfy Dehn-Sommerville relations. Graphs for which the unit spheres are Dehn-Sommerville manifolds of dimension $d-1$ can be seen as generalized d-manifolds. We wondered whether for odd dimensional manifolds, also higher curvatures K_k are constant 0. It turns out that this is not the case in general.

6.2. The Poincaré-Hopf story on k-dimensional simplices is a tale about divisors on k-dimensional parts of space. One can see them as integer-valued curvatures. If we take a probability measure on the space of locally injective functions on G_k and take the expectation of
$$\sum_{x \in G_k} i_f(x) = \chi(G) \ .$$
We get on the left hand side sum of curvatures
$$K(x) = \mathrm{E}[i_f(x)]$$
on G_k. Index expectation links the discrete and continuum. if G is a finite simple graph with Euler characteristic $\chi(G) = \sum_k (-1)^k f_k(G)$, where $f_k(G)$ counts the number of k-dimensional complete sub-graphs in G, the Levitt curvature $K(v) = 1 - \sum_k (-1)^k f_k(S(v))/(k+1)$ satisfies Gauss-Bonnet $\sum_{v \in V} K(v) = \chi(G)$. Allowing the probability measure to vary gives more flexibility. We can for example embed the network G into a high dimensional linear space E and the space of linear functions on E (parametrized by vectors on the unit sphere equipped with the homogeneous measure) as probability space, then $\mathrm{E}[i_f(x)]$ is a curvature which is close to the continuum if the network is close to the physical space we want to model. If we take random IID values $f(x)$ on G_k with uniform distribution in some interval for example, then the index expectation is the curvature $K_k(x)$ considered above.

6.3. For K_0, there is a generalization of Gauss-Bonnet in which one does not work with Euler characteristic but more generally with the simplex generating function
$$f_G(t) = \sum_{k=0}^{\infty} f_k(G) t^k$$
which for $t = -1$ relates to the Euler characteristic by $\chi(G) = 1 - f_G(-1)$. The Levitt Gauss-Bonnet formula now reads more naturally as
$$f_G'(t) = \sum_{v \in V} f_{S(v)}(t)$$

9

which after integration and evaluating at $t = -1$ gives $1 - \chi(G)$ and so Gauss-Bonnet. Since the Levitt formula is an expectation of Poincaré-Hopf indices, this Levitt curvature is the discrete analog of the Gauss-Bonnet-Chern curvature K which involves the Riemann curvature tensor $R_{i,j,k,l}$. It is

$$K = (-1)^n \sum_{i,j} \epsilon_{i_1,\ldots,i_{2n}} \epsilon_{j_1,\ldots,j_{2n}} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2n-1} i_{2n} j_{2n-1} j_{2n}}$$

for Riemannian manifolds. In the discrete, curvature is zero for odd-dimensional manifolds, in the continuum, the intrinsic curvature of odd-dimensional manifolds is not even considered.
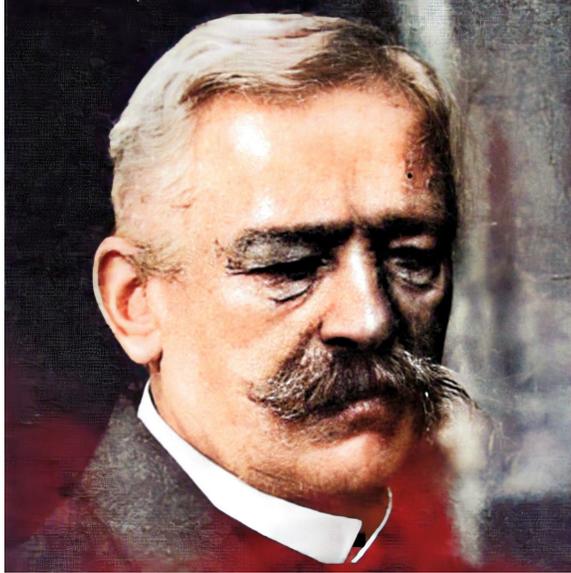

FIGURE 6. Victor Eberhard (1861-1927) in 1922. Eberhard lost his eye sight as a 13 year old and showed enormous resilience, developing strong intuition, memory, humor and positivity to overcome the handicap. The photo source is the biography [41] about Eberhard. We colorized the photograph electronically using a tool provided by MyHeritage.com.

7. Puiseux relation

7.1. In [20], we considered the Puiseux type curvature $K(v) = 2|S(v)| - |S_2(v)|$ for flat 2-dimensional planes, where we proved $\sum_{p \in \delta G} K(p) = 12\chi(G)$. We had felt at that time that a good curvature should be a second order notion, meaning that it should involve neighborhoods of distance 2. This was motivated by the classical Puiseux formula $K(v) = \lim_{r \to 0} 3 \frac{2\pi r - |S_r|}{\pi r^3}$ which is equivalent to the R2-D2 formula (**R**adius times two in **D**imension 2)

$$K(v) = \lim_{r \to 0} \frac{2|S_r| - |S_{2r}|}{2\pi r^3} .$$

For example, for a sphere of radius R, where $r = R\phi$ and $|S_r| = 2\pi R \sin(r/R)$. The R2-D2 formula gives $4\pi R \sin(r/R) - 2\pi R \sin(2r/R) = 2\pi r^3/R^2 - 2\pi r^5/(4R^4) + \ldots$ so that $K = 1/R^2$. Similarly, for a hyperbolic plane with $|S_r| = 2\pi R \sinh(r/R)$, we get $K = -1/R^2$.

7.2. If C is the boundary of a planar region G and $|S_r|$ is the arc length of S_r, the signed curvature is given by the **R2-D2** formula for $v \in C$

$$K(v) = \lim_{r \to 0} \frac{2|S_r(v)| - |S_{2r}(v)|}{2r^2} .$$

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For a disk of radius R one has $|S_r(v)| = 2r\arccos(r/(2R))$ for a point on the boundary, so that $(2|S_r(v)| - |S_{2r}(v)|)/(2r^2) = 1/R + O(r^2)$ is also intrinsic and parameter independent.
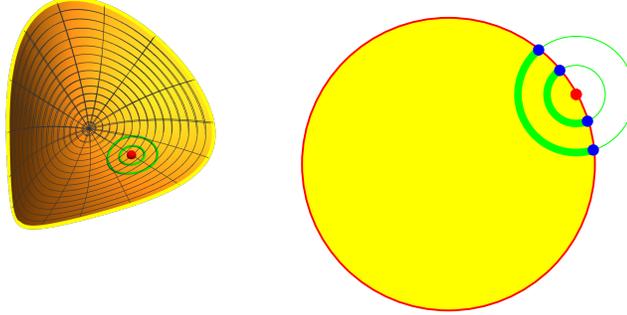

FIGURE 7. The R2-D2 formula is a classical expression for the curvature of a 2-dimensional manifold. It holds in the interior and on the boundary. Unlike the classical **Bertrand-Diquet-Puiseux formula** [4] is $K = \lim_{t\to 0} \frac{3(2\pi r - |S_r|)}{r^3\pi}$ the R2-D2 formulas do not refer the arc length $2\pi r$ in the flat case.

7.3. In the discrete, Puiseux fails for general 2-manifolds already. But we can get something related now using curvatures on k-forms: it follows from the edge curvature that for a 2-manifold we have the curvature

$$K(v) = 1 - \frac{d(v)}{6} + \frac{1}{2}\left[-1 + \sum_{w \in S(v)} \frac{1}{d(w)}\right]$$

on vertices. A similar curvature appear in [14]. Since $\sum_v 1 - \frac{d(v)}{6} = \chi(G)$ and $\sum_v K(v) = \chi(G)$ we see $\sum_v \sum_{w \in S(v)} \frac{1}{d(w)} = V$. Compare the **Euler handshake formula** $\sum_{v \in V} \sum_{w \in S(v)} 1 = 2E$ which by the way is the Gauss-Bonnet formula for the valuation $E(G) = |E|$. [8]

8. CODE

8.1. The following Mathematica code allows to compute the k-curvatures also after a wave deformation. The case without deformation is obtained by plugging in $t = 0$.

```
Generate[A_]:=If[A=={},{},Sort[Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]]];
Cl[G_]:=Union[Sort[Map[Sort,G]]];L=Length;sig[x_]:=Signature[x];Bin=Binomial;Rk=MatrixRank;
Whitney[s_]:=Cl[Generate[FindClique[s,Infinity,All]]];nu[A_]:=If[A=={},0,L[A]-Rk[A]];
F[G_]:=Module[{l=Map[L,G]},If[G=={},{},Table[Sum[If[l[[j]]==k,1,0],{j,L[l]}],{k,Max[l]}]]];
sig[x_,y_]:=If[SubsetQ[x,y]&&(L[x]==L[y]+1),sig[Prepend[y,Complement[x,y][[1]]]]*sig[x],0];
Dirac[G_]:=Module[{f=F[G],b,d,n=L[G]},b=Prepend[Table[Sum[f[[l]],{l,k}],{k,L[f]}],0];
 d=Table[sig[G[[i]],G[[j]]],{i,n},{j,n}];  {d+Transpose[d],b}];
Hodge[G_]:=Module[{Q,b,H},{Q,b}=Dirac[G];H=Q.Q;Table[Table[H[[b[[k]]+i,b[[k]]+j]],
 {i,b[[k+1]]-b[[k]]},{j,b[[k+1]]-b[[k]]}],{k,L[b]-1}]];        omega[x_]:=(-1)^(L[x]-1);
Betti[s_]:=Module[{G},If[GraphQ[s],G=Whitney[s],G=s];Map[nu,Hodge[G]]];
Fvector[A_]:=Delete[BinCounts[Map[L,A]],1];        Euler[A_]:=Sum[omega[A[[k]]],{k,L[A]}];
UnitSphere[s_,v_]:=VertexDelete[NeighborhoodGraph[s,v],v];
OpenStar[G_,x_]:=Select[G,SubsetQ[#,x] &];  deg[G_,x_,k_]:=F[OpenStar[G,x]][[k+1]];
Curvatures[s_,k_,t_]:=Module[{G,A,B,Q},If[GraphQ[s],G=Whitney[s],G=s];
 If[t>0,Q=First[Dirac[G]];U=Re[MatrixExp[t*I*Q]],U=IdentityMatrix[Length[G]]];
 A=Select[G,(L[#]==k+1)&];  If[L[Union[Flatten[A]]]<L[Union[Flatten[G]]],Print["No-cover"]];
 Table[y=A[[l]];  Sum[x=G[[i]];omega[x]*U[[i,i]]* If[SubsetQ[y,x],1/deg[G,x,k],
                     If[SubsetQ[x,y],1/Bin[L[x],k+1],0]],{i,L[G]}],{l,L[A]}]];
Curvatures[s_,k_]:=Curvatures[s,k,0];
```

[8]Euler handshake has been called "fundamental theorem of graph theory". It is a Gauss-Bonnet relation for the valuation $f_1(G)$. Gauss-Bonnet can be formulated for general valuations.

```
s = RandomGraph[{20, 50}];{Total[Curvatures[s,1]], Euler[Whitney[s]]}
s = ToGraph[Whitney[ToGraph[Whitney[CompleteGraph[3]]]]];
Table[CurvatureGemPlot[s, 1.0*k/4], {k,1,6}];
```

8.2. Lets look at some 3 manifolds. We start with a non-standard 3-sphere, the homology sphere. It can be found on [35]. See also [36, 6].

```
s=HomologySphere=Generate[
{{0,1,5,11},{0,1,5,17},{0,1,7,13},{0,1,7,19},{0,1,9,15},{0,1,9,21},{0,1,11,13},
{0,1,15,19},{0,1,17,21},{0,2,6,12},{0,2,6,18},{0,2,7,13},{0,2,7,19},{0,2,10,16},
{0,2,10,22},{0,2,12,13},{0,2,16,19},{0,2,18,22},{0,3,8,14},{0,3,8,20},{0,3,9,15},
{0,3,9,21},{0,3,10,16},{0,3,10,22},{0,3,14,22},{0,3,15,16},{0,3,20,21},{0,4,5,11},
{0,4,5,17},{0,4,6,12},{0,4,6,18},{0,4,8,14},{0,4,8,20},{0,4,11,12},{0,4,14,18},
{0,4,17,20},{0,11,12,13},{0,14,18,22},{0,15,16,19},{0,17,20,21},{1,2,5,11},{1,2,5,17},
{1,2,8,14},{1,2,8,20},{1,2,10,16},{1,2,10,22},{1,2,11,14},{1,2,16,20},{1,2,17,22},
{1,3,6,12},{1,3,6,18},{1,3,7,13},{1,3,7,19},{1,3,8,14},{1,3,8,20},{1,3,12,20},{1,3,13,14},
{1,3,18,19},{1,4,6,12},{1,4,6,18},{1,4,9,15},{1,4,9,21},{1,4,10,16},{1,4,10,22},{1,4,12,16},
{1,4,15,18},{1,4,21,22},{1,11,13,14},{1,12,16,20},{1,15,18,19},{1,17,21,22},{2,3,5,11},
{2,3,5,17},{2,3,6,12},{2,3,6,18},{2,3,9,15},{2,3,9,21},{2,3,11,15},{2,3,12,21},{2,3,17,18},
{2,4,7,13},{2,4,7,19},{2,4,8,14},{2,4,8,20},{2,4,9,15},{2,4,9,21},{2,4,13,21},{2,4,14,15},
{2,4,19,20},{2,11,14,15},{2,12,13,21},{2,16,19,20},{2,17,18,22},{3,4,5,11},{3,4,5,17},
{3,4,7,13},{3,4,7,19},{3,4,10,16},{3,4,10,22},{3,4,11,16},{3,4,13,22},{3,4,17,19},
{3,11,15,16},{3,12,20,21},{3,13,14,22},{3,17,18,19},{4,11,12,16},{4,13,21,22},{4,14,15,18},
{4,17,19,20},{11,12,13,23},{11,12,16,23},{11,13,14,23},{11,14,15,23},{11,15,16,23},
{12,13,21,23},{12,16,20,23},{12,20,21,23},{13,14,22,23},{13,21,22,23},{14,15,18,23},
{14,18,22,23},{15,16,19,23},{15,18,19,23},{16,19,20,23},{17,18,19,23},{17,18,22,23},
{17,19,20,23},{17,20,21,23},{17,21,22,23}}];
Curvatures[s,0]
Curvatures[s,1]
Curvatures[s,2]
Curvatures[s,3]
```

8.3. The f-vector of the homology manifold is $(24, 154, 260, 130)$. The 24 vertex curvatures K_0 are constant 0 as for any 3-manifold. The 154 edge curvatures K_1 take values in $\{-67/462, -5/132, 0, 5/84, 13/22\}$, the 260 face curvatures K_2 take values $\{-1/30, -1/60, -1/90, 0, 1/18\}$ and the 130 chamber curvatures K_3 take values $\{-1/40, -1/90, 0, 7/180\}$.

8.4. The following projective 3-space $\mathbb{RP}^3 \sim SO(3)$ is also from [35].

```
s=RP3=ProjectiveThreeSpace = Generate[
{{1,2,3,4},{1,2,3,5},{1,2,4,6},{1,2,5,6},{1,3,4,7},{1,3,5,7},{1,4,6,7},{1,5,6,7},
{2,3,4,8},{2,3,5,9},{2,3,8,9},{2,4,6,10},{2,4,8,10},{2,5,6,11},{2,5,9,11},{2,6,10,11},
{2,7,8,9},{2,7,8,10},{2,7,9,11},{2,7,10,11},{3,4,7,11},{3,4,8,11},{3,5,7,10},{3,5,9,10},
{3,6,8,9},{3,6,8,11},{3,6,9,10},{3,6,10,11},{3,7,10,11},{4,5,8,10},{4,5,8,11},{4,5,9,10},
{4,5,9,11},{4,6,7,9},{4,6,9,10},{4,7,9,11},{5,6,7,8},{5,6,8,11},{5,7,8,10},{6,7,8,9}}];
Curvatures[s, 0]
Curvatures[s, 1]
Curvatures[s, 2]
Curvatures[s, 3]
```

8.5. For the projective 3-space the f-vector is $(11, 51, 80, 40)$. The Euler characteristic is 0, the betti vector is $(1, 0, 0, 1)$. The 11 vertex curvatures are zero. The 51 curvatures K_1 on edges take values in $\{-2/15, -1/9, -1/15, 1/30, 2/45\}$. The 80 curvatures K_2 on faces take values in $\{-1/30, -2/105, -11/840, 1/40, 13/420\}$. The 40 curvatures K_3 on chapters take values in $\{-3/80, -9/280, 1/56, 33/560\}$.

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8.6. Here is an example of a small 4-manifold $\mathbb{CP}^2$.

```
s=CP2=Generate[{{1,2,3,4,5},{1,2,3,4,7},{1,2,3,5,8},{1,2,3,7,8},{1,2,4,5,6},
{1,2,4,6,7},{1,2,5,6,8},{1,2,6,7,9},{1,2,6,8,9},{1,2,7,8,9},{1,3,4,5,9},
{1,3,4,7,8},{1,3,4,8,9},{1,3,5,6,8},{1,3,5,6,9},{1,3,6,8,9},{1,4,5,6,7},
{1,4,5,7,9},{1,4,7,8,9} {1,5,6,7,9},{2,3,4,5,9},{2,3,4,6,7},{2,3,4,6,9},
{2,3,5,7,8},{2,3,5,7,9},{2,3,6,7,9},{2,4,5,6,8},{2,4,5,8,9},{2,4,6,8,9},
{2,5,7,8,9},{3,4,6,7,8},{3,4,6,8,9},{3,5,6,7,8},{3,5,6,7,9},{4,5,6,7,8},{4,5,7,8,9}}];
Curvatures[s, 0]
Curvatures[s, 1]
Curvatures[s, 2]
Curvatures[s, 3]
```

8.7. The f-vector is $(9, 36, 84, 90, 36)$. The Euler characteristic is 3. The Betti numbers are $(1, 0, 1, 0, 1)$. The K_0 curvature is constant $1/3$. The edge curvature K_1 is constant $1/12$. The face curvatures K_2 take value in $\{-31/140, -1/14, 11/140, 8/35\}$. The 3-dimensional curvatures K_3 take values in $\{0, 1/20, 1/30\}$. The 4-dimensional curvatures finally are all positive again taking values in $1/30, 1/10$. The 4-manifold CP^2 is an example of a $1/4$ pinched positive curvature manifold.

8.8. Finally lets mention some questions: What does it mean for G to have $K_1(x) > 0$ for all x? Or does it mean for G to have $K_2(x) > 0$ for all x? We originally looked into the face curvature because it felt like something close to **sectional curvature**, a notion a bit tricky to define in the discrete. As we see in the examples, positive sectional curvature can not just be replaced by positive K_2 curvature. We had been searching for index expectation curvatures K (also in the discrete) which have the property that if a manifold has positive sectional curvature, then this index expectation K is positive. That would settle the Hopf conjecture. The Gauss-Bonnet-Chern curvature used in the Gauss-Bonnet-Chern theorem which was developed from 1927 (Hopf) to 1944 (Chern) [16, 3, 12, 2, 7] works only until dimension 4 [8] but fails for 6-manifolds already [13, 17]. One of our motivations to look at curvatures supported on higher dimensional parts of space was to explore whether there are natural discrete versions of sectional curvature (which is not just a numerical scheme coming from the continuum) and (a very long shot) to get a discrete curvature that is positive for sufficiently nice discretizations of positive curvature manifolds. Index expectation curvatures allow more experimentation. Using index expectation curvatures on higher dimensional simplices allows to widen the search.

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