ENERGIZED SIMPLICIAL COMPLEXES

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ABSTRACT. For a simplicial complex with n sets, let $W^-(x)$ be the set of sets in G contained in x and $W^+(x)$ the set of sets in G containing x. A function $h : G \to \mathbb{Z}$ defines for every $A \subset G$ an energy $E[A] = \sum_{x \in A} h(x)$. The function energizes the geometry similarly as divisors do in the continuum, where the Riemann-Roch quantity $\chi(G) + \deg(D)$ plays the role of the energy. Define the $n \times n$ matrices $L = L^{--}(x, y) = E[W^-(x) \cap W^-(y)]$ and $L^{++}(x, y) = E[W^+(x) \cap W^+(y)]$ as well as $L^{+-}(x, y) = E[W^+(x) \cap W^-(y)] = (L^{-+})^T$. With the notation $S(x, y) = 1_n \omega(x) = \delta(x, y)(-1)^{\dim(x)}$ and $\mathrm{str}(A) = \mathrm{tr}(SA)$ define $g = SL^{++}S$. The results are: $\det(L) = \det(g) = \prod_{x \in G} h(x)$ and $E[G] = \sum_{x,y} g(x, y)$ and $E[G] = \mathrm{str}(g)$. The number of positive eigenvalues of g is equal to the number of positive energy values of h. In special cases, more is true: A) If $h(x) \in \{-1, 1\}$, all four homoclinic and heteroclinic matrices $L^{\pm\pm}$ are unimodular and $L^{-1} = g$. B) In the constant energy $h(x) = 1$ case, L and g are isospectral, positive definite matrices in $SL(n, \mathbb{Z})$ [9]. Actually, even if G is just a set of sets we get so isospectral multi-graphs defined by adjacency matrices L^{++} or L^{--} which have identical spectral or Ihara zeta function. The positive definiteness holds for effective divisors $h > 0$ in general. C) In the topological case $h(x) = \omega(x)$, the energy $E[G] = \mathrm{str}(L) = \mathrm{str}(g) = \sum_{x,y} g(x, y) = \chi(G)$ is the Euler characteristic of G and $\phi(G) = \prod_x \omega(x)$ [10], a product identity which holds for arbitrary set of sets. D) For $h(x) = -t^{|x|}$ with some parameter t we have $E[H] = 1 - f_H(t)$ with $f_H(t) = 1 + f_0 t + \cdots + f_d t^{d+1}$ for the f-vector of H and $L(x, y) = (1 - f_{W^-(x) \cap W^-(y)}(t))$ and $g(x, y) = \omega(x)\omega(y)(1 - f_{W^+(x) \cap W^+(y)}(t))$. Now, the inverse of g is $g^{-1}(x, y) = 1 - f_{W^-(x) \cap W^-(y)}(t)/t^{\dim(x \cap y)}$ and $E[G] = 1 - f_G(t) = \sum_{x,y} g(x, y)$.

1. INTRODUCTION

1.1. For a finite set G of sets, the connection matrix L is defined by $L(x, y) = 1$ if x and y intersect and $L(x, y) = 0$ if not. If G is a finite abstract simplicial complex, L is unimodular. We look here at the case where $L(x, y)$ is an "energy" of $x \cap y$. In the connection matrix case, the energy is the Euler characteristic of $x \cap y$.

1.2. Energizing a geometry G using a function $h(x)$ is not only motivated by physics or probability density functions $h(x)$, it also produces an affinity to algebraic geometry, as an integer-valued function with finite energy on an algebraic curve is a divisor. A combinatorial application is the construction of a large sample of isospectral multi-graphs and related, the construction of isospectral, positive definite, non-negative symmetric matrices L^{++}, L^{--} in $SL(n, Z)$. A set of sets G defines

a quadratic form, unimodular integral lattices in $\mathbb{R}^n$ and isometric inner products with isometric ellipsoids $(x, L^{++}x) = 1, (x, L^{--}x) = 1$.

1.3. Given a set of sets G, an energy function $h : G \to \mathbb{R}$ assigns an energy $\sum_{x \in A} h(x)$ to subsets A of G and in particular defines the total energy $E[G] = \sum_{x \in G} h(x)$. If G is a simplicial complex and $h(x) = \omega(x) = (-1)^{\dim(x)}$, the matrix L is unimodular [10]. In this case, the energy $E[A]$ is the Euler characteristic of a sub complex A of G. We generalize this here first to $\{-1, 1\}$-valued functions h. By allowing h to be arbitrary real numbers and also relaxing G to be an arbitrary set of sets, we explore also boundary cases, where things start to be different.

1.4. The subject ties in with a variety of other topics [8]. The unimodularity result has relations with hyperbolic dynamical systems, spectral theory, quadratic forms, integral lattices, zeta functions and the representation theory of graph arithmetic. Bringing in the discrete divisors in the form of energy relates to some discrete algebraic geometry: Riemann-Roch in the form of Baker-Norine theory [1] generalized to multigraphs in [5] tells that the total energy $E[G]$ of a divisor is a signed distance $l(G) - l(K - G)$ with canonical divisor K and where $l(G)$ is the least energy value one has to add so that the structure can be made effective (non-negative everywhere) modulo changing the energy using principal divisors.

1.5. We comment more on the Riemann-Roch situation elsewhere but for now just want to mention that in the discrete, Riemann-Roch is very concrete. While in the continuum, adding a principal divisor is done via rational functions, the principal divisors in the discrete are just the image of a discrete Laplacian. Adding such a divisor is physical in the energy picture: it means taking part of the energy on one node of the geometry and distributing the energy locally using the Laplacian given by the incidence graph of G, an operation which is energy preserving. Some call the process the chip firing game.

1.6. Also in the energized situation, the matrix entries L and its inverse g relate to a hyperbolic dynamical system in which stable and unstable manifolds intersect in a "heteroclinic tangle". A bit surprisingly, this structure appears in the simplest possible frame work of mathematics, where one just takes a finite set of sets. One does not even need to have a simplicial complex. But having the later structure makes things nicer: the gradient system to the dimension functional is then Morse in the sense that every $x \in G$ is a hyperbolic critical point with index $\omega(x) \in \{-1, 1\}$. The stable and unstable manifolds $W^-(x), W^+(x)$ are localized and the energy of their intersections define the matrices.

1.7. As for $\{-1, 1\}$-valued energies, one has unimodularity so that $g = L^{-1}$ is integer valued. This includes not only the original topological frame work $h(x) = \omega(x)$ but also in a situation close to quantum physics: in the manifold case, where the kernels g of the inverse of Laplacians are singular and the potential $V_x(y) = g(x, y)$ is long range; now, in the discrete, the interaction is finite range and can be of service in relativistic frame works. We remain here in a elementary combinatorial and linear

algebra set-up. Much even generalizes to sets of sets G, a situation obtained by dropping the only axiom.

1.8. We still should stress that having a mass gap, an interval around 0 without spectrum of the Laplacian is rather special as it also can survive the van Hove limit in which we look at an infinite structure like $\mathbb{Z}^d$. This is not automatically a consequence of a discretization. Indeed, the most obvious discrete Laplacian H in mathematics does not have this property: if d is the exterior derivative defined on G and if $H = dd^* + d^*d = (d + d^*)^2$ is the Hodge Laplacian on the set of differential forms, then $H = \oplus_k H_k$ is a $n \times n$ matrix which is block diagonal and the nullity of H_k is the Betti number b_k of G by Hodge theory. Even when inverting H_k on the ortho-complement of the harmonic forms, the pseudo inverse matrix entries $H_k^{-1}(x, y)$ are long-range, meaning that two points x, y in general interact, even if they are arbitrarily far apart.

1.9. Beside the situation where x has an energy $h(x) = \omega(x)$, we have also seen the case where the energy is constant 1. In this "effective divisor" case, the matrices L and g are in $SL(n, \mathbb{Z})$ and are positive definite as well as isospectral. It is a bit surprising that the result still holds also in the case when G is just a set of sets. The matrices L^{--} and L^{++} are then integer valued symmetric matrices which are isospectral. Each of them can be seen as the adjacency matrix of a multi-graph Γ^{++} and Γ^{--}. The fact that the matrices are isospectral means than that the graphs are isospectral. There are not many general tools to generate isospectral geometries, one is by Sunada (like for example used in [4]) which is relevant in structural chemistry [11].

2. The inverse

2.1. A finite abstract simplicial complex is a set of finite non-empty sets closed under the operation of taking finite non-empty subsets. For $x \in G$, the star $W^+(x)$ of x is the set of simplices which contain x (including x). The core $W^-(x)$ is the set of simplices contained in x (including x). Define $\omega(x) = (-1)^{\dim(x)}$, where $\dim(x) = |x| - 1$ and $|x|$ is the cardinality of x.

2.2. Define the matrix $S(x, y) = \delta(x, y)\omega(x)$ which has as trace the Euler characteristic $\chi(G)$ and which defines a super trace $\text{str}(A) = \text{tr}(SA)$ for any $n \times n$ matrix A. A function $h : G \to \mathcal{R}$ defines an energy of subsets A of G:

$$E[A] = \sum_{x \in A} h(x) \ .$$

2.3. Define the $n \times n$ matrices defined by homoclinic connections

$$L^{--}(x, y) = E[W^-(x) \cap W^-(y)], L^{++}(x, y) = E[W^+(x) \cap W^+(y)] \ .$$

For completeness, one can also define the heteroclinic connection matrices

$$L^{+-}(x, y) = E[W^+(x) \cap W^-(x)], L^{-+}(x, y) = E[W^-(x) \cap W^+(x)]$$

even so do not need them here. The later are upper and lower triangular if the simplicial complex G is ordered from smaller to larger dimensional simplices. To

be closer to the previous covered special cases [10, 9] we can define $L = L^{--}$ and $g = SL^{++}S$, which is conjugated to L^{++} as $S = S^{-1}$. All these matrices are integer-valued if h is integer-valued.

2.4. For the next result, we assume that G is sorted so that if $x \subset y$, then x comes before y in the listing of the sets in G. An order of G together defines the basis in which the matrices are written.

Theorem 1. *The product Lg is a lower triangular matrix with diagonal entries $h(x)^2$. The entries of Lg below the diagonal are all of the form $h(u)^2 - h(v)^2$.*

Proof. a) To check the diagonal entries, we have to verify

$$\sum_y L^{--}(x, y)\omega(y)\omega(x)L^{++}(y, x) = h(x)^2 .$$

Since $L^{--}(x, y)$ is the energy of all the sets in $x \cap y$ $L^{++}(y, x)$ is the energy of all sets containing $x \cup y$ the contribution of $y = x$ is $h(x)^2$. The claim therefore is

$$\sum_{y \neq x} L^{--}(x, y)\omega(y)\omega(x)L^{++}(y, x) = 0 .$$

As any set u contributing to $L(x, y)$ must be strictly inside x and any set u contributing to $g(y, x)$ must contain x strictly, the statement follows.
b) If $x \subset z$ but not $x = z$, the upper triangular entry must be

$$\sum_y L^{--}(x, y)\omega(y)\omega(z)L^{++}(y, z) = 0 .$$

To see this, note that set u contributing to $L^{--}(x, y)$ must be contained in x and any set u contributing to $L^{++}(y, z)$ must contain z. There is no such set and therefore, the answer is 0.
c) If $z \subset x$ then, the entry

$$\sum_y L^{--}(x, y)\omega(y)\omega(z)L^{++}(y, z)$$

is a sum of of differences $h(u)^2 - h(v)^2$.
To see this, look at a set u which is a subset of x and contains z. Show that they appear in pairs. The reason is that the complement of z in x is a complete complex including the empty complex. The even and odd ones appear with the same cardinality. □

2.5. This leads immediately to the corollary:

Theorem 2. *If h takes values in $\{-1, 1\}$, then g is the inverse of L.*

2.6. Any ordering of G defines a basis. An other ordering would lead to a conjugated matrices which would in general no more be lower triangular. For example, if we would order the simplices with the largest simplices first, the matrix Lg would be upper triangular. We see from the proof that if the energy h takes values $\{-a, a\}$, with some real non-zero a, then Lg is a^2 times the identity matrix, implying the corollary.

3. DETERMINANT

3.1. The next result deals with determinants. For any real-valued function $h : G \to \mathbb{R}$ define its Fermi characteristic

$$\phi(G) = \prod_{x \in G} h(x) \ .$$

It is a multiplicative version of the total energy $E[G] = \sum_{x \in G} h(x)$. All the homoclinic or heteroclinic matrices $L^{++}, L^{--}, L^{+-}, L^{-+}$ have the same determinant. The result is a bit more general now as it does not require G to be a simplicial complex. It could be a finite topology or finite Boolean algebra for example.

Theorem 3 (Determinant). *If G is a finite set of sets, then $\det(L) = \det(g) = \prod_x h(x)$.*

Proof. The matrix L^{++} has only entries 0 or $h(x)$ in the last column because $L^{++}(x, y) = h(x)$ if $y \subset x$ and $L^{++}(x, y) = 0$ else.

$$g = \begin{bmatrix} g(1,1) & g(1,2) & . & . & . & g(1,n) & b_1 h(x) \\ g(2,1) & g(2,2) & . & . & . & . & b_2 h(x) \\ . & . & . & . & . & . & b_3 h(x) \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ g(n,1) & . & . & . & . & g(n,n) & b_n h(x) \\ b_1 h(x) & b_2 h(x) & \ldots & \ldots & \ldots & b_n h(x) & h(x) \end{bmatrix}$$

where $b_i \in \{0, 1\}$. Look at all the possible paths in the determinant which enter the neighborhood of x. These interaction paths come in pairs one which has x as a fixed point and goes yz and which which does not and goes yxz. These pairs cancel. The only paths left are the paths which do not interact with x. But that means that we have the situation where x is separated. If x is gone, then any two pair y, z of $S(x)$ are not connected in g as $W^+(y) \cap W^-(y) = 0$. $\square$

3.2. This implies in the case $h(x) \in \{-1, 1\}$ that both matrices L, g are unimodular. If $h(x)$ is constant 1 [9], the case where the energy simply counts simplices, we additionally know that L, g are in $SL(n, \mathbb{Z})$.

Corollary 1 (Unimodularity). *If G is a finite set of sets and h takes values in $\{-1, 1\}$ then $\det(L) = \det(g) = \prod_x h(x)$ takes values in $\{-1, 1\}$. The matrices L^{++}, L^{--}, L and g are then all unimodular.*

3.3. When $h(x) = (-1)^{\dim(x)}$ we interpreted $\sum_{x \in G} h(x) = \mathrm{E}[G]$ as a Poincaré-Hopf result, and interpreted $h(x)$ as an index. The formula $\sum_{x \in G} \omega(x) E[S(x)] = \mathrm{E}[G]$ was a dual Poincaré-Hopf result used in the original proof of the energy theorem.

3.4. Similarly, $\prod_x h(x) = \det(L)$ can be seen as a multiplicative Poincaré-Hopf result. Actually, one can see the identity $\mathrm{E}[G] = \sum_x \omega(x) E[W^+(x)]$ as a Poincare-Hopf result in general.

4. The eigenvalues

4.1. Related it determinants is the next result on eigenvalues. For a symmetric invertible matrix A, define the Morse index of A as the number of negative eigenvalues of A. Define also the Morse number of the energy function h as the number of negative eigenvalue entries. There is some relation between the function h and the eigenvalues. We could formulate it for the matrices $L^{++}, L^{--}, L^{+-}, L^{-+}$, where it is true also but do it for the two matrices $L = L^{--}$ and $g = SL^{++}S$:

Theorem 4. *The number of negative eigenvalues of $L = L^{--}$ is equal to the number of negative values of the energy function h. The same is true for the matrices L^{++} or g.*

Proof. This goes by induction. Let x be the latest cell added to the CW complex. Again look at the matrix g and multiply the last entry with t:

$$
g = \begin{bmatrix}
g(1,1) & g(1,2) & . & . & . & g(1,n) & tb_1 h(x) \\
g(2,1) & g(2,2) & . & . & . & . & tb_2 h(x) \\
 & . & . & . & . & . & tb_3 h(x) \\
 & . & & . & . & . & . \\
 & . & & . & . & . & . \\
g(n,1) & . & & . & . & g(n,n) & tb_n h(x) \\
tb_1 h(x) & tb_2 h(x) & . & . & . & \ldots & tb_n h(x)
\end{bmatrix}
$$

For $t = 0$ we have the determinant $E = \prod_{k=1}^{n} e(k)$, for $t = 1$ we have the determinant $\prod_{k=1}^{n+1} e_k = Eh(x)$. Assume there exists a t_0 such that the determinant of $g(t_0) = 0$. It is linear in t. A linear function between two positive values is never 0. $\square$

4.2. This implies:

Corollary 2. *If h is takes only positive values, then both L, g are positive definite.*

4.3. It also implies that an old result [7] that if

$$
h(x) = \omega(x) = (-1)^{\dim(x)} = (-1)^{|x|-1}
$$

for which the total energy is the Euler characteristic, that the Euler characteristic is the number of positive eigenvalues of g minus the number of negative eigenvalues of g.

4.4. Having positive definite integer quadratic forms is always exciting in mathematics. It leads to other topics like lattice packings. The matrices L, g are integral quadratic forms which could serve as a metric in $\mathbb{R}^n$. Some questions are asked at the end of this document. One can for example ask number theoretical questions like how large the set of sets is for which the quadratic form are universal in the context of the Conway-Schneeberger's 15 theorem.

5. The potential energy

5.1. We now relate the total energy $E[G] = \sum_x h(x)$ of the complex with the total potential theoretical energy is $\sum_{x,y} g(x,y)$ of G. The next result tells that these two quantities agree. The theorem requires G to be a simplicial complex. It does not work for sets of sets.

Theorem 5 (Energy theorem for energized complexes). *For any simplicial complex G and any energy function h, the total energy is $E[G] = \sum_{x,y} g(x,y)$.*

5.2. The key is to show that $\omega(x)g(x,x) = \sum_y g(x,y)$ for every x and using the spectral energy result in the next section. But this is

$$\omega(x)g(x,x) = \sum_y \omega(x)\omega(y)E[W^+(x) \cap W^+(y)]$$

which is equivalent to

$$E[W^+(x) \cap W^+(x)] = \sum_y \omega(y)E[W^+(x) \cap W^+(y)]$$

which is the statement "total energy = spectral energy" in the case $G = W^+(y)$ as the left hand side is the total energy of $W^+(x)$ and $E[W^+(x) \cap W^+(y)]$ s the diagonal entry $g(y,y)$ if $W^+(x)$ is the total geometry.

6. The spectral energy

6.1. There is a third energy, the spectral energy which is the sum of the eigenvalues of Sg. Note that in general the eigenvalues of Sg are complex because Sg is not symmetric. But also this spectral energy agrees with the energy. We interpret it as a McKean-Singer statement which classically is $\mathrm{str}(e^{-tH}) = \chi(G)$ where $H = (d+d^*)^2$ is the Hodge Laplacian.

Theorem 6 (Mc Kean Singer). $E[G] = \mathrm{tr}(Sg) = \mathrm{str}(g)$.

Proof. Every simplex x has an energy $h(x)$. The function $\dim(x)$ is a locally injective function on the Barycentric graph in which G are the vertices and where two sets x, y in G are connected, if one is contained in the other. For every simplex x, let $v(x)$ be a choice of the $\dim(x) - 1$-dimensional simplices in x. Subtracting the energy from x and adding it to $v(x)$ does not change the total energy nor does it change the super trace. After moving all the energy down to the vertices, the statement is obvious as $E[W^+(x)]$ is now $(-1)^{\dim}(x)h(x)$. □

6.2. The fact that the energy is the super trace of g is a discrete analogue of the super trace formula $\chi(G) = \mathrm{str}(e^{-H})$ for the Hodge Laplacian $H = (d + d^*)^2$ and Euler characteristic. This is an important step in the proof as the diagonal entries of $Sg(x,x) = \omega(x)g(x,x)$ can be interpreted as the total potential energy $\sum_{y \in G} V_x(y)$ with $V_x(y) = \omega(x)\omega(y)g(x,y)$. The proof actually will see this then as a curvature and the theorem as a Gauss-Bonnet statement for the total energy functional.

7. Spectral symmetry

7.1. Finally, in the constant energy case $h == 1$, there is more symmetry. This has been mentioned in [9] already. We want to say here more about the proof which is inductive but still a bit technical even-so we use duality to half the difficulty and use a continuous deformation argument.

Theorem 7 (Spectral symmetry). *If the energy function is constant 1 then L and g are positive definite. They are inverse to each other and iso-spectral. The iso-spectral property of L^{++} and L^{--} result holds for arbitrary finite sets of sets G.*

7.2. It helps to notice that the statement about L^{++} and L^{--} is true even if G is an arbitrary set of sets and not only true for simplicial complexes. For sets of sets, there is a duality between stable and unstable parts. The axiom of simplicial complexes introduces an asymmetry in that we require the structure to be invariant under inclusions and not the inclusion of complements. Now, in the more general frame-work taking a structure G allows to look at the dual structure $\hat{G}$. The matrices L^{++}, L^{--} for G become then the matrices L^{--}, L^{++} for the Boolean dual $\hat{G}$. Everything which only concerns determinants or spectra is true in general for sets of sets. The spectral symmetry is that g and L have the same coefficients in their characteristic polynomial.

7.3. As mentioned in [9], the proof is a deformation argument. But rather than using a deformation in which the coefficients of the characteristic polynomial are quadratic functions in t (leading to an "artillery picture"), we use here a deformation in which the coefficients change in a linear manner if one parameter is changed.

7.4. The basic idea is to use induction and change the energy of one of the sets x from $h(x) = 0$ to $h(x) = 1$. In the case if x is not contained in any other set, the deformation of L^{--} is easier to describe than the deformation of L^{++}. For L^{--} only one column changes, while for L^{++} all entries $L(x, y)$ with $y \subset x$ change. If we go to the dual picture $\hat{G}$, which is the set of complement sets of G, the matrices L^{--} and L^{++} interchange and the analysis for L^{--} for $\hat{G}$ which is analogue to the analysis for L^{++} for G is the same.

7.5. We split the deformation into two parts. In a first part, the energies of the energies "outgoing" from x are throttled. We start with the situation $t = 0$, where by induction assumption, the coefficients of the characteristic polynomials of L^{--} and L^{++} are both palindromic. This deformation does not preserve the palindromic property of $L^{++}(t)$ at $t = 1$ but does preserve the palindromic property of $L^{--}(t)$ for $t = 1$.

7.6. Since L, g are both symmetric, they are both diagonalizable using orthogonal matrices. It follows that L and g are conjugated by orthogonal matrices. By computing eigenbasis this "scattering matrix" $Q \in SO(n)$ satisfying $g = Q^{-1}LQ = L^{-1}$. This matrix Q is defined uniquely up to permutation and signature change coordinate changes. If the order with which the set of sets G is build-up is given we still have the choice of two directions in each eigenvector. There is then a natural

ordering of the eigenvalues λ_k of both L and g. If $ULU^T = \text{Diag}(\lambda_1, \cdots, \lambda_n)$ and $VgV^T = \text{Diag}(\lambda_1, \cdots, \lambda_n)$, where the λ_i are paired with the sets x_i. This pairing also works if there should be multiple eigenvalues. Then $O = U^{-1}V$ conjugates L and g.

7.7. Example:
1) For the ""komma structure" $G = \{\{1\}, \{1,2\}\}$, which is not a simplicial complex,
$L = L^{--} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. $L^{++} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. The matrix $O = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ conjugates

$OL^{--} = L^{++}O$. 2) For $G = \{\{1,2,3\}, \{1,2\}, \{1\}\}$, we have $L^{--} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$L^{++} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $O = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

7.8. What happens is that the palindromic property in the case n becomes an anti-palindromic property for $n+1$ and vice versa. The artillery function

$$A_k(t) = p_k(t) * p_{k+1}(t) - p_{n+2-k}(t)^2 - (-1)^n(p_{k+1}(t)p_{n+2-k}(t) - p_k(t) * p_{n+2-k}(t))$$

is a quadratic function in t. The goal to show that if it is zero for $t = 0$ it is zero for $t = 1$ is a bit difficult even so we know that the coefficients are sums of minors.
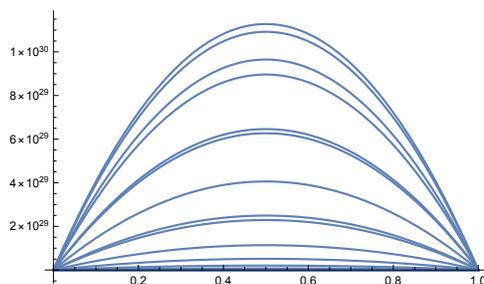

FIGURE 1. The artillery functions $A_k(t)$ is a quadratic function in t. The isospectral property in the induction step from n to $n+1$ cells means that $A_k(t) = 0$ for $t = 1$. This means that the characteristic polynomial remains palindromic.

7.9. We know already that L and g are positive definite and that L and g are inverse of each other. In order to show that L and g are isospectral we have to show that if λ_k is an eigenvalue, then $1/\lambda$ is an eigenvalue. This means that the characteristic polynomials $\det(L - \lambda)$ and $\det(g - \lambda)$ agree. Alternatively, this means that the coefficients

$$p(x) = \det(g - x) = p_0(-x)^n + \cdots + p_k(-x)^{n-k} + \cdots + p_n$$

of the characteristic polynomial satisfies the palindromic property

$$p_k = \sum_{|P|=k} \det(g_P) = (-1)^n \sum_{|P|=n-k} \det(g_P) = (-1)^n p_{n-k}$$

where $\det(g_P)$ is a minor, the determinant of a pattern. (Proof: $p(\lambda^{-1} = \det(g - \lambda^{-1}) = \text{prod}(\lambda_k - \lambda^{-1})$ is equal to $\det(\lambda_k^{-1} - \lambda^{-1}) = (-\lambda^n \prod_k \lambda_k)\text{prod}\lambda_k - \lambda = (-\lambda^n)\det(g)\det(g - \lambda) = (-1)^n\lambda^n p(\lambda)$.)

7.10. To prove the result, we look at the characteristic polynomials of L^{++} and L^{--} together and show that they remain palindromic after adding a new cell. For the proof we deform the matrices L^{++} and L^{--} with two parameters. One parameter parameter T scales the column of the cell. The second parameter H is the energy of the cell x_i If the set of sets G is ordered in such a way that the last element is not contained in any other set, the parameter T can be interpreted as a throttle to release energy from the last element to the others.

$$L_{T,R} = \begin{bmatrix} L(1,1) & L(1,2) & . & . & . & L(1,n) & TL(1,n+1) \\ L(2,1) & L(2,2) & . & . & . & . & TL(2,n+1) \\ . & & . & . & . & . & TL(3,n+1) \\ . & & . & . & . & . & . \\ . & & . & . & . & . & . \\ L(n,1) & & . & . & . & L(n,n) & TL(n,n+1) \\ L(n+1,1) & L(n+1,2) & \ldots & \ldots & \ldots & \ldots & TL(n+1,n+1) \end{bmatrix}.$$

7.11. In a dual way, we can throttle the energy coming to the last element. As the last set is not contained in any other this is equivalent to parametrize the energy $h(n) = H$ with $H \in [0,1]$. The two situations and deformations are analog because of duality. If we go from G to $\hat{G}$, then this changes the subset ordering to supset ordering and changes the L^{++} to L^{--}.

7.12. In order to go from a situation with n sets to a situation with $n+1$ sets, we first turn on the parameters T and H from 0 to 1. Each part only preserves the palindromic property of one of the characteristic polynomials as well as the difference between suitably shifted sequences.

7.13. The coefficients $p_k(T,H)) = q_k(H,T)$ of the characteristic polynomials of $L^{++}(T,H)$ and $L^{--}(T,H)$ are all multi-linear functions in H and T. The difference $p_k(T,H) - q_k(T,H)$ is symmetric in T,H and it remains to show that $Tp_k(T,1/T)$ is palindromic for all times. This is a statement which holds for any energy change of any cell. The $H = 1/T$ of the outgoing energy compensates with the energy T of the in incoming energies.

7.14. The choice of $H = 1/T$ is motivated by the fact that with $H = 1/T$ the deformed matrices remain in $SL(n,\mathbb{Z})$. (We currently believe that this would lead to a more general result besides the case $h(x) = 1$ in which every node has in and out-going energies which multiply to 1.) We get then a quadratic function in T, but we only need to show that the first and second derivatives are palindromic at

$T = 1$. Since shooting the artillery leads for $T = 0$ to a palindromic situation and the palindromic property holds along the way, we also have a palindromic situation at $T = 1$. Now, taking the derivative with respect to T is by the Laplace expansion of the minors a sum of smaller dimensional situations in which cells y_k have their energy modified allowing induction on the number n of cells.

8. ILLUSTRATING THE DEFORMATION PROOF

8.1. To illustrate the proof, we work with

$$G = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\} .$$

We have

$$L^{--} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 3 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 3 \end{bmatrix} ,$$

$$L^{++} = \begin{bmatrix} 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

Both matrices have the palindromic characteristic polynomial

$$p_L(\lambda) = \det(L - \lambda) = \lambda^8 - 16\lambda^7 + 95\lambda^6 - 268\lambda^5 + 380\lambda^4 - 268\lambda^3 + 95\lambda^2 - 16\lambda + 1 .$$

The coefficient list of $p_L(-\lambda)$ is $p = (1, 16, 95, 268, 380, 268, 95, 16, 1)$. We have taken $-\lambda$ in order not having to bother with negative signs and palindromic or anti-palindromic situations.

8.2. Adding an other cell $\{2, 3, 4\}$ gives the complex

$$G = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}\}$$

leading to the matrices

$$L^{--} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 3 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 3 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 & 1 & 3 & 1 & 3 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 3 & 3 \\ 0 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & 7 \end{bmatrix}$$

and

$$L^{++} = \begin{bmatrix} 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 2 & 2 & 1 & 2 & 2 & 1 & 1 \\ 0 & 2 & 4 & 2 & 0 & 2 & 1 & 2 & 1 \\ 0 & 2 & 2 & 4 & 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 & 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} .$$

Both matrices have palindromic characteristic polynomials

$$p_L(\lambda) = \det(L-\lambda) = -\lambda^9 + 23\lambda^8 - 176\lambda^7 + 628\lambda^6 - 1167\lambda^5 + 1167\lambda^4 - 628\lambda^3 + 176\lambda^2 - 23\lambda + 1 .$$

The coefficient list of $p_L(-\lambda)$ is the palindrome $q = (1, 23, 176, 628, 1167, 1167, 628, 176, 23, 1)$.

8.3. In our case, as $\{2, 3, 4\}$ is not contained in any other set, the matrix L^{--} has not affected the first 9 rows and columns of L^{--} before the addition. But the matrix L^{++} has changed every entry $L^+(x, y)$ if $y \subset x$. The energy of these sets y changed because they are contained in a common cell of x and y.

8.4. Now lets change the energy of the cell x and denote it with $H = h(x)$. The other energy entries remain 1. Now we have

$$L^{--} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 3 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 3 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 & 1 & 3 & 1 & 3 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 3 & 3 \\ 0 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & H+6 \end{bmatrix}$$

and

$$L^{++} = \begin{bmatrix} 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & H+4 & H+1 & H+1 & 1 & H+1 & H+1 & H & H \\ 0 & H+1 & H+3 & H+1 & 0 & H+1 & H & H+1 & H \\ 0 & H+1 & H+1 & H+3 & 0 & H & H+1 & H+1 & H \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & H+1 & H+1 & H & 0 & H+1 & H & H & H \\ 0 & H+1 & H & H+1 & 0 & H & H+1 & H & H \\ 0 & H & H+1 & H+1 & 0 & H & H & H+1 & H \\ 0 & H & H & H & 0 & H & H & H & H \end{bmatrix}.$$

8.5. The coefficient list of the characteristic polynomial of L^{--} as a function of H is $p(H) = (H, 7+16H, 81+95H, 360+268H, 787+380H, 899+268H, 533+95H, 160+16H, 22+H, 1)$ and for L^{++} it is $q(H) = (H, 1+22H, 16+160H, 95+533H, 268+899H, 380+787H, 268+360H, 95+81H, 16+7H, 1)$. While $q(0), q(1) = p(1)$ are palindromic, the characteristic polynomial of $L^{--}(H = 0)$ is not but since H only appears in the corner, the derivative $p'(H)$ is constant $q(0)$ and palindromic. While the induction step is harder to handle in the case L^{--}, the $L^{+}+$ picture has a better deformation property in the sense that $q(0)$ and $q(1)$ are both palindromic. The reason that $p(0)$ is not palindromic is that in the case $H = 0$, we still have contributions of the new cell even so its energy has been turned off.

8.6. We therefore have to turn off also the flux of the energy to the new cell x We do this with an other parameter T. There is now a two parameter family of deformations

$$L^{--} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & T \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & T \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & T \\ 1 & 1 & 0 & 0 & 3 & 1 & 1 & 0 & T \\ 0 & 1 & 1 & 0 & 1 & 3 & 1 & 1 & 3T \\ 0 & 1 & 0 & 1 & 1 & 1 & 3 & 1 & 3T \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 3 & 3T \\ 0 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & (H+6)T \end{bmatrix}.$$

It has the characteristic polynomial coefficients

$$\begin{aligned} p_{T,H} = \ & (HT, 1 + 6T + 16HT, 16 + 65T + 95HT, 95 + 265T + 268HT, 268 + 519T + 380HT, \\ & 380 + 519T + 268HT, 268 + 265T + 95HT, 95 + 65T + 16HT, 16 + 6T + HT, 1). \end{aligned}$$

When $T = 0, H = 0$, we have the palindromic property from the case without the cell x but shifted due to an other eigenvalue 0.

$$T = 0, H = 0, p_{00} = (0, 1, 16, 95, 268, 380, 268, 95, 16, 1)$$

Now, $T = 1, H = 0$ is not palindromic but $T = 0, H = 1$ gives the palindromic characteristic polynomial coefficients:

$$T = 0, H = 1, p_{01} = (0, 1, 16, 95, 268, 380, 268, 95, 16, 1).$$

Finally, when $T = 1, H = 1$, when we allow energy to go into the new cell x, we have the new situation

$$T = 1, H = 1, p_{11} = (1, 23, 176, 628, 1167, 1167, 628, 176, 23, 1) .$$

8.7. To see that the transition from $T = 0, H = 1$ to $T = 1, H = 1$ preserves the palindromic property we look at

$$L^{++} = \begin{bmatrix}
2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & H+4 & H+1 & H+1 & 1 & H+1 & H+1 & H & HT \\
0 & H+1 & H+3 & H+1 & 0 & H+1 & H & H+1 & HT \\
0 & H+1 & H+1 & H+3 & 0 & H & H+1 & H+1 & HT \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & H+1 & H+1 & H & 0 & H+1 & H & H & HT \\
0 & H+1 & H & H+1 & 0 & H & H+1 & H & HT \\
0 & H & H+1 & H+1 & 0 & H & H & H+1 & HT \\
0 & H & H & H & 0 & H & H & H & HT
\end{bmatrix}$$

which has the coefficients

$$
\begin{aligned}
q_{T,H} \quad = \quad & (HT, 1 + 6H + 16 * HT, 16 + 65H + 95HT, 95 + 265H + 268HT, \\
& 268 + 519H + 380HT, 380 + 519H + 268HT, 268 + 265H + 95HT, 95 + 65H + 16HT, 16 + 6H + HT, 1)
\end{aligned}
$$

which satisfies $q_{T,H} = p_{H,T}$. The difference $q_{T,H} - p_{T,H}$ is palindromic

$$
\begin{aligned}
q_{T,H} - p_{T,H} \quad = \quad & (0, -6H + 6T, -65H + 65T, -265H + 265T, -519H + 519T, -519H + 519T, \\
& -265H + 265T, -65H + 65T, -6H + 6T, 0) .
\end{aligned}
$$

We have

$$
\begin{aligned}
q_{T,T} \quad = \quad & p_{T,T} = (T^2, 16T^2 + 6T + 1, 95T^2 + 65T + 16, 268T^2 + 265T + 95, 380T^2 + 519 \\
& T + 268, 268T^2 + 519T + 380, 95T^2 + 265T + 268, 16T^2 + 65T + 95, T^2 + 6T + 16, 1) .
\end{aligned}
$$

The palindromic property holds for $T = 0$ to $T = 1$. If we look at

$$
\begin{aligned}
T q_{T,1/T} \quad = \quad & T p_{T,1/T} = (T, 6T^2 + 17T, 65T^2 + 111T, 265T^2 + 363T, 519T^2 + 648T, 519T^2 + 648T, 265 \\
& T^2 + 363T, 65T^2 + 111T, 6T^2 + 17T, T) .
\end{aligned}
$$

The palindromic property for $q_{1,1}$ follows from the statement that $T q_{T,1/T}$ is palindromic for all T. The statement for $p_{1,1}$ then follows from duality. Let us look at this next:

8.8. When going from G to the dual

$$\hat{G} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$$

we reverse L^{++} and L^{--}. If we would have added the largest cell, we get just a reverse of L^{--} and L^{++}. Without resorting $\hat{G}$, we have

$$
L^{--} = \begin{bmatrix}
2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & H+4 & H+1 & H+1 & 1 & H+1 & H+1 & H & HT \\
0 & H+1 & H+3 & H+1 & 0 & H+1 & H & H+1 & HT \\
0 & H+1 & H+1 & H+3 & 0 & H & H+1 & H+1 & HT \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & H+1 & H+1 & H & 0 & H+1 & H & H & HT \\
0 & H+1 & H & H+1 & 0 & H & H+1 & H & HT \\
0 & H & H+1 & H+1 & 0 & H & H & H+1 & HT \\
0 & H & H & H & 0 & H & H & H & HT
\end{bmatrix}
$$

and

$$
L^{++} = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & T \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & T \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & T \\
1 & 1 & 0 & 0 & 3 & 1 & 1 & 0 & T \\
0 & 1 & 1 & 0 & 1 & 3 & 1 & 1 & 3T \\
0 & 1 & 0 & 1 & 1 & 1 & 3 & 1 & 3T \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 3 & 3T \\
0 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & (H+6)T
\end{bmatrix}
$$

So, if we make the induction assumption that for all sets of sets G with n elements, the palindromic property for L^{--}, L^{++} implies the palindromic property for L^{--} in the case $n+1$, then we also have that in the dual picture the palindromic property for L^{++}. We can therefore focus on one case.

8.9. So, the proof reduced to show that $Tp(T, 1/T) = Tq(1/T, T)$ is a palindromic sequence of quadratic functions in T for all T. This is shown if $Tp(T, 1/T)$

9. ZETA FUNCTION

9.1. As L, g are now positive definite quadratic forms which are isospectral one can define a spectral zeta function

$$
\zeta(s) = \sum_{k=1}^{n} \lambda_k^{-s} .
$$

One can also look at the Ihara zeta function $\zeta_I(s) = 1/\det(1 - sL)$.

Corollary 3. *Both the spectral zeta function as well as the Ihara zeta function satisfy in the constant energy case $h = 1$ the functional equation $\zeta(a + ib) = \zeta(-a + ib)$ respectively $\zeta_I(-a + ib) = \zeta_I(a + ib)$.*

Proof. Since the eigenvalues λ_k are real, we have $\zeta(a+ib) = \zeta(a-ib)$ and $\zeta_I(a+ib) = \zeta_I(a - ib)$. $\qquad\square$

9.2. Of particular interest is the case when $h(x) = 1$ for all x as then, we not only have the spectral symmetry, but additional know that the matrices are in $SL(n, \mathbb{R})$. They define positive definite quadratic quadratic forms, so that:

Corollary 4. *For every set of sets G, the matrices L^{++} and L^{--} define unimodular integral lattices defined by a positive definite quadratic form of determinant 1.*

10. Isospectral multigraphs

10.1. An other consequence of the spectral symmetry concerns multigraphs as illustrated in Figure (10.4), Figure (10.4), Figure (10.4) and Figure (10.4). If G is a set of sets, we can define the multi graphs Γ^{--} and Γ^{++} in which the sets are the nodes and where two sets x, y are connected using exactly $L^{--}(x, y)$ or $L^{--}(x, y)$ connections from x to y. As the adjacency matrices L^{++} and L^{--} are isospectral, the graphs are isospectral and especially, the number of closed paths of length k or length $n - k$ are the same.

Corollary 5. *For any set of sets G, the multigraphs defined by the adjacency matrices L^{--} and L^{++} are isospectral.*

10.2. The picture for the random walk on the graph Γ^{--} is that we can hop from set to set but take on the way from x to y a side step to a set in the intersection of x and y. This is encoded in the matrix entry $L^{--}(x, y)$ which count the total energy of the sets contained in $x \cap y$. The dual picture is that we can jump from x to y while side-stepping onto a set containing both x and y. These two situations can be looked at also in the dual case where $\hat{G}$ is the set of complements of sets in G. It is a primitive model in which x, y are entities and the sets in $x \cap y$ or $x \cup y$ model common shared interactions or (in the dual case) common bridge operations.

10.3. Various zeta functions have been defined for graphs. One can look at the sums $\sum_k \lambda_k^{-s}$ where λ_k are the eigenvalues of a Laplacian defined on the graph. Example of "Laplacians" are Kirchhoff Laplacians, Hodge Laplacian, connection Laplacians. The adjacency matrix works if it is positive definite. One can also look at the Ihara zeta function $\zeta(s) = 1/\det(1 - sL)$.

Corollary 6. *The graphs Γ^{++} and Γ^{--} have the same spectral and the same Ihara zeta function.*

10.4. The coefficients p_k of the characteristic polynomial of L or g have a geometric interpretation as the number of closed prime paths of length k, meaning that these are simple paths. In contrary, $\text{tr}(L^k)$ super counts the number of all closed paths of length k. The spectral symmetry result implies that both for $\Gamma = \Gamma^{++}$ and $\Gamma = \Gamma^{--}$ we have:

Corollary 7. *The super count p_k of all closed simple paths of length k in Γ either has the symmetry $p_k = p_{n-k}$ or $p_k = -p_{n-k}$.*
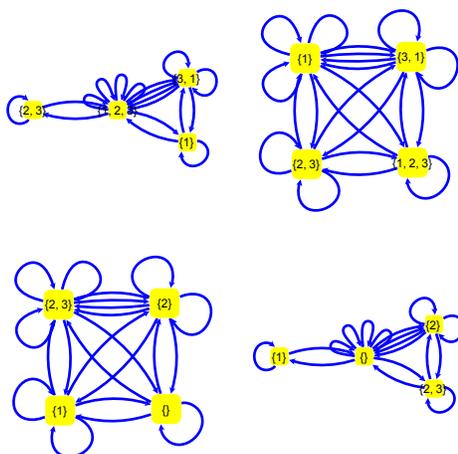
FIGURE 2. The graphs Γ^{--} and Γ^{++} for the set of sets $G = \{\{1\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\}$ have the sets as nodes and $L^{--}(x, y)$ rsp $L^{++}(x, y)$ connections from x to y. Below we see the dual set $\hat{G} = \{2, 3\}, \{1\}, \{2\}, \emptyset\}$ and its corresponding graphs $\hat{\Gamma}^{--}$ and $\hat{\Gamma}^{++}$.
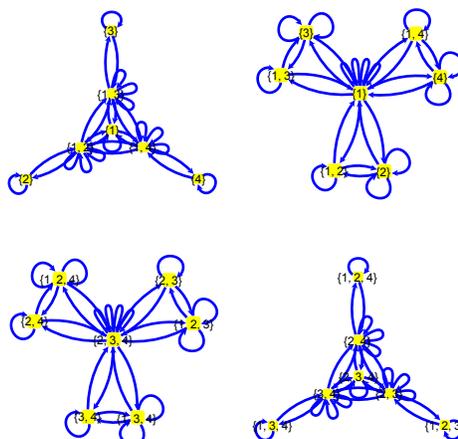

FIGURE 3. The graphs Γ^{--} and Γ^{++} for $G = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}\}$ and the dual structure. For any set of sets we get isospectral multi-graphs.

10.5. Finally, we should mention that the construction allows to construct many periodic and so also almost periodic or even random graphs which are isospectral. Just start with a sequence $\{0, 1\}$ and encode this into a graph by attaching little dangles at the points where we have a 1 and none else.
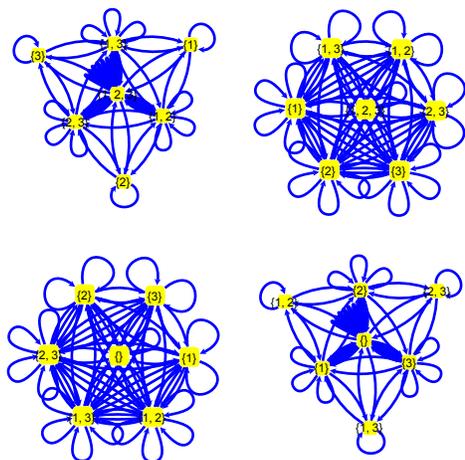
FIGURE 4. The graphs Γ^{--} and Γ^{++} for the complete complex G of all non-empty subsets of $\{1, 2, 3\}$.
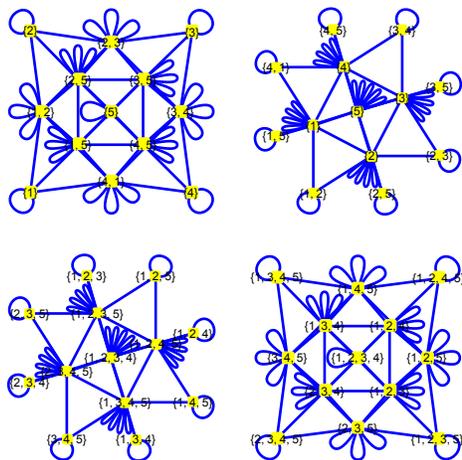

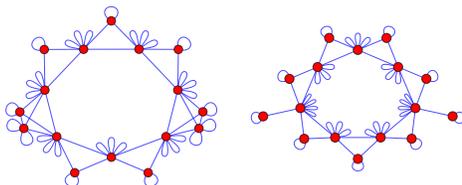FIGURE 5. The graphs Γ^{--} and Γ^{++} for wheel graph.

FIGURE 6. Constructing periodic and almost periodic matrices which are isospectral. Here is a small periodic isospectral pair of graphs
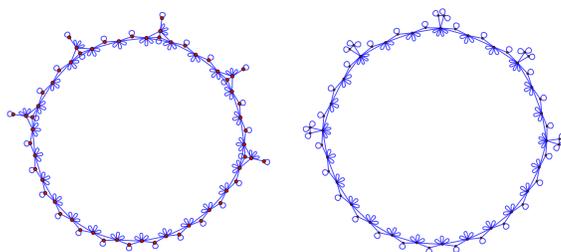
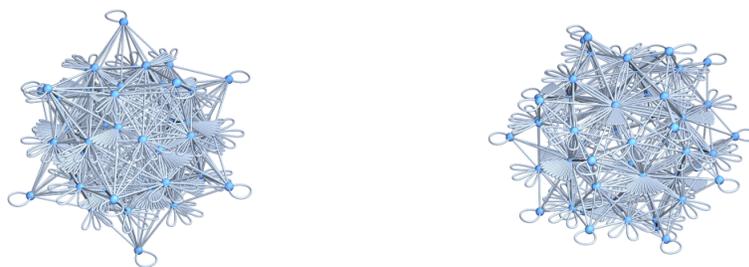FIGURE 7. A larger isospectral pair of multi-graphs

FIGURE 8. The multigraph and dual multigraph for the Whitney complex of the icosahedron graph.
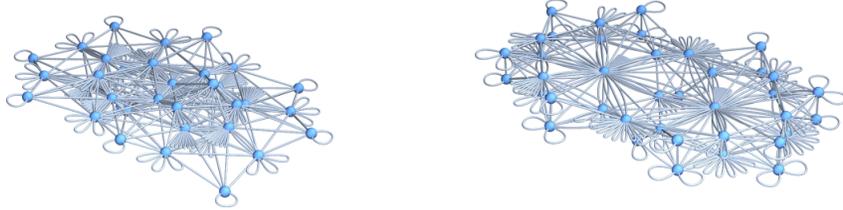
FIGURE 9. The multigraph and dual multigraph for the Whitney complex of a simple lattice region. We can take the van Hove limit and get pairs of periodic infinite matrices which are isospectral and have continuuous spectrum.

FIGURE 10. The same in any dimension. Any simplicial complex G defines two isospectral multi-graphs Γ^+, Γ^-. If G is a Barycentric refinement, then G, Γ^+, Γ^- are homotopic.

11. Examples

11.1. 1) Let $G = \{\{1\}, \{2\}, \{1, 2\}\}$ and let x, y, z be the energy values $h(\{1\}) = x, h(\{2\}) = y, h(\{z\}) = z$. Then

$$L = L^{--} = \begin{bmatrix} x & 0 & x \\ 0 & y & y \\ x & y & x + y + z \end{bmatrix}, L^{++} = \begin{bmatrix} x + z & z & z \\ z & y + z & z \\ z & z & z \end{bmatrix}$$

with determinants $\det(L) = \det(g) = xyz$ and

$$Lg = \begin{bmatrix} x^2 & 0 & 0 \\ 0 & y^2 & 0 \\ x^2 - z^2 & y^2 - z^2 & z^2 \end{bmatrix}.$$

This shows that if x, y, z take values in $\{-1, 1\}$, then L, SgS are inverses of each other. An other special case is if $x = y = z = a$ is constant, then the L and g have the same characteristic polynomial $p_L(t) = p_g(t) = a^3 - 5a^2t + 5at^2 - t^3$ and are therefore isospectral.

11.2. 2) For $G = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, and energy values $\{a, b, c, d, x, y, z\}$, we have

$$L = L^{--} = \begin{bmatrix} a & 0 & 0 & a & a & 0 & a \\ 0 & b & 0 & b & 0 & b & b \\ 0 & 0 & c & 0 & c & c & c \\ a & b & 0 & a + b + d & a & b & a + b + d \\ a & 0 & c & a & a + c + x & c & a + c + x \\ 0 & b & c & b & c & b + c + y & b + c + y \\ a & b & c & a + b + d & a + c + x & b + c + y & a + b + c + d + x + y + z \end{bmatrix}$$

and

$$L^{++} = \begin{bmatrix} a + d + x + z & d + z & x + z & d + z & x + z & z & z \\ d + z & b + d + y + z & y + z & d + z & z & y + z & z \\ x + z & y + z & c + x + y + z & z & x + z & y + z & z \\ d + z & d + z & z & d + z & z & z & z \\ x + z & z & x + z & z & x + z & z & z \\ z & y + z & y + z & z & z & y + z & z \\ z & z & z & z & z & z & z \end{bmatrix}$$

which both have the determinant $abcdxyz$. The sum of the matrix elements of

$$g = \begin{bmatrix} a + d + x + z & d + z & x + z & -d - z & -x - z & -z & z \\ d + z & b + d + y + z & y + z & -d - z & -z & -y - z & z \\ x + z & y + z & c + x + y + z & -z & -x - z & -y - z & z \\ -d - z & -d - z & -z & d + z & z & z & -z \\ -x - z & -z & -x - z & z & x + z & z & -z \\ -z & -y - z & -y - z & z & z & y + z & -z \\ z & z & z & -z & -z & -z & z \end{bmatrix}$$

is the total energy $a + b + c + d + x + y + z$. The matrix

$$Lg = \begin{bmatrix} a^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c^2 & 0 & 0 & 0 & 0 \\ a^2 - d^2 & b^2 - d^2 & 0 & d^2 & 0 & 0 & 0 \\ a^2 - x^2 & 0 & c^2 - x^2 & 0 & x^2 & 0 & 0 \\ 0 & b^2 - y^2 & c^2 - y^2 & 0 & 0 & y^2 & 0 \\ a^2 - d^2 - x^2 + z^2 & b^2 - d^2 - y^2 + z^2 & c^2 - x^2 - y^2 + z^2 & d^2 - z^2 & x^2 - z^2 & y^2 - z^2 & z^2 \end{bmatrix}$$

shows that if $x \in \{-1, 1\}$, then L and SgS are inverse matrices. If all energy values are equal to a, then both L and SgS have the characteristic polynomial $a^7 - 19a^6t + 102a^5t^2 - 228a^4t^3 + 228a^3t^4 - 102a^2t^5 + 19at^6 - t^7$.
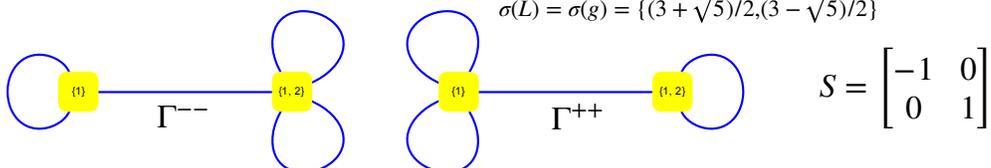

The Comma

{1}

{1,2}
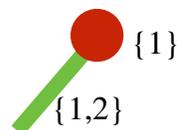
$G = \{\{1,2\}, \{1\}\}$ not a simplicial complex $h(x) = 1, \forall x$ energy

$$L = L^{--} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad L^{++} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad L^{-1} = g = SL^{++}S = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$\sigma(L) = \sigma(g) = \{(3 + \sqrt{5})/2, (3 - \sqrt{5})/2\}$

Γ^{--} Γ^{++}

$$S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

No Energy theorem $\displaystyle\sum_{x,y \in G} g(x,y) = E[G]$ only holds for simplicial complexes

FIGURE 11. A figure illustrating the story in one of the simplest cases. G is not a simplicial complex here. In the case of a constant energy 1, the matrices L^{--} and L^{++} define isospectral multi-graphs. The matrix $g = SL^{++}S$ is the inverse of $L = L^{--}$. The energy theorem does not hold here.

The 1-simplex

$G = \{\{1,2\}, \{1\}, \{2\}\}$ is a simplicial complex $h(x) = 1, \forall x$ energy

$$L = L^{--} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad L^{++} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad L^{-1} = g = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Γ^{--}

$\sigma(L) = \sigma(g) = \{2+\sqrt{3}, 2-\sqrt{3}, 1\}$

isospectral
multi-graphs Γ^{++}

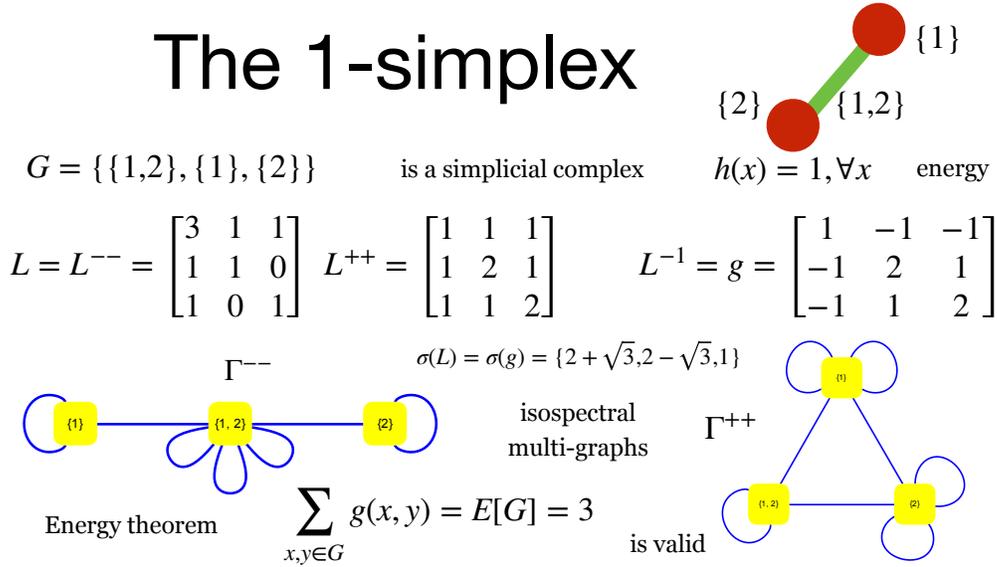Energy theorem $\displaystyle\sum_{x,y \in G} g(x,y) = E[G] = 3$ is valid

FIGURE 12. In this case G is a simplicial complex with three sets. In the case of a constant energy 1, the matrices L^{--} and L^{++} define isospectral multi-graphs. The matrix $g = SL^{++}S$ is the inverse of $L = L^{--}$. The energy theorem does hold here. The total energy is 3, the number of sets. With the energy $\omega(x) = (-1)^{\dim(x)}$ the total energy would have been $\chi(G) = 1$, the Euler characteristic of G.

12. THE PARAMETER CASE

12.1. Define $\dim(x) = |x| - 1$ and $\omega(x) = (-1)^{\dim(x)}$ and the connection matrix

$$L_{xy}(t) = t^{-\dim(x \cap y)}(1 - f_{W^-(x) \cap W^-(y)}(t))$$

which has rational expressions in t. Define the Green function matrix which is a polynomial in t:

$$g_{xy}(t) = \omega(x)\omega(y)(1 - f_{W^+(x) \cap W^+(y)}(t)) .$$

These definitions are triggered by the case $t = -1$, where we have seen that the inverse of $L = L_{xy}(-1)$, a matrix for which $L_{xy} = 1$ if x intersects with y and $L_{xy} = 0$ else.

12.2.

Theorem 8 (Determinant). $\det(L_t) = (-1)^{f_G(1)-1} t^{f'_G(1)} = \prod_x (-t)^{|x|}$.

Proof. The result holds in general for discrete CW complexes. Proceed by induction on the number of cells. Every time we add a cell, the determinant gets multiplied by $(-t)^{|x|}$.   □

12.3. There are two values, $t = 1$ and $t = -1$ for which the matrix $L_{xy}(t)$ is unimodular. For $t = -1$, the determinant of L_{-1} is the Fermi number $(-1)^f = \prod_{x \in G} \omega(x)$, where f is the number of odd dimensional simplices. For $t = 1$, the determinant of $-L_1$ is $\prod_{x \in G} 1 = 1$.

Theorem 9 (Green Star). $g_t = L_t^{-1}$.

Proof. See the computation before.   □

12.4. The f-function of a complex G is defined as $f_G(t) = 1 + f_0 t + \cdots + f_d t^{d+1}$. We have

Theorem 10 (Energy theorem). $\sum_{x,y} g_t(x,y) = 1 - f_G(t)$ *for all* t.

Proof. The parametrized Poincaré-Hopf theorem tells that $1 - f_G(t) = -t \sum_x f_{S_g(x)}(t)$. Define the potential $k(x) = \sum_y g_t(x,y)$. It is enough to show that $k(x) = -t f_{S_g(x)}(t)$ for some function g. It seems that $g = -\dim$ seems to work.   □

12.5. The case $t = 1$ is of interest because, then L is conjugated to its inverse g. The matrices $L_1, g_1 \in SL(n, \mathbb{Z})$ are isospectral and negative definite. Let λ_k be the eigenvalues of $-L_1$ which are also the eigenvalues of $-g_1$. Define the parametrized zeta-function of the complex G as

$$\zeta_t(s) = \sum_{k=1}^{n} \lambda_k^{-s} .$$

It is an entire function from $\mathbb{C} \to \mathbb{C}$ and unambiguously defined as $\lambda_k^{-s} = e^{-\log(\lambda_k)s}$ with $\lambda_k > 0$. Because $\lambda(s) = \lambda(\overline{s})$ one has $\zeta_1(a + ib) = \zeta_1(-a + ib)$.

13. REPRESENTATION

13.1. The disjoint union of simplicial complexes forms a monoid which completes to a group. Given an energy function h on G and an energy function k on H, there is a natural energy function h_{G+H} on $G + H$ defined as $h_{G+H}(x) = h(x)$ if $x \in G$ and $(h + k)(x) = k(x)$ if $x \in H$. To extend the energy to the Grothendieck completion define $h_{-G}(x) = -h_G(x)$. The Cartesian product $G * H$ has the energy $h_{G*H}(x \times y) = h_G(x) h_H(y)$. The total energy functional $E : G \to E[G] = \sum_{x \in G} h(x)$ is now a ring homomorphism almost by definition.

13.2. To stay within a finite frame work assume h to be integer valued also to stay closer to the story to divisors. In the continuum, an integer valued function with finite total energy $\sum_{x \in N} h(x)$ must to have a finite set as support, which means it is a divisor. With the disjoint union and Cartesian product, the class $\mathcal{X} = \{X = (G, h)\}$ of simplicial complexes is a ring $\mathcal{X}$ which naturally extends the ring $\mathcal{G}$ of simplicial complexes. The point is that we think of the energized complex as a geometric object similarly as in the continuum, a vector bundle is a geometry object.

Proposition 1. *a) The map* $X = (G, h) \to G$ *from* $\mathcal{X}$ *to* $\mathcal{G}$ *is a ring homomorphism.* *b) The map* $X = (G, h) \to E[X] = \sum_x h(x)$ *from* $\mathcal{X} \to \mathbb{Z}$ *are ring homomorphisms.*

13.3. The prototype example is $h(x) = \omega(x)$ where the energy $E[G] = \chi(G)$ is the Euler characteristic of G. An other example is $h(x) = 1$, in which case $E[G] = |G|$ is the number of elements. A third example is $h(x) = t^{|x|}$ in which case the total energy is a ring homomorphism from $\mathcal{N}$ to the polynomial ring $\mathcal{R}[t]$. Energy functions h of the form $h(x) = H(|x|)$ with multiplicative $H(n*m) = H(n)H(m)$ now defines a representation in a tensor ring of matrices. Examples are $h(x) = \omega(x)$ or $h(x) = 1$ or $h(x) = |x|$.

13.4. Now, if h is an arbitrary energy function on complexes, then with the extension $h_{G*H}(x \times y) = h_G(x)h_H(y)$, we have multiplicity and so:

Theorem 11 (Representation). *The map which assigns to $X = (G, h) \in \mathcal{X}$ the matrix K is a ring homomorphism from $\mathcal{X}$ to the tensor ring of finite matrices.*

13.5. One can assign to $X = (G, h)$ also a graph divisor $(\Gamma = (V, E), h)$, where $V = G, E = \{(a, b) \mid a \cap b \neq \emptyset\}$. The graph Γ is the connection graph of G. The addition is the disjoint union still and the multiplication is the Sabidussi muliplication. Also this is a ring. The graph complement operation maps this ring into subring of the Zykov ring of graphs with join and dual Sabidussi multiplication, but now also equipped with the energy. This is an interesting link as the analysis shows that on the geometric side, there is a natural norm $|\cdot|$, which is the independence number of Γ. The integer ring $\mathcal{X}$ is therefore an arithmetic object with a norm $|(X, h)| = \sqrt{|X|^2 + |h|^2}$ satisfying the inequality $|(X*Y, h*k)| \leq |(X, h)||(Y, k)|$ and therefore can as $\mathcal{G}$ already be extended to a Banach algebra $\mathcal{R}$.

14. Gauss-Bonnet

14.1. The classical Gauss-Bonnet theorem writes the Euler characteristic as a sum of curvatures. It generalizes to valuations X, real valued maps from sub structures having the property $X(G \cup H) + X(G \cap H) = X(G) + X(H)$, but it holds also in a Hamiltonian set-up. If Γ is a graph with Whitney complex G equipped with a Hamiltonian h define the parametrized energy $E_G(t) = \sum_{x \in G} h(x)t^{|x|}$ which is for $t = 1$ the energy. Now $K(v) = \int_0^1 E_{S(v)}(s)\, ds$ defines a curvature for the graph and the energy can be written as

Proposition 2. $E(G) = \sum_v K(v)$.

Proof. A simplex of cardinality $|x|$ distributes the energy equally to $|x| + 1$ points so that each point gets $h(x)/(|x| + 1)$ which is $\int_0^1 h(x)t^{|x|}dt$. $\square$

14.2. Also here, we note already that this does simple principle does not even require the simplicial complex structure. We just have to define $|x|$ in general as the number of atoms in x, where an "atom" is a set in G which does not have a proper non-empty subset. The curvature will then be supported on atoms of the structure.

14.3. Poincaré-Hopf deals with a locally injective function g on G. Define $S_g(v) = \{y \in G \mid g(y) < g(v)\}$. Now, we also have additionally an energy funtion h given. This function h does not need to have any thing to do with g. But h could be the index of g.

Proposition 3. $E_G = \sum_{v \in V} E_{S_g(v)}$,

14.4. We will discuss this a bit more in a follow-up. The upshot is that both Gauss-Bonnet, as well as Poincaré-Hopf hold not only for simplicial complexes but in a rather general set-up of energized sets of sets.

15. REMARKS

15.1. We have seen already that many of the results can be adapted to the situation where G is just a set of finite non-empty sets. This is especially true for results about determinant and spectrum. Having a more symmetric category, where one can also take the Boolean dual $\hat{G}$ of G produces more symmetry. The matrices L^{++} and L^{--} are then completely dual each other. Of course, there is still a reason to look at simplicial complexes. One of them is cohomology, an other is to be closer to classical geometries and especially manifolds. The escape to the larger category of sets of sets is illuminating however.

15.2. The nomenclature $L^{++} = E[W^+ \cap W^+]$, and $L^{--} = E[W^- \cap W^-]$ and $L^{+-}(x) = E[W^+(x) \cap W^-(y)]$ and $L^{-+} = E[W^-(x) \cap W^+(y)]$ is more symmetric. In the topological case $h(x) = \omega(x)$, the matrix $w = SLS$ is of interest as $\sum_{x,y} w(x,y) = \omega(G)$ is the Wu characteristic of the complex defined by $\sum_{x \sim y} \omega(x)\omega(y)$ summing over intersecting pairs $x \sim y$. This suggests to look in general at the sum $W[G] = \sum_{x,y} SLS(x,y)$. This is the symmetric analogue of the sum $E[G] = \sum_{x,y} g(x,y)$. We have not yet investigated this. A first interesting case is in the constant energy case $h(x) = 1$, where $W[G] = \sum_{x,y} \omega(x)\omega(y)(2^{|x \cap y|} - 1)$.

15.3. We could also look at the polynomial version $-g_{xy}(t)/t^{dim(x)}$ which is not symmetric but has constant determinant 1 and has the rational inverse $L_{xy}(t) = ((1+t)^{d+1} - t^{d+1})/t^d$, where $d = \dim(x \cap y)$. For $t = 0$, these are triangular matrices.

15.4. The palindromic coefficient list p_k of the characteristic polynomial $\det(g - \lambda)$ in the constant energy case $h(x) = 1$ is very regular. If we plot the log of the coefficients, it approaches a parabola. But this regularity is only present if we chose constant energy values. Already if we replace $h(x) = 1$ with $h(x) = \omega(x)$, the parabola is less smooth.

15.5. The choice of using integers $h(x)$ as energy values has reasons. The assumption is not necessary for the theorems covered here. But it allows to see h as a divisor, which is an integer valued function on a geometry with a finite set as support, the prototype is a divisor of a rational function on a variety which as principal divisors define an equivalence class of all divisors. Riemann-Roch expresses the energy $E(G)$ as a signed distance to $\{E(G) = \chi(G)\}$. In the continuum, where the base energy $E[X] = \omega(x)$ is included, the energy is written as $\chi(G) + \deg(D)$. In the discrete, where one can chose space itself is part of the divisor (in the form $h(x) = \omega(x)$ with total energy the Euler characteristic) and rather than writing $l(D) - l(K - D)$, we just write $l(D) - l(-D)$ and see $l(D)$ and $l(-D)$ as the analogous to $l(x) = \max(x, 0)$

and $l(-x) = \max(-x, 0)$ which gives $|x| = l(x) - l(-x)$ which is Riemann-Roch for a 1-point space.

15.6. It is the privilege of finite geometries that we can chose a "clean slate" zero energy $K = 0$ as the canonical divisor K. In the continuum, (where we can not access lower dimensional "atoms of space" as points, we need sheave theoretical constructs like differential forms) this is not possible and we need a canonical divisor (a global meromorphic function which has as the degree the Euler characteristic of the curve). Furthermore, in the continuum, the discreteness of energy forces an energy function to be located on a finite set of points and where writing the Euler characteristic as an energy of a natural divisor. In the finite, the geometry is implemented in the form of the divisor $h(x) = (-1)^{\dim(x)}$ which assigns to a set an energy so that the total energy is Euler characteristic.

15.7. Alternatively, one can think of $h(x)$ as an energy excitement level of a quantum harmonic oscillator. While in physics, one might prefer $1/2 + h(x)$ with integer $h(x)$, a geometer would just write $h(x) = \omega(x) + D(x)$ so that the energy is $\chi(G) + \deg(D)$ as in Riemann-Roch. That theorem motivates to see a divisor (similarly as a vector bundle) as a geometry by itself and extend the category of simplicial complexes to simplicial complexes which are "energized". Essentially everything done in geometry works in the energized frame work. Examples are Gauss-Bonnet, Euler-Poincaré. Riemann-Roch is a form of Euler-Poincaré for a cohomology. Changing the divisor by adding principal divisors behaves like homotopy deformations. The dimension value $l(G)$ appearing in Riemann Roch is a homotopy invariant in this picture.

16. Code

16.1. The following Mathematica code generates the matrix K and its inverse K^{-1}, and the zeta function for a random complex according to the definitions and illustrates the results in examples. As usual, the code can be grabbed from the ArXiv. It should serve as pseudo code also:

```
Generate[A_]:=Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1];
RandomSets[n_,m_]:=Module[{A={},X=Range[n],k},Do[k:=1+Random[Integer,n-1];
  A=Append[A,Union[RandomChoice[X,k]]],{m}];Sort[Generate[A]]];
G=RandomSets[6,9];n=Length[G];      e=Table[RandomChoice[{1,-1}],{k,n}];
S=Table[-(-1)^Length[G[[k]]]*If[k==l,1,0],{k,n},{l,n}];  (* Super matrix  *)
energy[A_]:=If[A=={},0,Sum[e[[Position[G,A[[k]]][[1,1]]]],{k,Length[A]}]];
star[x_]:=Module[{u={}},Do[v=G[[k]];If[SubsetQ[v,x],u=Append[u,v]],{k,n}];u];
core[x_]:=Module[{u={}},Do[v=G[[k]];If[SubsetQ[x,v],u=Append[u,v]],{k,n}];u];
Wminus      = Table[Intersection[core[G[[k]]],core[G[[l]]]],{k,n},{l,n}];
Wplus       = Table[Intersection[star[G[[k]]],star[G[[l]]]],{k,n},{l,n}];
Wminusplus  = Table[Intersection[star[G[[k]]],core[G[[l]]]],{k,n},{l,n}];
Wplusminus  = Table[Intersection[core[G[[k]]],star[G[[l]]]],{k,n},{l,n}];
Lminus      = Table[energy[Wminus[[k,l]]],    {k,n},{l,n}];  L =      Lminus;
Lplus       = Table[energy[Wplus[[k,l]]],     {k,n},{l,n}];  g =   S.Lplus.S;
Lminusplus  = Table[energy[Wminusplus[[k,l]]],{k,n},{l,n}];
Lplusminus  = Table[energy[Wplusminus[[k,l]]],{k,n},{l,n}];
Potential   = Table[Sum[g[[k,l]],{l,n}],{k,n}];
Superdiag   = Table[S[[k,k]]*g[[k,k]],{k,n}];
Print[{
Total[Flatten[g]] == Total[e],               (* Potential energy        *)
```

```
Tr[S.g] == Total[e],                        (*  Spectral  energy         *)
Det[g]==Det[L]==Product[e[[k]], {k, n}],    (*  Unimodularity  Theorem   *)
Sort[Sign[Eigenvalues[1.0 L]]]==Sort[Sign[e]],  (*  eigenvalues  of  L    *)
Sort[Sign[Eigenvalues[1.0 g]]]==Sort[Sign[e]],  (*  eigenvalues  of  g    *)
Superdiag == Potential,                     (*  selfinteraction=potential *)
g.L ==IdentityMatrix[n]}]                    (*  only  true  in  spin  case  *)
```

16.2. Here is the code to draw the multi-graphs Γ^{++}, Γ^{--} defined by their adjacency matrices L^{--}, L^{++}. In this example, G is not a simplicial complex and the graphs have the adjacency matrices

$$
L^{--} = \begin{bmatrix} 5 & 2 & 2 & 1 & 1 \\ 2 & 2 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}, L^{++} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 & 3 \end{bmatrix}.
$$

```
RandomSets[n_,m_]:=Module[{A={},X=Range[n],k},Do[k:=1+Random[Integer,n-1];
  A=Append[A,Union[RandomChoice[X,k]]],{m}];Sort[Generate[A]]];
G=RandomSets[4,6]; n=Length[G]; e=Table[1,{n}];
G = {{1,2,3}, {1,2}, {2,3}, {1}, {3}}; n = Length[G];  e = {1, 1, 1, 1, 1};
energy[A_]:=If[A=={},0,Sum[e[[Position[G,A[[k]]]][[1,1]]]],{k,Length[A]}]];
star[x_]:=Module[{u={}},Do[v=G[[k]]; If[SubsetQ[v,x],u=Append[u,v]],{k,n}];u];
core[x_]:=Module[{u={}},Do[v=G[[k]]; If[SubsetQ[x,v],u=Append[u,v]],{k,n}];u];
Wminus      = Table[Intersection[core[G[[k]]], core[G[[l]]]],{k,n},{l,n}];
Wplus       = Table[Intersection[star[G[[k]]], star[G[[l]]]],{k,n},{l,n}];
Lminus      = Table[energy[Wminus[[k,l]]],     {k,n},{l,n}];
Lplus       = Table[energy[Wplus[[k,l]]],      {k,n},{l,n}];
U=Graph[Flatten[Table[Table[G[[k]]<->G[[l]],{Lminus[[k,l]]}],{k,n},{l,k,n}]]];
V=Graph[Flatten[Table[Table[G[[k]]<->G[[l]],{Lplus[[k,l]]}] ,{k,n},{l,k,n}]]];
DrawGraph[s_]:=Graph[s,GraphStyle->"SmallNetwork"];
Print["Isospectral:____", Eigenvalues[Lminus] == Eigenvalues[Lplus]];
{DrawGraph[U],DrawGraph[V]}
```

16.3. And here is the parametrized version where a set with k elements gets the energy t^k. In this case, we scale the entries of L^{--} a bit so that it matches with the inverse of $L^{++} = g$. The energy of a set of sets is now the genus function $1 - f_A(t)$. We definitely need a simplicial complex for the parametrized result. The function $F[A]$ in the code is now the f-vector f of A and $f_A(t) = 1 - \sum_k f_k(A)t^k$ so that $1 - f_A(t) = \sum_k f_k(A)t^k$. This is equivalent to assigning the energy $-t^k$ to a set with k elements.

```
Generate[A_]:=Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]; Clear[t];
RandomSets[n_,m_]:=Module[{A={},X=Range[n],k},Do[k:=1+Random[Integer,n-1];
  A=Append[A,Union[RandomChoice[X,k]]],{m}];Sort[Generate[A]]];
F[A_]:=If[u=Map[Length,A];A=={},A,Table[Length[Position[u,k]],{k,Max[u]}]];
f[A_,t_]:=Module[{R=F[A],k},If[A=={{}},1,1+Sum[R[[k]]*t^k,{k,Length[R]}]]];
G=RandomSets[4,6]; G=Generate[{{1,2},{2,3}}]; n=Length[G];
S=Table[-(-1)^Length[G[[k]]]*If[k==l,1,0],{k,n},{l,n}];energy[A_]:=1-f[A,t];
star[x_]:=Module[{u={}},Do[v=G[[k]]; If[SubsetQ[v,x],u=Append[u,v]],{k,n}];u];
core[x_]:=Module[{u={}},Do[v=G[[k]]; If[SubsetQ[x,v],u=Append[u,v]],{k,n}];u];
Wminus      = Table[Intersection[core[G[[k]]], core[G[[l]]]],{k,n},{l,n}];
Wplus       = Table[Intersection[star[G[[k]]], star[G[[l]]]],{k,n},{l,n}];
M=Table[Intersection[G[[k]],G[[l]]],{k,n},{l,n}];
Lminus      = Table[energy[Wminus[[k,l]]],     {k,n},{l,n}];
Lplus       = Table[energy[Wplus[[k,l]]],      {k,n},{l,n}];
```
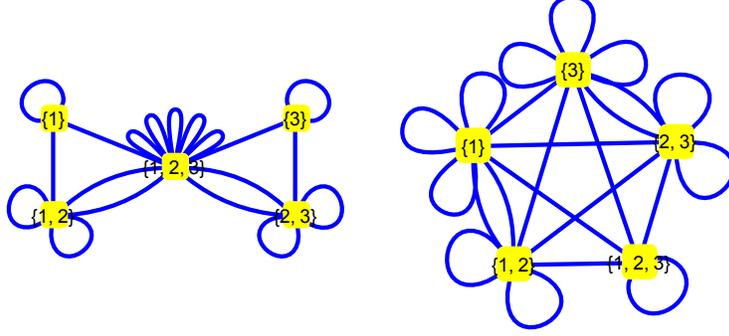
FIGURE 13. We see the isospectral multi-graphs Γ^{++}, Γ^{--} produced in the code example, where $G = \{\{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{1\}, \{3\}\}$, which is a set of sets and not a simplicial complex. In the first case, two nodes are connected by links encoding a common subset, in the second case, two nodes are connected by links encoding a common superset.

```
g=S.Lplus.S;    L=Table[1−f[Wminus[[k,l]],t]/t^Length[M[[k,l]]],{k,n},{l,n}];
Potential    = Table[Sum[g[[k,l]],{l,n}],{k,n}];
Superdiag    = Table[S[[k,k]]*g[[k,k]],{k,n}];
Print[{
Total[Flatten[g]]  == energy[G] ,               (* Potential energy          *)
Simplify[Tr[S.g]  == energy[G] ] ,              (* Spectral energy           *)
Det[g]==(−1)^(1−f[G,1]) t^Total[Map[Length,G]] , (* Unimodularity Theorem     *)
Superdiag == Potential ,                         (* selfinteraction=potential *)
Simplify[g.L ==IdentityMatrix[n]]}]              (* g and L are inverse       *)
```

16.4. In the given case, where $G = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$, we have

$$
g = \begin{bmatrix}
-t^2 - t & -t^2 & 0 & t^2 & 0 \\
-t^2 & -2t^2 - t & -t^2 & t^2 & t^2 \\
0 & -t^2 & -t^2 - t & 0 & t^2 \\
t^2 & t^2 & 0 & -t^2 & 0 \\
0 & t^2 & t^2 & 0 & -t^2
\end{bmatrix}
$$

and

$$
L = \begin{bmatrix}
1 - \frac{t+1}{t} & 0 & 0 & 1 - \frac{t+1}{t} & 0 \\
0 & 1 - \frac{t+1}{t} & 0 & 1 - \frac{t+1}{t} & 1 - \frac{t+1}{t} \\
0 & 0 & 1 - \frac{t+1}{t} & 0 & 1 - \frac{t+1}{t} \\
1 - \frac{t+1}{t} & 1 - \frac{t+1}{t} & 0 & 1 - \frac{t^2+2t+1}{t^2} & 1 - \frac{t+1}{t} \\
0 & 1 - \frac{t+1}{t} & 1 - \frac{t+1}{t} & 1 - \frac{t+1}{t} & 1 - \frac{t^2+2t+1}{t^2}
\end{bmatrix}.
$$

The sum over all matrix elements $\sum_{x,y} g(x, y)$ in g is $1 - f_G(t) = -3t - 2t^2$ which agrees with the super trace of g.

17. Some questions

17.1. By choosing an energy function $h(x)$ with k negative values -1 and $n - k$ positive values 1 produces non-negative symmetric $n \times n$ integral matrices L with L^{-1} integral valued of that L has k negative and $n - k$ positive eigenvalues. This solves inverse problems for non-negative matrices like for example Corollary 2.1 in [12]. Can one characterize the spectra of $n \times n$ matrices $L = L^{--}$ which appear for a set of n sets G and a function $h \in \{-1, 1\}^n$? An inverse problem is to get back G from the spectrum. An easier task might be to reconstruct G from knowing all the spectra for all possible energy functions $h \in \{-1, 1\}^n$.

17.2. One can wonder for which set G of sets the matrices L^{++} and L^{--} are completely positive definite in the sense of [2]: a matrix is completely positive if $A = BB^T$, where B is a non-negative matrix. Given a simplicial complex G with n elements, the Hodge matrix $H = (d + d^*)^2$ by definition is completely positive because the Dirac operator $D = d + d^*$ is a $0 - 1$ matrix and so non-negative. For the connection Laplacians this is still unclear as there is no cohomology yet associated (except in one dimensions).

17.3. An other question is the multiplicity $m(G)$ of the eigenvalue 1 in $L = L^{--}$ in the case of constant energy 1. It often correlates with the Betti numbers. For simplicial complexes obtained by one-dimensional grid graphs for example, it is always $b_0 + b_1$. For complete complexes G, we often have $m(K_n) = 1$ but $m(K_4) = 5$ and $m(K_6) = 19$ and $m(K_8) = 69$. Investigating the structure of the eigenvalues of $L^{--}(G)$ which are the eigenvalues of $L^{++}(G)$ is in general not yet done.
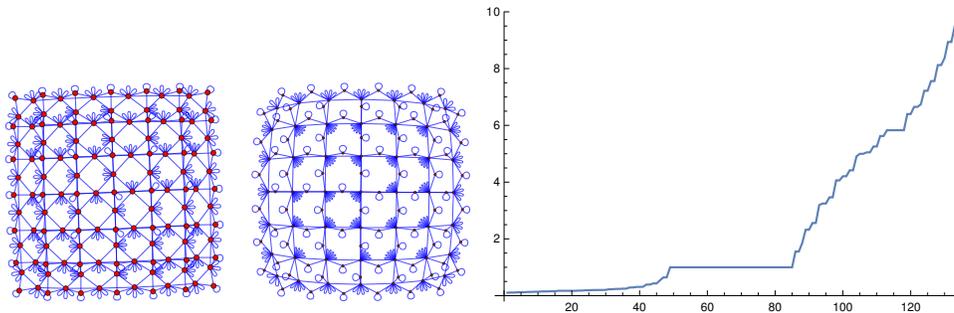

FIGURE 14. The Whitney complex G of the grid graph $G(7, 7)$ defines matrices Γ^{++}, Γ^{--} defined by the adjacency matrices L^{++}, L^{--}. In the figure below we see the eigenvalues of L^{++}. The flat large plateau consists of 37 eigenvalues 1. For grid graphs, we see that the number of eigenvalues 1 is the sum of the Betti numbers of G.

17.4. The matrix L defines a self-dual lattice packing in $\mathbb{R}^n$. Every set of sets G defines such a packing. The packing density is the same than than of the lattice $\mathbb{Z}^n$ as the minimal lattice distance is 1. In the context of packings and more generally,

for integral quadratic forms one looks also at the θ-function

$$\theta(z) = \sum_{m \in \mathbb{Z}^n} e^{2\pi i z L m \cdot m} .$$

It is an other analytic way to describe the connection multi graph with adjacency matrix L similarly as the spectral zeta function $\zeta(s) = \sum_{k=1}^{n} \lambda_k^{-s}$ defined by the eigenvalues of L or the Ihara zeta function $\zeta_I(s) = 1/\det(1-sL)$, a discrete analogue of the Selberg zera function, do.

17.5. It should be obvious that we get matrices L^{++}, L^{--} also if we start with differential complexes which are infinite. We can look at the infinite lattice $\mathbb{Z}^d$ for example and decorate various parts of the lattice to get periodic, almost periodic or random Schrödinger type matrices [3]. What remains true is that L^{++}, L^{--} remain isospectral. While in the finite case, the matrices are diagonalizable, this is not necessarily the case in the infinite case as this requires a complete set of eigenfunctions which one can not expect in general. In the periodic case for example the spectrum is continuous. In general, we expect singular continuous spectrum to appear. All we know from general principles is that the density of states of L^{++} and L^{--} agree. A first thing to study is the one-dimensional Schrödinger case where we start with the simplicial complex of $\mathbb{Z}$. In that case we know L^{++}, L^{--} in the limit as they are Barycentric limits which is completely understood in one dimension but open in higher dimensions [6].

17.6. In the periodic case $G = C_5$ for example with

$$G\{\{1\}, \{1,2\}, \{2\}, \{2,3\}, \{3\}, \{3,4\}, \{4\}, \{4,5\}, \{5\}, \{5,1\}\} \ ,$$

we have the isospectral periodic Schrödinger operators "on a strip" (special Töplitz matrices):

$$L^{--} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 3 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3
\end{bmatrix}$$

and

$$L^{++} = \begin{bmatrix} 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

What happens here is that in the limit where the matrices are bounded operators on $l^2(\mathbb{Z})$, the two matrices are the same and invertible. Unlike in the usual discrete Schrödinger case (which is the Hodge case), where the spectral interval has 0 at the boundary which leads in dynamical system cases to KAM situations as one needs then a strong implicit function theorem to perturb, the connection case with a mass gap is hyperbolic.

17.7. In the above case, the isospectral property in the limit was trivial. If we take limits of periodic graphs which are a bit fatter
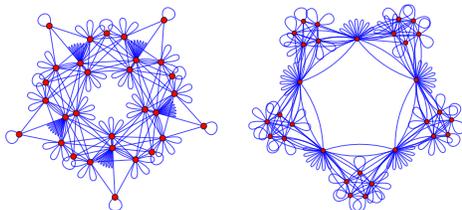

FIGURE 15. Two isospectral periodic graphs Γ^{++}, Γ^{--} defined by a simplicial complex from the connection graph of C_{10} are non-trivially isospectral in the limit. The Jacobi matrices L^+ and L^- on $l^2(\mathbb{Z})$ have the same density of states.

17.8. For which set of sets G does the 15 theorem kick in? Here are two examples: for the set of sets $G = \{\{1,2\}, \{1\}\}$ the "komma" for which the connection Laplacian for constant energy $h(x) = 1$ gives the matrix $L = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ which is the Arnold cat matrix, the quadratic form is $Q(x,y) = 2x^2 + 2xy + y^2$. It is not universal because the value 3 is not attained. However, for $G = \{\{1\}, \{2\}, \{3\}, \{1,3\}\}$, we have $L = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$, which leads to a quadratic form $Q(x,y,z,w) = 3w^2 + 2w(x+z) + x^2 + y^2 + z^2$ which is universal as all values from $\{1, \cdots, 15\}$ are attained.

Preliminary experiments show that only very small sets of sets lead to non-universal quadratic forms.

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