MORSE AND LUSTERNIK-SCHNIRELMANN FOR GRAPHS

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Abstract. Both Morse theory and Lusternik-Schnirelmann theory link algebra, topology and analysis in a geometric setting. The two theories can be formulated in finite geometries like graph theory or within finite abstract simplicial complexes. We work here mostly in graph theory and review the Morse inequalities $b(k) - b(k-1) + ... + b(0) \leq c(k) - c(k-1) + ... + c(0)$ for the Betti numbers b(k) and the minimal number c(k) of Morse critical points of index k and the Lusternik-Schnirelmann inequalities $\text{cup} + 1 \leq \text{cat} \leq \text{cri}$ between the algebraic cup length cup, the topological category cat and the analytic number cri counting the minimal number of critical points of a function.

In memory of Frank Josellis (1959-2022)

1. Introduction

1.1. Let $G = (V, E)$ be a **finite simple graph** with **vertex set** V and **edge set** E. A **scalar function** g on G is a map $g : V \to \mathbb{R}$. The **unit sphere** $S(v)$ of $v \in V$ is the sub graph generated by all vertices $w \in V$ with $(v, w) \in E$. The function g is called **locally injective** if $g(v) \neq g(w)$ for all $(v, w) \in E$). The **stable unit sphere** $S_g^-(v)$ is the sub-graph of $S(v)$ defined by $S_g^-(v) = \{w \in S(v), g(w) < g(v)\}$. A graph is **contractible** if there is $v \in V$ such that both $S(v)$ and $G \setminus v$ are contractible, where $G \setminus v$ is the graph in which v has been removed. A vertex v is a **critical point** of g if $S_g^-(v)$ is not contractible.

1.2. The **Lusternik-Schnirelman** category $\text{cat}(G)$ of G is the minimal number of contractible graphs that cover G. The minimal number of critical points which a locally injective function can have is denoted by $\text{cri}(G)$. Graphs define a simplicial complex, an exterior derivative d, a **Hodge Laplacian** $L = (d + d^*)^2$. The kernel of L is the space of **harmonic forms** which represents **cohomology**. It carries a **graded multiplication** called the **cup product**. The maximal number of positive degree forms which can be multiplied to get something non-zero is the cup length $\text{cup}(G)$. **Lusternik-Schnirelmann's theorem** is $\boxed{\text{cup}(G) + 1 \leq \text{cat}(G) \leq \text{cri}(G)}$.

1.3. A scalar function $g : V \to \mathbb{R}$ is **Morse** if at every critical point v, the graph $S_g^-(v)$ is a k-sphere for some k. The value $k+1$ is called the Morse index of such a critical point v. If v is a minimum of g for example, then $S_g^-(v) = 0$ is the empty graph 0, which is a -1-sphere so that $k = 0$. A **k-sphere** G is k-manifold for which $G \setminus v$ is contractible for some v. A **k-manifold** is a graph for which every unit sphere $S(v)$ is a $(k-1)$-sphere. If $c_k(G)$ denotes the minimal number of critical points that a Morse function g can have on G, and $b_k(G)$ is the k'th **Betti number**, the dimension of the kernel of the Laplacian L restricted to k-forms, then the **strong Morse inequalities** are $\boxed{b_k(G) - b_{k-1}(G) + \cdots + (-1)^k b_0(G) \leq c_k(G) - c_{k-1}(G) + \cdots + (-1)^k c_0(G)}$.

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By adding the inequalities for k and $k-1$, one gets the **weak Morse inequalities** $b_k(G) \leq c_k(G)$. These inequalities work for all Morse functions on a finite simple graph.

1.4. Morse functions do exist on many graphs but not for all graphs. The **cube graph** or the **dodecahedron graph** for example are 1-dimensional (triangle free) graphs. They both do not admit any Morse function. The reason is that for any g, the maximum has as $S_g^-(v) = S(v) = \overline{K_3}$ (the graph complement of the complete graph K_3 is a 0-dimensional graph with 2 vertices and no edges), which is not contractible so that the maximum is a critical point but $S_g^-(v)$ is not a sphere. More generally, any vertex regular graph for which the unit sphere (which are all isomorphic graphs) is not a sphere, can not carry a Morse function.

1.5. Having seen this example, one could suspect that G needs to be a manifold in order to be able to carry a Morse function. This is not the case: the graph (V, E) defined by any finite abstract simplicial complex G (in which $V = G$ and E are the pairs (x, y) with $x \subset y$ or $y \subset x$) always has a Morse function. An example is the **dimension function** $g(x) = \dim(x) = |x| - 1$ which is locally injective because if x is contained in y, then $g(x) < g(y)$. It has the property that $S_g^-(x)$ is a sphere, the Barycentric refinement of the boundary of the k-simplex x is a $(k-1)$-sphere. To see that g is Morse, we build up the graph brick by brick. Start with the 0-dimensional part, the vertices. These are all critical points of Morse index 0 because the unit spheres are empty and so (-1)-dimensional spheres. Then we add edges which are critical points of Morse index 1. Now add triangles, which are points of Morse index 2 etc. The number $c_k(G)$ of critical points of index k in G agrees with $f_k(G)$, the number of k-dimensional **simplices** (=complete subgraph = **clique**) x in G. The vector $(f_0, f_1, \dots, f_d)$ is called the f**-vector** of G.

1.6. We will look at Poincaré-Hopf and the Morse inequalities in the next section. But the just considered example illustrates the theme. We have just seen that any simplicial complex has already a natural Morse complex built in. Each cell is a "handle". If d is the maximal dimension of G, then the **Poincaré-Hopf** formula $\sum_{k=0}^d (-1)^k c_k = \chi(G)$ rephrases with **Euler-Poincaré** $\chi(G) = \sum_k (-1)^k b_k$ as the special case $\sum_{k=0}^d (-1)^k b_k = \sum_{k=0}^d (-1)^k c_k$ of the Morse inequalities. The general Morse inequality appears by building up all simplices up to dimension k. At each stage, **Betti number** $b_k(G)$ is the **nullity** of the k**-form Laplacian** L_k defined by G. If the simplicial complex of the graph G has n elements, then all exterior derivatives are encoded in one lower triangular $n \times n$ matrix d and $D = d + d^*$ is the **Dirac matrix** of G. The **Hodge Laplacian** $L = \oplus_{k=0}^d L_k$ is block diagonal and the block L_k is the k-form Laplacian.

1.7. The frame work of graphs is **intuitive** and so also a paradigm for **clarity**. Every finite simple graph defines a finite geometry which allows to define discrete manifolds and so covers quite a bit of classical geometry. They are familiar because street networks, genealogy trees, social networks, computer networks or flow diagrams are graphs. We could start also with a **finite abstract simplicial complex**, which stands for **simplicity** as there is only one axiom: it is a finite set of non-empty sets closed under the operation of taking finite non-empty subsets. Every graph defines such a complex. It is known as the **Whitney complex**, **flag complex** or **order complex**. While we could start with a simplicial complex without requiring it to come from a graph, it is more intuitive and so clear to use the language of graphs. Restricting to graphs is almost no loss of generality because we can from a complex G get a graph $(G, \mathcal{E})$, where $\mathcal{E} = \{(x, y), x \subset y \text{ or } y \subset x\}$. There is a generalization of simplicial complexes called a **delta set**. It is a paradigm for **generality** as delta sets form a topos, a mathematical object that is closed under any operation like product, quotient, level sets. It includes many structures not covered by simplicial complexes like **quivers**, graphs for which multiple-connections or loops are allowed.

1.8. Composing the **Whitney map** with the **graph formation map** produces the **Barycentric refinement**. According to Dieudonné, the Barycentric refinement is one of the three pillars of **combinatorial topology**; the other two are **incidence matrices** and **duality**. Examples of complexes that are not Whitney complexes are **k-skeleton complexes** obtained by restricting a simplicial complex to simplices of dimension $\leq k$. The simplest example is the 1-skeleton complex of $K_3 = \{\{1,2,3\},\{1,2\},\{2,3\},\{1,3\},\{1\},\{2\},\{3\}\}$ which is $C_3 = \{\{1,2\},\{2,3\},\{1,3\},\{1\},\{2\},\{3\}\}$. While no more the Whitney complex of a graph, it is still a 1-dimensional complex. Its Barycentric refinement is C_6 which is now is the Whitney complex of the cyclic graph C_6. Every complex which contains such a C_3 (without containing the corresponding K_3) fails to be the Whitney complex of a graph.

1.9. A simplicial complex $G = (G_0, \dots, G_n)$ defines **face maps** $d_{i,k} : G_k \to G_{k-1}$, $x \mapsto (x_0, \dots, \hat{x}_i, \dots, x_k)$ and so a **delta set**, which is an element of a **finite topos**. A delta set is a presheaf over the simplex category Δ with strict inclusions as morphisms on Δ. If all inclusions are chosen as morphisms, more conditions need to be satisfied for the functors and we get the smaller category of **simplicial sets**. Simplicial sets are more complicated in that there are two type of structure maps which need more conditions. A **delta set** is a geometry that is more general than simplicial complexes in that it is possible to do products, build quotients, level sets of functions, where we can factor out symmetries or build Whitney type cell complexes from quivers. It even allows to model divisor structures (geometric structures with additional integer valued structures telling about multiplicity).

1.10. Every delta set has an exterior derivative d and so has a cohomology, a Laplacian $L = (d + d^*)^2$, leading to discrete partial differential equations like **wave equation** $u_{tt} = -Lu$ or the **heat equation** $u_t = -Lu$. The set $G = \bigcup G_k$ can be the vertex set of a graph, where two elements are connected if there is a sequence of face maps getting from one to the other. (The Dirac operator which contains the incidence matrices encodes this completely). This produces a finite simple graph which in general has similar topological properties than G, at least if G is a simplicial complex. As a data structure, we can encode a delta set as a triple (G, D, r), where G is a finite set of n elements, D is a $n \times n$ matrix and $r : G \to \mathbb{N}$ is a dimension function.

1.11. It is more convenient to ignore the face maps in the definition of a delta set and directly go to the exterior derivative d. A delta set is just a finite set of n elements, a Dirac matrix $D = d+d^*$ and a dimension function which gives the dimension to each element $G_k = \dim^{-1}(k)$ so that the elements of G belonging to blocks of $L = D^2$ have fixed dimension. We need the dimension function r because (unlike for simplicial complexes), for delta sets, some of the dimension blocks L_k of L can be empty. The delta set $(G, D, r) = (\{1,2\},\{0\},\{1\}\}$ for example has only one element in G, the Dirac matrix is a 1×1 matrix and the element has the assigned dimension 1. No 0-dimensional elements are present. The name "Dirac matrix" has been chosen because it is a square root of the Hodge Laplacian L. It stands for a symmetric $n \times n$ matrix $D = d+d^*$ such that d (and so the transpose) is nilpotent $d^2 = 0$. The trivial case $d = 0$ reduces the delta set to a set. But r can give a bit more structure. The open set $U = \{1,2\}$ in K_2 for example is a one-dimensional delta set with Betti vector $(0,1)$. It is the complement of the closed set $K = \overline{K}_2$ in $G = K_2$ with Betti vector $(2,0)$. Now $b(U)+b(K) = (2,0)+(0,1) = (2,1) > b(G)$ illustrates one of the simplest cases for the fusion inequality.

1.12. A **quiver** is a graph in which also multiple-connections and loops are allowed. Quivers also carry an **exterior derivative** d. One can use see them as 1-dimensional delta sets. If a quiver has n nodes and m edges, the **gradient** d maps a 0-form to a 1-form. It is a $m \times n$ matrix. A quiver is a delta set with face maps $d_{0,0}, d_{0,1} : E \to V$. The category of quivers is a **topos** too, as it is a presheave category on the Kronecker quiver Q. Quivers are natural as they can also be seen as a functor category. One can see them as 1-dimensional delta sets but there are still 1-dimensional delta sets which do not come from quivers. Examples are $G = \{\{1,2\}\}$ or $G = \{\{1\},\{1,2\}\}$ which both are open sets in the simplicial complex K_2. There is also a functor from **quivers** to higher dimensional delta sets which extends the Whitney functor from **finite simple graphs** to finite abstract simplicial complexes. [1] Having multiple loops allowed in a graph was useful for spectral estimates, like in [64] because principal submatrices of quiver Laplacians remain quiver Laplacians.

1.13. Delta sets are especially useful when looking at the **topology** on a simplicial complex. The open sets in a simplicial complex are the complements of sub-simplicial complexes. Open sets are no more simplicial complexes. But they are still delta sets. The cohomology of an open set is in general different from the cohomology of its Barycentric refinement or closure: for $U = \{\{1,2,3\}\}$ for example, the **Betti vector** is $(0,0,1)$, while the closure $G = K_3$ has Betti vector $(1,0,0)$. The complement K of the open set U is the skeleton complex C_3 which has Betti vector $(1,1,0)$ (it is a simplicial complex which is the 1-skeleton complex of K_3 but not a Whitney complex of a graph). The relation $(1,1,1) = b(C_3) + b(U) \geq b(K) = (1,0,0)$ for Betti vectors is an example of the **Fusion inequality** [65] which tells that $b(U) + b(K) \geq b(G)$ if U, K are open-closed pairs in G, meaning that U is open, K is closed and $U \cup K = G$ and $U \cap K = \emptyset$. The Barycentric refinement coming from the graph defined by U is K_1 which has cohomology $b = (1,0,0)$.

1.14. In this review, which was written mostly in the fall of 2022 triggered by the passing of Frank Josellis (who had collaborated on this with me in [32]), we keep it within graph theory and especially do not attempt to generalize the Lusternik-Schnirelmann theorem nor the Morse inequalities to delta sets. Already when reformulating the results in the category of simplicial complexes, one has to adapt some definitions and make choices, like whether to define the category as the minimal number of **open sets** covering a simplicial complex or the minimal number of **closed sets** covering it. The revision of the presentation benefited much from the work of Jennifer Gao [23] who wrote in the spring of 2024 a senior thesis about topological aspects of graph theory.

1.15. Note that there are various different definitions of Lusternik-Schnirelmann category in the literature in the continuum. There are different ways one can proceed to carry them over to the discrete. When working on [32], we started with other definitions than what is used here and tried to get to notions that are homotopy invariant. We eventually decided not to worry about homotopy invariance as the notion of category is a rather topological notion. A difficulty is that category as defined here is not homotopy invariant. The **dunce hat** is not contractible

[1]When talking about functors, one also has to take into account the morphisms. The standard morphisms in graph theory are graph homomorphisms, the usual morphisms in simplicial complexes are simplicial maps.

so that the category is larger than 1. But it is homotopic to 1. There must therefore have been a homotopy step which changed the Lusternik-Schnirelmann category as defined here.

1.16. When defining Lusternik-Schnirelmann category one has to decide first of all, what elements should be used as coverings. One can use open contractible sets, closed contractible sets, open sets that have closures homotopic to 1 or closed sets that are homotopic to 1. The decision to pick the notion "contractible" was because it made the notion computable. One could also look at the notion of **contractible within an other host space**: an example is a closed path in a graph. It is intrinsically never contractible as the cohomology of a circle is $b = (1, 1)$ and not zero; but a closed loop in a contractible space is considered to be **homotopically trivial** like when defining the **fundamental group**. For us, **contractible parts** of a cover always means intrinsically contractible.

1.17. We had attempted to summarize some aspects of geometry and calculus in graphs in expository documents [41, 44, 52]. The definitions we have used in the last decade have fluctuated a bit, even for basic things like the notion of a **manifold**, the definition of **Morse function** or the choice of definition for Lusternik-Schnirelmann category. The current choice of definitions has remained pretty stable. It is guided by three "functionals": **the simplicity of the definition** and that theorems should work **as close as possible to the continuum** without much modification. Finally, the **proofs should be simple**.

1.18. We realized over the years that many graph theorists think about graphs differently. A common view is to see them as **1-dimensional simplicial complexes**. It should be clear however, that the language of graphs is just one of many approaches to cover material which usually is described using the language of simplicial complexes or notions from the continuum. Already in older texts like [3], one can see that graphs were drawn to visualize continuum manifolds. In computer graphics, surfaces are drawn as triangulations and the graphics complexes only need vertices and edges as well as the position of the vertices in space.

1.19. In many texts that use simplicial complexes, also describes "usual simplicial complexes" meaning that one looks at notions in the continuum. Or then one uses the **geometric realization** functor giving from an abstract version a Euclidean realization. Even Dehn and Heegaard who introduced abstract simplicial complexes, use also the continuum [17]. In this text, we never look at geometric realizations. The continuum is not wanted. We want to avoid the infinity axiom. Everything is combinatorial. While we mention the real line $\mathbb{R}$ like when talking about $\mathbb{R}$-valued functions, it should be clear that one could very well also use integer-valued functions or functions taking values in a finite totally ordered set. We always can remain in a finitist setting.

2. Poincaré-Hopf

2.1. The **Euler characteristic** of a graph G is defined as the Euler characteristic of its Whitney complex: $\sum_{x \in G} \omega(x)$ with $\omega(x) = (-1)^{\dim(x)}$. The 1-dimensional Euler characteristic $|V| - |E|$ would just take into account the 1-dimensional skeleton complex. The historically best known situation is the 2-dimensional $|V| - |E| + |F|$, where F is the set of 2-dimensional faces. This formula has been discovered by Descartes [1]. For convex polyhedra (to which the majority of books on polyhedra restrict to [27, 93]), there is no difficulty to see the highest dimensional facets. Especially in the case of 2-dimensions, where we deal with planar graphs.

2.2. The simplicial complex for the graph (V, E) has as sets the vertex sets of complete sub-graphs of the graph. We denote it again G. For a general delta set G, the Euler characteristic is defined in the same way. It is $\chi(G) = \sum_{j=1}^{n} (-1)^j |G_j|$, where G_j are the elements of dimension j.

2.3. Given a locally injective function g on the vertex set V of a graph, define $S_g^-(v)$ as the sub-graph generated by all vertices in the sub-graph $S(v)$, where $g(w)$ is smaller than $g(v)$. One can now define **Poincaré Hopf indices** $i_g(v) = 1 - \chi(S_g^-(v))$. As an integer-valued function on the set V of vertices of G, it is a **divisor** in the sense of the **discrete Riemann-Roch theory** [5].

2.4. Note that we look here at functions on the **vertex set** V of the graph G and not at functions on the simplicial complex G, which is the vertex set of the Barycentric refinement graph G_1. An example for the later is the function $\omega(x) = (-1)^{\dim(x)}$. It can be seen as the **Poincaré-Hopf index function** of the dimension function $g(x) = \dim(x)$. The later is a locally injective function on the Barycentric refinement G_1. The sum over all these Poincaré-Hopf indices on G_1 is the definition of the Euler characteristic of G. The statement that the total Poincaré-Hopf indices of g adds up to the Euler characteristic is already an example of the Poincaré-Hopf theorem.

2.5. By induction in the number of vertices, one immediately gets the **Poincaré-Hopf formula**. It is a discrete analog of the continuum [81, 28, 89], in the special case if the vector field is a gradient field. We will see below how to generalize this to "vector fields" like a digraph structure on the graph having the property that there is no closed cycle in each simplex. We formulated the following result in [38] with a more complicated proof. It matured while working on Lusternik-Schnirelmann in [32].

Theorem 1 (Poincaré-Hopf)**.** *If g locally injective then* $\chi(G) = \sum_{v \in V} i_g(v)$

Proof. The induction foundation is the one-point graph $1 = K_1$, with $\chi(K_1) = 1$, where $i_g(v) = 1$ because the unit sphere $S_g^-(v)$ is empty independent of the function g. Now to the induction step: if the result is known for all graphs with $(n-1)$ vertices and all colorings (locally injective functions on V) on such graphs, and G is a graph with n vertices with a coloring g, identify a local maximum v of g. The graph $G \setminus v$ has $n - 1$ elements and induction assumption can be used. Then $\chi(G) = \chi(G \setminus v) + \chi(B(v)) - \chi(S(v)) = [\sum_{w \neq v} i_g(w)] + i_g(v)$, because every unit ball $B(v)$ has $\chi(B(v)) = 1$ and $S_g^-(v) = S(v)$ if v is a local maximum of g. Note that adding the new vertex v has not altered the graphs $S_g^-(w)$ of the other vertices so that the indices $i_g(w)$ stayed the same. This proves the induction step. □

2.6. There are various way how one can generalize this result [38, 56, 58, 57]. Instead of the Euler characteristic, one can look at the **f-function** $f_G(t) = 1 + f_0 t + f_1 t^2 + \cdots f_d t^{d+1}$ which satisfies $\chi(G) = 1 - f_G(-1)$. The statement is $f_G(t) = 1 + t \sum_{x \in V} f_{S_g^-(x)}(t)$, where $S_g^-(x) = \{y \in S(x), g(y) < g(x)\}$ One can also replace the Euler characteristic by an **energy sum** $H(G) = \sum_{x \in G} h(x)$, if h is a ring-valued function on the simplicial complex. This generalizes the situation when $h(x) = \omega(x)$.

2.7. Savana Ammons [4] informed us in 2023 about a previously overlooked Poincaré-Hopf result for graphs on classical 2-manifolds [24]. It can be generalized to discrete 2-dimensional CW complexes (which can be seen as delta sets) and which does not involve the continuum. In two dimensions, the traditional view is to see G embedded in a 2-dimensional surface. One has then a notion of "face". We note however that Theorem 1 works in arbitrary dimensions and is close to the traditional Poincaré-Hopf theorem for gradient fields. We have formulated [56] also with digraphs without circular parts in any simplex as this produces a total order on each simplex and so a **vector field** $F : G \to V$, where G is the simplicial complex and V the vertex set. Directing every energy $h(x)$ contributing to the Euler characteristic $\chi(G) = \sum_{x \in G} h(x)$ to its vertex produces a function $i_F(v)$ on vertices which satisfies $\sum_v i_F(v) = \chi(G)$.

2.8. The just given argument works also for the above proven theorem, if g is a locally injective function on the vertices. Being locally injective makes sure that g defines a **total order** on each simplex $x \in G$. We can now move every energy value $\omega(x)$ entering the Euler characteristic $\sum_{x \in G} \omega(x)$ to the vertex in x, where g is maximal. The sum of the indices is the same; indeed, no $\omega(x)$ has been lost nor double counted.

2.9. The frame work can be extended a bit more. We can think of an assignment which maps a simplex x to a vertex $v = F(x) \in x$ as a **vector field**. It is a generalization of a **direction** because on a 1-dimensional simplicial complex, we can see F as defining an orientation of the edges. For 1-dimensional simplicial complexes, meaning a graph with the 1-dimensional skeleton complex a vector field is the same than assigning a direction to each edge. The vector field structure so upgrades the graph to a **directed graph**.

2.10. The just given argument for Poincaré-Hopf is in the spirit of Morse theory: pick the vertex v on which the function g is maximal, use that $\chi(B(v)) = 1$ for any **unit ball** $B(v)$ of G and the **valuation formula** $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ which is valid for simplicial complexes as well as for sub-graphs A, B of G that generate themselves in G. An injective function g defines a **Morse type build up** of the graph: we have a set of graphs $c \to G(c)$ generated by vertices $V(c) = \{v, g(v) \leq c\}$. Points v, where $\chi(S_g^-(v)) \neq 1$, are automatically critical points because the Euler characteristic changes by $1 - \chi(S_g^-(v))$. If $S_g^-(v)$ are k-spheres, meaning that f is a **Morse function**, the changes are by $(-1)^{m(v)}$, where $m(v) - 1$ is the dimension of the sphere S_f^-. For a minimum v of f for example, $S_f^-(v) = 0$, the (-1)-dimensional sphere so that the **Morse index** $m(v) = 0$ and the Poincaré-Hopf index $i_g(v) = 0$.

2.11. Let us look a bit more at the **Glass theorem** [24], which is the special case of a 2-dimensional complex but where one has also faces coming from an embedding in a surface. We can see the embedded graph structure as a delta set (V, E, F), where F are the **faces**, connected regions enclosed by edges. If we prefer to stay in combinatorics and avoid the continuum, we could just add the faces and require that the Barycentric refinement of this structure is a discrete 2-manifold, a graph for which every unit sphere is a circular graph with 4 or more elements. A given (fixed but arbitrary) orientation on each of the unit spheres $S(v)$ now allows to redistribute the values $\omega(x)$ for each of the simplices $V \cup E \cup F$. Keep the values $\omega(x)$ from V and F where they are then move the value $\omega(e)$ of an edge $e = (a, b)$ either to the vertex set b or then to the face $f = (a = a_0, b = a_1, \dots, a_{n-1})$ defined by the orientation. For a vertex $x = v$ or face $x = v$, the **Glass index** is $1 - R(x)/2$, where $R(x)$ is the number of orientation changes of edges entering or surrounding x. The division by 2 comes from the fact that every edge e induces exactly two sign changes on neighboring part of $V \cup F$.

2.12. Let us reproof quickly the Poincaré-Hopf version for **digraphs** (directed graph) $G = (V, E)$ as given in [58]. Let us call a **digraph locally non-circular** if no closed path exists in any any of its cliques x (=simplex=face=complete subgraph). This especially means that if $(a, b) \in E$, then (b, a) is not in E. The relation E now defines a **total order** on each simplex. Now think $\omega(x) = (-1)^{\dim(x)}$ as an **energy** attached to the simplex x. The Euler characteristic is the total energy. One can now in any simplex identify the maximal vertex v given by the total order and assign the energy $\omega(x)$ to v. If this is done for all simplices, we get a function $i(v)$ on the vertex set and naturally $\sum_{v \in V} i(v)$ is equal to $\sum_x \omega(x) = \chi(G)$. We have shown:

Theorem 2 (Poincaré-Hopf for digraphs)**.** *For a locally non-circular digraph,* $\sum_{v \in V} i(v) = \chi(G)$.

2.13. The paper [58] generalized this to more abstract vector fields on a graph (V, E) with simplex set G. A **vector field** is just a map $F : G \to V$ with $F(x) \in x$. One can enhance this with an additional map $i : V \to G$, where $v \in i(v)$ and so get a **permutation** $T = i \circ F : G \to G$. A vector field then not only defines the Poincaré-Hopf indices but also a **dynamics**. Equilibria are simplices x for which $T(x) = i \circ F(x) = x$. The permutations defined by the vector field F model now more closely a vector field dynamics in the continuum because $T(x)$ intersects with x.

3. Gauss-Bonnet

3.1. Poincaré-Hopf indices are **divisors** on the graph (maps $V \to \mathbb{Z}$). We will see curvature as an average of such indices. It is in general not a divisor because curvature values are in general rational. **Integral geometry** links Poincaré-Hopf indices with curvature. It uses probability theory to define geometric quantities like length, volume or curvature. It is a classical topic [90, 10, 84, 25, 6, 7]. Given any **probability space** $(\Omega, \mathcal{A}, m)$ on the set of **locally injective functions** Ω, one gets a **curvature** $K(x) = \mathrm{E}[i_g]$ as an **index expectation** [40, 39, 63, 53, 62, 43]. Because of Fubini's theorem, the expectation commutes with summation and curvature automatically satisfies the **Gauss-Bonnet formula**. "Gauss-Bonnet is an expectation of Poincaré-Hopf".

Theorem 3 (Gauss=Bonnet)**.** *For any index expectation curvature, we have* $\chi(G) = \sum_{v \in V} K(v)$.

Proof. This directly follows from Fubini: we can switch the finite summation and expectation procedures: $\sum_v K(v) = \sum_{v \in V} \mathrm{E}[i(v)] = \mathrm{E}[\sum_{v \in V} i(v)] = \mathrm{E}[\chi(G)] = \chi(G)$. □

3.2. We stepped into this topic first with [37, 36, 41]. For graphs which define 2-dimensional manifolds, the curvature is $K(v) = 1 - deg(v)/6$ which is a century old theme, especially in the graph coloring literature (see [45]) and going back to [18]. If the measure is the counting measure on all locally injective functions for example, one gets the **Levitt curvature** $K(v) = 1 - f_0(S(v))/2 + f_1(S(v))/3 - \dots$, where $S(v)$ is the **unit sphere** of v, the graph generated by all vertices attached to v [73]. For 2-dimensional manifolds, where $f_0(S(v)) = f_1(S(v)) = \deg(v)$, we have $K(v) = 1 - \deg(v)/6$. For 1-dimensional graphs, graphs without triangles, we have $K(v) = 1 - \deg(v)/2$. [2]

[2] We noticed [73] only after [36] was written. [73] does not stress the Gauss-Bonnet connection.

3.3. The Poincaré-Hopf theorem can be understood as a rule which pushes the signed measure (= energy) $\omega(x) = (-1)^{\dim(x)}$ from the simplex x to a vertex v in x, where f is largest ending up with a weight $i_g(v)$ on the vertex x. The symmetric Gauss-Bonnet formula of Levitt can then be understood as distributing $\mu(x)$ equally to all $(k+1)$ vertices that are present in the k-simplex x. All this works in more generality, when $\omega(x)$ can be a quite arbitrary function to a ring and especially for functions that take values in the units of real division algebras. [59, 54].

3.4. Having a probability measure on locally injective functions is a way to **deform** the geometry. We think about it as a way to impose a **discrete Riemannian metric**. The reason is that a probability space on functions also defines a distance function on V. Let v, w be two vertices in a graph, its **Crofton distance** is the probability of the set of functions g for which $g(v)$ and $g(w)$ have different signs. This is part of integral geometry [84, 85, 35, 86]. The idea of index expectation in the context of Gauss-Bonnet probably first appeared in [6, 7] even-so one can see the ideas of integral geometry in the Gauss-Bonnet-Chern theme in general. In the discrete, integral geometry paves a way to add a metric and curvature structure which is **deformable**. Similarly as one can deform the Riemannian metric in the continuum, we can tune the probabilities on functions. This allows more flexibility and hope that using suitable notions of "sectional curvature" in discrete manifolds, one could get a Gauss-Bonnet curvature which is positive for positive curvature manifold. See [53, 63, 62].

3.5. The general **integral geometric definition of distance** on a Riemannian manifold goes as follows: if $\gamma : [a, b] \to M, t \to r(t)$ is a smooth curve in a smooth connected manifold M, denote by $L(\gamma) = \int_a^b |r'(t)| \, dt$ its length (where $ds = |r'(t)|dt$ is measured using the Riemannian metric). Given a probability space $(\Omega, ,P)$ of smooth Morse functions ω on M, we can look at the random variable $N_\gamma(\omega)$ counting the number of intersections of the level surface $\{\omega = 0\}$ with γ. More precisely, we count the number of transitions from $f \leq 0$ to $f > 0$. This defines a **pseudo metric**

$$d(x, y) = \inf_{\gamma(x,y), N_\gamma \in L^1(\Omega, P)} \mathrm{E}[N_\gamma] ,$$

where the infimum is taken over all curves connecting x with y with the understanding that $d(x, y) = \infty$, if there should be no γ for which N_γ is in L^1. The **Kolmogorov quotient** (M_P, d_P) consists of all equivalence classes, with the equivalence relation given by $x \sim y$ if $d(x, y) = 0$. This is a generalization of Riemannian metric because if we take a Riemannian manifold and isometrically embed it into an ambient space [70] and use a rotationally symmetric measure, then we recover the standard metric.

4. Homotopy

4.1. The **unit sphere** $S(v)$ of a vertex v is the sub-graph of G generated by all $w \in V$ with $(v, w) \in E$. In the language of **metric spaces**, it is the unit sphere with respect to the **geodesic metric** on G, where adjacent nodes have distance 1. But in graph theory, it is understood as a graph and not just as a set of vertices.

4.2. A graph $G = (V, E)$ is called **contractible** if there exists $v \in V$, such that both the unit sphere $S(v)$ and the graph $G \setminus v$, the sub-graph of G generated by all vertices of G different from v, are contractible. This inductive definition is primed by the assumption that the **one-point graph** K_1 is contractible. The empty graph is not considered contractible. Note that the definition also depends on the simplicial complex we have chosen. If we would look at graphs as one-dimensional simplicial complexes, then already a triangle K_3 would not be contractible.

4.3. What is here called contractible is often also called collapsible. If we can use homotopies (=contraction and extension steps) to get from a graph to K_1, we call this **homotopic to** 1. [3]

4.4. This definitions of manifold was chosen so that the notion of "sphere" and so the notion of "manifold" is **constructive**. One could replace "contractible" with "homotopic to 1", but that would be harder to check and lead to a definition which can not be checked with a fixed algorithm The inductive definition of sphere goes back to Evako, formerly Ivashchenko [30, 19, 31]. The process of removing a vertex v with contractible $S(x)$ or the reverse process of linking a new point to a contractible part of a graph are what constitutes **homotopy steps**. Two graphs are called **homotopic**, if a sequence of homotopy steps allows to get from one to an other. Unlike checking whether a graph is contractible, checking whether a graph is **homotopic to** $1 = K_1$ can be hard. The **dunce hat graph** is an example of a graph which is not contractible but which is **homotopic to** 1: it is a graph which needs to be expanded first before it can be contracted. For more about the difficulty of deciding whether a structure is a sphere or not, see [80, 14, 33].

4.5. Homotopy in discrete finite frame work has a long history. Historically, combinatorial notions of homotopy were put forward by J.H.C. Whitehead [91]. For graphs, the process of homotopy has been defined in [29] and was refined in [13] (we consider the later a significant step as it simplified the notion considerably). These are definitions which work when we look at the Whitney complex. Homotopy deformations of graphs produce deformations of the simplicial complexes which would produce homotopy deformations of geometric realizations. We can see homotopy deformations also as the process of adding a disjoint open set U of trivial cohomology $\vec{b}(U) = \vec{0}$ to a closed set [65]: $G \to G \cup U$ produces again a simplicial complex which by the **trivial cohomology** of U must have the same cohomology. The space $U = \{\{1\}, \{1, 2\}\}$ for example has trivial cohomology (something which is not possible for simplicial complexes). Attaching this to the leaf of a one dimensional graph produces a homotopy extension of the graph. See [75, 76] or finite topological spaces.

4.6. To summarize, unlike the straightforward **contractibility**, **homotopy** is an equivalence relation on the category of finite simple graphs that is computationally difficult. To decide whether two graphs are homotopic or not might, can need some creativity. To see for example that the graphs C_5 and C_6 are homotopic, we first have to "fatten" it. It needs a few homotopy steps. But like all discrete manifolds (a notion covered in the next section), these graphs C_5 and C_6 are not contractible. It is impossible to contract C_6 to C_5 but it is possible to make a homotopy deformation from C_6 to C_5. In a general manifold, one can not even remove one single vertex, as every unit sphere is a sphere and is not contractible. Already the (-1) sphere 0 and the **0-sphere**, the 2-point graph $\overline{K_2}$, the graph complement of the complete graph K_2), are not contractible.

5. Manifolds

5.1. A graph $G = (V, E)$ is a **discrete** d**-manifold** if every **unit sphere** $S(v)$ is a $(d-1)$-sphere. Inductively, a d**-sphere** is a discrete n-manifold if there exists a vertex v such that

[3] Rather than using the "gotcha" nomenclature of contractible and collapsible which is used inconsistently in the literature anyway, we chose to identify contractible and collapsible and use "homotopic to 1" for the wider equivalence relation. It makes sense to stress this point and repeat it. The difference between contractible and "homotopic to 1" could not be bigger, justifying to distinguish it also with nomenclature.

$G \setminus v$ is **contractible**. The empty graph 0 is declared to be the (-1)-sphere. So, the 2-vertex graph $\overline{K_2}$ without any edge is the **0-sphere** and a cyclic graph with 4 or more vertices is a **1-sphere**. A graph is called a **d-ball** if it is of the form $G \setminus v$, where G is a d-sphere. A graph is a **d-manifold with boundary**, if **(i)** every point has a unit sphere that is either a $(d-1)$-sphere or a $(d-1)$-ball and **(ii)** both cases do appear. A complete graph K_{d+1} for example is not a d-manifold with boundary but its Barycentric refinement is an d-ball. The refinement of the 2-simplex K_3 for example is a wheel graph with 7 vertices. The simplicial complex of K_3 is $\{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,3\},\{1,2,3\}\}$, a set with seven elements.

5.2. The **boundary operation** $G \to \delta G$ can be defined for a general graph G. The graph δG is the sub-graph generated by the set of vertices for which the unit sphere $S(v)$ is contractible. The remaining vertices are called **interior points**. A graph is called a **d-manifold with boundary** if every interior point v has a unit sphere $S(v)$ that is a $(d-1)$-sphere and every boundary point v has a unit sphere that is a $(d-1)$**-ball**, (which by definition is a **punctured** $(d-1)$ **sphere**). An example of a 2-manifold with boundary is a wheel graph W_n with boundary C_n.

5.3. The complement of the boundary in G is the **interior** of the manifold. It it the graph generated by the interior points. There is a more sophisticated notion of **interior** in a simplicial complex by taking the union of all stars of vertices away from the boundary and where the boundary is the set of simplices for which $S(x) = \overline{U(x)} \setminus U(x)$ with star $U(x)$ is a contractible subcomplex.

5.4. Here, in a graph theoretical setting, the **interior** is the set of points v for which the unit sphere $S(v)$ is not contractible. In a manifold with manifold setting, it is the set of v for which $S(v)$ is a sphere. We always assume that a manifold with boundary both has a non-empty interior and a non-empty boundary. The unit sphere $S_{\delta G}(v)$ of every $v \in \delta G$ is the boundary of the ball $S_G(v)$ and so a $(d-2)$-sphere. Therefore, if G is a d-manifold with boundary δG, then its boundary δG is a $(d-1)$-manifold without boundary.

5.5. A d-**ball** B is defined to be a graph which is of the form $S \setminus v$ for some d-sphere S and vertex v, where $S \setminus v$ is the subgraph generated by all vertices different from v. Every ball B is contractible because by definition, $B = S \setminus v$ for **some** v is contractible. If S is a d-sphere, then also $B = S \setminus w$ for any other vertex w is contractible: to see this, we only have to show that if $S \setminus v$ is contractible and w is a neighboring vertex, then $S \setminus w$ is contractible. For any ball B, we can by definition remove a boundary vertex, unless it is the **unit ball** $B(v) = S(v) \cup \{v\}$ of a single vertex v. The unit ball $B(v)$ is contractible but removing any vertex on the boundary produces a graph without interior.

5.6. We can check by induction that every contractible graph A has Euler characteristic $\chi(A) = 1$ and that every d-sphere A satisfy the **Euler gem formula** $\chi(A) = 1+(-1)^d$. See [72, 83]. The induction step uses the **valuation formula** $\chi(A \cup B) = \chi(A)+\chi(B)-\chi(A \cap B)$ which holds for two arbitrary sub-graphs A, B that generate themselves in G. Here is the induction step: if A is a d-sphere, then $S(v)$ is a $(d-1)$-sphere that generates itself and $B(v)$ is a d-ball that generates itself. Therefore $\chi(A) = \chi(A \setminus v) + \chi(B(v)) - \chi(S(v)) = 1 + 1 - (1 + (-1)^{d-1}) = 1 + (-1)^d$.

5.7. A discrete d-manifold is never contractible because no vertex can be removed: indeed, every unit sphere $S(v)$ is a $(d-1)$-sphere and since $\chi(S(v)) \in \{0,2\}$, we would remove a point with non-zero index 1 or -1. There is no homotopy reduction possible because a homotopy reduction step would remove a point with contractible unit sphere. But the unit sphere of a d-manifold is a $(d-1)$-sphere and so not contractible. That a d-sphere is not contractible follows from the just computed Euler gem formula for Euler characteristic (either $\{0,2\}$) and that a contractible graph has Euler characteristic 1.

6. Dimension

6.1. The **maximal dimension** $\dim_{max}(G)$ of a graph (G,V) is k, if the graph contains a complete sub-graph $x = K_{k+1}$ but does not contain any larger complete sub-graph K_{k+2}. Complete sub-graphs x are also called **faces**, **cells**, **simplices** or **cliques** (where the later terminology is mostly used in graph theory). The set of all vertex sets of complete sub-graphs x in G is a **finite abstract simplicial complex** G, a finite set of sets x closed under the operation of taking finite non-empty subsets. The integer $\dim_{max}(G)+1$ is also known as the **clique number** of G. The maximal simplices are also known as **facets**. The maximal dimension of the **empty graph** 0 is assumed to be (-1). The empty graph is the **initial object** in the category of graphs; it is an object of Euler characteristic 0. The 1-point graph 1 is the **terminal object** in the category.

6.2. The **inductive dimension** of a vertex $v \in V$ in a graph G is defined as $\dim_G(v) = 1 + \dim_{ind}(S(v))$, where $\dim_{\rm ind}(A)$ is the average of all inductive dimensions of vertices in A. With this definition, $\dim_{ind}(G) = \frac{1}{|V|}\sum_{v \in V} \dim_{ind}(S(v))$, one has $\dim_{ind}(A) \leq \dim_{max}(A)$. Examples, where equality holds are discrete manifolds with or without boundary, regular graphs and especially complete graphs. The inductive dimension behaves a bit like the Hausdorff dimension for metric spaces. The inductive dimension is bounded above by the maximal dimension but the discrepancy can be arbitrarily large as one can see by adding as many zero dimensional parts to a given graph (which lowers the average dimension but does not change the maximal dimension.)

6.3. The **augmented dimension** is defined as the inductive dimension incremented by 1: $\dim^+(G) = 1+\dim(G)$. In [8] it was shown that the Zykov join operation $+$ on graphs satisfies $\dim^+(G+H) = \dim^+(G)+\dim^+(H)$. Here is a short proof: call $h = |G+H|^{-1}$. $\dim^+(G+H) = 1+h\sum_{x\in G}\dim^+ S_{G+H}(x)+h\sum_{y \in H}\dim^+ S_{G+H}(y)$. This is $1+h\sum_{x\in G}\dim^+(S_G(x)+H)+|G+H|^{-1}\sum_{y\in H}\dim^+(G+S_H(y))$. Induction gets this to $1+h\sum_{x\in G}\dim^+(S_G(x))+\dim^+(H))+h\sum_{y\in H}\dim^+(G)+\dim^+(S_H(y)))$. This reduces further to $\dim^+(G)+\dim^+(H)$.

6.4. There is also the notion of **topological dimension** which is natural if one looks at the **finite topology of a graph** and which is discussed more in the next section. The topological dimension is not only a discrete analog of the **Lebesgue covering dimension**, but it has the same definition, just used for finite topological spaces. The classical definition uses the notion of refinement: a **refinement of a cover** $\mathcal{U}$ is a new cover $\mathcal{V}$ such that every $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$. A sub-cover of a cover is an example of a refinement. The **topological dimension** now is the smallest $n \geq 0$ such that for every open cover $\mathcal{U}$ of G, there is a refinement such that every vertex is in no more than $n+1$ sets of the covering.

6.5. The classical topological dimension of the Euclidean space with the topology coming from the Euclidean distance metric is d because for every cover, there is a sub-cover such that only $d+1$ sets intersect. For $\mathbb{R}$ for example, we can look at covers by open intervals. We can now constructively get a cover in which no point is covered by more than 2 intervals. As for an example in the discrete, the topological dimension of the complete graph $G = K_{n+1}$ is n. Any cover has a refinement in which the sets are the stars $U(v_i)$ where v_i are the vertices of G. This cover consists of $n+1$ sets.

6.6. To motivate the next section, let us point out that if we take a finite topological space coming from a metric, then there are open covers of order 1 and the topological dimension is zero. That for a finite topology coming from a metric space, the topological dimension is always zero can also be seen from the fact that it produces discrete topology where all sets are **clopen**, open and closed. In order to have a reasonable topology in which connectivity and dimension works as expected, we need topologies which are non-Hausdorff.

7. Topology

7.1. The **topology of a graph** G is a classical **finite topology** on the simplicial complex G of G. The collection of **stars** $U(x) = \{y \in G, x \subset y\}$ together with the empty set $\emptyset$ define the **basis of this topology**. The basis is already closed under intersections. Every open set in the topology can be written as a union of elements in the basis. [4] The **topology** $\mathcal{O}$ is the closure of this basis under the operation of taking unions and intersections. The topology is **Alexandrov** [2, 75], meaning that any set contains a minimal non-empty open subset. It is not Hausdorff in general however. An **open cover** is a finite set of open sets whose union is all of G. An open cover of G necessarily must contain all stars $U(v)$ with $v \in V$. The stars $U(v)$ play the role of the "atoms of space".

7.2. The topology of a graph has parallels to the **Zariski topology** in algebraic geometry, where the closed sets of an algebraic set are the algebraic subsets. The closed sets of the graph topology are the sub-simplicial complexes. A slightly rougher topology only uses the closed sets which come from sub-graphs of G. This has some disadvantages like already that for Euler characteristic $\chi(A \cup B) \neq \chi(A) + \chi(B) - \chi(A \cap B)$ holds only for sub-simplicial complexes and not for graphs as the union of two graphs can generate simplices of larger dimension. K_3 as a union of K_2 and a path graph P_3 intersecting in a zero-dimensional $\overline{K_2}$. But the union of the simplicial complexes of K_2 and P_3 is the simplicial complex of C_3 which is not generated by a graph.

Lemma 1. *The topological dimension is the same than the maximal dimension.*

Proof. We follow a proof given in [23]: the topological dimension is less or equal than the maximal dimension n because for **any open cover** $\{U_j\}$, the set of all stars $\{U(v)\}_{v \in V}$ of the vertices of G is always a open refinement of $\{U_j\}$ because for all $v \in V$ the star $U(v)$ must be in one of the open sets U_j. There is no $x \in G$ with more than $n+1$ vertices. The order of the open cover $\{U(v)\}_{v \in V}$ is therefore $n+1$ or less. We have shown that every open cover has a subcover of order $n+1$. (ii) to see that the topological dimension is larger or equal than the maximal dimension n note that the vertex cover $\{U(v)\}_{v \in V}$ of G of order $n+1$ because $(n+1)$ sets are needed to cover K_{n+1}, the largest complete subgraph of G. ☐

[4] A set of sets closed under intersection is also called a π-system.

7.3. In general, a basis for the set of closed sets are the stars **stable manifolds** $U(x) = \{y \in G, x \subset y\}$ of simplices x. These are subsets of the simplicial complex. One can see a star also as the **unstable manifold** of a simplex x. The **core** $C(x) = \{y \in G, y \subset x\}$ is a sub simplicial complex which can also be seen as the **stable manifold**. [5] Seen in this way, every simplicial complex and in particular, every graph has a **hyperbolic structure**.

7.4. Not all subsets of the set of simplices are open or closed. For the cyclic graph C_5 for example, only 12 percent of all set of simplices are open. And for K_4, only half a percent of all subsets are open. Cohomology can be extended from closed sets to open sets as they are delta sets [66, 65]. We considered in October 2016 the topology on the simplicial complex of a graph, where the closed sets come from sub-graphs which are generated by their vertex sets. Not all simplicial sub-complexes are closed in this topology so that this is a rougher topology. The topology in which all simplicial subcomplexes define closed sets is slightly finer. The topology having subgraphs as closed sets does not work well with counting because the union of two subgraphs might generate a larger complex than the union of the complexes.

7.5. With a topology comes a **Borel σ-algebra** obtained by closing $\mathcal{O}$ under taking complements, unions and intersections. A functional like $\mu_k(A) = f_k(A)/f_k(G)$ is now an example of a **probability measure** on this σ-algebra. **Measurable sets** are more general objects than graphs or simplicial complexes. Open sets were relevant already in in **connection calculus** because $g(x,y) = \omega(x)\omega(y)\chi(U(x) \cap U(y))$ were the **Green function entries** of the matrix $g = L^{-1}$, where $L(x,y) = 1$ if $x \cap y$ is non-empty and $L(x,y) = 0$ else. We wrote $L(x,y)$ also as the Euler characteristic of a closed set $\chi(W^-(x) \cap W^-(y))$. See [60, 59, 61, 68]. Any function $h : G \to R$ on a ring extends to a R-valued measure on the σ algebra by defining $h(A) = \sum_{x \in A} h(x)$. An **energized simplicial complex** defines then a measure on this algebra.

7.6. The **stable manifold** $W^-(x) = \{y \subset x\}$ of a simplex $x \in G$ is closed because it is a sub-complex the full complex G. Also its **boundary** $S^-(x) = \delta W^-(x)$, as the intersection of $W^-(x)$ and the complement of the open set $U(x)$, is closed. The boundary $S^-(x)$ of $W^-(x)$ is always a sphere of dimension 1 less than x. One can check this by induction with respect to dimension. The **unstable manifold** $W^+(x)$ of $x \in G$ is open by definition, because it is a star and stars have been defined as basis elements generating the topology. The open set $U(x) = W^+(x)$ has a boundary a closed set $S^+(x)$ that is not always a sphere. We write often just $S(x)$ for this closed set in G if we work in a simplicial complex. But in graph theory, if v is a vertex in a graph, we write $S(v)$ for the **unit sphere in the graph** which is the sub-graph induced by the set of vertices attached to v. The Whitney complex of the graph unit sphere $S(v)$ is isomorphic to the simplicial complex unit sphere $S(x)$ in the case $x = \{v\}$ has zero dimension.

7.7. If the maximal dimension is positive, the topology of a graph is **non-Hausdorff**: we can not separate two non-maximal simplices x, y for example. [6]. The topology of a graph has properties we like: for example, the topology is **connected** if and only if the graph is connected. Note that the topology on V coming from the **geodesic metric** on a graph would be completely disconnected, because every point $\{v\}$ is both open and closed. An open set U

[5] We need to distinguish x, which is a single element in the simplicial complex G and $W^-(x)$ which is a sub-simplicial complex of G.

[6] A simplex is called non-maximal, if it is contained in a larger simplex

is declared to be **contractible** if the graph generated by the union of all vertices which are contained in simplices $x_j \in U$ is contractible. Alternatively, we could state that a complex G is contractible if its graph $G = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = G$ and $\mathcal{E} = \{(x,y), x \subset y \text{ or } y \subset x\}$ is contractible.

7.8. A finite set $\{H_j\}$ of sub-graphs H_j of G is a **graph cover** of a finite simple graph G, if every simplex x in G is a sub-graph of at least one of the graphs H_j and every of the graphs H_j is contractible. The notion of a graph cover is graph theoretical and does not require the structure of simplicial complex. But the notion is a bit more tricky. It depends on the notion of contractibility for open sets. We could define an open set to be contractible, if its closure is contractible. We could also define an open set to be contractible if the subgraph generated by its vertices is contractible. In both cases, every open cover U_j of G with contractible U_j defines a graph cover, where H_j is the graph generating the smallest closed set containing U_j. On the other hand, a graph cover defines the open cover $U_j = \bigcup_{v \in H_j} B(v)$.

7.9. From the point of view of covering graphs with contractible sets, it does not matter whether we deal with **graph covers** or with **open covers**. The minimal number of contractible sub-graphs that cover G is the same than the number of contractible open sets which cover G. There is no proof required if we define an open set to be contractible if its closure is contractible. Graph covers are more intuitive while open covers are closer to what we deal with in the continuum.

7.10. In the continuum, one sometimes covers a space also with closed sets. This happened in particular in Lusternik-Schnirelmann theory. [22] changed it to open sets. Most of the time, like [88] (page 279), one sees closed sets in the definition of Lusternik-Schnirelmann category.

7.11. Having a topology associated to a graph, we can define a map between two graphs $f : V(G) \to V(H)$ to be **continuous**, if the inverse of an open set is open. This is equivalent to that the inverse of a closed set is closed. Continuous maps are by no means the same than graph homomorphism and also not the same than simplicial maps as both require vertices to map vertices into vertices. A constant map $f(G) = c$ for example is not a graph homomorphism (which by definition must map edges to edges), but it is continuous map from $G \to G$.

7.12. A continuous map on graphs also does not define a **simplicial map** of their Whitney complexes because for a simplicial map, simplices must be mapped into simplices. Simplicial maps map vertices to vertices so that not all continuous maps on the topology of G are simplicial maps. A simplicial map maps sub-simplicial complexes into simplicial complexes and so, open sets into open sets. One calls this an "open map". A simplicial isomorphism is also called a homemorphism. If we have a simplicial map, it must preserves order. This means the inverse of a closed set must be closed. In other words, it must be continuous.

7.13. The **Barycentric refinement** G_1 of a graph G is a new graph for which the vertices V_1 are the elements in the simplicial complex associated to G. The edges are given by connecting two (x,y) if either x is contained in y or y is contained in x. The Barycentric refinement of G_1 is denoted G_2 etc. The Barycentric refinement is defined for much more general spaces like cell complexes or delta sets and produces a graph where the elements are the nodes and two are connected if one is contained in the other. (For delta sets, the Barycentric refinement G_1 can have other topological properties than G. Take an open set $U = \{x\}$ in a simplicial complex for example, where x is a facet, a maximal simplex. Now U is a delta set by itself. Its Barycentric refinement would be the 1-point graph. The Barycentric refinement construction however shows that graphs not much of a loss of generality. If one does not mind taking a refinement, one can stay within graph theory.

7.14. In the following, we look at a notion of homeomorphic that is motivated by traditional notion of homeomorphism of graphs. Two one-dimensional simplicial complexes are called **homeomorphic**, if they have a common Barycentric refinement. This notion is used in topological graph theory but does not take into account 2 or higher dimensional structures in the graph. A more general notion of homeomorphism for graphs is by allowing single edge refinements (which are by nature local refinements). Every Barycentric refinement in one dimensions is a composition of local edge refinements. The graph C_5 is obtained from C_4 by one edge refinement. The graph C_8 is the Barycentric refinement of C_4.

7.15. In **topological graph theory**, where graphs are embedded in some 2-dimensional surface, the notion of homeomorphism for graphs is enough because the connected components on the surface can then also be considered homeomorphic as all faces (2-dimensional cells) are silently assumed to be balls. Topological graph theory goes beyond combinatorics because Euclidean spaces and so the concept of infinity is involved. We are looking for a notion of homeomorphism of graphs or finite abstract simplicial complexes which does not involve geometric realizations or infinity and which also can constructively be checked in a reasonable amount of time. The notion should be effective.

7.16. Here is a variant of a proposal from [46]. Two graphs G, H are declared to be **pre-homeomorphic** if there exists a continuous map from some Barycentric refinement G_n to H and a continuous map from some Barycentric refinement H_n to G. By definition, a graph G is pre-homeomorphic to its n'th Barycentric refinement G_n. When considering one-dimensional cases only (triangle free graphs) or graphs with a 1-dimensional skeleton complex imposed, then the notion of pre-homeomorphic is what homeomorphic means in the topological graph theory literature like for example in [26].

7.17. In [46] we made the assumption a bit stronger. Because we can not prove yet that a graph pre-homeomorphic to a manifold must be a manifold, we declared f to be a **homeomorphism** if it is a pre-homeomorphism and additionally, for every atom $U(y)$ of a maximal simplex $y \in H_n$, the open set $f^{-1}(U(y))$ is an open ball and for every atom $U(x)$ of a maximal simplex $x \in G_n$ the open set $f(U(x))$ is an open ball. With this definition, homeomorphic graphs in which one is a d-manifold forces also the other to be a manifold.

7.18. The **Čech graph** of the basis $\mathcal{B}$ of the topology $\mathcal{O}$ of a graph G is a new graph which in many cases has the same topological features than the graph itself. The **Čech graph** of the cover defined by all $U(v) = S^+(\{v\})$ with $v \in V$ is the graph G itself. So, G itself can be seen Čech graph of an open cover of G. The **order** of an open cover is the smallest m such that each point in the space belongs to at most $m + 1$ open sets in the cover. The **topological dimension** is the smallest $k \geq 0$ such that for every open cover $\mathcal{U}$ of G, there is a refinement of order k. We have seen that the topological dimension is the maximal dimension.

8. Category

8.1. The **Lusternik-Schnirelmann category** $\text{cat}(G)$ of a graph G is defined as the minimal cardinality k of a graph cover $\{U_j\}_{j=1}^k$ of G using contractible sub graphs U_j of G. Note that the sub-graphs U_j are not required to generate themselves within G. There are many contractible sub-graphs of a given graph that do not generate themselves. For example, every **spanning**

tree of G covers all the vertices of G and is contractible. It does generate the entire graph G but a spanning tree does not cover the entire graph in general as there are edges which might be missing. We want the union of the U_j to cover not only all the vertices but also all the edges.

8.2. The Lusternik-Schnirelmann category is of interest in graph theory also because it is related in spirit to a class of functionals which are classical and well studied in graph theory. For example, the **edge arboricity** tells how many forests are needed to cover a graph. By the **Nash-Williams theorem** [78], this arboricity of a graph (V, E) is the maximum of $|E_W|/|W|$, where W ranges over all subsets of V and E_W is the number of edges generated by the induced graph of W. The **vertex arboricity** (=point arboricity) is the maximal number of forests partitioning V such that each forest generates itself. The vertex arboricity is of interest because the **chromatic number** is sandwiched between the vertex arboricity and twice the vertex arboricity. See [69, 67].

8.3. Instead of graph covers, we could also use (like Fox classically did in [22]) use **open covers** of the topology and the category would be the same: for an open set U, we can look at the graph generated by the vertices appearing in U and define U to be contractible, if that graph is contractible. With this definition, we also could have used **closed coverings** U, the reason being that the closure of a contractible open set must by definition be contractible. Unlike for closed sets, where the notion of contractible is intrinsic and does not depend on where it is embedded into, the notion of contractible for open sets depends on the embedding as we look at the closure of the open set. We chose here the more graph theoretical definition and use covers by subgraphs.

8.4. The topological closure of an open set is not necessarily contractible as the example of a punctured sphere = open ball shows. An open ball should with this definition not be considered contractible. Contractibility for open sets is a bit tricky is because the cohomology of an open set is in general much different than the cohomology of a closed set. An open ball has only a non-trivial maximal cohomology. Nevertheless, if we wanted to define Lusternik-Schnirellmann category using open sets, we would have to define an open set to be contractible, if its closure is contractible.

8.5. By induction, a contractible graph G has Euler characteristic 1. The Lusternik-Schnirelmann category of a contractible graph by definition is equal to 1. A d-sphere G of dimension $d \geq 0$ almost by definition has Euler characteristic $1 + (-1)^d$ because removing an open d-simplex of Euler characteristic $(-1)^d$ produces a closed ball of Euler characteristic 1. A d-sphere almost by definition also has category 2 because the unit ball $B(v)$ together with the graph $G \setminus v$ cover G. Now, the unit ball $B(v)$ and $G \setminus v$ are contractible and cover the graph.

8.6. Also, almost by definition, the Lusternik-Schnirelmann category of a non-connected graph G is the sum of the categories of its connected components. Because a discrete manifold is never contractible - we can not even remove a single point - the category of a manifold is always ≥ 2, with the single exception of the **1-point graph** K_1 which is the only connected 0-dimensional manifold. The 1-point graph K_1 is the **terminal object** in the category of graphs.

8.7. Lets look at some examples:
1) $\text{cat}(K_n) = 1$ for all $n \geq 1$.
2) $\text{cat}(C_n) = 2$ for all $n \geq 4$.
3) $\text{cat}(T) = 1$ if T is a tree as a tree is contractible.
4) $\text{cat}(F) = b_0(F)$ if F is a **forest** a disjoint union of trees. The cover is given by the maximal subtrees.
5) The graph complement $\overline{G}$ can have a very different Lusternik-Schnirelmann category than G. For K_n, the graph $\overline{K_n}$ has no edges an $\text{cat}(K_n) = 1, \text{cat}(\overline{K_n}) = n$.
6) For any d-sphere, $\text{cat}(G) = 2$ as $B(x), G \setminus x$ are both balls and so contractible.

9. Operations

9.1. The **Stanley-Reisner product** $G \times_1 H$ of two graphs can be defined as the Barycentric refinement of the Cartesian product $G \times H$ of its simplicial complexes. It is best defined in terms of graphs. Let $V = G \times H$ be the vertex set of a graph and E the set of pairs (a,b) for which either $a \subset b$ or $b \subset a$. The Whitney complex of this graph is now declared to be the **Stanley-Reisner product** or topological product. This product is not associative because $G \times 1 = G_1$ is the Barycentric refinement of G.

9.2. The **Stanley-Reisner picture** writes the product in terms of a multiplication of polynomials. We have an associative product within polynomials. But this does not mean associativity for the product. The reason is that we need to returning from a ring element to a simplicial complex. If G is a complex and p the Stanley-Reisner polynomial then the complex of p is the Barycentric refinement G_1.

9.3. Since the Cartesian product of two simplicial complexes (as sets) is not a simplicial complex in general, we must use a product in a larger category of cell complexes or even the larger **topos of delta sets** The later is a natural frame work to do finite geometry [87]. The Barycentric refinement always could bring us back to graphs. The upshot is that a Cartesian product for graphs which works in higher dimension and satisfies the dimension and Kuenneth requirements can be done either with the Stanley-Reisner product (no associativity), abstractly by extending the frame work or by using delta sets. Both the Cartesian product of graphs or the categorial product (small product) of graphs are not suited for higher dimensional considerations. A good product in graph theory is the Shannon product, but it does not preserve manifolds.

9.4. If we want to use graphs and stay in the category of graphs and have associativity, we need to pick one of the few products available: small product, Categorical product, large product (Sabidussi) and strong product (Shannon). None of these products really fits all bills like that the product of manifolds is manifolds. The Shannon product has many nice properties, the Shannon product of two graphs is homotopic to the topological Stanley-Reisner product. The Shannon product together with disjoint union as addition defines a semi-ring that extends to a Shannon ring of graphs. But the Shannon product of two manifolds is only homotopic to a manifold and not a manifold by itself in general.

9.5. Classically, the Cartesian product of two (Euclidean) k manifolds without boundaries is known to have category at least $(k + 1)$. We will show this later. A **k-torus** for example has category $k + 1$. Category is **not a homotopy invariant**: the **dunce hat** G for example is homotopic to $1 = K_1$ but G has category 2 as the dunce hat is not contractible. We could define the homotopy invariant $\overline{cat}(G)$ as the smallest category of a space homotopic to G.

[7]

9.6. In the following, we mean with the product the Cartesian product. The **join** (or **Zykov join**) of two graphs G, H is the disjoint union with additional connections between any vertex of $V(G)$ and $V(H)$. For simplicial complexes G, H the join is the disjoint union $G \cup H$ together will all elements $\{x \cup y, x \in G, y \in H\}$.

9.7. We denote with $G * H$ the **Shannon product** of G and H. It is the graph for which the vertex set is the Cartesian product of the vertex sets of G and H and where (x, y) is connected with (a, b) if one of the projections is an edge in G or H.

Lemma 2. *a)* $\text{cat}(G_1) = \text{cat}(G)$ *if* G_1 *is the Barycentric refinement.*
b) For the disjoint union $+$, $\text{cat}(G + H) = \text{cat}(G) + \text{cat}(H)$.
c) The join $G \oplus H = \overline{\overline{G} + \overline{H}}$ *gives* $\text{cat}(G \oplus H) = \min(\text{cat}(G), \text{cat}(H))$.
d) $\text{cat}(G \times H) \leq \text{cat}(G) \cdot \text{cat}(H)$. *e)* $\text{cat}(G * H) \leq \text{cat}(G) \cdot \text{cat}(H)$.

Proof. a) The Barycentric refinement of a contractible graph is contractible. If $\{U_j\}$ is a cover of G by contractible graphs then the Barycentric refinements $\{V_j = (U_j)_1\}$ form a cover of G_1 by contractible graphs.
b) If $\{U_i\}$ is a cover of G and $\{V_j\}$ is a cover of H, then $\{U_i\} \cup \{V_j\}$ is an open cover of $G \cup H$. We can not take less.
c) If U is a contractible graph and H is an arbitrary graph, then $U \oplus H$ is contractible. Assume that G is the graph with minimal category k. This means we have a cover $U_1, U_2, \dots, U_k$ of G. The graphs $U_j \oplus H$ are contractible. The graph $G \oplus H$ therefore can be covered with k contractible graphs.
d) If $\{U_i\}$ is a cover of G and $\{V_j\}$ is a cover of H then $\{U_i \times_1 V_j\}$ is a cover of $G \times H$.
e) If $\{U_i\}$ is a cover of G and $\{V_j\}$ is a cover of H then $\{U_i * V_j\}$ is a cover of $G \times H$ and $U_i * V_j$ are contractible. ☐

9.8. The example of the torus $\mathbb{T}^2 = \mathbb{T}^1 \times \mathbb{T}^1$ shows that part d) can not be an equality in general because $\text{cat}(\mathbb{T}^2) = 3$. However, if $\text{cat}(G) = 1$, then $\text{cat}(G \times H) = \text{cat}(H)$.

9.9. Let us add a remark to e). Since $\text{cup}(A * B) \geq \text{cup}(A)\text{cup}(B)$ we have $\text{cat}(A * B) \geq \text{cup}(A * B) + 1 \geq \text{cup}(A)\text{cup}(B)$ For example, the category of a Shannon product of a non-simply connected orientable manifold is $\geq k + 1$ because $\text{cup}(G) + 1 \leq \text{cat}(G)$ and the fact that every discrete manifold that is orientable and non-simply connected has a degree 1 differential form.

10. Functions

10.1. When looking at functions on a graph, we want to impose some non-degeneracy condition in general. A most natural one is that if (x, y) is an edge then $f(x) \neq f(y)$. One calls this **locally injective** or a **coloring**. This local continuity condition was needed in Poincaré-Hopf. In the continuum, one imposes more regularity on maps by assuming that the functions to be Morse, meaning that at all critical points, the Hessian is invertible. In the continuum, one also works in a smooth setting, which means that at a critical point x, the stable spheres $S_r^-(x) = S_r(x) \cap \{f \leq f(x)\}$ are spheres. This is what one can adopt also in the discrete.

[7]If we would define $\overline{cri}(G)$ is the minimum of all $\text{cri}(H)$ with H homotopic to G, the main theorem of Lusternik-Schnirelmann theory would then mean $\text{cup}(G) + 1 \leq \overline{\text{cat}}(G) \leq \overline{\text{cri}}(G)$ relating three homotopy invariants $\text{cup} = \overline{\text{cup}}$, $\overline{\text{cat}}$ and $\overline{\text{cri}}$.

At regular points, the stable sphere $S_r^-(x)$ is a ball for small enough r. We adapt this to the discrete.

10.2. For a general finite simple graph $G = (V, E)$ and a locally injective function f, a vertex $x \in V$ is called a **regular point** if $S_f^-(x)$ is contractible. If $S_f^-(x)$ is not contractible, it is called a **critical point**. The minimal number of critical points of a general locally injective function is denoted $\text{cri}(G)$.

10.3. As defined in the introduction already, a function $f : V \to \mathbb{R}$ is a **Morse function** if every $S^-(x)$ for $x \in V$ is either a k-sphere for some $-1 \leq k \leq n-1$ or contractible. If $S^-(v)$ is a contractible, the point is called a **regular point**, otherwise it is considered to be a **critical point** of f. A function could be called **strongly Morse** if both f and $-f$ are Morse. The later condition is hard to achieve if we have not a manifold. It does not required it however: take a manifold M and make a homotopy extension by adding one vertex w to a single vertex of M. If g was a strongly Morse function on M, then extend it to the additional point by setting $g(w) = \max_{v \in V(M)} g(v) + 1$.

10.4. An example of a strongly Morse function is the **dimension function** on the Barycentric refinement G_1 of a graph G. If v is a critical point of a Morse function, then $S^- f(x)$ is a $(k-1)$-sphere for some k and k is called the **Morse index** of x. Because of the **Euler-Gem formula** $\chi(A) = 1 + (-1)^k$ for any k-sphere A, the Poinaré-Hopf index of a point v of Morse index k is $i_g(x) = (-1)^k$. If c_k is the number of critical points of Morse index k in G, then the **Poincaré-Hopf** theorem reads as $\sum_k (-1)^k c_k = \chi(G)$, where $\chi(G) = \sum_k (-1)^k f_k(G)$, and $f_k(G)$ counts the number of k-dimensional simplices.

10.5. For a Morse function f on a discrete manifold, the general critical points agree with critical points of the Morse function because for a Morse function, all $S^-(x)$ are either a ball and so contractible or spheres and so non-contractible. The minimal number of critical points which some Morse function can achieve on a discrete manifold is denoted by $c(G)$. We have $\text{cri}(G) \leq c(G)$. Proof: for a Morse function f with c_f critical points, $\text{cri}(G) \leq c_f(G)$. Now minimize the right hand side over all Morse functions f to get $\text{cri}(G) \leq c(G)$.

10.6. The inequality can be strict like for any 2-torus graph $\mathbb{T}^2$, where $\text{cri}(G) = 3$ and $c(G) = 4$. Indeed, for a Morse function, the indices are ± 1 and by Poincaré-Hopf they add up to 0 so that there must be an even number of critical points for a Morse function on a 2-torus. There can not be 2 because that would mean that it is a sphere.

10.7. The vertex sets G of all complete sub-graphs of (V, E) is a **finite abstract simplicial complex**, a finite set of non-empty sets closed under the operation of taking non-empty subsets. The finite structure was introduced in 1907 by Dehn and Heegaard [17, 12]. We can equip each of these sets x with an **orientation**, a fixed order of its vertices. An example is to enumerate the vertices of G, then use these labels on each simplex. The simplicial complex of the kite graph $K_{1,2,1} = (V, E) = (\{1, 2, 3, 4, 5\}, \{(12), (13), (23), (24), (34)\})$ for example is $G = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\}$. Writing this down in lexicographic order already gives an example of an orientation.

10.8. A k-**form** is a function on G_k, the set of sets in G with $k+1$ elements. [8] A differential form f is nothing else than a scalar function on $G = \cup_{k=0} G_k$, once the elements of G are equipped with orientations. The restriction of f to G_k is the vector space of **k-forms**. The dimension of this vector space is f_k.

10.9. The total set of forms is a $n = \sum_{k=0}^{d} f_k$ -dimensional vector space Ω, where n is the number of simplices of G and $f_k = f_k(G)$ the number of k-dimensional simplices in G. The set of k-forms Ω_k is a f_k-dimensional real vector space. Define **the exterior derivative**

$$df(x) = \sum_{y \subset x, \dim(y)=\dim(x)-1} \operatorname{sign}(x,y) f(y) ,$$

where $\operatorname{sign}(x,y)$ is 1 if the orientation of y matches the orientation of x on y and (-1) else. [9] For example, $\operatorname{sign}(\{1,2,3\},\{1,3\}) = -1$ and $\operatorname{sign}(\{1,2,3\},\{2,3\}) = 1$.

10.10. If G has n elements, the exterior derivative map d is represented in an explicit way as a lower triangular $(n \times n)$-matrix satisfying $d^2 = 0$. If d^* is the **transpose** of d, the matrix $D = d + d^*$ is called the **Dirac operator** of G [42] and $L = D^2 = dd^* + d^*d$ is the **Hodge operator** of G. Note that the matrix entries of both matrices D and L depend on the orientation choice of the simplices but that any different choices of the orientation just corresponds to an orthogonal change of coordinates in the finite dimensional Hilbert space $\mathbb{R}^n$ or $\mathbb{C}^n$ on which D and L operate. The reason for the name "Dirac operator" is because D is a square root of L which shares properties of the Dirac operator in the continuum. The operator $D = d + d^*$ also appears in the continuum [16].

10.11. The map d maps Ω_k to Ω_{k+1} and d^* maps Ω_{k+1} to Ω_k. Because L leaves the f_k-dimensional space Ω_k invariant, L decays into blocks $L_k = d_k^* d_k + d_k d_k^*$, for $k \geq 0$. Each L_k is a $f_k \times f_k$ matrix. Changing the orientation of a simplex is a unitary transformation. The non-negative numbers $b_k(G) = \operatorname{dimker}(L_k)$ are called the **Betti numbers**. They do not depend on the choice of orientation. The kernel $\ker(L_k) = H^k(G)$ are **k-harmonic forms** and represent cohomology classes of G. It is isomorphic to the traditionally defined as the space $\ker(d_k)/\operatorname{im}(d_{k-1})$. This is the same because of the **Hodge decomposition** $\operatorname{im}(d_k) \oplus \operatorname{im}(d_{k+1}^*) \oplus \ker(L_k)$ which follows directly from the **rank-nullity theorem** applied to the block side diagonal matrix L_k with d_k, d_{k+1} blocks so that $\operatorname{im}(d_k) \oplus \operatorname{im}(d_{k+1}^*)$ is $\operatorname{im}(L_k)$.

10.12. The zero'th block $K = L_0$ of the Hodge Laplacian L is known as the **Kirchhoff matrix**. It can be written as $B - A$, where B is the diagonal **vertex degree matrix** with $B(v,v)$ being the **vertex degree** of the vertex v and A is the **adjacency matrix** of G which is a $0-1$ matrix with $A(v,w) = 1$ if and only if $(v,w) \in E$. From this representation $K = B - A$, one can see that K does not depend on the choice of the orientation used on vertices. The definition goes over to quivers, graphs for which multiple connections and self-loops are allowed [64].

[8] The function could be symmetrized by defining $f(\pi(x)) = \operatorname{sign}(\pi) f(x)$ if π is a permutation. It would then be anti-symmetric but there is no need to do that and just consider f as a function on the ordered element.

[9] One usually writes this as $df(x) = \sum_{j=1^k} (-1)^{k-1} f(x_1, \dots, \hat{x}_j, \dots, x_k)$ which is the same because if y is the simplex with the j'th vertex removed, then y has the same orientation than x if and only if $(j-1)$ is even.

10.13. A small example, where some matrix entries of L can depend on the orientation is the kite graph $G = K_{1,2,1}$ obtained by taking away a vertex from K_4. The off diagonal entries of L_2 can depend on the orientation. But in general, the change of orientation of the simplices only induces an orthogonal change of basis in the linear space Ω. It is like choosing a coordinate system in traditional Euclidean geometry.

10.14. The **Shannon product** of two graphs $G = (V, E), H = (W, F)$ has as vertex set the **Cartesian product** $V \times W$ and as edge set $\{((a, b), (c, d)), (a = c)$ or $(a, c) \in E)$ and $(b = d$ or $(b, d) \in F)\}$. Two pairs in the Cartesian product are connected if the projection on each of the factors is either a vertex or edge and at least one is an edge. A k-form as a function on the set of signed k-simplices. Given a p-form f on G and a q-form g on H, define a $(p+q)$-form on the Shannon product $G * H$ by first considering $f * g(x, y) = f(x)g(y)$ which is a $(k + l + 1)$-form, then take the divergence. When restricted to cohomology classes, we get the Künneth formula [71] (see [47] for graphs). It is a function on $(k + l)$-simplices on $G * H$ and so is a $(k + l)$ form on $G \times H$. This gives the product $f \otimes g = d^*(f * g)$.

10.15. In order to have an associative product on simplicial complexes, one has to leave the category of simplicial complexes. There is no Cartesian product of simplicial complexes that is associative, has the natural dimension properties and preserves manifolds. One can go beyond simplicial complexes by using cell complexes like discrete CW complexes. The most convenient one is the even larger class of **delta sets**. Delta sets form a **functor category** and are **presheaves**. It is usually given by a finite set of sets G_i and face maps $d_i^n : G_{n+1} \to G_n$ satisfying $d_i^n d_j^{n+1} = d_{j-1}^n d_i^{n+1}$ leading to the exterior derivative which maps $f \in \Omega_n$ into $df(x) = \sum_{i=0}(-1)^i f(d_i^n x) \in \Omega_{n+1}$. The commutation relation in the axiom for delta set assures that d is an exterior derivative.

10.16. It is more convenient to encode a delta set as a finite set G with n elements, a single $n \times n$ matrix $D = d + d^*$ and a dimension function $r : G \to \mathbb{N}$ which selects the cells $G_n = r^{-1}(n)$ of dimension n. The information (G, D, r) encodes everything we need to know. The category of delta sets is powerful because it is a **topos** and so Cartesian closed. The same tensor product construction works for differential forms, but instead of bringing down the dimension, we just declare elements (x, y) as $(p + q)$-dimensional if x has dimension p and y has dimension q. For example, if x, y are both 0-dimensional, then (x, y) is a 0-dimensional point even, so the data structure represents it as a 0-dimensional object.

11. Cup product

11.1. As Hassler Whitney first realized [92], the definition of the cup product is a bit puzzling at first in a discrete setting. If we take a k-simplex x and a l-simplex y, then $x \oplus y$ is a $k + l + 1$-simplex. For example, if $x = (a, b)$ and $y = (c, d)$ are 1-dimensional, then $x \oplus y = (a, b, c, d)$ is a tetrahedron and 3 dimensional. We would like however to have tensor product definition as in the continuum and get a 2-dimensional object. The trick is to take pointed simplices and join them along this point. For example, if (a, b, c) is a triangle in a graph, then fix a point a. A 1-form cocycle on the triangle is now fixed by giving $f(a, b)$ and $f(b, c)$. Given two 1-form cocycles, define $f \otimes g(a, b, c) = f(a, b)g(a, c)$. The wedge product is now $f \wedge g(a, b, c) = f(a, b)g(a, c) - f(a, c)g(a, b)$. This parallels what the cross product does in $\mathbb{R}^2$. The vector space of 1-cocycles on (a, b, c) is two dimensional and can be described by a vector $[x, y]$ where $x = f(a, b), y = f(a, c)$. Given two such 1-cocycles $[x_1, y_1], [x_2, y_2]$, the cross product is $x_1 y_2 - x_2 y_1$.

11.2. If we do the same thing on a 3-simplex (a tetrahedron) (a,b,c,d) then the space of 1-cocycle is 3-dimensional. It can naturally be identified as $\mathbb{R}^3$ containing elements $[x,y,z]$ where $x = f(a,b), y = f(a,c), z = f(a,d)$. The exterior product is now a 2-cocycle which is determined by the values on the three triangles containing a. The 2-cocycles also form a 3-dimensional vector space $\mathbb{R}^3$ containing elements $[x,y,z]$ where $x = f(a,c,d), y = f(a,d,b), z = f(a,b,c)$. The exterior product is now the naturally the cross product after identifying the vector spaces of 1-forms and 2-forms. Lets now look at the general case:

11.3. If x is a $(k+l)$-simplex in G, it can be written as $x = (x_0, x_L, x_R)$. Define $f \cup g = f \otimes g + (-1)^{|x_l|} g \otimes f$ which defines a $(k+l)$-form on G. For example $x = (x0, x_1, x_2)$ is $1+1$ simplex if (x_0, x_1) and (x_0, x_2) are 1-simplices. Then, $f \otimes g(x) = f(x_0, x_1) g(x_0, y_1)$ and $f \wedge g(x) = f(x_0, x_1) g(x_0, y_1) - g(x_0, x_1) f(x_0, y_1)$. Now $d(f \otimes g) = df \otimes g + (-1)^k f \otimes dg$.

11.4. The **exterior product** can be defined cocycles of any simplicial complex G. It is associative and super commutative so that it is a graded **super algebra**. In the continuum, it is also known as the **Grassmann algebra**. Since cocycles are mapped by d into cocycles and coboundaries into coboundaries, different cohomology classes are mapped into different cohomology classes. This produces the **cup product** on the space of harmonic forms $H^p \times H^q \to H^{p+q}$.

11.5. There are different ways to define the exterior product in the discrete. In [32], we chose to symmetrized version of the product to make it orientation independent. Here are some properties of the exterior product.

Lemma 3. *a) Leibniz rule:* $d(f \wedge g) = df \wedge g + (-1)^p f \wedge dg$.
b) Associativity $(f * g) * h = f * (g * h)$ *and see that it is a coboundary.*
c) Super commutativity $f \wedge g = g \wedge f (-1)^{pq}$.
d) If $df = 0$ *and* $dg = 0$, *then* $d(f * g) = 0$.
e) If $f = dh$ *and* $g = dk$ *gives* $f * g = dh * dk = d(h * dk) = -d(dh * k)$
f) The one-element is the constant harmonic 0*-form which is 1 everywhere.*

11.6. A simplex is **locally maximal** if it is not strictly contained in an other simplex. A graph is **orientable** if one can orient the locally maximal simplices in such a way that $\text{sign}(x,y) = 1$ for all $x \subset y$ as well as for $y \subset x$. Not all graphs are orientable. The smallest non-orientable graph is the graph complement of C_7 which is a discrete **Moebius strip** with 7 vertices, 14 edges and 7 triangles. Fixing an orientation of a triangle fixes the orientation of its neighbors but going around the closed loop changes the orientation. An orientable complex, we can define a Hodge dual.

11.7. In an orientable discrete 2-manifold, the exterior product of two 1-forms in different cohomology classes is a 2-form, a **volume form**. In three dimensions, the exterior product defines **cross product** when identifying 2-forms and 1-forms. In classical calculus, where we associate both 1-forms or 2-forms as **vectors**, we can build a cross product $v \wedge g$ which is considered a scalar function but in more advanced set-ups like differential geometry the cross product can be related to the exterior product obtained from two 1-forms. For example, if $G = K_3$ is oriented cyclically and f assigns the values $(1,0,0)$ to the three edges and g the values $(0,1,0)$ to the three edges, then $f \wedge g$ gives the value 1 on the 2-simplex.

11.8. In an orientable 3-manifold, we also have **dual edges** E'. They are given by pairs (x, y), where x is a 2-simplex and y is a 0-simplex such that x, y span a 3-simplex. We can think of a dual edge as an **altitude** from the vertex y to the center of x. Since in a 3-manifold every 2-simplex y bounds two 0-simplices x, z, and if the maximal simplices define the orientation, then there is for every triangle a unique altitude (x, y) such that y is compatible with the orientation of the tetrahedron $x \cup y$. A 2-form therefore can be visualized as attaching values to these dual vectors. If we chose a coordinate system $(i, j, k = i \times j)$ in each tetrahedron and postulate the relations of Hamilton $i^2 = j^2 = k^2 = ijk = -1$, the multiplication can be extended from edges E to the union of edges and dual edges $E \cup E'$. This means that in every tetrahedron, we have a quaternion algebra. The exterior product makes any orientable 3-manifold into a Lie algebra, once we join 1 and 2-forms.

11.9. The closest to the continuum is to define a **Hodge dual** to a 2-form f as a 1-form g which has the property that $f \wedge g$ is the volume form. The Hodge dual can be defined if one has a volume form. The Hodge dual is well defined if to $y \subset x$ is given defines a unique z with $y + z = x$ overlapping in the smallest element. Having this, allows us to define a cross product of 1-forms in a graph given by an orientation.

11.10. Let G is a d-manifold and f is a locally injective function on G. If c is a value not taken by f, we can look at the sub-graph $A = \{f = c\}$ of G_1 generated by all simplices x of G on which $f - c$ changes sign. [48] That this graph is a $(d-1)$ manifold is proven by showing that for every simplex x, $\{f = c\} \cap S(x)$ is by induction assumption a $(d-1)$ manifold. Locally injective functions which have the property that $f(x) = \sum_{v \in x} f(v)/|x|$ produces locally injective functions on the Barycentric refinements can now be used to define "varieties" $\{f_j = c_j, j = 1, \dots, k\}$ in the k'th Barycentric refinement. These are all vertices in G_k on which all functions $f_j - c_j$ change sign. It is always a $(n-k)$-manifold or empty.

11.11. For example, If G is a 3-manifold, then at every vertex we can look at $\{f = c\} \cap S(x)$ which is a disjoint union of circular graphs. If G is a 4-manifold, then for any two functions f_1, f_2 we can look at the 3-sphere $S(x)$ and have a **link** $\{f_1 = c_1, f_2 = c_2\}$, a one-dimensional sub-manifold of the 3-sphere. For a given n manifold, almost all functions f we can look at manifolds $f = c$ with $minf < c < maxf$. We have so a probability space of sub-manifolds. We can look at expectation values. We could look for example what the average k'th Betti number is or what the average Euler characteristic is Or we can look how many of the functions are Morse functions.

12. Morse Inequality

12.1. We get now to the Morse inequality. For the classical theory, see [77, 74, 79]. A semi-classical analysis approach to the Morse inequalities using Witten deformation [**?**] can be found in [16]. For Forman's discrete Morse theory [20, 21]. Morse theory competes with Lusternik-Schnirelmann theory. We will later see a comparison of L-S with a Picasso painting. Morse theory was promoted in [11] by: *And the term " critical point " of course brings me to my topic proper of this morning: "Morse Theory Indomitable ". I think Morse would have approved the title for when I first met him, he preached the gospel of critical point theory first, last and forever, to such an extent, that we youngsters would wink at each other whenever he got started.*

12.2. Unlike in the continuum, the theorem works for any graph, whether it is a manifold or not. It works for example for the Barycentric refinement of a simplicial complex G and the function $f(x) = \dim(x)$, which is a really special case, where $b_k = c_k$, but still where G is not necessarily a manifold. The result works also for simplicial complexes. In that case, the Morse extensions have to be done by adding new simplices by joining a given simplicial complex (a closed set) with a star (an open set). An example of a homotopy extension $K_2 \to K_3$ adds a new vertex to the contractible K_2. In the simplicial complex picture, we would add an open cone $U = \{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ to $\{\{1\}, \{2\}, \{1,2\}\}$. One can then see $U = K_3 \setminus K_2$ as the complement of the closed set K_2 in K_3.

Examples:
1) The complete multipartite graph $K_{(2,2,2)}$ is the octahedron graph, a 2-sphere. Its Betti vector is $(1,0,1)$ like for all 2-spheres. There are Morse functions on the 2-sphere with exactly two critical points. One with index 0 (the minimum) and one with index 2, the maximum.
2) The complete multipartite graph $K_{(2,2,2,2)}$ is the smallest 3-dimensional sphere. Its Betti vector is $(1,0,0,1)$. Also here, there is always a Morse function which has exactly two critical points. This works in arbitrary dimensions and is related to a theorem of Reeb in the continuum [82].
3) A 2-torus can be realized as a graph with 9 vertices. Its Betti vector is $(1,2,1)$. There is a Morse function which has 4 critical point, one with index 0, two with index 1 and one with index 2.
4) A Klein bottle has the Betti vector is $(1,1,0)$. Like the Torus, its Euler characteristic is zero but it is non-orientable ($b_2 = 0$). There is no Morse function with 2 critical points. But there is a Morse function with 4 critical points.
5) The Betti vector of a projective plane is $(1,0,0)$. There is no Morse function with 1 critical point however. The minimal number of Morse critical points is 3.

12.3. We first look what happens with the cohomology if S is a sub-graph of a graph G that is a $(k-1)$-sphere and we make an extension $G \to G +_S v$ along S. There are two possibilities: either S is the boundary of a contractible graph B in G. In that case, we increase b_k because we add a new k-sphere. If S is not a boundary of a contractible part of G, it carries a cohomology (let the heat flow act to get a class). There are therefore two possibilities when attaching a k-ball along a $(k-1)$-sphere:

Lemma 4. *If a k ball $B = S + x$ (handle) is attached to a $(k-1)$ sphere S, then either b_k increases by 1 or b_{k-1} decreases by 1.*

Theorem 4 (Strong Morse inequalities)**.** *For any Morse function f on a graph, we have $b_k - b_{k-1} + b_{k-2} - \cdots + (-1)^k b_0 \leq c_k - c_{k-1} + c_{k-2} - \cdots + (-1)^k c_0$.*

Proof. For each fixed k, we use induction with respect to the number n of vertices in the graph. For $k = 0$, the inequality $b_0 \leq c_0$ could also be proven directly. The left hand side is the number of connectivity components the right the minimal number of minima which a Morse function can have.
Lets now look at the induction step with respect to the number of vertices. When adding a new vertex, b_0 can only change if the number of connected components changes. But then also c_0 changes. In general, c_0 the number of local minima can increase in the Morse build-up.
For $k = 1$, we have the inequality $b_0 - b_1 \leq c_0 - c_1$. It again is clear for one vertex. If we look at the Morse build-up for f, the numbers can change only at critical values. If a vertex v with

Morse index k is added, the Euler characteristic gets augmented by $(-1)^k$. Because we add a $(k+1)$-ball (handle) along a k-sphere, the cohomology changes.
The Euler-Poincaré formula and Poincaré-Hopf show that b_k can increase by 1 if c_k increases by 1 or that b_{k-1} decreases by 1. The later happens if we cover an existing open $(k-1)$-sphere with a k ball. The former happens if we cover an existing k-ball along the boundary sphere with a k-ball, forming a new non-contractible k-sphere. (see the lemma). In both cases, the inequalities remain valid. It is the second possibility which is the reason why we do not have equality in general. Every time we remove a cohomology class, the total (unsigned) sum $c = \sum_k c_k$ increases, while the total Betti number (unsighed) $\sum_k b_k$ decreases. □

12.4. By adding successive equations in this list, we get the **weak Morse inequalities** $b_k \leq c_k$. An other consequence is obtained for $k = d$, we have equality by the Poincaré-Hopf formula: $b_d - b_{d-1} + b_{d-2} - \cdots + (-1)^d b_0 = c_d - c_{d-1} + c_{d-2} - \cdots + (-1)^d c_0$ because both sides are then the Euler characteristic.

12.5. Examples:
1) For the 1-torus (circle), we have $b_0 = 1$, $b_1 = 1$, $c_0 = 1$, $c_1 = 1$. Lets take $G = C_4 = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{3,4\},\{4,2\}\}$ where we numbered the points so that $g(v) = v$ is a Morse function. Start with building up the graph $\{\{1\}\}$ which is a critical point of Morse index 0. When adding the points 2 and 3 we have homotopy extensions where $S^- g(v)$ is K_1 and so contractible. When adding the 4th point, we add a 1-handle to a 0-sphere $S = \{\{2\},\{3\}\}$.
2) If we take the C_4 from above and add an other 1-ball to $S = \{\{2\},\{3\}$, then b_1 is increased by 1 and the Betti vector becomes $(1,2)$.
3) We we take the $G = C_4$ from above an add a 2-ball to the 1-sphere $S = G$ then b_1 decreases by 1.
4) For the 2-torus we have $b_0 = 1$, $b_1 = 2$, $b_2 = 1$, $c_0 = 1$, $c_1 = 2$, $c_2 = 1$. The minimal number of critical points which a Morse function can have is 4.

13. Category Theorem

13.1. In this section, we look at the fundamental theorem of Lusternik-Schnirelmann theory. [15] write: *LS category is like a Picasso painting. Looking at category from different perspectives produces completely different impressions of category's beauty and applicability..*

13.2. Recall that cri(G) is the minimal number of critical points which a locally injective function g on G can have. This means that the function g does not necessarily have to be a Morse function. And cat(G) is the minimal number of contractible graphs which cover the graph, which means both vertices and edges of course as a spanning tree already covers all vertices. First to the lower bound:

Theorem 5. $\mathrm{cat}(G) \leq \mathrm{cri}(G)$.

Proof. We use induction with respect to the minimal number k of critical points of G. Without loss of generality, we can assume that the function g is injective. The reason is that a general locally injective function can be modified to be injective without changing the critical points. The function values of g now produce a total order on the vertices V and so a stratification, where G_j is the sub-graph of G generated by the vertices $\{v_1, \dots, v_j\}$.
Let us start with the induction foundation: if $k = 1$, then by definition, G is contractible, where the minimum is the only critical point. Because no further critical point occurs, all other extensions $G_j \to G_{j+1}$ are homotopy steps keeping G_{j+1} contractible if G_j is contractible.

Now lets look at the induction step. Lets assume that every graph with maximally k critical points can be covered with k contractible graphs. Now take a graph with $k+1$ critical points and that v_{m_k} is the largest critical point. Now all G_{m_k+j} are contractible until we reach $G_{m_{k+1}-1}$. The next expansion gives us $v_{m_{k+1}}$, the last critical point. As $G_{m_{k+1}-1}$ is contractible and the unit ball $B(v_{k+1})$ is contractible we could get over the $k+1$'th critical point by adding an other contractible graph. Now, there are no critical points any more and the rest of the extensions G_j for $j = m_{k+1} + 1$ to the last vertex n are all homotopy extensions not changing the number of contractible parts. □

13.3. One could also argue closer to the continuum: define the set S_k of sub-graphs of G of category larger or equal than k in M is not empty. We can assume that each graph in S_k is maximal in the sense that it is not a subgraph of a larger graph with the same category k. Take any enumeration function g. The value $c_k = \min_{S \in S_k} \max_{x \in S} g(x)$ is a critical value because if it were not, S would not be maximal and we could extend S without changing category. It follows that the number of critical points is bigger or equal than $\text{cat}(M)$.

13.4. It follows that if there is a function with only one critical point, then it produces a contractible graph. Lets move to the lower cohomological bound:

Theorem 6. $\text{cup}(G) + 1 \leq \text{cat}(G)$.

13.5. First a lemma:

Lemma 5. *Given any $p \geq 1$-form f and any contractible sub-graph U of G, there is a coboundary $h = dg$ such that $f - h$ is zero on U.*

Proof. Because the cohomology of U is trivial, every cocycle is a coboundary. That is, f restricted to U is a coboundary when we look at it as a k-form on U. This means $f|U = dg$,where $g|U$ means g restricted to U. So, $f - h = f - dg$ is zero on U. □

13.6. Now to the proof of the lower bound through cup length:

Proof. Assume that $\text{cup}(G) = k$. Take a maximal set of forms $f_1, \dots, f_k$, where f_j is a p_j-form with $p_j \geq 1$ and such that $f_1 \wedge f_2 \wedge \cdots \wedge f_k \neq 0$. We want to show that $\text{cap}(G) \geq k + 1$. Assume this is not true, then we could find sets $U_1, \dots, U_k$ that are contractible and cover G. By the above lemma we can deform each f_k by adding coboundaries such that $f_k \in H^1(G, U_k)$, meaning that that f_k are zero in U_k. But this means that the cup product of the f_k is zero. □

13.7. The following theorem is adapted from the continuum, where it applies to manifolds. Remember that **order** of an open cover is the least integer k such that there are $k+1$ members of the covering with non-zero intersection. The **topological dimension** was the smallest $k \geq 0$ such that for every open cover $\mathcal{U}$ of G, there is a refinement of order k. We have seen that the topological dimension of a graph agrees with the maximal dimension of the graph.

Theorem 7. *If G is a connected manifold then* $\text{cat}(G) \leq \dim(G) + 1$.

Proof. Let us denote the dimension with d. In order to prove the inequality we have to construct $(d + 1)$ contractible graphs which cover G. We use induction with respect to k. Let us look at the case $k = 1$. It can be covered with two linear graphs. As for the induction step: we can cover each unit sphere $S(x)$ with d contractible graphs of dimension one less. Let $U(x, i)$ denote these graphs. Now build a spanning tree T_i connecting all $\{U(v, i), v \in V\}$ and form the d graphs $U_i = T_i \cup \{U(v, i), v \in V\}$. These graphs are contractible and their union cover all unit spheres simultaneously. Additionally take an other spanning tree U_0, hitting all vertices. Also this is contractible. The union $U_0 \cup \{U_i\}_{i=1}^d$ covers the entire graph. ☐

13.8. The proof does not extend to general graphs. Already the induction foundation fails. For a one dimensional graph, the category is the **edge arboricity**, the minimal number of trees covering the graph. By the Nash-Williams theorem, this is bounded below by the smallest integer larger or equal than $(|E|-1)/|V|$. Lets take K_n with n vertices and $n(n-1)/2$ edges. Make a one dimensional refinement. Then we have $n+n(n-1)/2$ vertices and $n(n-1)$ edges. For one dimensional graphs without triangles, the arboricity and so category can already become arbitrarily large.

13.9. As mentioned, category is in graph theory close to **arboricity**. The **Nash-Williams theorem** gives the arboricity in terms of the maximal $|E_H|/(|V_H|-1)$ taken over the set of all **induced sub-graphs** H of G. We can also look at coverings by edges which is an **edge coloring problem**. The **edge chromatic number** or **edge index** is by Vizing's theorem between $\Delta(G)$ and $\Delta(G)+1$, where $\Delta(G)$ is the maximal vertex degree of the graph. We have worked on arboricity and chromatic number questions elsewhere [67, 69].

13.10. There are some higher dimensional discrete manifolds, for which we can compute the Lusternik-Schnirelmann category. Since $\text{cup}(G) = n$ for a n-dimensional torus, we have $\text{cat}(G) \geq n+1$. We need therefore at least $n+1$ contractible sub-graphs to cover G. To get the upper bound construct a function with $n+1$ critical points. We especially have $\text{cat}(T^2) \geq 3$. On the 2-torus one can get 3 critical points so that $\text{cri}(G) \leq 3$. The book [15] gives the example $f(x,y) = \sin(x)\sin(y)\sin(x+y)$ which has only 3 critical points. Can we explicitly construct a function with $n+1$ critical points on a discrete n torus?

13.11. Every tree, a connected simply connected graph has category 1. A connected graph $G = (V,E)$ without any triangles defines a one-dimensional simplicial complex $G = \{\{v\}, v \in V\} \cup \{\{(a,b),(a,b) \in E\}$. The Lusternik-Schnirelmann category of any such graph is 1 if it is simply connected and larger if it is not simply connected. Since the maximal dimension of G is 1, the cup -length is 0 or 1. What is the minimal number of critical points? For a small figure 8 graph, we can just brute force all functions and see that there are always at least 3 critical points.

13.12. It would be nice to get a **Morse cohomology** in the discrete. There is hope: if G is a graph and G_1 is the Barycentric refinement in which the complete sub-graphs are the vertices and two are connected if one is contained in the other, then $S^-(x)$ for $f(x) = \dim(x)$ is a Morse function on G_1 because $S_f^-(x)$ is a $k-1$-dimensional sphere if the dimension of x was k. This means that the Morse index is k and $i_g(x) = (-1)^{\text{ind}(x)} = \omega(x)$. So, $\chi(G_1) = \sum_x \omega(x)$ by Poincaré Hopf. But this is the definition of $\chi(G)$. We see $\chi(G) = \chi(G_1)$. In this case, $S^-(x) + S^+(x) = S(x)$, where $A+B$ is the **join** of two graphs A, B defined as $A+B = (V(A) \cup V(B), E(A) \cup E(B) \cup \{(a,b), a \in A, b \in B\})$. Every $v \in G_1$ is a critical point. The Morse cohomology boundary agrees now with the usual exterior derivative. $d(x,y) = \text{sign}(x,y)$ which is 1 if $y \subset x$ has dimension 1 less, and matches orientation, -1 if $y \subset x$ has dimension 1 less and does not match orientation and 0 else.

13.13. Here is a nice example showing that counting is a Morse buildup [9, 49] [10]. The set of integers $\{2, 3, \dots, n\}$ define a graph in which two nodes=integers are connected if one divides the other. We can see this graph as the Barycentric refinement of the **complete prime graph** P, Adding a new number is a Morse extension. Critical points are integers n that have no square prime factors. The Morse index is then the value of the Moebius function $-\mu(n)$. It follows from the Poincaré-Hopf formula that $\chi(G) = 1 - M(n)$ is the Euler characteristic, where $M(n)$ is the **Mertens function** $M(n) = \sum_{k=1}^{n} \mu(k)$.

13.14. We should point out **some variations** about Lusternik-Schnirelmann theory in the literature: classically, one sometimes uses the **reduced category** $\text{cat}(A) - 1$ or one then one uses uses $\text{cup}(G) + 1$ and calls this **augmented cup length** the **algebraic category**. Classically, Lusternik-Schnirelmann category is often silently defined only for connected spaces and with respect to an ambient space in that one defines $\text{cat}_G(A)$. The inequality $\text{cat}(G) \leq \dim(G) + 1$ for example only can hold for connected spaces. Example: take a zero-dimensional space with k connected components. This space has category k but is zero dimensional so that $\text{cat}(G) \leq \dim(G) + 1$ definitely can not hold. As for contractibility with respect to an ambient space or contractibility independent of a background, one classically defines $\text{cat}(A) = 1$ if the injection $i : A \to X$ is **homotopic to a constant**. A circle A in a sphere G with that definition has category $\text{cat}_G(A) = 1$, while a circle $\text{cat}_A(A) = 2$ in itself has category 2. The relative category $\mathbf{cat}_G(A)$ can also be defined for graphs by declaring it to be the minimal number of contractible sets U_i in G which cover A.

13.15. In order to push the dimension result $\text{cat}(A) \leq \dim(A) + 1$ to the discrete, we need a topology for which connectedness match the definitions for graphs. In 2017 we defined a Zariski type topology on the simplex set G where the **closed sets** are the simplicial complexes of sub-graphs A of G. The open sets includes the **void** $\emptyset$ which is also the simplicial complex of the empty graph and so also closed. [11] We here defined the graph topology as the topology with basis $B^+(x) = S^+(x) \cup \{x\}$ with $x \in G$. Closing it under intersections and union produces a **topology** on the set of simplices. The closed sets are generated by the basis $B^-(x) = S^-(x) \cup \{x\}$. In a cyclic graph C_n for example, a set $S^+(\{b\}) = \{\{b\}, \{a, b\}, \{b, c\}\}$ is an example of an open set as is $\{\{a, b\}\}$ the intersection of two such $B^+(x) \cap B^+(y)$ with $(x, y) \in E$. The set $S^-(\{a, b\}) = \{\{a\}, \{b\}, \{a, b\}\}$ is a closed set. The open basis elements play the role of open intervals the closed basis elements the role of closed intervals. Two different open sets intersect in an open set and two closed sets intersect in a closed set.

13.16. Let us take a locally injective function f and a critical point v for f on a general finite simple graph G. Let us assume that the index $i_g(v)$ of the vertex v is zero. This implies that $S_f^-(v)$ has Euler characteristic 1. Sometimes we can modify f locally without changing the nature of other critical points so that v becomes a regular point, sometimes we have an essential singularities and the singularity is not a removable singularities. Let us take the example of a large graph G which contains a vertex v on which $S(v)$ is a **dunce hat** and such that within G, we can make a homotopy deformation of $S(v)$ to a contractible graph. If we assume that f is a locally injective function on G such that v is a local maximum then v is by definition a critical point because $S(v)$ is not contractible. But it is also a removable critical point because we can modify f along the homotopy deformation of $S(v)$ to a contractible space to a function

[10]We found [9] only after writing [49]

[11]A simplicial complex does not contain the empty set. The axiom applies for the void $\emptyset$ as it does not contain the empty set.

g without introducing new critical points (we can assure this by making G large enough) such that g has now v as a regular point. In general this is not possible. Even if we take the cone $G = D + \{v\}$ over the dunce hat D (meaning that v is attached to D such that $S(v) = D$ in G), we have an essential singularity. An other example is the suspension G of a discrete projective plane P. If v is one of the two new vertices with $S(v) = P$ and f is a maximum on v then $S_f^-(v) = S(v) = P$ but no other function on G can render $S_f(v)$ contractible.

13.17. The minimal length of a non-contractible closed simple path in a metric space is known as a **systole**. [12] Gromov showed that for fixed d, all d-manifolds M satisfy $\text{sys}^d(M) \leq C\text{vol}(M)$ and showed also this is sharp as the Fubini-Study metric on $\mathbb{CP}^n$ gives $C = n!$. The **systole category** is defined as the longest product of systoles which give a curvature-free lower bound for the total curvature. This systole category is a lower bound for Lusternik-Schnirelmann category. In the set-up considered here, the systole category is smaller or equal than the cup length. **Gromov's systolic inequality** goes over to discrete d-manifolds. One can take as **volume** the number of maximal simplices. The discrete version of Gromov's inequality in graph theory would readily follow from the continuum by geometric realization. We are convinced that a finite proof would not be too difficult. For more information about the constant C see [34].

13.18. Morse theory has grown in the last decades even more. **Morse cohomology** is computationally often less complex than simplicial cohomology. The reason is that the differential complex is smaller and the matrices become smaller and more workable. Given a Morse function f, look at the vector space of all functions on the set of critical points $v \in V$ of a Morse function. Define $dg(p) = \sum_{q,\text{ind}(q)=\text{ind}(p)-1} i_g(p,q)g(q)$, where for $p, q \in V$, the index $i_f(p,q)$ is the signature of the intersection $W^+(p) \cap W^-(q)$. It is defined as $\sum_{x \in W^+(p) \cap W^-(q)} \text{sign}(x, W^+(p))\text{sign}(x, W^-(q))\omega(x)$ and where $\text{sign}(x, W^-(v))$ is 1 if the orientation of x agrees with the orientation of q propagated along $W^-(q)$ and (-1) else. One has to check $d^2 = 0$ to see that there is a cohomology, called **Morse cohomology**. In the case $g = \dim$ on a Barycentric refinement, the Morse cohomology agrees with simplicial cohomology because $W^+(p) \cap W^-(q) = \text{sign}(p, q)$.

13.19. The map which assigns to a signed graph G in the Shannon ring its cohomology ring is a functor from the category of signed graphs to the category of commutative rings. Under addition in the Shannon ring, we get the direct product of rings. Under multiplication in the Shannon ring, we get the tensor product of rings. The functor therefore extends to a ring homomorphism from the ring of graphs to a ring of cohomology rings of graphs. We should add to [50, 51, 55], that the finite topos of delta sets can naturally be extended to an associative, commutative ring with 1. Introducing "negative space" can be done in canonical way. The disjoin union which is the coproduct in the category defines a monoid and so extends naturally to a group. Every ring element now can be written as $A - B$, where A, B are topos elements. The product extends. Euler characteristic and cohomology extends, where the Betti numbers just flip signs. Notions like Poincaré-Hopf index or curvature extend.

[12] For 1-dimensional complexes, this is known as **girth**. With that definition, if there is triangle present, the girth would be 3. For us, a triangle is contractible so that its systole is 0, the systole of a d-sphere 0 for $d > 1$.

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References

[1] A. Aczel. *Descartes's secret notebook, a true tale of Mathematics, Mysticism and the Quest to Understand the Universe*. Broadway Books, 2005.

[2] P. Alexandroff. Diskrete Räume. *Mat. Sb. 2*, 2, 1937.

[3] P. Alexandroff and H. Hopf. *Topologie*, volume 45 of *Grundlehren der Mathematischen Wissenschaften*. Springer, 1975.

[4] S. Ammons. A Discrete Analogue for the Poincare-Hopf Theorem. HMC Senior Theses. 237, 2020.

[5] M. Baker and S. Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Advances in Mathematics*, 215:766–788, 2007.

[6] T. Banchoff. Critical points and curvature for embedded polyhedra. *J. Differential Geometry*, 1:245–256, 1967.

[7] T. Banchoff. Critical points and curvature for embedded polyhedral surfaces. *Amer. Math. Monthly*, 77:475–485, 1970.

[8] K. Betre and E. Salinger. The knill graph dimension from the minimum clique cover. https://arxiv.org/abs/1903.02523, 2019.

[9] A. Björner. A cell complex in number theory. *Advances in Appl. Math.*, 46:71–85, 2011.

[10] W. Blaschke. *Vorlesungen über Integralgeometrie*. Chelsea Publishing Company, New York, 1949.

[11] R. Bott. Morse theory indomitable. *Publications mathématiques de l'I.H.E.S*, 68:99–114, 1988.

[12] G. Burde and H. Zieschang. Development of the concept of a complex. In *History of Topology*. Elsevier, 1999.

[13] B. Chen, S-T. Yau, and Y-N. Yeh. Graph homotopy and Graham homotopy. *Discrete Math.*, 241(1-3):153–170, 2001. Selected papers in honor of Helge Tverberg.

[14] A.V. Chernavsky and V.P. Leksine. Unrecognizability of manifolds. *Ann. Pure Appl. Logic*, 141:325–335, 2006.

[15] O. Cornea, G. Lupton, J. Oprea, and D. Tanré. *Lusternik-Schnirelmann Category*. AMS, 2003.

[16] H.L. Cycon, R.G.Froese, W.Kirsch, and B.Simon. *Schrödinger Operators—with Application to Quantum Mechanics and Global Geometry*. Springer-Verlag, 1987.

[17] M. Dehn and P. Heegaard. Analysis situs. *Enzyklopaedie d. Math. Wiss*, III.1.1:153–220, 1907.

[18] V. Eberhard. *Morphologie der Polyeder*. Teubner Verlag, 1891.

[19] A.V. Evako. Dimension on discrete spaces. *Internat. J. Theoret. Phys.*, 33(7):1553–1568, 1994.

[20] R. Forman. Morse theory for cell complexes. *Adv. Math.*, page 90, 1998.

[21] R. Forman. A user's guide to discrete Morse theory. *Séminaire Lotharingien de Combinatoire*, 48, 2002.

[22] R. Fox. On the Lusternik-Schnirelmann category. *Ann. of Math. (2)*, 42:333–370, 1941.

[23] J. Y. Gao. Topology in graph theory: A new perspective on classical concepts. Senior Thesis, Harvard University, 2024.

[24] L. Glass. A combinatorial analog of the Poincare index theorem. *Journal of combinatorial theory*, 15:264–268, 1973.

[25] A. Gray. *Tubes*. Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1990.

[26] J.L. Gross and T.W. Tucker. *Topological Graph Theory*. John Wiley and Sons, 1987.

[27] B. Grünbaum. *Convex Polytopes*. Springer, 2003.

[28] H. Hopf. Über die Curvatura integra geschlossener Hyperflaechen. *Mathematische Annalen*, 95:340–367, 1926.

[29] A. Ivashchenko. Contractible transformations do not change the homology groups of graphs. *Discrete Math.*, 126(1-3):159–170, 1994.

[30] A. V. Ivashchenko. Representation of smooth surfaces by graphs. transformations of graphs which do not change the euler characteristic of graphs. *Discrete Math.*, 122:219–233, 1993.

[31] A.V. Ivashchenko. Graphs of spheres and tori. *Discrete Math.*, 128(1-3):247–255, 1994.

[32] F. Josellis and O. Knill. A Lusternik-Schnirelmann theorem for graphs. http://arxiv.org/abs/1211.0750, 2012.

[33] M. Joswig, D.Lofano, and M. Tsuruga F.H.Lutz. Frontiers of sphere recognition in practice. *Journal of Applied and Computational Topology*, 6:503–527, 2022.

[34] M.G. Katz. *Systolic Geometry and Topology*, volume 137 of *Mathematical Surveys and Monographs*. AMS, 2017.

[35] D.A. Klain and G-C. Rota. *Introduction to geometric probability.* Lezioni Lincee. Accademia nazionale dei lincei, 1997.
[36] O. Knill. A graph theoretical Gauss-Bonnet-Chern theorem. http://arxiv.org/abs/1111.5395, 2011.
[37] O. Knill. A discrete Gauss-Bonnet type theorem. *Elemente der Mathematik*, 67:1–17, 2012.
[38] O. Knill. A graph theoretical Poincaré-Hopf theorem. http://arxiv.org/abs/1201.1162, 2012.
[39] O. Knill. An index formula for simple graphs . http://arxiv.org/abs/1205.0306, 2012.
[40] O. Knill. On index expectation and curvature for networks. http://arxiv.org/abs/1202.4514, 2012.
[41] O. Knill. The theorems of Green-Stokes,Gauss-Bonnet and Poincare-Hopf in Graph Theory. http://arxiv.org/abs/1201.6049, 2012.
[42] O. Knill. The Dirac operator of a graph. http://arxiv.org/abs/1306.2166, 2013.
[43] O. Knill. The Euler characteristic of an even-dimensional graph. http://arxiv.org/abs/1307.3809, 2013.
[44] O. Knill. Classical mathematical structures within topological graph theory. http://arxiv.org/abs/1402.2029, 2014.
[45] O. Knill. Coloring graphs using topology. http://arxiv.org/abs/1410.3173, 2014.
[46] O. Knill. A notion of graph homeomorphism. http://arxiv.org/abs/1401.2819, 2014.
[47] O. Knill. The Künneth formula for graphs. http://arxiv.org/abs/1505.07518, 2015.
[48] O. Knill. A Sard theorem for graph theory. http://arxiv.org/abs/1508.05657, 2015.
[49] O. Knill. On primes, graphs and cohomology. https://arxiv.org/abs/1608.06877, 2016.
[50] O. Knill. On the arithmetic of graphs. https://arxiv.org/abs/1706.05767, 2017.
[51] O. Knill. The strong ring of simplicial complexes. https://arxiv.org/abs/1708.01778, 2017.
[52] O. Knill. The amazing world of simplicial complexes. https://arxiv.org/abs/1804.08211, 2018.
[53] O. Knill. Constant index expectation curvature for graphs or Riemannian manifolds. https://arxiv.org/abs/1912.11315, 2019.
[54] O. Knill. Energized simplicial complexes. https://arxiv.org/abs/1908.06563, 2019.
[55] O. Knill. More on numbers and graphs. https://arxiv.org/abs/1905.13387, 2019.
[56] O. Knill. More on Poincaré-Hopf and Gauss-Bonnet. https://arxiv.org/abs/1912.00577, 2019.
[57] O. Knill. A parametrized Poincare-Hopf theorem and clique cardinalities of graphs. https://arxiv.org/abs/1906.06611, 2019.
[58] O. Knill. Poincaré-Hopf for vector fields on graphs. https://arxiv.org/abs/1911.04208, 2019.
[59] O. Knill. Division algebra valued energized simplicial complexes. https://arxiv.org/abs/2008.10176, 2020.
[60] O. Knill. The energy of a simplicial complex. *Linear Algebra and its Applications*, 600:96–129, 2020.
[61] O. Knill. Green functions of energized complexes. https://arxiv.org/abs/2010.09152, 2020.
[62] O. Knill. Integral geometric Hopf conjectures. https://arxiv.org/abs/2001.01398, 2020.
[63] O. Knill. On index expectation curvature for manifolds. https://arxiv.org/abs/2001.06925, 2020.
[64] O. Knill. Eigenvalue bounds of the Kirchhoff Laplacian. https://arxiv.org/abs/2205.10968, 2022.
[65] O. Knill. Discrete algebraic sets in discrete manifolds. https://arxiv.org/abs/2312.14671, 2023.
[66] O. Knill. Finite topologies for finite geometries. https://arxiv.org/abs/2301.03156, 2023.
[67] O. Knill. On the arboricity of manifolds. https://arxiv.org/abs/2309.01869, 2023.

[68] O. Knill. The sphere formula. https://arxiv.org/abs/2301.05736, 2023.
[69] O. Knill. The three tree theorem. https://arxiv.org/abs/2309.01869, 2023.
[70] H. W. Kuhn and S. Nasar. *The Essential Nash*. Princeton University Press, 2002.
[71] H. Künneth. Über die Bettischen Zahlen einer Produktmannigfaltigkeit. *Math. Ann.*, 90(1-2):65–85, 1923.
[72] I. Lakatos. *Proofs and Refutations*. Cambridge University Press, 1976.
[73] N. Levitt. The Euler characteristic is the unique locally determined numerical homotopy invariant of finite complexes. *Discrete Comput. Geom.*, 7:59–67, 1992.
[74] Y. Matsumoto. *An introduction to Morse theory*. AMS, 2002.
[75] J.P. May. Finite topological spaces. Notes for REU, Chicago, 2003-2008, 2008.
[76] J.P. May and E. Pishevar. Finite spaces and larger contexts. Book project, Last accessed 2024.
[77] A.P. Morse. The behavior of a function on its critical set. *Ann. of Math. (2)*, 40(1):62–70, 1939.
[78] C.St.J. A. Nash-Williams. Decomposition of finite graphs into forests. *Journal of the London Mathematical Society*, 39:12–12, 1964.
[79] L. Nicolaescu. *An invitation to Morse Theory*. Springer, 2018.
[80] P.S. Novikov. On the algorithmic insolvability of the word problem in group theory. *Izdat. Akad. Nauk SSSR Moscow: Trudy Mat, p. 44. Inst, Steklov.*, 1955.
[81] H. Poincaré. Sur les courbes definies par les equation differentielle III. *Journal de Mathematique pures et appliquées*, pages 167–244, 1885.
[82] G. Reeb. Sur certaines proprietes topologiques des varieties feuilletes. *Publ.Inst.Math.Univ.Strasbourg*, pages 91–154, 1952.
[83] D.S. Richeson. *Euler's Gem*. Princeton University Press, Princeton, NJ, 2008. The polyhedron formula and the birth of topology.
[84] L.A. Santalo. *Introduction to integral geometry*. Hermann and Editeurs, Paris, 1953.
[85] L.A. Santalo. *Integral Geometry and Geometric Probability*. Cambridge University Press, second edition, 2004.
[86] R. Schneider. *Integral geometric tools for stochastic geometry*, volume 1892 of *Lecture Notes in Math.* Springer, 2007.
[87] S.Eilenberg and J. Zilber. On products of complexes. *Amer. J. Math.*, 75:200–204, 1953.
[88] E.H. Spanier. *Algebraic Topology*. Springer Verlag, 1966.
[89] M. Spivak. *A comprehensive Introduction to Differential Geometry I-V*. Publish or Perish, Inc, Berkeley, third edition, 1999.
[90] H. Weyl. On the volume of tubes. *American Journal of Mathematics*, 61:461–472, 1939.
[91] J.H.C. Whitehead. Simplicial spaces, nuclei and m-groups. *Proc. London Math. Soc.*, 45(1):243–327, 1939.
[92] Hassler Whitney. *Moscow 1935: Topology Moving Toward America*, pages 1–21. Birkhäuser Boston, 1992.
[93] G.M. Ziegler. *Lectures on Polytopes*. Springer Verlag, 1995.

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