ON FREDHOLM DETERMINANTS IN TOPOLOGY

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Abstract. For a finite simple graph G with adjacency matrix A, the Fredholm determinant $\zeta(G) = \det(1 + A)$ is $1/\zeta_G(-1)$ for the Bowen-Lanford zeta function $\zeta_G(z) = \det(1 - zA)^{-1}$ of the graph. The connection graph G' of G is a new graph which has as vertices the set V' of all complete subgraphs of G and where two such complete subgraphs are connected, if they have a non-empty intersection. More generally, the connection graph of an abstract finite simplicial complex or even CW complex G has as vertices the simplices or cells in G, where two are connected if they intersect. We prove that for any G, the Fredholm characteristic $\psi(G) = \zeta(G')$ is equal to the Fermi characteristic $\phi(G) = (-1)^{f(G)}$, where $f(G)$ is the number of odd dimensional cells in G; the functional $f(G)$ is a valuation for which Poincaré-Hopf and Gauss-Bonnet formulas hold. Given $\omega(x) = (-1)^{\dim(x)}$, we can see the Fermi characteristic $\phi(G) = \prod_x \omega(x)$ as a cousin of the Euler characteristic $\chi(G) = \sum_x \omega(x)$ which sums the signatures of simplices. The main result is the unimodularity theorem $\psi(G) = \phi(G)$ which relates an algebraic and a combinatorial quantity. We illustrate this with prime graphs, where $\omega(x) = -\mu(x)$ is the Möbius function of an integer. A key proposition for the proof of the theorem is that if $i(x) = 1-\chi(S(x))$ is the Poincaré-Hopf index of x, where $S(x)$ is the unit sphere of x, then $\psi(G \cup \{x\}) = i(x)\psi(G)$. If $S(x)$ is a graph theoretical sphere, then $i(x) \in \{-1, 1\}$ proving inductively that ψ is $\{-1, 1\}$-valued. The unimodularity theorem follows then by induction by building up the simplicial complex cell by cell, using that spheres have Euler characteristic 0 or 2. Having established that the Fredholm matrix $1+A(G')$ of a simplicial complex G is unimodular, the entries of the Green function $g_{ij} = [(1 + A(G'))^{-1}]_{ij}$ are always integers. They appear of interest as experiments indicate that the range of g is a combinatorial invariant for G: we conjecture that global or local Barycentric refinements of simplicial complexes G do not change the range of g.

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1. The Fredholm determinant of a graph

Fredholm matrices appear naturally in graph theory. They arise most prominently in the **Chebotarev-Shamis forest theorem** [19, 20] which tells that $\det(1+L)$ is the number of rooted forests in a graph G, if L is the Kirchhoff Laplacian of G. This forest theorem follows readily from the **generalized Cauchy-Binet formula** [9] $\det(1 + F^T G) = \sum_P \det(F_P)\det(G_P)$ which holds for any pair F, G of $n \times m$ matrices and were the right hand side is a dot product of the minor vector giving all possible minors of the matrix, defined by the index set P which can be empty in which case $\det(A_P) = 1$. The forest theorem uses this with $L = d^T d$, where $d =$ grad and $d^T =$ div are Poincaré's incidence matrices. The Fredholm determinant $\det(1 + L)$ is then $\sum_P \det(d_P)^2$, which directly counts rooted forests.

Similarly, any Fredholm determinant $\det(1 + A)$ of a matrix A can be written as $\det(1 + A) = \sum_P \det(A_P)\det(1_P)$ which implies the well known formula $\det(1 + A) = \sum_{k=0}^{\infty} \mathrm{tr}(\Lambda^k(A))$ where $\Lambda^k(A)$ is the k'th exterior power of A. That expansion is at the heart of extending determinants to Fredholm determinants in infinite dimensions, in particular if A is trace class [22]. In this paper, we look at the adjacency matrix A, which unlike the Laplacian L of a graph is not positive semi-definite so that one can not write $A = F^T F$ for some other matrix. Indeed, the determinant of an adjacency matrix $A(G)$ of a graph G appears rather arbitrary in the following strong sense: experiments show that a limiting distribution emerges when looking at the random variables $X(G) = \det(A(G))$ on probability spaces of graphs. See Figure (12).

Given a finite simple graph G with adjacency matrix A, we call $1 + A$ the **Fredholm adjacency matrix** and $\zeta(G) = \det(1 + A)$ the **Fredholm determinant** of G. The Fredholm determinant of a graph can be pretty arbitrary as the following examples show: for complete graphs $G = K_d$, where the eigenvalues of the adjacency matrix are -1 with multiplicity n and n with multiplicity 1 and the Fredholm determinant is 0 for $d > 1$. For cyclic graphs C_n, the Fredholm determinant is 6-periodic in n, zero for $n = 6k$, and 3 for $n = 2k$ not divisible by 3 and -3 for $n = 2k + 1$ not divisible by 6. For wheel graphs W_n with $n + 1$ vertices, it is $(n - 3)(-1)^n$ if n is not divisible by 3 and 0 else. Measuring the statistical distribution of Fredholm determinant on classes of random graphs suggests that the normalized distribution of the adjacency or Fredholm determinants produces an absolutely continuous limit with singularity. Also this appears not yet explored theoretically but it illustrates that the Fredholm determinant of a general graph can be pretty arbitrary.

We can think of $\det(A)$ as a partition function or a "path integral", in which the underlying paths are fixed-point-free permutations of the vertices of the graph. The determinant generates therefore **derangements** and $x \to \pi(x) \to \pi(\pi(x)) \dots$ defines the oriented paths. On the other hand, the Fredholm determinant $\zeta(G) = \det(1 + A)$ is a partition function for all oriented paths in the graph as $x \to \pi(x)$ can now also have pairs $(a, b) \in E$ as transpositions and vertices $v \in v$ as fixed points. One can see the effect of changing from determinants to Fredholm determinants well when replacing the determinants with the **permanent**, the Bosonic analogue of the determinant: $\mathrm{per}(A)$ is the number **derangements** of the vertex set of the graph while $\mathrm{per}(1 + A)$ is the number of all **permutations** of the vertex set honoring the connections. For the complete graphs $G = K_n$ in particular, $\mathrm{per}(1 + A(K_n))$ generates the **permutation sequence** $1, 2, 6, 24, 120, 720, \dots$ while the permanent of the adjacency matrix $\mathrm{per}(A(K_n))$ generates the **derangement sequence** $0, 1, 2, 9, 44, 265, \dots$. It is therefore not surprising that Fredholm determinants are natural.

The distribution of Fredholm determinants changes drastically if we evaluate them on the set of **connection graphs**, graphs which have the set of simplices of a graph G as subgraphs and where two simplices are connected if they intersect. Connection graphs are also defined for abstract finite simplicial complexes or even finite CW complexes. If we talk about a graph G, we usually understand it equipped with the Whitney complex, the set of complete subgraphs of G. But any simplicial complex structure or CW complex structure on the graph works. It does not even have to come from simplices. The **graphic matroid** of a graph is an example where the connection graph has forests in G as vertices and has two forests connected if some trees in it share a common branch.

For the smaller **Barycentric refinement** G_1 of G, two simplices are connected only if and only if one is contained in the other. The graph G_1 has the same vertices than G' but is a subgraph of G'. Connection graphs are in general much higher dimensional than the graph G or even the Barycentric refinement G_1: for a triangle $G = K_3$ already, G' a graph which contains the complete graph K_4. Small spheres like the octahedron are examples where G and G' are not homotopic because G' has Euler characteristic 0 while the octahedron, as a 2-sphere has

Euler characteristic 2. However this only is the case because the sphere is too small. For the icosahedron G already, the connection graph G' a two sphere again. In general, the Barycentric refinement G_1 of a graph G has a connection graph G_1' which is homotopic to G_1 and so to G. This implies then that all cohomology groups of G and G_1' agree. While G and G_1 are not homeomorphic as already their dimension is different in general, they can be useful in geometry, like Barycentric refinements. They can be used for example to **regularize singularities** as they "homotopically fatten" the " discrete manifolds" or simplicial complexes and still have the same homotopy type after applying one Barycentric refinement.

We got interested in connection graphs in the context of "connection calculus", a calculus where differential forms are not functions on simplices but on pairs or k-tuples of connecting simplices in the simplicial complex. The corresponding cohomology is compatible with calculus in the sense that common theorems like Gauss-Bonnet [6], Poincaré-Hopf [7], Euler-Poincaré or Kuenneth [11] or Brouwer-Lefschetz fixed point theorem [8] generalize when Euler characteristic is replaced by **Wu characteristic** but for which the cohomology is finer. The cohomology already allows to distinguish spaces which classical simplicial cohomology can not, like the Möbius strip and the cylinder [15].

The Bowen-Lanford **zeta function** [3] of a graph G with adjacency matrix A is defined as the complex function $\zeta(z) = 1/\det(1 - zA)$, from $\mathbb{C}$ to $\mathbb{C}$, where A is the adjacency matrix of A. If r is the spectral radius of A, the absolute value of the largest eigenvalue of A, then the function ζ is analytic in $|z| < 1/r$. The Fredholm determinant of A is then $1/\zeta(-1)$, which if -1 is an eigenvalue of A is defined as 0. Zeta functions are of interest as they relate with topology. We have a Taylor expansion $\zeta(z) = \exp(\sum_{k=1}^{\infty}(N_k/k)z^k)$ for small $|z|$, where N_k is the number of rooted closed paths of length k in the graph. The zeta function is therefore a generating function for a dynamical property of the graph, the dynamical system being the Markov chain defined by the graph. It is in particular an **Artin-Mazur zeta function** and a special case of the **Ruelle zeta function** [21]. Since $\zeta(z)$ is a rational function, the sum can be understood for general z by analytic continuation. While for general graphs, $\zeta(-1)$ can be quite arbitrary, we will see that for connection graphs G', the analytic continuation of the divergent series $\sum_{k=1}^{\infty}(N_k/k)z^k$ at $z = -1$ is either 0 or πi and that the case 0 appears if and only if there is an even number of odd-dimensional complete subgraphs of the original graph G. If $v_k(G)$

is the number of k-dimensional simplices, then the Euler characteristic $\chi(G) = \sum_{k=0}^{\infty}(-1)^k v_k(G)$ is the difference of $b(G) - f(G)$, where $b(G) = \sum_{k=0}^{\infty} v_{2k}(G)$ and $f(G) = \sum_{k=0}^{\infty} v_{2k+1}(G)$.

While these cardinalities $\psi(G)$ appear naturally when evaluating the zeta function of a connection graph at $z = -1$, concrete examples of zeta function of a connection graph of some of the simplest graphs indicate, that the result $\zeta_{G'}(-1) \in \{-1, 1\}$ is not that obvious: $\zeta_{K_1'}(z) = -1/z$, $\zeta_{K_2'}(z_) = -1/(z^3 - 2z)$, $\zeta'_{K_3}(z) = 1/(-z^7 + 15z^5 + 26z^4 - 3z^3 - 24z^2 - 2z + 6)$. $\zeta'_{C_5}(z) = 1/(z^{10} - 15z^8 - 10z^7 + 70z^6 + 78z^5 - 100z^4 - 160z^3 - 15z^2 + 30z - 4)$. These functions evaluated at -1 either have the value 1 or -1. By the way, the topic of Fredholm determinants has appeared in the movie "Good will hunting" as one of the problems involves the Bowen-Lanford function: the last blackboard problem in that movie asks for the generating function for walks from a vertex i to a vertex j in concrete graph. The answer is $[(1 - zA)^{-1}]_{ij}$ which by Cramer is expressed by the **adjugate matrix** as the rational function $\det(1 - zA(j,i))/\det(1 - zA) = \zeta(z)\det(1 - z(-1)^{i+j}A(i,j))$, where $A(i,j)$ is the matrix obtained by deleting row i and column j in A.

In search of a proof of the theorem, it can be helpful to see the connection graph as a geometric space and see a permutation π of its vertices as a one-dimensional oriented "submanifold", a collection of disjoint cyclic oriented paths or "strings". Since a permutation compatible with the graph as a "flow" on the geometry, the Fredholm determinant sums over all possible "measurable continuous dynamical systems T". They can be considered flows in G' in the sense that for every vertex x, the pair $(x, T(x))$ is an edge in G'. We call them measurable because T is not continuous in the geodesic distance metric of G'.

The signature $\omega(\pi)$ of a flow is the product of the signatures of the individual connected cyclic components of the flow=permutation π. The Fredholm determinant $\psi(G) = \det(1 + A')$ of the adjacency matrix A' of the connection graph G' is then a **path integral** $\psi(G) = \sum_{\pi} \omega(\pi)$, where π runs over all possible flows in G'. The unimodularity theorem tells then that the Fredholm determinant $\psi(G)$ is equal to the **Fermi characteristic** $\phi(G) = \prod_x \omega(x)$, where x runs over all complete subgraphs of G, showing so that ψ is a **multiplicative valuation** $\psi(G \cup F) = \psi(G)\psi(H)/\psi(G \cap H)$. We can compare $\psi(G)$ with the additive valuation $\chi(G)$ on graphs which is the **Euler characteristic** $\chi(G) = \sum_x \omega(x)$ or with the **Wu characteristic** $\omega(G) =$

$\sum_{x \sim y} \omega(x)\omega(y)$, summing over all edges (x, y) of the connection graph G' [24, 12]. But unlike Euler characteristic χ or Wu characteristic ω, the functional ψ is not a combinatorial invariant, as ψ is constant 1 on Barycentric refinements.

Functionals like the range of the unimodular Green function $g_{ij}[(1 + A')^{-1}]_{ij}$ values appear to be combinatorial invariants - at least in experiments. We have not proven this observation yet. The closest analogy which comes to mind is an invariant found by Bott [2] who coined the term **combinatorial invariant** as a quantity which is invariant under Barycentric subdivision. By Cramer, $g_{ii}\psi(G)$ is the Fredholm characteristic of the geometry in which cell i is removed and $g_{ij}\psi(G)$ a Fredholm characteristic of a geometry, where outgoing connections from cell i and incoming connections to cell j are snapped.

2. Connection graphs

If $G = (V, E)$ is a finite simple graph, we denote by V_1 the set of all complete subgraphs of G. Also named **simplices** or **cliques**, these subgraphs are points of the **Barycentric refinement** G_1 of G, which has as a vertex set V_1 the set simplices and where two such simplices are connected if one is contained in the other. The larger **connection graph** G' has the same vertex set like G_1. In that graph, two simplices are connected, if they have a non-empty intersection. Unlike for G_1, for which the maximal dimension of G and G_1 are the same, the graph G' is in general “fatter”: for a one-dimensional circular graph for example, the graph G' has triangles attached to each edge. More generally, G' contains complete subgraphs K_{n+1} if there is a vertex of x which is contained in n simplices: the unit ball of a simplex $x' = (x)$ belonging to an original vertex is a complete graph.

Also if we primarily want to analyze the graph case, it is convenient to look at more general simplicial complex structures on the graph. Assume G is an **abstract finite simplicial complex**, a finite set V equipped with a collection V' of finite non-empty subsets of V such that for every $A \in V'$ and every subset B of A also $B \in V'$, then the connection graph $G' = (V', E')$ is the finite simple graph for which two elements in V' are connected, if they intersect. An abstract simplicial complex is not only a generalization of a graph, we can see it as a **structure** imposed on a graph similarly as a **topology**, an **order structure** or **σ-algebra** is imposed on a set. Much of the graph theory literature sees a graph $G = (V, E)$ by default equipped with the

one-dimensional **skeleton complex** $V \cup E$. The largest complex is the **Whitney complex** on G, which is the set of all complete subgraphs. Graphs can handle many simplicial complexes as given an abstract finite simplicial complex G, the Barycentric refinement $G_1 = (V_1, E_1)$ is a graph which has as vertex set V_1 the set of elements in G and has $E_1 = \{(a, b) \mid a \subset b \text{ or } b \subset a\}$.
Given an abstract finite simplicial complex G given by a finite set V equipped with a collection V' of finite subsets, the **connection graph** of G is the graph with vertex set V', where two vertices x, y are connected, if they intersect as subsets of V.

More general than simplicial complexes are discrete **CW complexes**. This structure is built up inductively. It recursively defines also the notion of contractibility and a notion of sphere in this structure. This generalizes the Evako setup in the graph case (see [10]) Start with the empty set, which is declared to be the (-1)-sphere. It does not contain any cells. A **CW-complex** is declared to be a d-sphere if when punctured becomes contractible and which has the property that every unit sphere $S(x)$ of a $(d-1)$ sphere. The **unit sphere** of a cell x is the CW-sub complex of G containing all cells which are either part of x or which contain x. Also inductively, a CW complex G is **contractible** if there exists a cell x such that both $S(x)$ and G without x are contractible. Inductively, if G is a CW-complex one can build a larger complex by choosing a sphere H in G, then do an extension over H with a new cell x, producing so a larger CW complex. The unit ball of x has $S(x) = H$ as a boundary and the new cell is declared to be $1 + \dim(H)$. Starting with the empty set, one can build up like this structures which are more general than finite simplicial complexes but which still do not (unlike the classical definition of CW complexes) invoke the infinity axiom in Zermelo-Frenkel. The **connection graph** of a CW complex is the finite simple graph (V, E), where V is the set of cells and where two cells are connected if they intersect. The Fermi characteristic and Fredholm characteristic of a CW complex are defined in the same way as before.

Remarks.
1) If one looks at the unit balls of cells as a "cover" of the CW complex, then the connection graph plays the role of the **nerve graph** in Čech setups. The elements in the cover have however more structure because their boundaries are always graph theoretical spheres. Every simplicial subcomplex of the Whitney graph is a CW complex in the just given sense and any finite classical CW complex can be described

combinatorially as such. There is a more general notion of **abstract polytope** given as a poset satisfying some axioms but the unimodularity theorem won't generalize to that. The **Barycentric refinement** of a CW complex is a graph containing the cells as vertices and where two cells are connected if one is contained in the other. This graph is equipped with the Whitney complex structure which is again a CW complex, but which is much larger. But again, like for the notion of simplicial complex, we see that it can be implemented as a graph and that we can see therefore a CW complex as a **structure imposed on a graph**. There is still a reason to keep the notion of CW complexes: the connection graph of a CW complex has the same number of vertices than cells and the unimodularity theorem applies to it. The connection graph of the graph attached to the CW complex would be much larger. This will be relevant when we look at prime graphs. It is also relevant in general: if we look at a Barycentric refinement of a CW complex, then it is a graph which has the same number of vertices than the connection graph of this CW complex. The prime graph and prime connection graph considered below are then a Morse filtration of the Barycentric refinement and Connection graphs of the simplest simplicial complex one can imagine: the set of primes equipped with the set of all subsets as complex.

2) As the Barycentric refinement of a simplicial complex or CW complex encodes most essential topological features in G, there is not much loss of generality by looking at graphs rather than simplicial complexes. Still, the slightly increased generality can make the result more transparent. But applying it to graphs is more intuitive. The structure of a CW complex is very natural and practical: look at the cube graph for example. Since the unit spheres are graphs without edges, it is a one-dimensional graph when equipped with the Whitney complex. If we stellate the 6 faces, then we have a larger graph with 14 edges, the stellated cube. Its Whitney complex is large as there are already 24 triangles present. But adding 6 cells with C_4 boundaries, we get the familiar picture of a cube with f=6 two dimensional faces, e=12 one dimensional edges and v=8 vertices. This is how already Descartes counted the Euler characteristic $v - e + f = 2$ [1]. History shows [17] how difficult it has been to get to a good notion of "polyhedron" (see also [4]) and one usually refers to Euclidean embeddings, using through notions like convexity to define it properly [5].

3) The notion of CW complex in the discrete allows (without using any Euclidean notions) to give a decent definition of polyhedron as a

CW complex which is a sphere in the sense that removing one cell renders the CW complex contractible and that every unit sphere of any cell is a sphere. The essential foundation to that is in Whitehead [25] already in the 1930ies. What was new in the 90ies is the realization that one can do all this on graphs without using the continuum. There are three reasons why the language of finite graphs is more convenient: it is an intuitive structure which small kids can grasp already; it is a data structure which exists in many higher level programming languages. All results discussed here can be explored with a few lines of code (provided below). Finally, it is a finite structure; finite mathematics works also in a framework of finitists like Brouwer or strict finitist, which many computer scientists are, wanting to implement the complete structure faithfully.

4) Whitehead **homotopy** has been ported to discrete structures by first defining **contractibility** inductively: it is either the 1 vertex graph or a graph for which there exists a vertex x such that both the unit sphere $S(x)$ is contractible and such that the graph without x is contractible. Contractible graphs have Euler characteristic 1. A **homotopy step** is the process of an addition or removal of a vertex x, for which $S(x)$ is contractible. Two graphs are homotopic, if one can get one from the other by applying a sequence of homotopy steps. A graph can be homotopic to a 1-point graph without being contractible. An example is the **dunce hat** which shows that one first has to enlarge the graph before it becomes contractible. Homotopic graphs have the same cohomology and Euler characteristic however.
While the Barycentric refinement G_1 is homotopic to G, G' is not homotopic to G in general but this happens only if **very small** homotopically non-trivial spheres are present. For the octahedron G for example, the graph G' has Euler characteristic 0 while G has Euler characteristic 2 so that G and G' can not be homotopic. This only happened because the geometry was too small and the H^2 cohomology of G collapsed in G'. For smooth enough graphs, G' is homotopic to G it is homotopic and has the same cohomology. The connection graph of the Barycentric refinement G_2 of G_1 for example is always homotopic to G_1 and so to G as loosing the additional connection bonds does allows to morph from G_1' to G_2 without changing the topology.

5) The connection graphs emerged for us in the context of the **Wu characteristic**

$$\omega(G) = \sum_{(x,y) \in E'} (-1)^{\dim(x)+\dim(y)}$$

of a graph G which is a "second order" Euler characteristic. The Wu characteristic shares all important properties of Euler characteristic: it is multiplicative and additive with respect to multiplication or addition of the geometric structures; there is a compatible calculus, cohomology and theorems like Gauss-Bonnet, Poincaré-Hopf, Euler-Poincaré, Kuenneth or Lefschetz generalize. A relation of the Fredholm adjacency matrix and the Wu characteristic is given by

$$\omega(G) = \text{tr}((1 + A')J) ,$$

where J is the **checkerboard matrix** $J_{ij} = (-1)^{i+j}$.

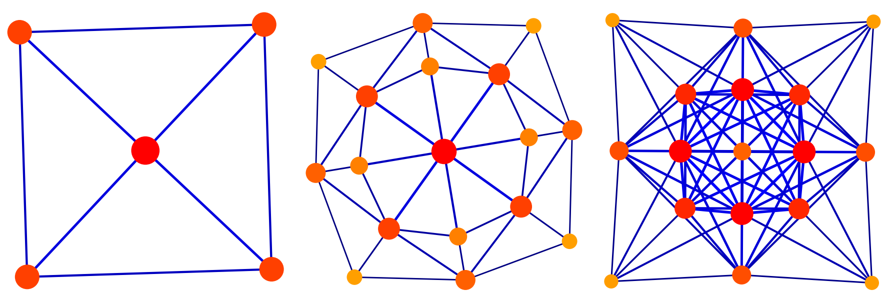

FIGURE 1. The wheel graph G, its Barycentric refinement G_1 and the connection graph G'.

3. The Unimodularity theorem

An integer matrix M is called **unimodular** if its determinant is either 1 or -1. By the explicit **Cramer-Laplace inversion formula**, unimodularity is equivalent to the fact that its inverse M^{-1} is an integer-valued matrix. Because an unimodular matrix M is non-singular in the ring $M(n,Z)$ of integer matrices, a unimodular matrix M is an element in $GL(n,Z)$.

Let G be a graph equipped with a simplicial complex or a CW-complex. With the Fermi number $f(G)$ giving the number of odd dimensional simplices of cells in G, the Fermi characteristic is defined as $\phi(G) = (-1)^{f(G)} = \prod_x \omega(x) = \prod_x (-1)^{\dim(x)}$. The **Fredholm characteristic**

$$\psi(G) = \det(1 + A(G'))$$

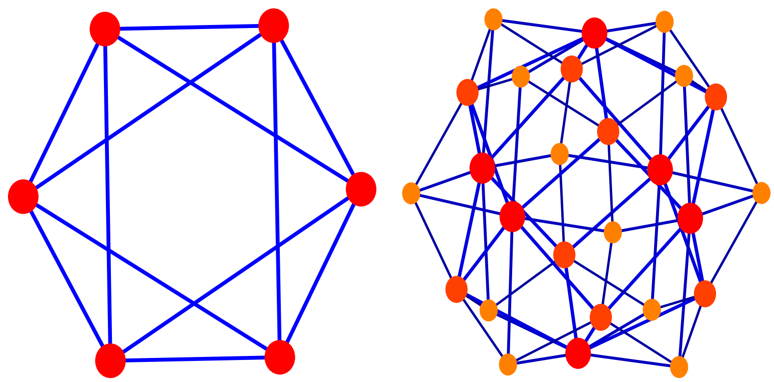

FIGURE 2. The octahedron graph G is an example of a discrete 2-sphere. Unlike G or its Barycentric refinement G_1, the connection graph G' is not a sphere any more. The graph was so small that the connections modified the 2-sphere and made the graph simply connected.

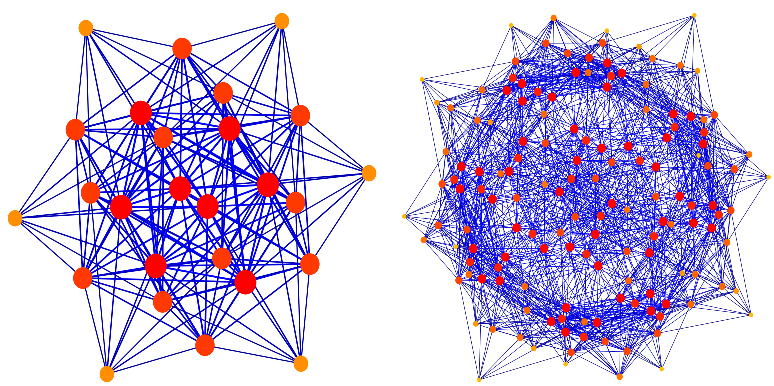

FIGURE 3. The Barycentric refinement G_1 of the octahedron is again a 2-sphere. Its connection graph G'_1 is now homotopic to G.

is the Fredholm determinant of the adjacency matrix of the connection graph G' of the complex.

Theorem 1 (Unimodularity)**.** *For any graph G equipped with a simplicial or CW structure:*

$$\psi(G) = \phi(G) \ .$$

The following corollary of the theorem is actually equivalent to the theorem, at least on simplicial complexes. As usual, if $G = (V, E)$ and $H = (W, F)$ are two finite simple graphs, the **intersection graph** $G \cap H = (V \cap W, E \cap F)$ and **union graph** $G \cup H = (V \cup W, E \cup F)$

are both finite simple graphs. Also in the slightly more general case, if G, H are finite abstract simplicial complexes or CW complexes, one can look at the intersection $G \cap H$ and union complex $G \cup H$.

Corollary 1. *For any finite simple graphs G, H, simplicial complexes or CW complexes, the determinant formula*

$$\psi(G)\psi(H) = \psi(G \cup H)\psi(G \cap H) \tag{1}$$

holds and $\psi(K_d) = \phi(G) = (-1)^{f(G)} = -1$ for $d > 1$ and $\psi(K_1) = 1$.

Proof. The first statement follows directly from the theorem and uses the fact that the explicit knowledge of $f(G)$ outs it is a **valuation**, an integer-valued functional f on the set of graphs or simplicial complexes satisfying $f(G \cup H) = f(G) + f(H) - f(G \cap H)$. The second statement follows from the fact that the number of odd-dimensional simplices in K_d is odd if $d > 1$. The reason is that the f-vector of a complete graph K_{d+1} is explicitly given in terms of Binomial coefficients $(B(d+1,1), \dots, B(d+1, d+1))$ so that $f(G) = 2^d - 1$ for $d > 0$. In the case $d = 0$, we have $f(G) = 0$, otherwise $f(G)$ is odd. □

Let $b(G)$ denote the number of even-dimensional complete subgraphs of G. The **join** of two graphs $G = (V, E), H = (W, F)$ is defined as the graph $(V \cup W, E \cup F \cup \{(v, w)\ v \in V, w \in W\ \})$. The join operation has the same properties as in the continuum: the join of two discrete spheres for example is again a discrete sphere. The join of a graph G with a 0-sphere $S_0 = (V, E) = (\{a, b\}, \emptyset)$ is the **suspension** of G. The octahedron from example is the suspension of the cyclic graph C_4 and repeating the suspension construction on the octahedron and beyond produces all cross polytopes. An other consequences of the unimodularity theorem is:

Corollary 2. *If G is the topological join of K_1 with a graph H, then*

$$\psi(G) = \psi(H)(-1)^{b(H)} = (-1)^{\chi(H)}\psi(H)\ .$$

Especially, if G is the suspension of H, then $\psi(G) = \psi(H)$.

Proof. Each newly added odd-dimensional simplex in G corresponds to an even-dimensional simplex in H. To see the second part, note that the suspension is obtained by performing an second join operation over H so that we get again $\psi(H)$. □

Remarks.

1) The unimodularity theorem is clear for disjoint graphs F, G or if F is a subgraph of G. A special case is if F, G intersect in a single vertex.

Then the Fredholm characteristics of F and G multiply. This special case is related to the known formula [22] (Corollary 8.7)

$$\det(1+A+B) \leq \det(1+A)\det(1+B)$$

because in the case of two subgraphs of a large complete host graph, intersecting in a point, the adjacency matrix of $F \cup G$ is $A+B$. This is no more true for the connection graph: if we join F, G at a point, there are many simplices x, y from different graphs which join, so that $A' + B' = (A+B)'$ is no more true. Still, the unimodularity theorem implies in that case that

$$\det(1+A'+B') = \det(1+A')\det(1+B')$$

for the adjacency matrices of the two connection graphs F', G' if $F \cap G = K_1$. We see also that except for K_2, removing an edge from K_d removes an even number of odd-dimensional complete subgraphs. The reason is that 2^k is even for positive k and odd for $k = 0$. For example, when removing an edge from K_4, we remove a tetrahedron and an edge to get a kite graph. The f-vector encoding the cardinalities of the complete subgraphs changes from $(4, 6, 4, 1)$ to $(4, 5, 2, 0)$.

2) The unimodularity theorem implies for example that the path integral sum can be replaced with one single permutation: for example, we can enumerate the odd dimensional simplices as $x_1, x_2, \dots, x_k$, then define a map $x_i \to y_i$ from odd to even dimensional simplices by just dropping the first coordinate. Define $S_i(x_i) = y_i, S_i(y_i) = x_i$ and $S_i(x) = x$ for any other $x \neq x_i, x \neq y_i$. The transformation $S(x) = S_1(x) \cdots S_k(x)$ has the signature $\psi(G)$. While S is generated by transpositions, it is not a transposition itself in general. It is not a transposition for a kite graph for example. But it can be, like in the case C_4. While some permutations are continuous, the just mentioned one is not. The closed set $\{x\}$ of a vertex x is mapped into an open set $\{(xy)\}$. The map is not continuous. While on $G = K_2$, there are 3! different transformations, only two of them are continuous and only the identity is a continuous flow. The other two transpositions $x \to (xy)$ and $y \to (xy)$ are two signature -1 transformations but they are not continuous. The sum $1 + (-1) + (-1) = -1$ is $\psi(G)$.

The next corollary expresses the Fredholm characteristic of a ball by the Euler characteristic of its boundary:

Corollary 3 (Fredholm characteristic of unit ball)**.** *For any graph $G = (V, E)$ and every vertex $x \in V$, we have $\psi(B(x)) = (-1)^{\chi(S(x))}$.*

Proof. The number of odd-dimensional simplices in $B(x)$ not in $S(x)$ is equal to the number of even-dimensional simplices in $S(x)$. Therefore, $f(B(x)) = f(S(x)) + b(S(x)) = \chi(S(x)) + 2b(S(x))$. Now exponentiate

$$(-1)^{f(B(x))} = (-1)^{\chi(S(x))}(-1)^{2b(S(x))} = (-1)^{\chi(S(x))} .$$

(We could write this as $-(-1)^{i(x)}$, where $i(x) = 1 - \chi(S(x))$ is a Poincaré-Hopf index at x.) ☐

Examples:
1) If G is a graph for which every unit sphere is a discrete sphere of Euler characteristic 0 or 2, then the unit ball $B(x)$ has $\psi(B(x)) = 1$. For an icosahedron G for example, where every unit ball is a wheel graph W_5 with 5 spikes, there are 10 edges in each unit ball.

2) While the just mentioned corollary shows that for discrete spheres $S(x)$ with $\chi(S(x)) \in \{0, 2\}$, the unit balls always have $\psi(G) = 1$, there are spheres of arbitrary large dimension which can come both with $\psi(G) = -1$ or $\psi(G) = 1$. Here are examples: Start with one dimensional sphere C_n with odd n. It has $\psi(G) = -1$. Every suspension is again a sphere and ψ does not change under the suspension operation.

4. Refining the simplicial complex

Besides enlarging the base set V of a graph $G = (V, E)$, there is an other possibility to deform the geometry of a graph: we can **refine the simplicial complex structure** imposed on it. The simplest simplicial complex on a finite set is the **0**-skeleton, where the set of subsets of V is the set $\{\{x\} \mid x \in V\}$. Since there are no paths except the trivial path with signature 1, the Fredholm characteristic of such a 0-dimensional space is 1 and the unimodularity result is obvious in that case as no odd dimenional simplices exist then.

The next possibility is to take the **1**-skeleton complex $V \cup E$. This renders the graph one-dimensional; it is the structure which is often associated with a graph, when defining a graph as a one-dimensional simplicial complex. Also in this case, one can see the unimodularity theorem. But it is less obvious already. First of all, we have $f(G) = |E|$ as no higher dimensional simplices are present. In the case when the graph is a tree, then there are no closed paths beside K_2 paths. In order to analyze this, we have to see that for each edge there are two loops exchanging the edge with the attached vertices. What happens if we merge two edges is that the two transpositions merge to two circular loop depending on the order and that there is a new transposition exchanging edges. The functional ψ is multiplicative.

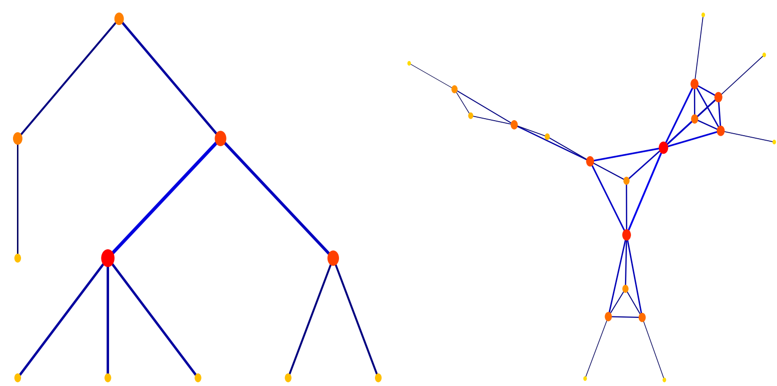

FIGURE 4. A tree and its connection graph.

If we start including two-dimensional simplices by "switching on" the triangle, the number of paths increases considerably. Lets call C_3 the triangular graph K_3 equipped with the one-dimensional simplicial complex. We have $\psi(C_3) = -1$ as there are three odd dimensional simplices present, the 3 edges. We also have $\psi(K_3) = -1$ as there are no new odd-dimensional simplices. The reason why the functional does not change when moving from C_3 to K_3 is because $\chi(S(x)) = 0$. But in the three dimensional case, because $\chi(S(x)) = 2$, the functional ψ changes by -2, becoming $\psi = -1$. Then again, the Euler characteristic of $S(x) = 0$ and there is no change in $\chi(G)$.

The unimodularity theorem holds more generally when the graph is equipped with a **finite CW complex structure**. If a finite simple graph $G = (V, E)$ is equipped with such a simplicial or CW structure, it still defines a **connection graph** and so a Fredholm determinant. The definition of a CW structure requires for a good notion of a "sphere" in graph theory. Traditionally, a CW complex is a Hausdorff space equipped with a collection of **structure maps** from k-balls to X. As we have seen, a **finite CW complex structure** works the same way, but cells don't need to be simplices any more. This is of practical value as we can work in general with smaller complexes when dealing with homotopy invariant notions like cohomology. It allows us to see the deformation of the simplicial structure as the process of adding or removing cells to the CW complex.

5. Extension proposition

The following key result will allow to see how the Fredholm determinant changes if we add a cell to a CW complex. The Poincaré-Hopf indices are not necessarily $\{-1,1\}$-valued any more in general, because $\chi(G)$ can take values different from 0 or 2. The Fredholm determinant extension formula needs a slightly more general extension process:

Let G be a finite simple graph and let H be a subgraph H of G. Define the **pyramid connection graph** $\tilde{G} = G' \cup_H \{x\}$ as the pyramid extension over the subgraph of G' generated by the simplices which intersect a vertex in H. This new graph $\tilde{G}$ is equipped not with the Whitney complex of the pyramid extension but with the union of the complex of G' together with all simplices $y \cup \{x\}$, where y is a simplex in H. One can build up any simplicial complex as such. Given for example the triangle graph $G = (V, E)$ equipped with the 1-skeleton complex $C = V \cup E = \{a, b, c, ab, ac, bc\}$. The pyramid connection complex $\tilde{G}$ has now as vertices the elements in C together with a new element x. Now x is connected to any element in C which intersects H additionally, any two elements in C are connected if they intersect. In this case, the graph $\tilde{G}$ agrees with the connection graph of K_3 equipped with the Whitney complex.

Here is the result:

Proposition 1 (Fredholm extension proposition)**.** *For any graph G and any subgraph H of G we have*

$$\psi(\tilde{G}) = \psi(G' \cup_H \{x\}) = (1 - \chi(H))\psi(G') \ .$$

The same holds for simplicial complexes or CW complexes G if H is a sub complex.

Proof. **(i)** The map $\eta : H \to \psi(G \cup_H x) - \psi(G)$ is an additive valuation. Proof. The functional super counts the number of new paths which appear by adding the cell x and if a path passes through x, then no other path can pass through it. For any two sub complexes H, K we therefore have

$$\eta(H \cup K) + \eta(H \cap H) = \eta(H) + \eta(K) \ .$$

(ii) It follows that also

$$X(H) \to 1 - \psi(G \cup_H x)/\psi(G)$$

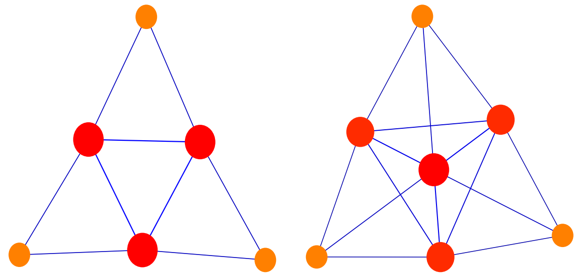

FIGURE 5. The 1-dimensional triangle or C_3 is the same graph as K_3 but it is equipped with the 1-skeleton complex (meaning that we disregard the 2-dimensional face which is present in the Whitney complex). It is a sphere and has Euler characteristic $\chi(G) = 3 - 3$ as we disregard the 2-simplex. We see first its connection graph G'. After having added an other two dimensional cell to $H = G$, the connection graph $\tilde{G}$ is now graph isomorphic to the connection graph of K_3 equipped with the usual Whitney complex. The proposition tells that $(-1) = \det(1+A(\tilde{G})) = (1-\chi(G))\det(1+A(G')) = (1-0)(-1)$.

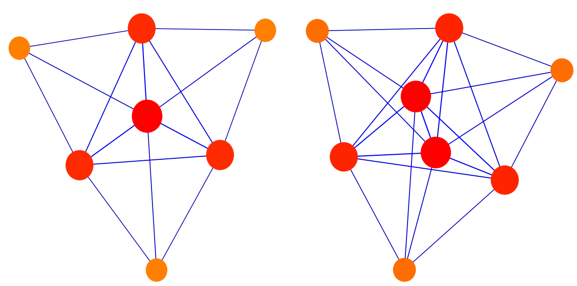

FIGURE 6. As a comparison, we see $G = K_3$ equipped with the Whitney complex so that $\chi(G) = 1$. After adding an other vertex x to $H = G$, we get the graph $\tilde{G}$. The proposition tells that $0 = \det(1 + A(\tilde{G})) = (1 - \chi(G))\det(1 + A(G')) = (1 - 1)(-1)$.

is a valuation: the value $\psi(G)$ is constant as we take H as an argument of the functional.

(iii) The claim of the proposition holds if H is a complete graph K_{d+1} inside G. We have to show $\psi(\tilde{G}) = \psi(G' \cup_H \{x\}) = 0$. Proof. In the

Fredholm matrix of $\tilde{G}$, the row belonging to the d-simplex belonging to H in G and the row belonging to the new vertex x all have entries 1. The matrix is singular.

(iv) Since by (iii), $\psi(X \cup_H x) = 0$ if H is a complete graph, the valuation X takes the value 1 for every complete subgraph $H = K_k$. Claim: a valuation with this property must be the Euler characteristic:

$$X(H) = \chi(H) .$$

Proof. By discrete Hadwiger, every valuation has the form $X(x) = a \cdot x$. We know $X((1)) = 1, X((2,1)) = 2X((1,0)) + X(0,1)$ implying $X((0,1)) = -1$. Inductively this shows that the vector a is $(1,-1,1,-1,1,\dots)$. An other, more elaborate argument is to see that the assumption implies that the Euler characteristic of a Barycentric refinement of a simplex is 1 too, showing that X is invariant under Barycentric refinement. Since one knows that the eigenvalues and eigenvectors of the Barycentric refinement operator and especially that Euler characteristic is the only fixed point of the Barycentric operator, we are done. □

Examples.
1) If $H = G = K_1$, then there is one empty path and one involution in the Leibnitz-Fredholm determinant, which is a path integral sum. The Fredholm characteristic is $1 - 1 = 0$.
2) If $H = G = K_2$, then there are 2 graphs of length 3, the empty graph and three involutions. Again we have $3 - 3 = 0$.
3) If $H = G = K_3$, then the Fredholm matrix

$$1 + A(\tilde{G}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} .$$

The proposition makes the path counting problem additive as $\chi(H)$ is a valuation. It shows that path integral of the set of new paths which appear when adding x.

Examples:
1) Assume that H has k vertices and no edges. For every of the vertices y we can look at the paths xy of length 2. Since we can not have

simultaneously two paths hitting x, the number of additional paths is $k\psi(G)$. They are counted negatively as they have even length.
2) If H is K_2 then we have three cycles of length 2, three cycles of length 3, and one path of length 4 paths passing through x. The last one cancels the empty path. We get zero.
3) If H is a complete graph of k vertices, then we have a unit ball in the connection graph which is a complete graph. The extended graph has Fredholm characteristic 0.
4) If H is a cyclic graph of $k \geq 4$ vertices. Now the sum over new path contributions is zero:

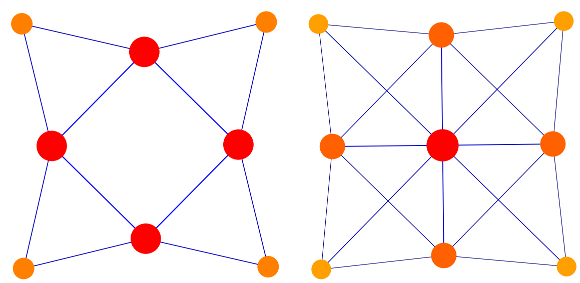

FIGURE 7. We take $G = H = C_4$ with $\chi(G) = 0$ and add a new cell x. The Poincaré-Hopf index is $i(x) = 1 - \chi(H) = 1$. The Fredholm characteristic of G_x remains unchanged by the proposition.

While $\psi(\tilde{G}) \in \{-1, 1\}$ is not true any more in general, the proposition shows why the unimodularity theorem works. We can interpret the pyramid extensions over k-spheres as adding a $(k+1)$-dimensional cell or simplex. The proposition allows to relate Fredholm characteristic to Euler characteristics of extensions. In the case of simplicial complex extensions, the $(1 - \chi(H))$ are always 1 or -1.

Examples:
1) Let G be a one point graph. The graph $\tilde{G}$ is K_2 and $\psi(K_2) = (1 - \chi(G))\psi(K_1) = 0$.
2) Let G be the two point graph without edges. Then $\tilde{G}$ is the line graph L_3 and $\psi(L_3) = (1 - \chi(G))\psi(G) = -1$. Indeed $L_3 = K_2'$.
3) If H is contractible, then $\psi(G' \cup_H \{x\}) = 0$. For example, if H is a one point graph $\{y\}$ we just assign a new cell for which every simplex connected to y is belonging. We have now a complete subgraph $B(x)$. We know already that if a connection graph has a unit ball which is a

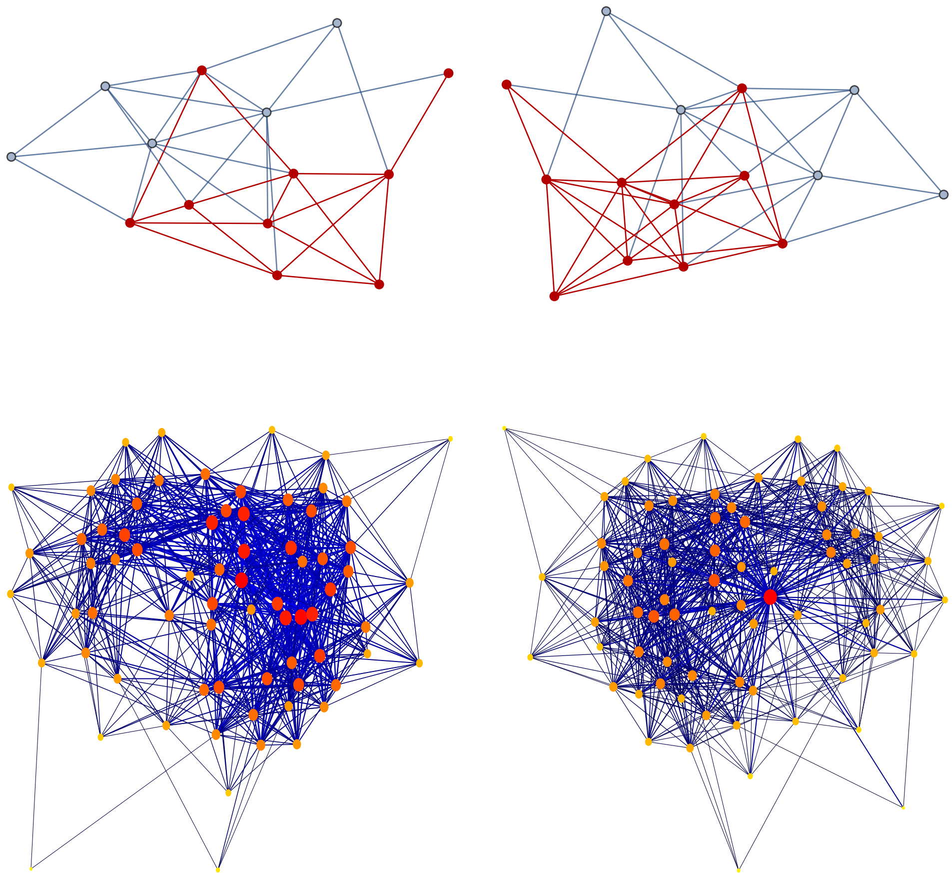

Figure 8. We see a random graph G with 14 vertices and Euler characteristic $\chi(G) = -4$. Inside is a random subgraph with 9 vertices of Euler characteristic $\chi(H) = -2$. The first picture shows H highlighted as a subgraph of G. The second picture shows the pyramid extension G_x of G in which an additional vertex has been added. The Poincaré-Hopf index of the new index is $1 - \chi(H) = 3$. The Fredholm characteristic of G_x is $3 \cdot (-4) = -12$ by the proposition.

complete graph, then the Fredholm characteristic is zero.

Corollary 4 (Adding odd or even dimensional cells)**.** *If G is an even dimensional k-sphere, then $\psi(G' \cup \{x\}) = -\psi(G')$ and if G is an odd dimensional k-sphere, then $\psi(G' \cup \{x\}) = \psi(G)$.*

This implies that if we add an odd dimensional simplex to a simplicial complex, the Fredholm characteristic changes sign and if we add an even dimensional simplex, then the Fredholm characteristic stays the same.

6. Prime and prime connection graphs

For ever integer $n \geq 2$ let V be the set of natural numbers in $\{2, 3, \dots, n\}$ which are square free. Connect two integers if one is a factor of the other [13]. The corresponding graph G_n is the **prime graph**. One can see it as the part $f \leq n$ of the Barycentric refinement of the complete graph on the spectrum P of the integers $\mathbb{Z}$, where $f(x) = x$ is the counting function. The **prime connection graph** H_n has the same vertex set V but now, two integers are connected if they have a common factor larger than 1 [14]. The two graphs allow us to illustrate some of the theorems. For the graph G_n, the Poincaré-Hopf theorem and the Euler-Poincaré relation to cohomology is interesting as it allows to express the Mertens function interms of Betti numbers. In the case of H_n, we can now formulate a consequence of the proposition which is the analogue of Poincaré-Hopf and the unimodularity result relating the Fermi characteristic with a Fredholm determinant of the adjacency matrix. We see G_n as part of the Barycentric refinement of the complete graph on the spectrum of the integers $\mathbb{Z}$ and see H_n as the connection graph of that complete graph. This point of view comes into play when seeing **square free integers** as simplices in a simplicial complex.

Despite the fact H_n is not directly the connection graph of an other graph (it is part of the connection graph of an finite graph, the complete graph with vectex set spec($\mathbb{Z}$)), lets still denote by $\psi(H_n)$ the Fredholm characteristic, the Fredholm determinant of the adjacency matrix of H_n. Let $i_f(x) = 1 - \chi(S_f^-(x))$ denote the Poincaré-Hopf index of the counting function f at x. It is $-\mu(x)$, the **Möbius function**. We have seen that Poincaré-Hopf implies that $\chi(G_n) = 1 - M(n)$, where $M(n)$ is the **Mertens function**.

We have now directly from the unimodularity theorem for CW complexes a multiplicative Euler characteristic formula analogous to

$$\chi(G_n) = \sum_{x \in V, x \leq n} \mu(x) \; .$$

Corollary 5. $\psi(G_n) = \prod_{x \in V, x \leq n} \mu(x)$.

Proof. The left hand side is the Fredholm characteristic of the CW complex G_n. The right hand side is the Fermi characteristic of G_n. $\square$

Here is a comparison between the graph G_n and H_n. Lets define $\pi(G) = \sum_k (-1)^k b_k(G)$ as the value of the Poincaré polynomial evaluated at -1, where $b_k(G)$ is the k'th Betti number. The cohomological data of course correspond exactly to the corresponding notions in the continuum. An Evako d-sphere for example has the Poincaré polynomial $1 + x^d$ and a contractible space has Poincaré polynomial 1.

Prime graph graph G_n	Prime connection graph H_n
introduced in [13]	introduced in [14]
relation is divisibility	relation is nontrivial GCD
in Barycentric refinement of spec($\mathbb{Z}$)	in connection graph of spec($\mathbb{Z}$)
Euler characteristic $\chi(G) = \sum_x \omega(x)$	Fermi characteristic $\phi(G) = \prod_x \omega(x)$
Poincaré-Hopf $\chi(G \cup x) - \chi(G) = i(x)$	proposition $\chi(G \cup x) = i(x)\chi(G)$
Euler-Poincaré $\chi(G) = \pi(G)$	Unimodularity $\phi(G) = \psi(G)$

Example. The prime connection graph H_{30} is a graph with 18 vertices $\{2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30\}$ and 39 edges. We can look at H_n as a CW complex, where the individual nodes represent the cells. The Fredholm matrix of H_n is

$$1 + A = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix} .$$

Its determinant is -1.

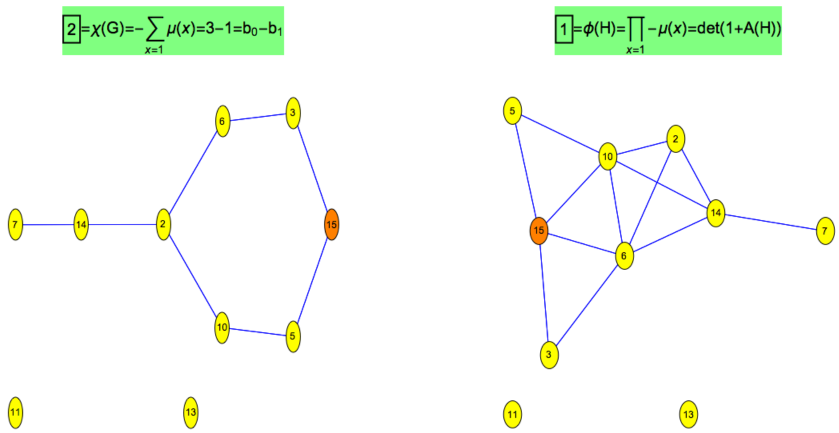

FIGURE 9. The prime graph G_{15} and prime connection graph H_{15}.

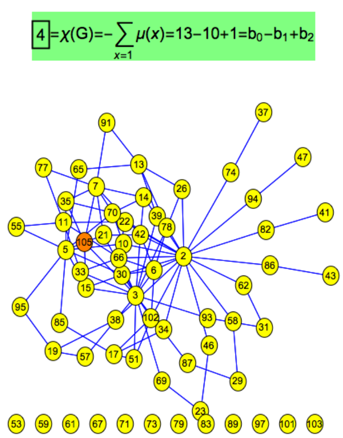

FIGURE 10. The prime graph G_{105} and prime connection graph H_{105}.

7. REMARKS

7.1. The unimodularity theorem does not hold for an arbitrary **generalized matrix function** as defined by Marcus and Minc [18]. It does not work for **permanents** for example as we can check that ψ is not a multiplicative valuation: the Fredholm permanent of the kite graph is 14. The Fredholm permanent of the intersection is 2. The Fredholm permanent of the two triangular pieces is 6. As mentioned in the introduction, the Fredholm permanent of a complete graph is $n!$, while the determinant is the number of derangements. The fact that the product property or the unimodularity result can not hold for permanents is similar to the fact is that the Euler characteristic $\sum_k (-1)^k v_k$ is constant 1 for a contractible graph and invariant under homotopy deformations, while the Bosonic analogue, the **Kalai number** $\sum_k v_k$ counting the number of simplices in a graph is a valuation but not invariant under homotopy deformations.

7.2. Why does the proof only work for the connection graphs and not for the graphs themselves? Already for small graphs like K_2 or K_3, the Fredholm matrices are not unimodular any more. Indeed, for complete graphs, the Fredholm adjacency matrix is the matrix which has 1 everywhere. It seems that what is needed is the high connectivity of the simplices in the connection graph: paths in the unit ball in G' of every original vertex x are in 1-1 correspondence to permutations of the vertices in the ball $B(x)$ of G'. This is important as the sum of the signatures over all non-identity transformations in $B(x)$ is -1.

7.3. We initially tried to prove the result by gluing two graphs F, K to a larger graph $G = F \cup H$ and intersection $H = F \cap H$ and take a permutation x of F' and y of K' and combine them to a permutation xy in G'. The most natural choice is a composition of the permutations. While

$$\psi(F)\psi(K) = (\sum_x \text{sign}(x))(\sum_y \text{sign}(y)) = \sum_{x,y} \text{sign}(xy)$$

holds, we are not able to match the composed transformations xy with a transformation zw, where z is a permutation in G' and w a permutation in H'. The product xy is not necessarily a permutation any more in G' and we need w in H' to achieve that. Now if $xy = x'y'$, then $x'^{-1}x = yy'^{-1}$ is a permutation in H. In one of the simplest cases, where $F = K = L_3$ and $H = F \cap K = K_2$ and $F \cup K = L_4$. Now the number of permutations in F' or K' are 11. There are 121 product transformations. There are 39 transformations in G' and 3 transformations in H'. There are only 117 products. We see, we can not get all the pairs xy with pairs zw. Cooking up a proof strategy which pairs the permutations up and then proves that that the signatures of the remaining sum up to zero has not yet worked.

7.4. Let $G = (V, E)$ be a finite simple graph. The determinant of the adjacency matrix A is a super count of cyclic permutations of V. The determinant of the Fredholm matrix $A+1$ is a super count of the permutations of V. The Pseudo determinant of the Laplacian $L = B - A$ counts the number of rooted trees in G and the determinant of the Fredholm Laplacian $L+1$ counts the number of rooted forests in G. In each case we see that the determinant counts some zero or one-dimensional directed subgraphs. What the significance of the unimodularity of the Fredholm matrix $1+A$ is remains to be seen. It will depend on properties of the Green functions, the entries of the matrix $g(A) = (1+A)^{-1}$ which is the Fréchet derivative of the map $A \to \det(1+A)$ from $n \times n$ matrices to the real axes [22] (Corollary 5.2).

7.5. It follows from the explicit formula for the f-vector that the Barycentric refinement G_1 of any graph G is always a positive graph. There is an upper triangular matrix A such that $\vec{v}(G_1) = A\vec{v}(G)$ for all simplicial complexes. The matrix is $A_{ij} = S(i,j)j!$, where where $S(i,j)$ are the **Stirling numbers** $S(i,j)$ of the second kind. Since $j!$ is even for $j > 1$ all rows of A beyond the first one are even. The Barycentric refinement matrix for simplicial complexes with maximal dimension 4 for example is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 & 30 \\ 0 & 0 & 6 & 36 & 150 \\ 0 & 0 & 0 & 24 & 240 \\ 0 & 0 & 0 & 0 & 120 \end{bmatrix} .$$

Having seen that $\phi(G_1) = 1$ for any graph G or simplicial complex if G_1 is the Barycentric refinement, it follows by the unimodularity theorem that ψ is **not** a combinatorial invariant.

7.6. Having discovered unimodularity experimentally in February 2016 and announced in October [16] it took us several attempts to do the proof presented here. Induction by looking algebraically at the matrices appeared difficult. An other impediment is the lack of an additive structure on the category of graphs. It would be nice to see $G \to \log(\psi(G))$ as a group homomorphism for some larger group and then prove it for a basis only. This has not yet worked: in the larger category of Z_2-chains we have also connection graphs but ψ does not extend. One can also look at the ring of graphs obtained by taking the connection graphs defined by a set of complete subgraphs (which form a Boolean ring). But ψ does not remain multiplicative on that larger structure.

7.7. A finite simple graph G with unimodular adjacency matrix A is called an **unimodular graph**. An example is the linear graph L_{2k} for which the adjacency matrix has determinant $(-1)^k$. As this is not the same than having the Fredholm matrix $1+A$ unimodular, we here did not use the terminology "unimodular graph". Experiments show that the Fredholm matrices $B = 1+A'$ of connection graphs often also have unimodular or zero determinant submatrices obtained by deleting one row and one column which is equivalent that B^{-1} is a $0, 1, -1$ matrix. In general, for a connection graph G, the inverse $(1+A)^{-1}$ has only few elements different from $0, 1, -1$. These appear to be interesting **divisors** to consider.

7.8. An **abstract finite simplicial complex** $\mathcal{X}$ on a finite set V is a set of subsets of V such that if $A \in \mathcal{X}$ then also any subset of A is in $\mathcal{X}$. Special simplicial complexes are **matroids**, simplicial complexes with the **augmention property** telling that if $\dim(x) > \dim(y)$ then one can enlarge x with and element in y to get an element in the simplex. Our point of view was to see simplicial complexes on a finite set V as a **geometric structure** imposed on V similarly as one imposes a **set theoretical topology** $\mathcal{O}$ in topology or a **σ-algebra** $\mathcal{A}$ in measure theory. Like any topology, σ-algebra or simplicial complex $\mathcal{X}$ on V defines a **connection graph** $\mathcal{X}'$ of the structure: the vertices of $\mathcal{X}'$ are the sets in $\mathcal{X}$ and two such vertices are connected, if they intersect as subsets of V. The deformation of the simplicial structure gave more freedom than the deformation of the graph. Besides complete graphs, one can define other simplicial complexes based on graphs. One can for example look at the matroid on the set of edges in which the faces are the forests. This simplicial complex defines what one calls a **graphic matroid**. The Fermi characteristic of this matroid is the number of forests with an odd number of edges. The connection graph of the matroid has as vertices the forests and connects two if they intersect in at least one edge. The universality theorem applies here too. For example, if $G = K_3$, there are 6 forests in G. The Fredholm matrix of the connection graph

$$1 + A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

has determinant -1 so that the graphic matroid has Fredholm characteristic $\psi(G) = -1$. There are 3 one-dimensional forests in G so that also the Fermi characteristic is -1.

7.9. A finite simple graph $G = (V, E)$ carries a natural associated simplicial complex, the **Whitney complex** $\mathcal{X}$. It consists of all subsets of V which are complete subgraphs of G. The **one skeleton complex** $\mathcal{X} = V \cup E$ is an other example of a simplicial complex. Simplicial complexes do not form a Boolean ring: take the Boolean addition of $\mathcal{X}_2$ with $\mathcal{X}_1 \subset \mathcal{X}_2$, then this is the complement of $\mathcal{X}_1$ in $\mathcal{X}_2$ which is no more a simplicial complex in general as $\mathcal{X}_1 = \{\{1,2\},\{1\},\{2\}\}$ and $\mathcal{X}_2 = \{\{1\}\}$ shows. The unimodularity theorem has no simple algebraic proof because the category of graphs or the category of simplicial complexes only forms a **Boolean lattice** and not a **Boolean ring**. An algebraic statement $\psi(G + H) = \psi(G)\psi(H)$ does therefore not work for the simple reason that the Boolean sum $G + H$ is not a graph. Also, since the result does not extend to the **ring of chains**, (the free Abelian group generated by the simplices), we had to proceed differently. The Boolean algebra of the set of simplices in G is a ring however so that for every element, we can define a connection graph. However, the functional ψ does not extend as one can build any graph like that.

7.10. The **line graph** of a graph G has the edges as vertices and two edges are connected, if they intersect. It is a subgraph of the connection graph G' of the 1-skeleton complex defined by the graph G. We mention this structure, as it is also sometimes called the **intersection graph** of G. It plays a role in topological graph theory. If $d^* = \text{div}$ is the incidence matrix of G which has as a kernel the cycle space, then $L_0 = d^*d = B - A$ is Kirchhoff Laplacian, where B is the vertex degree diagonal matrix. Now $|dd^*| + 2$ is the adjacency matrix of the line graph if $|A|_{ij} = A_{ij}$. We see that the Fredholm matrix of $|dd^*|/2$ is an adjacency matrix of some graph.

7.11. The formula

$$\log(\det(1-zA)) = \mathrm{tr}(\log(1-zA)) = -\sum_k \frac{\mathrm{tr}(A^k)}{k} z^k$$

which converges for small $|z|$ gives an interpretation of the Fredholm determinant in terms of closed paths, which can self intersect. Note that $\det(1-zA)$ is a polynomial of degree n, where n is the number of vertices in G. Since $\det(1-zA)^{-1} = \exp(\sum_k N_k z^k)$, where N_k is the number of closed paths in the graph of length k. There is a relation with Ihera zeta functions, which satisfies $\zeta(z)^{-1} = \det(1-zA)$, where A is the **Hashimoto edge adjacency matrix** of G.

7.12. Connection graphs could be defined for infinite, and even uncountable graphs. Its not clear yet which generalizations lead to Fredholm operators and so to Fredholm determinants. Take the points of a circle $\mathbb{R}/\mathbb{Z}$ for example as the vertex set, fix two irrational numbers α, β like $\alpha = \pi, \beta = e$ and $k \in \mathbb{N}$. Connect two points if there exists $n, m \in \mathbb{Z}$ with $|n|, |m| \leq k$ such that $x - y - n\alpha - m\beta = 0 \mathrm{mod} 1$. The points of the connection graph is then the set of orbit pieces of the commutative $\mathbb{Z}^2$ action as these are the "simplices" in the original graph $G_{k,\alpha,\beta}$.

7.13. Connection graphs have a high local connectivity: let G be a connected graph which is not a complete graph and let G' is its connection graph. For any vertex $x \in V(G')$ the Fredholm determinant of $B(x) \subset G'$ is zero. Proof. Assume first that x is not the maximal central simplex y containing x. Then, there are parallel columns x, y in the extended adjacency matrix. Let z be in $B(x)$. Then z intersects x and y. The only way, the unit ball $B(x)$ in a connection graph can have nonzero Fredholm determinant is if G is a complete graph and x is the maximal central simplex. For example, for $G = K_2 = \{\{a, b\ \} \mid \{(ab)\}\}$, the connection graph is L_3. The unit sphere of $x = (ab)$ is P_2 which has Fredholm determinant 1. For all other x, the unit ball is K_2, which has zero Fredholm determinant.
Benjamin Landon and Ziliang Che, members of the Harvard random matrix group tell me that current technology of matrix theory like [23] could be close in being able to prove a central limit theorem for determinants of random graphs but that the existence of a central limit in that graph case still remains to be done.

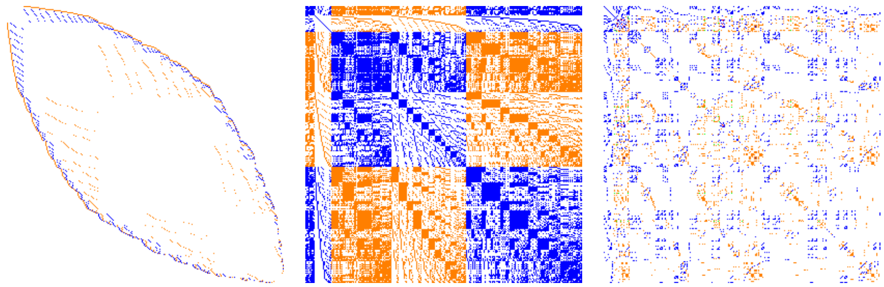

FIGURE 11. We see the Dirac matrix D, Fredholm matrix and its inverse. D is up to a sign the adjacency matrix of the Barycentric refinement. Its entries D_{xy} are nonzero whenever two different simplices x, y have the property that one is contained in the other. The Fredholm matrix $B = (1 + A')_{x,y}$ is nonzero if and only if x, y have a non-empty intersection. Because of the 1, the intersection is counted also when $x = y$. The last picture shows the inverse of that matrix B. Most of its entries are $0, 1, -1$ but there are a few larger entries.

8. Questions

1) Let $c(p) = \limsup_{n\to\infty} P_{n,p}[\psi = 1]$, where $P_{n,p}$ is the counting probability measure on the Erdős Rënyi space $E(n,p)$. We believe the limsup is actually a limit for every $p \in [0,1]$ and that the limit is $1/2$ for every $p \in (0,1)$. We know $c(0) = 1$ for $p = 0$ because a graph G without edges has $\psi(G) = 1$ and for $c(1) = 0$, because a complete graph G has $\psi(G) = -1$. Why do we think that the limit in general is $1/2$? Because of the central limit theorem for Z_2-valued random variables. What happens if we go from $E(n,p)$ to $E(n+1,p)$ is that we are given a distribution of a Z_2-valued random variable $(-1)^{b(H)}$ on subgraphs of G get the new graph by adding a pyramid over H. Since we expect the probability of picking a graph H with $\psi(H) = 1$ or $\psi(H) = -1$ is more and more $1/2$, we essentially multiply with some independent Z_2-valued random variable when adding a new vertex. By the central limit theorem of random variables taking values in compact groups with a Haar measure, the limiting distribution has to be the uniform distribution which means equal distribution on finite groups. The case is not yet settled because we convolute each time with a distribution which we want to show to converge. This requires some estimates. Of course, a situation with some nonzero thresholds with a probability switching from 0 to $1/2$ would have been more exciting, but it seems

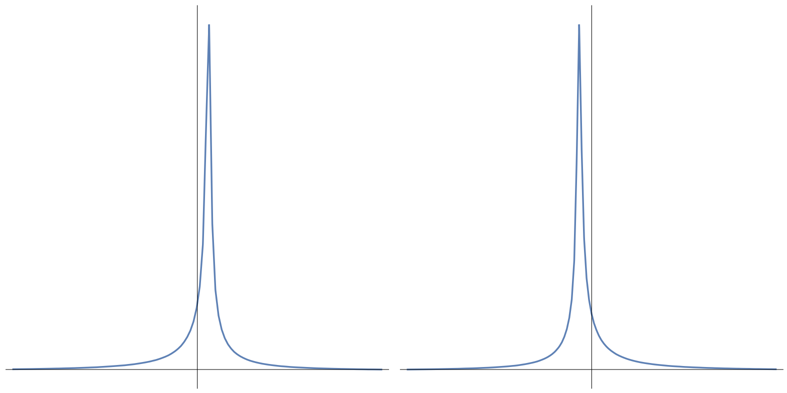

Figure 12. In the first picture we see the distribution of the determinant the adjacency matrix of a random graph with 70 nodes and edge probability $p = 1/2$. We computed it for 10^7 random graphs and plotted the distribution on $[-1, 1]$ after the random variable normalized so that the mean is zero and the standard deviation is 1. The right picture shows the same distribution but for the Fredholm determinant. Again the mean is zero and standard deviation 1 is the boundary of the interval. We see that both distributions appears to be singular in the limit but that there is no quantization which happens for connection graphs, where the value is either 1 or -1.

not to happen here; there is no reason in sight why the number of odd-dimensional subgraphs should pick a particular parity.

2) It appears that both the function $G \to \det(A(G))$ as well as $G \to \det(1+A(G))$ have a continuous limiting distribution on random graphs when the functions are normalized so that the mean is 0 and the standard deviation is 1. Establishing this would settle a central limit theorem for determinants. One usually looks also at the random variable $\log|\det(1 + A(G))|$ when looking at limits but that requires first establishing that the determinant 0 has asymptotically zero probability.

3) What is the structure of the inverse matrices of $1 + A(G')$, where $A(G')$ is the connection graph of the connection graph G' of G? Here are examples: for many graphs, including the complete graph, the wheel graphs, cross polytopes or cycle graphs, the inverse of the matrix $1 + A(G')$ only takes the values $-1, 0, 1$. For **star graphs** with n rays, we see that the values of the inverse of $1 + A(G)$ is $-(n-1), -1, 0, 1$.

For the utility graph, or the Petersen graph, the values are $-2, -1, 0, 1$. On the 13 Archimedian solids, we see 6 for which the value takes values in $\{-2, -1, 0, 1\}$ and 7, where the value takes value in $\{-1, 0, 1\}$. On the 13 Catalan solids, the value cases are $\{-1, 0, 1\}$, $\{-3, -2, -1, 0, 1\}$, $\{-4, -2, -1, -1, 0, 1\}$ and $\{-4, -3, -2, -1, 0, 1\}$ occur.

4) We measure that the **Green function values** given by the inverse matrix of

$$g_{ij} = [(1 + A(G'))^{-1}]_{ij}$$

is the same for G and the Barycentric refinement G_1 of G. We also measure that the maximal and minimal value among the diagonals, the **Green function values** are independent of Barycentric refinement. The diagonal entries G_{ii} are up to a sign the Fredholm determinants of the graph $G' \setminus x_i$, where the entry x_i has been removed. If the entry G_{ii} is 1 or -1, there is a chance that the modified graph is the connection graph of a CW complex. The Green function value g_{ij} is the Fredholm determinant of the structure obtained by snapping the ij connection between cell i and j. As the so lobotomized graph is no more a connection graph in general, the g_{ij} can be different. We have no explanation yet for these measurements. This would actually indicate that the minimal values taken for example are a **combinatorial invariants** of G. and that one could assign the invariant to the limiting continuum object. Following a suggestion of Noam Elkies, we looked also whether an edge refinement does not change the range of the Green functions $[(1 + A(G'))^{-1}]_{ij}$. Indeed also to be the case and a first thing which should be checked theoretically. The invariance under edge refinements would imply topological invariance under Barycentric refinements for graphs without triangles.

5) Because unimodular matrices form a group, we can define for all n, the group of all Fredholm connection matrices of subgraphs of a given graph G. Assume n is the number of simplices in G. If H is a subgraph of G, then the simplices are part of the simplices of G. All matrices therefore can be made the same size $n \times n$. The set of all these Fredholm connection matrices now forms a subgroup of the unimodular group $GL(n, Z)$. Can we characterize these groups somehow? The **tensor product** of two graphs G, H has the vertex set $V(G) \times V(H)$ and has $\{(a, b), (c, d)\} \in E(G \times H)$ if $(a, c) \in E(G)$ and $(b, d) \in E(H)$. We know that the tensor product of two graphs produces the **Kronecker tensor product** of their adjacency matrices. The tensor product of connection graphs therefore is a graph which has still an unimodular adjacency matrix.

6) We know that unit balls of connection graphs have Fredholm characteristic 0 and Euler characteristic 1. The structure of the unit spheres is less clear. We measure so far that unit spheres of connection graphs either have Euler characteristic 1 or 2. The later case is more rare but appears already for the octahedron graph G. Lets see: given a simplex x, then the unit sphere consists of all simplices which intersect x. Now, there are three possibilities: either y is a subgraph. The set of subgraphs is are all connected with each other and we can homotopically reduce them to a point. Then there are all graphs which contain x. Also all these are connected with each other and we can homotopically reduce them to a point. Finally, there are all graphs which intersect. We have to show that they are always homotopic to an even dimensional sphere.

7) We tried to correlate $\psi(G)$ with cohomological data, both for simplicial cohomology as well as connection cohomology. Simlilarly as homotopy deformations do change ψ but not cohomology, this also appears to happen for connection cohomology even so the later is not a homotopy invariant: the cylinder and the Möbius strip are homotopic but have different connection cohomology.

8) What is the relation between the f-vector of G and the f-vector of G'? Unless for Barycentric refinement, the f-vector of G does not determine the f-vector of G'. The Euler characteristic is not an invariant as the octahedron graph G for which the connection graph G' is contractible.

9. Illustrations

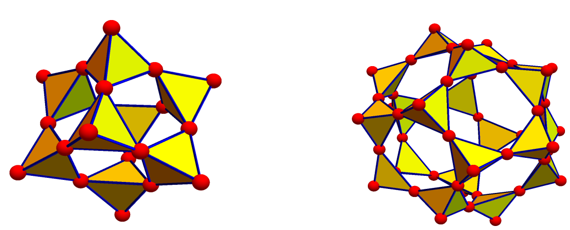

Figure 13. The connection graph of the cube graph and the dodecahedron graph.

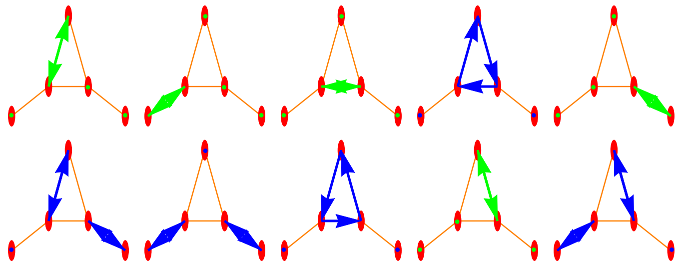

FIGURE 14. There are 11 one-dimensional oriented subgraphs of G' if G is the linear graph of length 2. There are 5 with odd signature and 6 with even signature including the empty path which is not shown. We have $\psi(G) = 6 - 5 = 1$ and $\phi(G) = (-1)^2 = 1$.

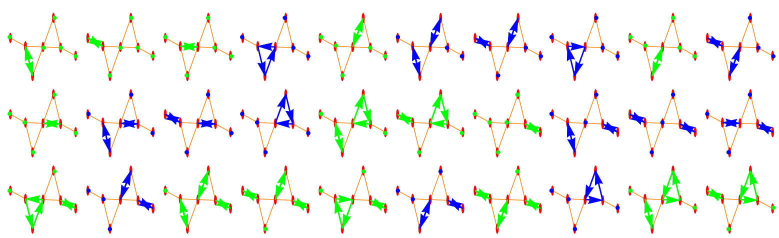

FIGURE 15. There are 39 one-dimensional oriented subgraphs of G' if G is the linear graph of length 3. There are 20 with odd signature and 19 with even signature. We have $\psi(G) = 19 - 20 = -1$ and $\phi(G) = (-1)^3 = -1$.

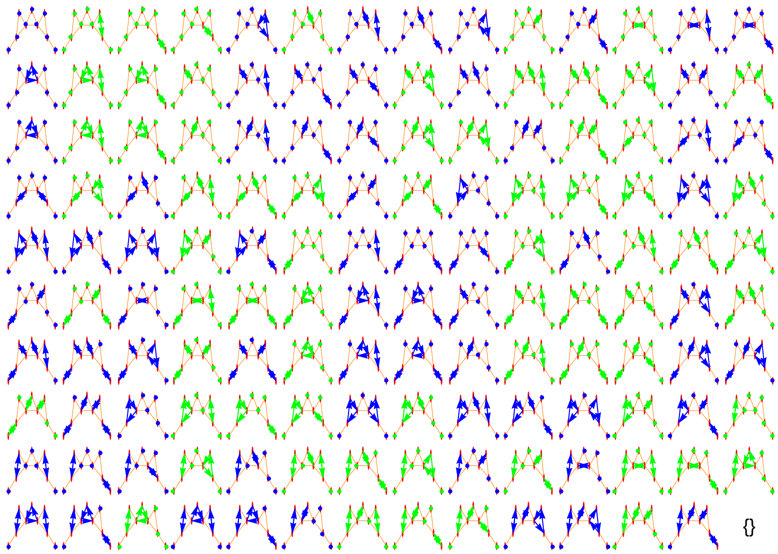

Figure 16. There are 139 one-dimensional oriented subgraphs of G' if G is the linear graph of length 4. There are 70 with even signature and 69 with odd signature. We have $\psi(G) = 70 - 69 = 1$ and $\phi(G) = (-1)^4 = 1$.

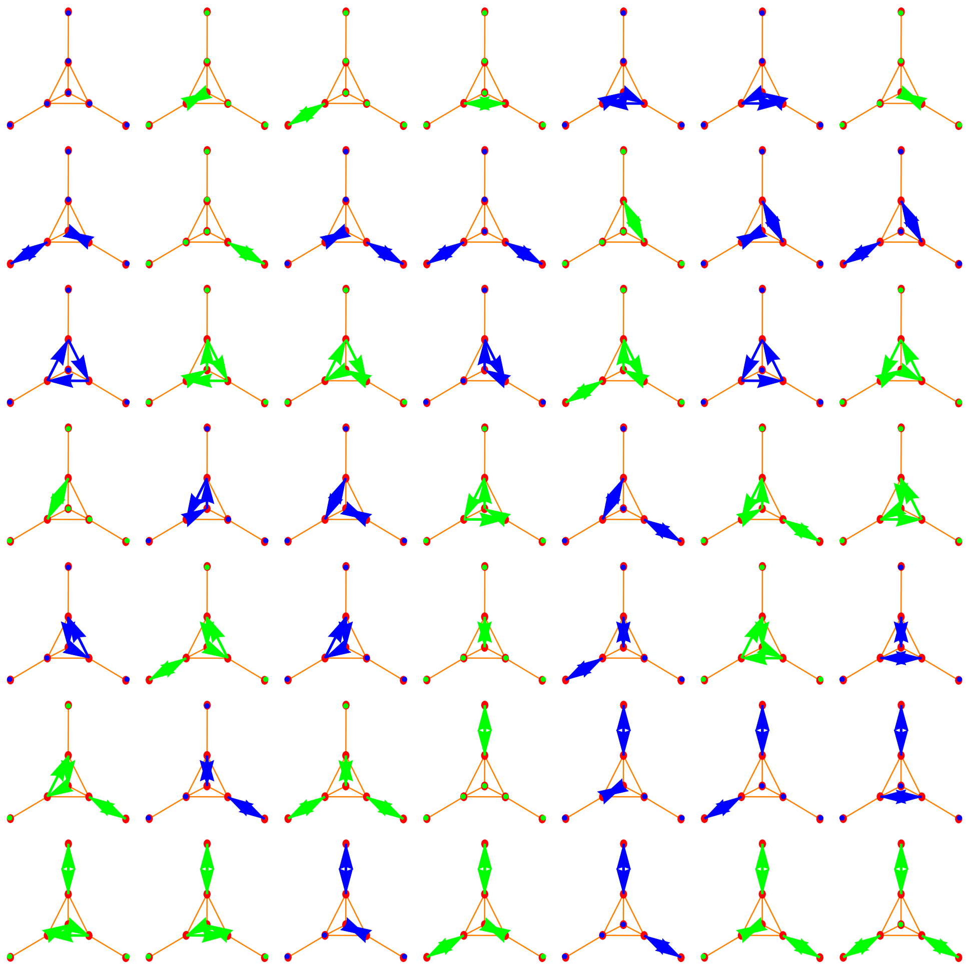

FIGURE 17. There are 49 one-dimensional oriented subgraphs of G' if G is the star graph with three spikes. 25 of them have odd signature. We have $\psi(G) = 24 - 25 = -1$ and $\phi(G) = (-1)^3 = -1$.

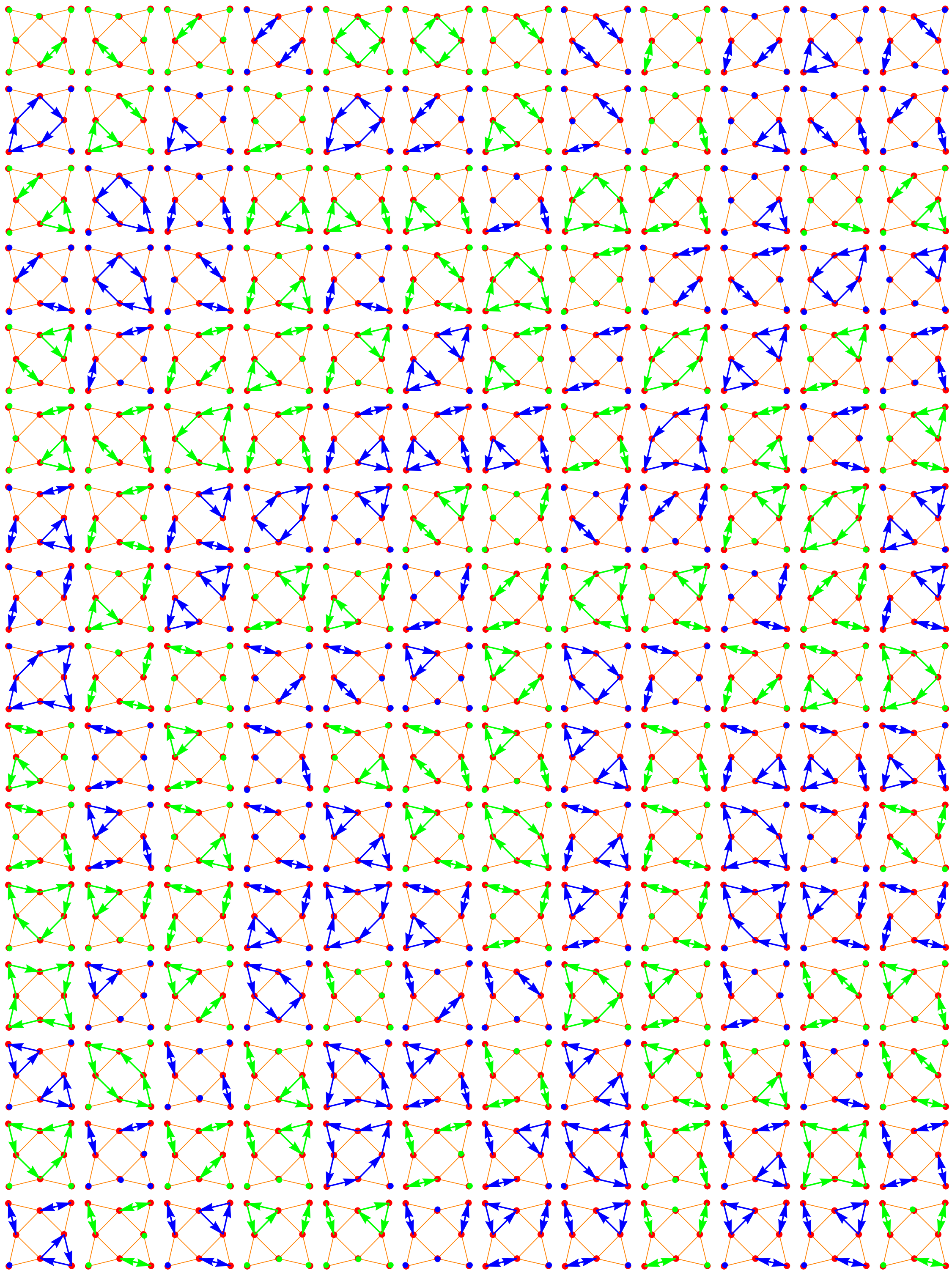

FIGURE 18. There are 193 one-dimensional oriented subgraphs of $G' = C_4'$ for $G = C_4$. The figure shows all except the empty path, 97 of of them have signature 1, and 96 have signature -1. We have $\psi(G) = 97 - 96 = 1$ and $\phi(G) = (-1)^4 = 1$.

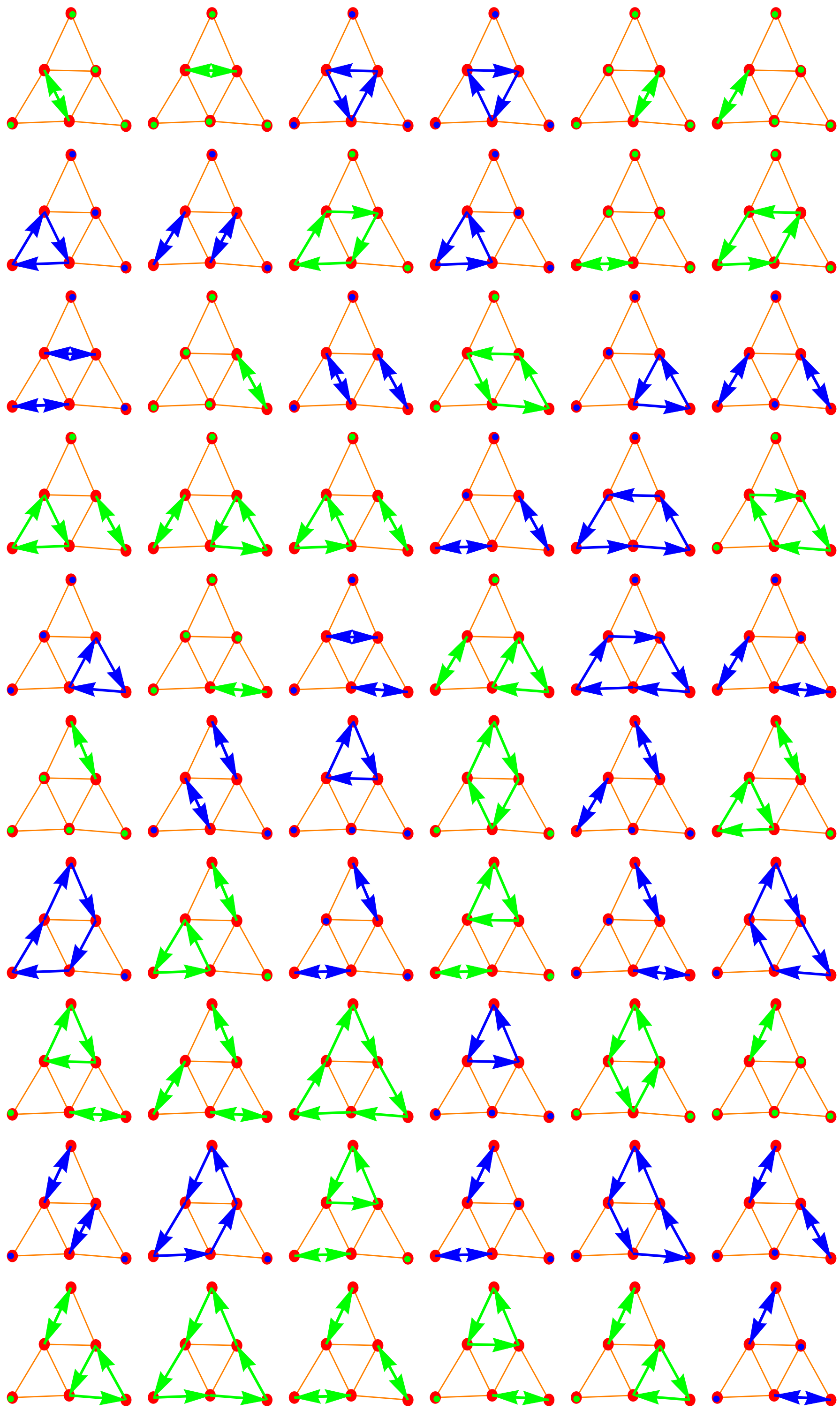

FIGURE 19. There are 61 one-dimensional oriented subgraphs of $G = K_3$ equipped with the one dimensional skeleton complex G'. We see the 60 paths which are different from the identity. 30 have signature 1, and 31 have signature -1. We have $\psi(G) = 30 - 31 = -1$ and $\phi(G) = (-1)^3 = -1$.

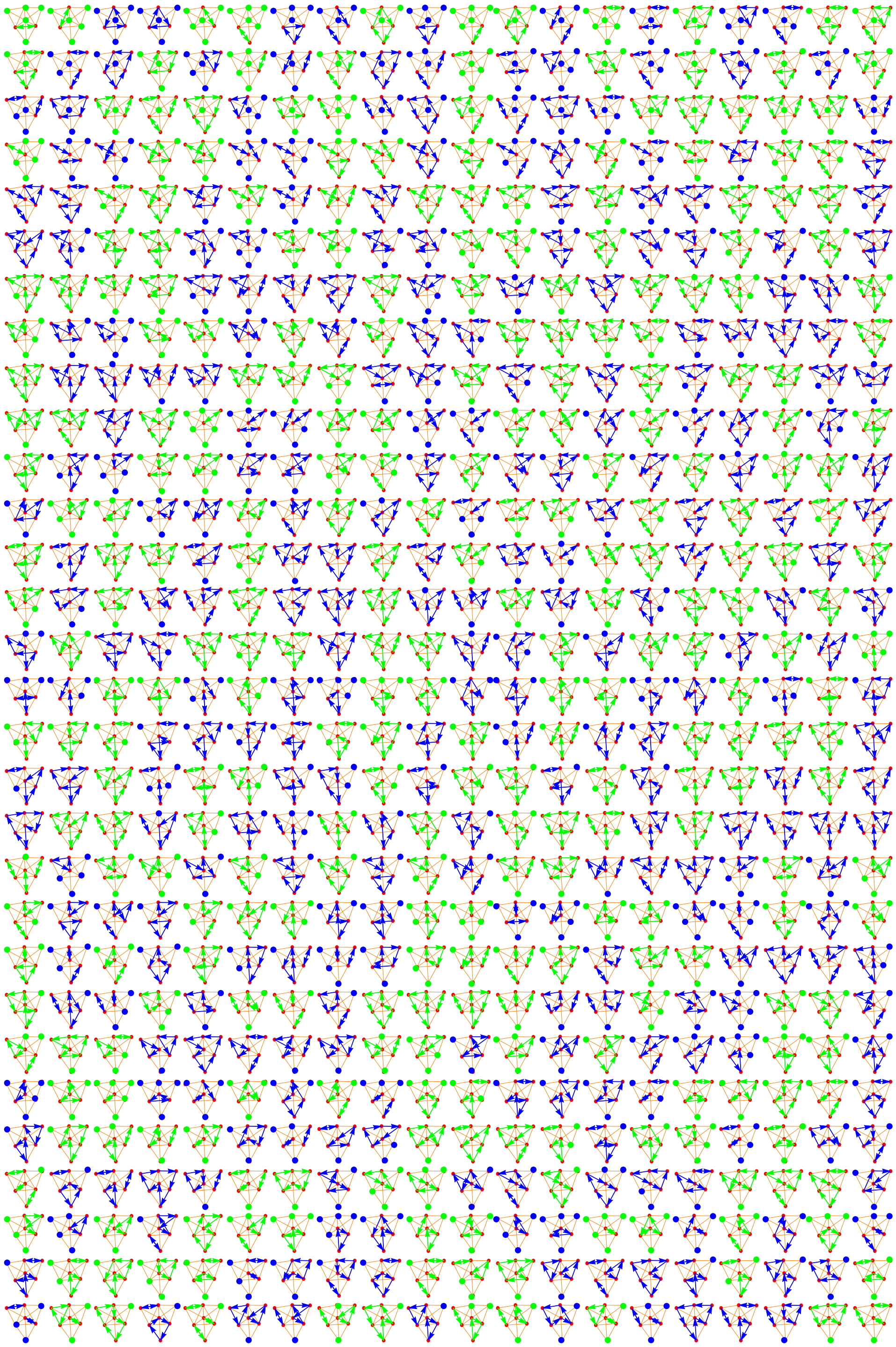

Figure 20. There are 601 one-dimensional oriented subgraphs of $G' = K_3'$ if $G = K_3$ is equipped with the Whitney complex. We see the 600 paths different from the identity. 300 permutations have signature 1, and 301 permutations have signature -1. We have $\psi(G) = 300 - 301 = -1$ and $\phi(G) = (-1)^3 = -1$.

10. Some code

The following code is written in the Mathematica language which knows graphs as a fundamental data structure. We first compute the Fredholm determinant $\det(1 + A)$ of a graph G with adjacency matrix A. The procedure "Whitney" produces the Whitney complex of a graph, the set of all complete subgraphs which is a finite abstract simplicial complex. This procedure is then is used to get the connection graph of a graph. We then compute $\psi(G)$ and $\phi(G)$ for 100 random graphs.

```
Fredholm[A_]:=A+IdentityMatrix[Length[A]];
FredholmDet[s_]:=Det[Fredholm[AdjacencyMatrix[s]]];
Experiment1=Table[FredholmDet[WheelGraph[k]],{k,4,20}]

CliqueNumber[s_]:=Length[First[FindClique[s]]];
ListCliques[s_,k_]:=Module[{n,t,m,u,r,V,W,U,l={},L},L=Length;
  VL=VertexList;EL=EdgeList;V=VL[s];W=EL[s]; m=L[W]; n=L[V];
  r=Subsets[V,{k,k}];U=Table[{W[[j,1]],W[[j,2]]},{j,L[W]}];
  If[k==1,l=V,If[k==2,l=U,Do[t=Subgraph[s,r[[j]]];
  If[L[EL[t]]==k(k-1)/2,l=Append[l,VL[t]]],{j,L[r]}]]];l];
Whitney[s_]:=Module[{F,a,u,v,d,V,LC,L=Length},V=VertexList[s];
  d=If[L[V]==0,-1,CliqueNumber[s]];LC=ListCliques;
  If[d>=0,a[x_]:=Table[{x[[k]]},{k,L[x]}];
  F[t_,l_]:=If[l==1,a[LC[t,1]],If[l==0,{},LC[t,l]]];
  u=Delete[Union[Table[F[s,l],{l,0,d}]],1]; v={};
  Do[Do[v=Append[v,u[[m,l]]],{l,L[u[[m]]]}],{m,L[u]}],v={}];v];
ConnectionGraph[s_] := Module[{c=Whitney[s],n,A},n=Length[c];
   A=Table[1,{n},{n}];Do[If[DisjointQ[c[[k]],c[[l]]]||
   c[[k]]==c[[l]],A[[k,l]]=0],{k,n},{l,n}];AdjacencyGraph[A]];
Fvector[s_] := Delete[BinCounts[Length /@ Whitney[s]], 1];
FermiCharacteristic[s_]:=Module[{f=Fvector[s]},
                     (-1)^Sum[f[[2k]],{k,Floor[Length[f]/2]}]];
FredholmCharacteristic[s_]:=FredholmDet[ConnectionGraph[s]];

Experiment2 = Do[s=RandomGraph[{10,30}];
  Print[{FermiCharacteristic[s],FredholmCharacteristic[s]}],
{100}];
```

Lets look at the proposition

```
ConnectionGraph[s_,V_]:=Module[{c=Whitney[s],n,A},n=Length[c];
  A=Table[1,{n+1},{n+1}];Do[If[DisjointQ[c[[k]],c[[l]]]  ||
  c[[k]]==c[[l]],A[[k,l]]=0],{k,n},{l,n}]; A[[n+1,n+1]]=0;
  Do[If[DisjointQ[V,c[[k]]],A[[k,n+1]]=0; A[[n+1,k]]=0],{k,n}];
  AdjacencyGraph[A]];
EulerChi[s_]:=Module[{c=Whitney[s]},
  Sum[(-1)^(Length[c[[k]]]-1),{k,Length[c]}]];
Experiment3 = Do[ s=RandomGraph[{30,50}]; v=VertexList[s];
   V=RandomChoice[v,Random[Integer,Length[v]]]; h=Subgraph[s,V];
   ss=ConnectionGraph[s]; sss=ConnectionGraph[s,V];
   u={FredholmDet[ss],FredholmDet[sss],1-EulerChi[h]};
   Print[u," ",u[[1]]  u[[3]]==u[[2]]],{100}  ];
```

And here is an other experiment. What is the probability of a random Erdős Rényi graph in $E(n,p)$ to have connection Fredholm determinant 1? In this example, we experiment with $n = 12$ and $p = 0.6$:

```
ErdoesRenyi[n_,p_]:=RandomGraph[{n,Floor[p n(n-1)/2]}];
G[n_,p_]:=FredholmDet[ConnectionGraph[ErdoesRenyi[n,p]]];
n=12;p=0.6;k=0.;m=0;
Experiment4 = Do[m++;If[G[n,p]==1,k++];Print[k/m],{Infinity}];
```

And here are the procedures producing the prime graphs G_n and prime connection graph H_n. In the first case we verify numerically $\chi(G_n) = \sum_k(-\mu(k))$, in the second case we verify numerically $\psi(H_n) = \prod_k(-\mu(k))$, where $\mu(k)$ is the Möbius function of the integer k.

```
PrimeGraph[M_]:=Module[{V={},e,s},
 Do[If[MoebiusMu[k]!=0,V=Append[V,k]], {k,2,M}];e={};
 Do[If[(Divisible[V[[k]],V[[l]]] || Divisible[V[[l]],V[[k]]]),
 e=Append[e,V[[k]]->V[[l]]]],{k,Length[V]},{l,k+1,Length[V]}];
 UndirectedGraph[Graph[V,e]]];

PrimeConnectionGraph[M_]:=Module[{V={},e,s},
 Do[If[MoebiusMu[k]!=0,V=Append[V,k]], {k,2,M}];e={};
 Do[If[GCD[V[[k]],V[[l]]]>1 || GCD[V[[l]],V[[k]]]>1,
 e=Append[e,V[[k]]->V[[l]]]],{k,Length[V]},{l,k+1,Length[V]}];
 UndirectedGraph[Graph[V,e]]];

Test5[M_]:=Module[{V,g,h,j}, h=PrimeConnectionGraph[M];
 g=PrimeGraph[M]; V=VertexList[g];
 j=Table[-MoebiusMu[V[[k]]],{k,Length[V]}];
 Print[{Total[j]                    ,EulerChi[g],
        Product[j[[k]],{k,Length[j]}],FredholmDet[h]}]];
Experiment5 = Do[Test5[k],{k,10,200}];
```

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