

ON A DEHN-SOMMERVILLE FUNCTIONAL FOR SIMPLICIAL COMPLEXES

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ABSTRACT. Assume G is a finite abstract simplicial complex with f -vector (v_0, v_1, \dots) , and generating function $f(x) = \sum_{k=1}^{\infty} v_{k-1} x^k = v_0 x + v_1 x^2 + v_2 x^3 + \dots$, the Euler characteristic of G can be written as $\chi(G) = f(0) - f(-1)$. We study here the functional $f'_1(0) - f'_1(-1)$, where f'_1 is the derivative of the generating function f_1 of G_1 . The Barycentric refinement G_1 of G is the Whitney complex of the finite simple graph for which the faces of G are the vertices and where two faces are connected if one is a subset of the other. Let L is the connection Laplacian of G , which is $L = 1 + A$, where A is the adjacency matrix of the connection graph G' , which has the same vertex set than G_1 but where two faces are connected they intersect. We have $f'_1(0) = \text{tr}(L)$ and for the Green function $g = L^{-1}$ also $f'_1(-1) = \text{tr}(g)$ so that $\eta_1(G) = f'_1(0) - f'_1(-1)$ is equal to $\eta(G) = \text{tr}(L - L^{-1})$. The established formula $\text{tr}(g) = f'_1(-1)$ for the generating function of G_1 complements the determinant expression $\det(L) = \det(g) = \zeta(-1)$ for the Bowen-Lanford zeta function $\zeta(z) = 1/\det(1 - zA)$ of the connection graph G' of G . We also establish a Gauss-Bonnet formula $\eta_1(G) = \sum_{x \in V(G_1)} \chi(S(x))$, where $S(x)$ is the unit sphere of x the graph generated by all vertices in G_1 directly connected to x . Finally, we point out that the functional $\eta_0(G) = \sum_{x \in V(G)} \chi(S(x))$ on graphs takes arbitrary small and arbitrary large values on every homotopy type of graphs.

1. SETUP

1.1. A finite abstract simplicial complex G is a finite set of non-empty sets with the property that any non-empty subset of a set in G is in G . The elements in G are called **faces** or **simplices**. Every such complex defines two finite simple graphs G_1 and G' , which both have the same vertex set $V(G_1) = V(G') = G$. For the graph G_1 , two vertices are connected if one is a subset of the other; in the graph G' , two faces are connected, if they intersect. The graph G_1 is called

Date: May 29, 2017.

1991 Mathematics Subject Classification. 05C50, 57M15, 68R10.

Key words and phrases. Mass gap.

the **Barycentric refinement** of G ; the graph G' is the **connection graph** of G . The graph G_1 is a subgraph of G' which shares the same topological features of G . On the other hand, the connection graph is fatter and be of different topological type: already the Euler characteristic $\chi(G)$ and $\chi(G')$ can differ. Both graphs G_1 and G are interesting on their own but they are linked in various ways as we hope to illustrate here. Terminology in this area of combinatorics is rich. One could stay within simplicial complexes for example and deal with “flag complexes”, complexes which is a Whitney complex of its 1-skeleton graphs. The complexes G_1 and G' are by definition of this type. We prefer in that case to use terminology of graph theory.

1.2. Let A be the adjacency matrix of the connection graph G' . Its Fredholm matrix $L = 1 + A$ is called the **connection Laplacian** of G . We know that L is unimodular [13] so that the **Green function operator** $g = L^{-1}$ has integer entries. This is the **unimodularity theorem** [15]. The Bowen-Lanford zeta function of the graph G' is defined as $\zeta(s) = \det((1 - sA)^{-1})$. As $\zeta(-1)$ is either 1 or -1 , we can see the determinant of L as the value of the zeta function at $s = -1$. We could call $H = L - L^{-1}$ the **hydrogen operator** of G . The reason is that classically, if $L = -\Delta$ is the Laplacian in R^3 , then L^{-1} is an integral operator with entries $g(x, y) = 1/|x - y|$. Now, $H\psi(y) = (L\psi)(y) - \psi(y)/|x - y|$ is the Hamiltonian of a Hydrogen atom located at x , so that H is a sum of a kinetic and potential part, where the potential is determined by the inverse of L . When replacing the multiplication operation with a convolution operation, then L^{-1} takes the role of the potential energy. Anyway, we will see that the trace of H is an interesting variational problem.

1.3. There are various variational problems in combinatorial topology or in graph theory. For the later, see [3]. An example in polyhedral combinatorics is the upper bound theorem, which characterizes the maxima of the discrete volume among all convex polytopes of a given dimension and number of vertices [19]. An other example problem is to maximize the Betti number $b(G) = \sum_{i=0} b_i$ which is bounded below by $\chi(G) = \sum_{i=0} (-1)^i b_i$ which we know to grow exponentially in general in the number of elements in G and for which upper bounds are known too [1]. We have looked at various variational problems in [10] and at higher order Euler characteristics in [12]. Besides extremizing functionals on geometries, one can also define functionals on the on the set of unit vectors of the Hilbert space H^n generated by the geometry. An example

is the free energy $(\psi, L\psi) - TS(|\psi|^2)$ which uses also entropy S and temperature variable T [15].

1.4. Especially interesting are functionals which characterize geometries. An example is a necessary and sufficient condition for a f -vector of a simplicial d -polytope to be the f -vector of a simplicial complex polytope, conjectured 1971 and proven in 1980 [2, 17]. Are there variational conditions which filter out discrete manifolds? We mean with a discrete manifold a connected finite abstract simplicial complex G for which every unit sphere $S(x)$ in G_1 is a sphere. The notion of sphere has been defined combinatorially in discrete Morse approaches using critical points [4] or discrete homotopy [5]. A 2-complex for example is a discrete 2-dimensional surface. In a 2-complex, we ask that every unit sphere in G_1 is a circular graph of length larger than 3. For a 2-complex the f -vector of G_1 obviously satisfies $2v_1 - 3v_2 = 0$ as we can count the number of edges twice by adding up 3 times the number of triangles. The relation $2v_1 - 3v_2 = 0$ is one of the simplest Dehn-Sommerville relations. It also can be seen as a zero curvature condition for 3-graphs [7] or then related to eigenvectors to Barycentric refinement operations [12, 11]. Dehn-Sommerville relations can be seen as zero curvature conditions for Dehn-Sommerville invariants in a higher dimensional complex.

1.5. One can wonder for example whether a condition like $\eta(G) = 2v_1 - 3v_2 = 0$ for the f -vector (v_0, v_1, v_2) of the Barycentric refinements G_1 of a general 2-dimensional abstract finite simplicial complex G forces the graph G_1 to have all unit spheres to be finite unions of circular graphs. For this particular functional, this is not the case. There are examples of discretizations of varieties with 1-dimensional set of singular points for which $2v_1 - 3v_2$ is negative. An example is $C_n \times F_8$, the Cartesian product of a circular graph with a figure 8 graph. An other example is a k -fold suspension of a circle $G = C_n + P_k$, where C_n is the circular graph, P_k the k vertex graph without edges and $+$ is the Zykov join which takes the disjoint union of the graphs and connects additionally any two vertices from different graphs. In that case, $\eta_0(G) = n(2 - k)$ which is zero only in the discrete manifold case $k = 2$ where we have a discrete 2-sphere, the suspension of a discrete circle.

1.6. Our main result here links a spectral property with a combinatorial property. It builds on previous work on the connection operator L and its inverse $g = L^{-1}$. We will see that $\eta(G_1) = \text{tr}(L - L^{-1})$, where L is the connection Laplacian of G , which remarkably is always

invertible. If G has n faces=simplices=sets in G , the matrix L is a $n \times n$ matrix for which $L_{xy} = 1$ if x and y intersect and where $L_{xy} = 0$ if $x \cap y$ is empty. We establish that the **combinatorial functional** $\eta(G_1) = 2v_1 - 3v_2 + 4v_3 - 5v_4 + \dots$ which is also the **analytic functional** $f'(0) - f'(-1)$ for a generating function $f(x) = \sum_{k=1}^{\infty} v_{k-1}x^k$ is the same than the **algebraic functional** $\text{tr}(L - L^{-1})$ and also equal to the **geometric functional** $\eta_1(G) = \sum_{x \in V(G_1)} \chi(S(x))$. The later is a Gauss-Bonnet formula which in general exists for any linear or multi-linear valuation [12].

1.7. The functional η_1 is a valuation like the Euler characteristic $\chi(G) = v_0 - v_1 + v_2 - \dots$ whose combinatorial definition can also be written as $f(0) - f(-1)$ or as a Gauss-Bonnet formula $\sum_x K(x)$ or then as the super trace of a heat kernel $\text{tr}(e^{-tL})$ by McKean-Singer [8]. The Euler curvature $K(x) = \sum_{k=0}^{\infty} (-1)^k v_{k-1}(S(x))/(k+1)$ [7] could now be written as $K(x) = F(0) - F(-1)$, where $F(t) = \int_0^t f(s) ds = \sum_{k=0}^{\infty} v_{k-1}x^{k+1}/(k+1)$ is the anti-derivative of the **reduced generating function** $1 - f$ of $S(x)$. We see a common theme that $F(0) - F(-1)$, $f(0) - f(-1)$, $f'(0) - f'(-1)$ all appear to be interesting.

1.8. Euler characteristic is definitely the most fundamental valuation as it is related to the unique eigenvector of the eigenvalue 1 of the Barycentric refinement operator. It also has by Euler-Poincaré a cohomological description $b_0 - b_1 + b_2 - \dots$ in terms of Betti numbers. The minima of the functional $G \rightarrow \chi(G)$ however appear difficult to compute [9]. From the expectation formula $E_{n,p}[\chi] = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} p^{\binom{k}{2}}$ [6] of χ on Erdős-Renyi spaces we know that unexpectedly large or small values of $\chi(G)$ can occur, even so we can not construct them directly. As the expectation of $\chi = b_0 - b_1 + b_2 - \dots$ grows exponentially with the number of vertices. Also the total sum of Betti numbers grows therefore exponentially even so the probabilistic argument gives no construction. We have no idea to construct a complex with 10000 simplices for which the total Betti number is larger than say 10^{100} even so we know that it exists as there exists a complex G for which $\chi(G)$ is larger than 10^{100} . Such a complex must be a messy very high-dimensional Swiss cheese.

1.9. After having done some experiments, we first felt that $\eta(G)$ must be non-negative. But this is false. In order to have negative Euler characteristic for a unit sphere of a two-dimensional complex, we need already to have some vertex for which $S(x)$ is a bouquet of spheres. A small example with $\eta(G) < 0$ is obtained by taking a sphere, then glue

in a disc into the inside which is bound by the equator. This produces a geometry G with Betti vector $(1, 0, 2)$ and Euler characteristic 3. It satisfies $\eta(G) = -8$ as every of the 8 vertices at the equator of the Barycentric refinement of G_1 has curvature $\chi(S(x)) = -1$ and for all the other vertices have $\chi(S(x)) = 0$.

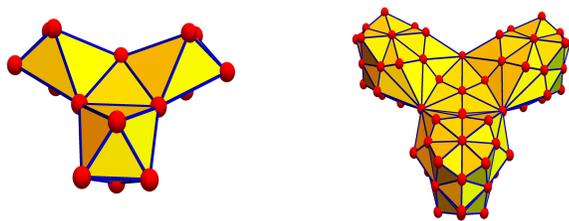


FIGURE 1. A 2-dimensional complex with $\eta(G) = -6 < 0$. The f -vector is $(v, e, f) = (15, 36, 25)$, the Betti numbers are $b_0 = 1, b_1 = 0, b_2 = 3$, the Euler characteristic $v - e + f = b_0 - b_1 + b_2 = 4$. In this case, $2e - 3f = (15, 36, 25) \cdot (0, 2, -3) = -3$. To the right, we see the refinement with f -vector $(76, 222, 150)$ and $\eta(G_1) = 2 \cdot 222 - 3 \cdot 150 = -6$. In general, the value doubles under refinement.

1.10. This example shows that $\eta(G)$ can become arbitrarily small even for two-dimensional complexes. But what happens in this example there is a one dimensional singular set. It is the circle along which the disk has been glued between three spheres. We have not yet found an example of a complex G for which G_1 has a discrete set of singularities (vertices where the unit sphere is not a sphere.) In the special case where G is the union a finite set of geometric graphs with boundary in such a way that the intersection set is a discrete set, then $\eta(G) \geq 0$.

2. OLD RESULTS

2.1. Given a face x in G , it is also a vertex in G_1 . The **dimension** $\dim(x) = |x| - 1$ with cardinality $|x|$ now defines a function on the vertex set of G_1 . It is locally injective and so a **coloring**. We know already $g(x, x) = 1 - \chi(S(x))$ [16] and that $V(x) = \sum_y g(x, y) =$

$(-1)^{\dim(x)}g(x, x)$ is **curvature**: $\sum_x V(x) = \chi(G)$. It is dual to the curvature $\omega(x) = (-1)^{\dim(x)}$ for which Gauss-Bonnet $\sum_x \omega(x)$ is the definition of Euler characteristic. Both of these formulas are just Poincaré-Hopf for the gradient field defined by the function \dim . The Gauss-Bonnet formula $\sum_x V(x) = \chi(G)$ can be rewritten as $\sum_{x,y} g(x, y) = \chi(G)$. We call this the **energy theorem**. It tells that the total potential energy of a simplicial complex is the Euler characteristic of G . By the way, $\sum_{x,y} L(x, y) = |V(G')| + 2|E(G')|$ by Euler handshake.

2.2. If v_k counts the number of k -dimensional faces of G , then $f(x) = \sum_{k=1}^{\infty} v_{k-1}x^k = v_0x + v_1x^2 + \dots$ is a **generating function** for the f -vector (v_0, v_1, \dots) of G . We can rewrite the Euler characteristic of G as $\chi(G) = -f(-1) = f(0) - f(-1)$. If G is a graph, we assume it to be equipped with the **Whitney complex**, the finite abstract simplicial complex consisting of the vertex sets of the complete subgraphs of G . This in particular applies for the graph G_1 . The f -vector of G_1 is obtained from the f -vector of G by applying the matrix $S_{ij} = i!S(j, i)$, where $S(j, i)$ are **Stirling numbers** of the second kind. Since the transpose S^T has the eigenvector $(1, -1, 1, -1, \dots)$, the Euler characteristic is invariant under the process of taking Barycentric refinement. Actually, as S has simple spectrum, it is up to a constant the unique valuation of this kind. Quantities which do not change under Barycentric refinements are called **combinatorial invariants**.

2.3. The matrices A, L, g act on a finite dimensional Hilbert space whose dimension is the number of faces in G which is the number of vertices of G_1 or G' . Beside the usual trace tr there is now a **super trace** str defined as $\text{str}(L) = \sum_x \omega(x)L(x, x)$ with $\omega(x) = (-1)^{\dim(x)}$. The definition $\chi(G) = \sum_x \omega(x)$ can now be written as $\text{str}(1) = \chi(G)$. Since L has 1's in the diagonal, we also have $\text{str}(L) = \chi(G)$. A bit less obvious is $\chi(G) = \text{str}(g)$ which follows from the Gauss-Bonnet analysis leading to the energy theorem. It follows that the Hydrogen operator H satisfies $\text{str}(H) = 0$, the super trace of H is zero. This leads naturally to the question about the trace of H . By the way, the super trace of the Hodge Laplacian $L = (d + d^*)^2$ where d is the exterior derivative is always zero by Mc-Kean Singer (see [8] for the discrete case).

2.4. The Barycentric refinement graph G_1 and the connection graph G' have appeared also in a number theoretical setup. If G is the countable complex consisting of all finite subsets of prime numbers, then the finite prime graph $G_1(n) \subset G_1$ has as vertices all square-free integers in $V(n) = \{2, 3, 4, \dots, n\}$, connecting two if one divides the other. The prime connection graph $G'(n)$ has the same vertices

than $G_1(n)$ but connects two integers if they have a common factor larger than 1. This picture interprets sets of integers as simplicial complexes and sees counting as a Morse theoretical process [14]. Indeed $\chi(G_1(n)) = 1 - M(n)$, where $M(n)$ is the Mertens function. If the vertex n has been added, then $i(n) = 1 - \chi(S(n)) = -\mu(n)$ with Möbius function μ is a Poincaré-Hopf index and $\sum_x i(x) = \chi(G_1(x))$ is a Poincaré-Hopf formula. In combinatorics, $-i(G) = \chi(G) - 1$ is called the **reduced Euler characteristic** [18]. The counting function $f(x) = x$ is now a discrete Morse function, and each vertex is a critical point. When attaching a new vertex x , a handle of dimension $m(x) = \dim(S(x)) + 1$ is added. Like for critical points of Morse functions in topology, the index takes values in $\{-1, 1\}$ and $i(x) = (-1)^{m(x)}$. For the connection Laplacian adding a vertex has the effect that the determinant gets multiplied by $i(x)$. Indeed, $\det(L) = \prod_x \omega(x)$ in general while $\chi(G) = \sum_x \omega(x)$, if $\omega(x) = (-1)^{\dim(x)}$.

3. THE FUNCTIONAL

3.1. Define the functional

$$\eta(G) = \text{tr}(H) = \text{tr}(L - L^{-1}) .$$

Due to lack of a better name, we call it the **hydrogen trace**. We can rewrite this functional in various ways. For example $\eta(G) = \sum_k \lambda_k - 1/\lambda_k$, where λ_k are the eigenvalues of L . We can also write $\eta(G) = \sum_k \mu_k \frac{2+\mu_k}{1+\mu_k}$ where μ_k are the eigenvalues of the adjacency matrix A of the connection graph G' . It becomes interesting however as we will be able to link η explicitly with the f -vector of the complex G_1 or even with the f -vector of the complex G itself.

3.2. We will see below that also the **Green trace functional** $\text{tr}(g)$ is interesting as $g = L^{-1}$ is the Green function of the complex. It is bit curious that there are analogies and similarities between the Hodge Laplacian $H = (d + d^*)^2$ of a complex and the connection Laplacian L . Both matrices have the same size. As they work on a space of simplices, where the dimension functional defines a parity, one can also look at the **super trace** $\text{str}(L) = \sum_{\dim(x)>0} L_{xx} - \sum_{\dim(x)<0} L_{xx}$ It is a consequence of Mc-Kean Singer super symmetry that $\text{str}(H) = 0$ which compares with the definition $\text{str}(L) = \chi(G)$ and leads to the McKean Singer relation $\text{str}(e^{-tH}) = \chi(G)$. We have seen however the Gauss-Bonnet relation $\text{str}(g) = \chi(G)$ which implies the energy theorem $\sum_{x,y} g_{xy} = \chi(G)$. It also implies $\text{str}(L - g) = 0$.

3.3. The invertibility of the connection Laplacian is interesting and lead to topological relations complementing the topological relations of the Hodge Laplacian to topology like the Hodge theorem telling that the kernel of the k 'th block of H is isomorphic the k 'th cohomology of G . Both L and H have deficits: we can not read off cohomology of L but we can not invert H , the reason for the later is exactly cohomology as harmonic forms are in the kernel of H . So, there are some complementary benefits of both L and H . And then there are similarities like $\text{str}(H) = \text{str}(L - L^{-1}) = 0$ and $\text{str}(e^{-H}) = \text{str}(L^{-1}) = \chi(G)$.

4. GAUSS-BONNET

4.1. The following **Gauss-Bonnet** theorem for η shows that its curvature at a face x is the Euler characteristic of the unit sphere $S(x)$ in the Barycentric refinement G_1 . We use the notation $\eta_0(G) = \sum_{x \in V(G)} \chi(S(x))$, $\eta_1(G) = \eta_0(G_1)$ and $\eta(G) = \text{tr}(L - L^{-1})$.

Theorem 1. *Let G be an arbitrary abstract finite simplicial complex. Then*

$$\text{tr}(L - L^{-1}) = \eta(G) = \eta_1(G) = \eta_0(G_1) = \sum_{x \in V(G_1)} \chi(S(x)) .$$

Proof. The diagonal elements of $g = L^{-1}$ has entries $(1 - \chi(S(x)))$. We therefore have $\text{tr}(g) = \sum_x (1 - \chi(S(x)))$. We also have $\text{tr}(G) = \sum_x 1$. \square

Examples.

- 1) If $G = C_n$, then $G_1 = C_{2n}$. Now, $\chi(S(x)) = 2$ for all vertices $x \in G_1$. We see that $\eta(C_n) = 4n$.
- 2) For a discrete two-dimensional graph G , a graph for which every unit sphere is a circular graph, we have $\eta(G) = 0$.
- 3) For a discrete three-dimensional graph G , a graph for which every unit sphere is a two dimensional sphere, we have $\eta(G) = 2V(G_1) = 2 \sum_{k=0}^{\infty} v_k(G)$. For example, for the 3-sphere, the suspension of the octahedron, which can be written as $G = 3P_2 = P_2 + P_2 + P_2$, we have $\eta(G) = 160$ because the f -vector of G is $\vec{v} = (8, 24, 32, 16)$.
- 4) For a graph without triangles, we have $\eta(G) = \sum_{x \in V(G_1)} \text{deg}(x)$ which is by handshaking $2v_1(G_1)$. Since Barycentric refinement doubles the edges, we have $\eta(G) = 4v_1(G)$. This generalizes the circular case discussed above.
- 5) For $G = K_n$ we have $\eta(G) = 2^n$ if n is even and $2^n - 2$ if n is odd. The numbers start as following: $\eta(K_1) = 0$ $\eta(K_2) = 4$, $\eta(K_3) = 6$, $\eta(K_4) = 16$, $\eta(K_5) = 30$ etc.

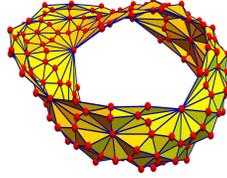


FIGURE 2. The Möbius strip is a 2-dimensional graph with connected boundary. As $\chi(S(x))$ is zero in the interior and $\chi(S(x)) = 1$ at the boundary we see that $\eta(G)$ is the length of the boundary. In the displayed example of the discrete Moebius strip, we have $\eta(G) = 32$.

5. GENERATING FUNCTION

5.1. Let $f_{G_1}(x) = 1 + v_1x + v_2x^2 + \dots = 1 + \sum_{k=1}^{\infty} v_{k-1}x^k$ be the (reduced) generating function for the Barycentric refinement G_1 of G . The **Zykov join** of two graphs $G_1 + H_1$ is defined as the graph with vertex set $V(G_1) \cup V(H_1)$ for which two vertices a, b are connected if they were connected in G_1 or H_1 or if a, b belong to different graphs. The generating function of the sum $G_1 + H_1$ is the product of the generating functions of G_1 and H_1 .

Since the Euler characteristic satisfies $\chi(G) = f(0) - f(-1) = \chi(G_1) = f_1(0) - f_1(-1)$, the following again shows that the functional η appears natural;

Corollary 1. $\eta(G) = f_1'(0) - f_1'(-1)$.

5.2. To prove this, we rewrite the Gauss-Bonnet result as a Gauss-Bonnet result for the second Barycentric refinement G_2 . Define for a vertex x in G_2 the curvature

$$k(x) = (-1)^{1+\dim(x)}(1 + \dim(x)) .$$

Lemma 1. $\eta(G) = \sum_{x \in V(G_2)} k(x)$, where the sum is over all vertices x in G_2 which have positive dimension.

Proof. This is a handshake type argument. We start with $\eta(G) = \sum_x \chi(S(x))$. Since every d -dimensional simplex in $S(x)$ defines a $(d+1)$ -dimensional simplex containing x , super summing over all simplices of $S(x)$ gives a super sum over simplices in G_2 where each simplex such y appears $\dim(y) + 1$ times. \square

Remark. This gives us an upper bound on the functional η in terms of the number of vertices in G_2 and the maximal dimension of G : $\eta(G) \leq |V(G_2)|(1+d)$. If G_1 has n elements, then G_2 has $\leq (d+1)n$ elements. We see:

Corollary 2. $\eta(G)$ is bounded above by $C_d|V(G_1)|$, where C_d only depends on the maximal dimension of G .

5.3. Now we can prove the result:

Proof. As $f_1(x) = v_0x + v_1x^2 + v_2x^3 + \dots$, we have $f'_1(x) = v_0 + 2v_1x + 3v_2x^2 + \dots$ and $f'_1(0) - h'_1(-1) = 2v_1 - 3v_2 + 4v_3$ which is the same than $\sum_{x, \dim(x) > 0} (-1)^{1+\dim(x)}(1 + \dim(x))$. \square

As an application we can get a formula for $\eta_0(G_1 + H_1)$, where $G_1 + H_1$ is the Zykov sum of G_1 and H_1 . The Zykov sum shares the properties of the classical join operation in the continuum. The Grothendieck argument produces from the monoid a group which can be augmented to become a ring [16].

Corollary 3. On the set of complexes with zero Euler characteristic, we have $\eta_0(G_1 + H_1) = \eta_0(G_1) + \eta_0(H_1)$.

Proof. We have $f_{G_1+H_1} = f_{G_1}f_{H_1}$. Now $\eta_0(G_1) = f'_{G_1}(0) - f'_{G_1}(-1)$ and $\eta_0(H_1) = f'_{H_1}(0) - f'_{H_1}(-1)$. By the product rule, $f'_{G_1+H_1} = (f_{G_1}f_{H_1})' = f'_{G_1}f_{H_1} + f_{G_1}f'_{H_1}$. we have now $f'_{G_1+H_1}(0) = f'_{G_1}(0) + f'_{H_1}(0)$ and $f'_{G_1+H_1}(-1) = f'_{G_1}(-1)(1 - \chi(H_1)) + (1 - \chi(G_1))f'_{H_1}(-1)$ so that $\eta_0(G_1 + H_1) = \eta(G_1) + \eta(G_2) + f'_{G_1}(-1)\chi(H_1) + f'_{H_1}(-1)\chi(G_1)$. \square

6. GEOMETRIC GRAPHS

6.1. We will see in this section that for graphs which discretized manifolds or varieties which have all singularities isolated and split into such discrete manifolds, the functional η is non-negative. A typical example is a bouquet of spheres, glued together at a point.

6.2. A d -**graph** is a finite simple graph for which every unit sphere $S(x)$ is a $(d-1)$ -graph which is a $(d-1)$ -sphere. A d -**sphere** is a d -graph which becomes collapsible if a single vertex is removed. The inductive definitions of d -graph and d -sphere start with the assumption that the empty graph is a -1 -sphere and -1 graph and that the 1 point graph K_1 is collapsible. A graph G is **collapsible** if there exists a vertex x such that both $G \setminus x$ and $S(x)$ are collapsible. It follows by induction that d -sphere has Euler characteristic $1 + (-1)^d \in \{0, 2\}$.

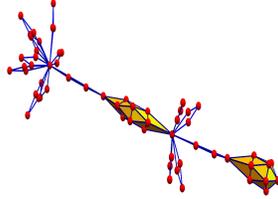


FIGURE 3. An example of a discrete variety. A graph for which all unit spheres are discrete spheres, with some exceptional but isolated points, the singularities.

6.3. A simplicial complex is called a d -complex if its refinement G_1 is a d -graph. We now see that for even-dimensional d -complexes, the functional η is zero. A graph is a **d -graph with boundary** if every unit sphere is either a sphere or contractible and such that the **boundary**, the set of vertices for which the unit sphere $S(x)$ is contractible is a $d - 1$ -graph. An example is the wheel graph G for which the boundary δG is a circular graph.

6.4. Since for an even dimensional d -graph with boundary the Euler characteristic of the unit spheres in the interior is zero, Gauss-Bonnet implies $\eta(G) = |V(\delta G)|$. In the case of an odd-dimensional d -graph with boundary, it leads to $\eta(G) = 2|\text{int}(G)| + |\delta G|$. This leads to the observation:

Lemma 2. *If G is a d -graph with boundary, then $\eta(G) \geq 0$. Equality holds if and only if G is an even dimensional graph without boundary.*

6.5. This can be generalized: if G is a union of finitely many d_k -graphs G_k such that the set of vertices which belong to at least 2 graphs is isolated in the sense that the intersection of any two G_k does not contain any edge, then $\delta(G) \geq 0$. Equality holds if G is a finite union of even dimensional graphs without boundary touching at a discrete set of points. The reason is that the unit spheres are again either spheres or then finite union of spheres of various dimension. Since the Euler characteristic of a sphere is non-negative and the Euler characteristic of a disjoint sum is the sum of the Euler characteristics, the non-negativity of η follows. We will ask below whether more general singularities are allowed and still have $\eta(G) \geq 0$.

6.6. Maybe in some physical context, one would be interested especially in the case $d = 4$ and note that among all 4-dimensional simplicial complexes with boundary the complexes without boundary minimize the functional η . In the even dimensional case, the curvature of η is supported on the boundary of G . If we think of the curvature as a kind of charge, this is natural in a potential theoretic setup. Indeed, one should think of $V_x(y) = g(x, y)$ as a potential [15]. In the case of an odd dimensional complex, there is a constant curvature present all over the interior and an additional constant curvature at the boundary. Again, also in the odd dimensional case, the absence of a boundary minimizes the functional η .

6.7. We should in this context also mention the **Wu characteristic** for which we proved in [12] that for d graphs with boundary, the formula $\omega(G) = \chi(G) - \chi(\delta G)$ holds. The Wu characteristic ω was defined as $\omega(G) = \sum_{x \cap y \neq \emptyset} \omega(x)\omega(y)$ with $\omega(x) = (-1)^{\dim(x)}$. The Wu characteristic fits into the topic of connection calculus as $\omega(G) = \text{tr}(LJ)$, where J is the **checkerboard matrix** $J_{xy} = (-1)^{\dim(i)+\dim(j)} = \omega^T \cdot \omega$ so that J/n is a projection matrix [15]. Actually, in the eyes of Max Born, one could see $\omega(G)/n = (\omega, L\omega)$ as the expectation of the **state** $\Omega = \omega/\sqrt{n}$.

7. THE SUM OF THE SPHERE EULER CHARACTERISTIC

7.1. We look now a bit closer at the functional

$$\eta_0(G) = \sum_{x \in V(G)} \chi(S(x))$$

on graphs. It appears to be positive or zero for most Erdős-Renyi graphs but it can take arbitrary large or small values. We have seen that $\eta(G) = \eta_0(G_1) = \text{tr}(L - L^{-1})$. But now, we look at graphs G which are not necessarily the Barycentric refinement of a complex.

Examples.

- 1) For a complete graph K_n we have $\eta_0(G) = n$.
- 2) For a complete bipartite graph $K_{n,m}$ we have $\eta_0(G) = 2nm$ and $\eta(G) = 4nm$.
- 3) For an even dimensional d -graph G , we have $\eta_0(G) = 0$.
- 4) For an odd dimensional d -graph G we have $\eta_0(G) = 2|V(G)|$.
- 5) For the product G of linear graph L_m of length m with a figure 8 graph $E = C_k \wedge_x C_k$ we have for $k \geq 4$ and $m \geq 1$ the formula $\eta_0(L_m \times (C_k \wedge_x C_k)) = 28 + (k - 4)8 - (m - 1)4 = 8k - 4m$.

- 6) So far, in all examples we have seen if G_1 is the Barycentric refinement, we see $|\eta_0(G_1)| \geq 2|\eta_0(G)|$.
 7) For the graph G obtained by filling in an equator plane, we have $\eta_0(G) = -4$ and $\eta(G) = -8$.

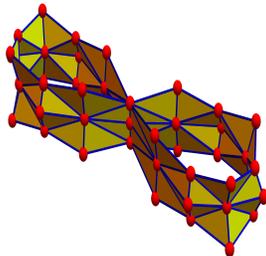


FIGURE 4. The graph $G = L_2 \times (C_4 \wedge_x C_4)$ has $\eta_0(G) = 28$. Increasing the length of C_n by 1 increases η_0 by 8; increasing the length of L_m decreases η_0 by 4.

Lemma 3. *The functional is additive for a wedge sum and for the disjoint sum.*

Proof. In both cases, the unit spheres $S(x)$ for vertices x in one of the graphs H, G only are not affected. For the vertex in the intersection, then $S_{G \cup H}(x)$ is the disjoint union $S_G(x) \cup S_H(x)$. \square

Corollary 4. *For every homotopy type of graphs, the functional η_0 is both unbounded from above and below.*

Proof. Take a graph G with a given homotopy type. Take a second graph $H = \eta_0(L_m \times (C_k \wedge C_k))$ with large m . It has $\eta_0(H) = 8k - 4m$. We can close one side of the graph to make it contractible. This produces a contractible graph with $\eta_0(\tilde{H}) = 8k - 2m$. The graph $\tilde{H} \wedge G$ now has the same homotopy type than G and has $\eta(\tilde{H} \wedge G) = 8k - 2m + \eta_0(G)$. By choosing k and m accordingly, we can make η_0 arbitrarily large or small. The addition of the contractible graph has not changed the homotopy type of G . \square

Let $C(k)$ denote the set of connected graphs with k vertices. On $C(2)$, we have $2 \leq \eta_0(G) \leq 2$, on $C(3)$, we have $3 \leq \eta_0(G) \leq 3$, on $C(4)$, we have $4 \leq \eta_0(G) \leq 8$, on $C(5)$, we have $4 \leq \eta_0(G) \leq 12$, and on $C(6)$ we have $0 \leq \eta_0(G) \leq 18$.

8. ABOUT THE SPECTRUM OF L

8.1. We have not found any positive definite connection Laplacian L yet. Since L has non-negative entries, we know that L has non-negative eigenvalues. By unimodularity [13] it therefore has some positive eigenvalue. The question about negative eigenvalues is open but the existence of some negative eigenvalues would follow from $\text{tr}(H) \geq 0$ thanks to the following formula dealing with the column vectors $A_i = L_i - e_i$ of the adjacency matrix of G' .

Lemma 4. $\eta(G) = -\sum_i (A_i, gA_i)$.

Proof. Let A be the adjacency matrix of the connection graph G' so that the connection Laplacian L satisfies $L = 1 + A$. As L has entries 1 in the diagonal only, we know $\text{tr}(L) = n$ and $g \cdot A = (1 + A)^{-1}A = -(1 - (1 + A)^{-1}) = g - 1$ so that $\eta(G) = \text{tr}(L) - \text{tr}(g) = \text{tr}(1 - g) = -\text{tr}(g \cdot A) = -\sum_i e_i g A e_i = -\sum_i e_i g A_i = -\sum_i A_i g A_i$. The reason for the last step is $\sum_i (e_i + A_i) g A_i = \sum_i L_i g A_i = \sum_i e_i A_i = 0$. \square

This immediately implies that g (and so $L = g^{-1}$) can not be positive definite if $\eta(G) \geq 0$. Indeed, if g were positive definite, then $A_i g A_i > 0$ for all A_i and so $\sum_i A_i g A_i > 0$ but $\eta(G) \geq 0$ implies $\sum_i A_i g A_i \leq 0$.

9. OPEN QUESTIONS

9.1. Since the entries of L are non-negative and L is invertible, the Perron-Frobenius theorem shows that L has some positive eigenvalue. A similar argument to show negative eigenvalues does not work yet, at least not in such a simple way. The next result could be easy but we have no answer yet for the following question:

Question: Is it true that every connection Laplacian L has some negative eigenvalue?

9.2. Lets call a graph a d -variety, if every unit sphere is a $(d - 1)$ -graph except for some isolated set of points, the singularities, where the unit sphere is allowed to be a $(d - 1)$ -variety. In every example of a d -variety with $\eta(G) < 0$ seen so far, we have the singularities non-isolated

Question: Is it true that $\eta(G) \geq 0$ if G is a d -variety?

The inequality holds for d graphs with boundary as for such graphs every unit sphere either has non-negative Euler characteristic 0, 2 (interior) or 1 (at the boundary). The example shown above with $\eta(G) < 0$

has some unit spheres which are not 1-graphs (disjoint unions of circular graphs) but 1-varieties.

It follows that also for patched versions, graphs which are the union of two graphs such that at the intersection, the spheres add up. This happens for example, if two disks touch at a vertex.

9.3. If we think of $\chi(S(x))$ as a **curvature** for the functional η , then a natural situation would be that zero total curvature implies that the curvature is zero everywhere. Here is a modification of the example with negative η for which $\eta(G) = 0$ and so $\eta(G_n) = 0$ for all n .

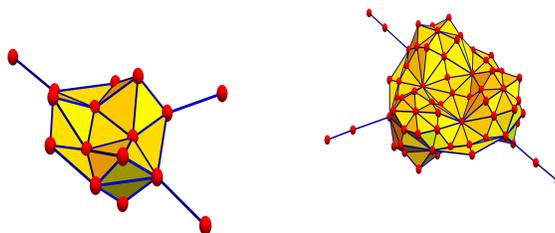


FIGURE 5. A 2-dimensional complex with $\eta(G) = 0$. The f -vector is $(v, e, f) = (18, 42, 28)$, the Betti numbers are still $b_0 = 1, b_1 = 0, b_2 = 3$, the Euler characteristic still $v - e + f = b_0 - b_1 + b_2 = 4$. In this case, $2e - 3f = (18, 42, 28) \cdot (0, 2, -3) = 0$. To the right, we see the refinement with f -vector $(88, 252, 168)$ and $\eta(G_1) = (88, 252, 168) \cdot (0, 2, -3)$. Of course $\eta(G_n) = S^n(18, 42, 28)(0, 2, -3) = 0$ for all n .

9.4.

Question: Is it true that for a d -variety, we have $\eta(G) = 0$ if and only if $\chi(S(x)) = 0$ for all $x \in V(G_1)$.

Examples:

- 1) For 1-dimensional graphs, we have $\eta(G) = 4v_1(G) > 0$.
- 2) For 2-dimensional graphs, graphs with K_3 subgraphs and no K_4 subgraphs, the functional is a Dehn-Sommerville valuation $\eta(G) = 2v_1 - 3v_2$. It vanishes if every edge shares exactly two triangles. See [12] for generalizations to multi-linear valuations.

3) For 3-dimensional graphs, graphs with K_4 subgraphs but no K_5 subgraphs, the functional is the Dehn-Sommerville valuation $2v_1 - 3v_2 + 4v_3$.

9.5. A bit stronger but more risky is the question whether zero curvature implies that G has the property that all unit spheres are unions of d -spheres:

Question: Does $\eta(G) = 0$ imply that G is a finite union of disjoint d_k -graphs wedged together so every unit sphere is a disjoint union of even dimensional d -graphs?

9.6. Besides discrete manifolds, there are discrete varieties for which $\eta(G) = 0$. Here is an example:

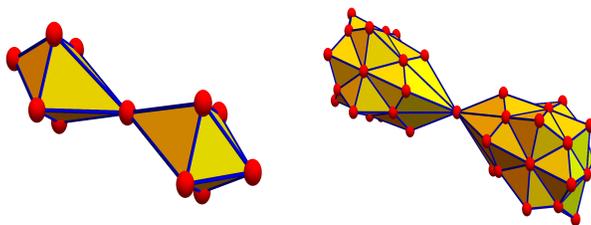


FIGURE 6. A graph for which one of the unit spheres is a disconnected union of circular graphs. It is a 2-variety but not a 2-graph. It satisfies $\chi(G) = 3$ and $\eta(G) = 0$. To the right we see the Barycentric refinement G_1 of G . The f-vector of G_1 is $\vec{v} = (51, 144, 96)$. We have $\chi(G) = \vec{v} \cdot (1, -1, 1) = 3 = b_0 - b_1 + b_2$ and $\eta(G) = \vec{v} \cdot (0, 2, -3) = 0$. Again, for all Barycentric refinements, we have $\eta(G) = 0$.

9.7. We could imagine for example that there are graphs which are almost d -graphs in the sense that the unit spheres can become discrete homology spheres, graphs with the same cohomology groups as spheres but whose geometric realizations are not homeomorphic to spheres. An other possibility is we get graphs for which the Euler characteristic is zero but for which are also topologically different from spheres. We could imagine generalized 4-graphs for example, where some unit spheres are 3-graphs. All odd-dimensional graphs have then zero Euler characteristic by Dehn-Sommerville (an incarnation of Poincaré duality). But we don't know yet of a construct of such graphs.

9.8. One can look at further related variational problems on graphs. One can either fix the number of elements in G or the number of elements in G_1 . In the later case, it of course does not matter whether one minimizes $\text{tr}(L - g)$ or maximizes the trace of the Green function $\text{tr}(g)$ as $\text{tr}(L) = f(0) = |V(G_1)| = n$.

Question: What are the minima of η on complexes with a fixed number n of faces? Equivalently, what are the maxima of the trace of the Green function on complexes with n faces?

This is a formidable problem if one wants to explore it numerically as the number of simplicial complexes with a fixed number n of faces grows very fast. A good challenge could be $n = 26$ already as the f -vector of the octahedron G is $(6, 12, 8)$. The simplex generating function of G_1 is $f(x) = 1 + 26x + 72x^2 + 48x^3$ with Euler's Gem $\chi(G) = f(0) - f(-1) = 2$. Since $f'(x) = 26 + 144x + 144x^2$ we have $f'(-1) = 26 = \text{tr}(g)$. It is a good guess to ask whether the trace of a Green function g of a simplicial complex G with 26 simplices can get larger than 26. In Figure (8) we look at the 26×26 matrices L and $g = L^{-1}$ of the Octahedron complex. Both matrices have 1 in the diagonal. In the Green function case, we know $g_{xx} = 1 - \chi(S(x)) = 1$ as all unit spheres $S(x)$ in G_1 are circles

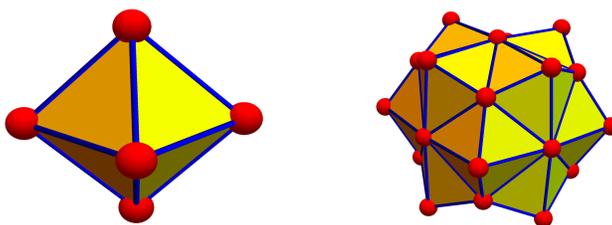


FIGURE 7. The octahedron complex G has $26 = 6 + 12 + 8$ faces. The connection graph of G is the Barycentric refinement of G . It is seen to the right. Is this the complex which minimizes the trace of the Green function among all complexes with 26 faces?

9.9. Finally, we have seen that the simplex generating function $f(x)$ and its anti derivative F can be used to compute Euler characteristic χ , the Euler curvature and functional η in a similar way

$$\chi(G) = f(0) - f(-1), K = F(0) - F(-1), \eta(G) = f'(0) - f'(-1) .$$

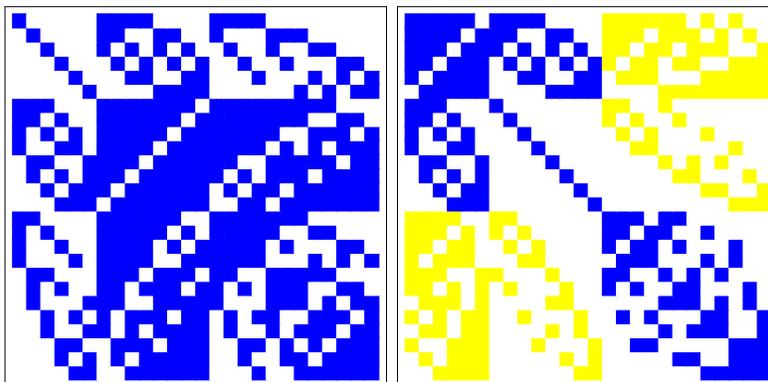


FIGURE 8. The 26×26 matrix L seen to the left is the connection Laplacian of the octahedron graph G . We have $L(x, y) = 1$ if two complete subgraphs of G intersect and $L(x, y) = 0$ else. The inverse g , the Green function is seen to the right. Also g has 1 in the diagonal as in the Barycentric refinement G_1 of G every unit sphere $S(x)$ is a circular graph. The complex G is a discrete manifold. We don't know of any other finite simplicial complex with 26 elements which has a larger trace of the Green function.

This of course prompts the question whether other similar function values of f are geometrically interesting. The two functionals were now linked also by the Gauss-Bonnet relation $\eta(G) = \sum_x \chi(S(x))$ as well as the old Gauss-Bonnet $\chi(G) = \sum_x K(x)$.

9.10. There are other algebraic-analytic relations like $\text{tr}(g) = f'(-1)$ and as a consequence of unimodularity, $\det(g) = (-1)^{1+(f_0(-1)+f_0(1))/2}$ but where f_0 is the f -vector generating function of G itself, not of the Barycentric refinement G_1 . The reason is that $\det(L) = \det(g)$ is what we called the Fermi characteristic $\phi(G) = \prod_x (-1)^{\dim(x)}$ which is 1 if there are an even number of odd dimensional simplices present in G and -1 if that number is odd. This number can change under Barycentric refinement unlike the Euler characteristic $\chi(G) = \sum_x (-1)^{\dim(x)}$, where $\chi(G) = f_0(0) - f_0(-1) = f(0) - f(-1) = \chi(G_1)$ is the same for the simplex generating function of G and G_1 .

10. CODE

As usual, the following Mathematica procedures can be copy-pasted from the ArXiv'ed LaTeX source file to this document. Together with the text, it should be pretty clear what each procedure does.

```

UnitSphere[s_ , a_]:=Module[{b=NeighborhoodGraph[s , a] ,
  If[Length[VertexList[b]]<2,Graph[{}],VertexDelete[b,a]]];
UnitSpheres[s_]:=Map[Function[x,UnitSphere[s,x]],VertexList[s]];
F[A_ , z_]:=A-z IdentityMatrix[Length[A]]; F[A_]:=F[A,-1];
FredholmDet[s_]:=Det[F[AdjacencyMatrix[s]]];
BowenLanford[s_ , z_]:=Det[F[AdjacencyMatrix[s],z]];
CliqueNumber[s_]:=Length[First[FindClique[s]]];
ListCliques[s_ , k_]:=Module[{n,t,m,u,r,V,W,U,l={},L},L=Length;
  VL=VertexList;EL=EdgeList;V=VL[s];W=EL[s];m=L[W];n=L[V];
  r=Subsets[V,{k,k}];U=Table[{W[[j,1]],W[[j,2]]},{j,L[W]}];
  If[k==1,l=V,If[k==2,l=U,Do[t=Subgraph[s,r[[j]]];
  If[L[EL[t]]==k(k-1)/2,l=Append[l,VL[t]],{j,L[r]}]];l];
Whitney[s_]:=Module[{F,a,u,v,d,V,LC,L=Length},V=VertexList[s];
  d=If[L[V]==0,-1,CliqueNumber[s]];LC=ListCliques;
  If[d>=0,a[x_]:=Table[{x[[k]]},{k,L[x]}];
  F[t_ , l_]:=If[l==1,a[LC[t,1]],If[l==0,{},LC[t,1]]];
  u=Delete[Union[Table[F[s,1],{1,0,d}],1];v={};
  Do[Do[v=Append[v,u[[m,l]],{1,L[u[[m]]]}],{m,L[u]}],v={};v];
BarycentricQ[s_]:=Module[{v={},c=Whitney[s]},Do[Do[If[c[[k]]!=c[[1]]
  &&(SubsetQ[c[[k]],c[[1]]||SubsetQ[c[[1]],c[[k]]]),
  v=Append[v,k->1]],{1,k+1,Length[c]},{k,Length[c]}];
UndirectedGraph[Graph[v]];
ConnectionGraph[s_]:=Module[{c=Whitney[s],n,A},n=Length[c];
  A=Table[1,{n},{n}];Do[If[DisjointQ[c[[k]],c[[1]]]||
  c[[k]]==c[[1]],A[[k,1]]=0,{k,n},{1,n}];AdjacencyGraph[A];
ConnectionLaplacian[s_]:=F[AdjacencyMatrix[ConnectionGraph[s]]];
FredholmCharacteristic[s_]:=Det[ConnectionLaplacian[s]];
GreenF[s_]:=Inverse[ConnectionLaplacian[s]];
Energy[s_]:=Total[Flatten[GreenF[s]]];
BarycentricOp[n_]:=Table[StirlingS2[j,i]i!,{i,n+1},{j,n+1}];
Fvector[s_]:=Delete[BinCounts[Length/@Whitney[s]],1];
Fvector1[s_]:=Module[{f=Fvector[s]},BarycentricOp[Length[f]-1].f];
Fv=Fvector;Fv1=Fvector1;
GFunction[s_ , x_]:=Module[{f=Fv[s]},1+Sum[f[[k]]x^k,{k,Length[f]}];
GFunction1[s_ , x_]:=Module[{f=Fv1[s]},1+Sum[f[[k]]x^k,{k,Length[f]}];
dim[x_]:=Length[x]-1;Pro=Product;W=Whitney;
Euler[s_]:=Module[{w=W[s],n},n=Length[w];Sum[(-1)^dim[w[[k]]],{k,n}];
Fermi[s_]:=Module[{w=W[s],n},n=Length[w];Pro[(-1)^dim[w[[k]]],{k,n}];
Eta[s_]:=Tr[ConnectionLaplacian[s]-GreenF[s]];
Eta0[s_]:=Total[Map[Euler,UnitSpheres[s]]];
Eta1[s_]:=Total[Map[Euler,UnitSpheres[Barycentric[s]]];
EtaG[s_]:=Module[{g=GFunction1[s,x]},f[y_-]:=g/.x->y;f'[0]-f'[-1]];
EulerG[s_]:=Module[{g=GFunction[s,x]},f[y_-]:=g/.x->y;f[0]-f[-1]];

s=RandomGraph[{23,60}];sc=ConnectionGraph[s];
{Euler[s],Energy[s],EulerG[s]}
{Fermi[s],BowenLanford[sc,-1],FredholmCharacteristic[s]}
{Eta1[s],Eta[s],EtaG[s]}

```

11. EXAMPLES

11.1. Let G be the house graph. It has the f -vector $(5, 6, 1)$, the generating function $f(x) = 1 + 5x + 6x^2 + x^3$ and the Euler characteristic $5 - 6 + 1 = 0$. The unit spheres have all Euler characteristic $\chi(S(x)) = 2$ except the roof-top where $\chi(S(x)) = 1$. We therefore have $\eta_0(G) = 9$. The Barycentric refinement G_1 of G has the f -vector $(12, 18, 6)$. The unit sphere Euler characteristic spectrum is $(0, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2)$ totals to $\eta_1(G) = \eta_0(G_1) = 18$. The connection graph G' of G already has dimension 4. Its f -vector is $(12, 29, 27, 12, 2)$ and $\chi(G') = 0$ still.

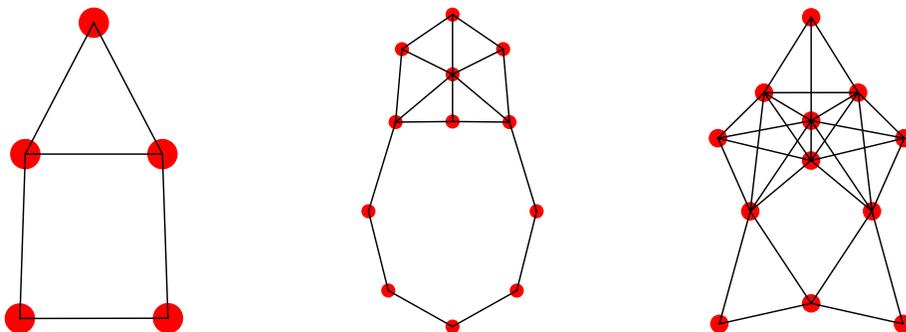


FIGURE 9. The house graph G , its barycentric refinement G_1 and its connection graph G' .

The connection Laplacian L and its inverse g are

$$L = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, g = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The trace of L is 12, the trace of g is -6 . The super trace of both L and g or the sum $\sum_{x,y} g(x, y)$ are all $\chi(G) = 0$. The spectrum of L is $\sigma(L) = \{ -1.30009, -0.827091, -0.646217, -0.528497, -0.338261, -0.255285, 0.245226, 1.20906, 1.72111, 2.9563, 3.17017, 6.59358 \}$. We see in most random graphs that about half of the eigenvalues are negative and that the negative spectrum has smaller amplitude.

11.2. The **double pyramid** G is a 2-variety with 7 vertices. It can be obtained by making two separate pyramid construction over a wheel graph. One can also write it as the Zykov join $P_3 + C_4$ of P_3 with C_4 or then $P_3 + P_2 + P_2 = P_3 + 2P_2$. While the octahedron $O = P_2 + C_4 = P_2 + P_2 + P_2 = 3P_2$ has $\eta(O) = 0$ and all unit spheres with Euler characteristic 0, now there are 4 vertices in G where $S(x)$ has Euler characteristic -1 . The graph G has f -vector $(7, 16, 12)$ and Euler characteristic 3 and Betti vector $(1, 0, 2)$. The f -vector of the Barycentric refinement is $(35, 104, 72)$. We have $\eta_0(G) = -4$. The Barycentric refinement has 8 vertices with Euler characteristic -1 and $\eta_1(G) = -8$.

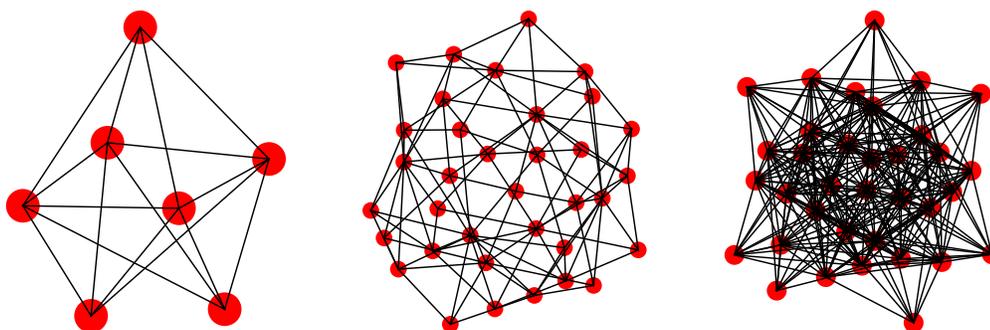


FIGURE 10. The double pyramid $G = P_3 + C_4 = P_3 + 2P_2$ is a triple suspension of a circle. Compare this with other Zykov sums: $P_1 + C_4 = W_4$, the wheel graph or $P_2 + C_4 = O$, the octahedron. We see also its Barycentric refinement G_1 and its connection graph G' .

REFERENCES

- [1] M. Adamaszek. Extremal problems related to Betti numbers of flag complexes. *Discrete Appl. Math.*, 173:8–15, 2014.
- [2] L.J. Biler and C.W. Lee. Sufficiency of McMullen’s conditions for f-vectors of simplicial polytopes. *Bull. Amer. Math. Soc.*, pages 181–185, 1980.
- [3] B. Bollobas. *Extremal Graph Theory*. Dover Courier Publications, 1978.
- [4] R. Forman. A discrete Morse theory for cell complexes. In *Geometry, topology, and physics*, Conf. Proc. Lecture Notes Geom. Topology, IV, pages 112–125. Int. Press, Cambridge, MA, 1995.
- [5] A.V. Ivashchenko. Graphs of spheres and tori. *Discrete Math.*, 128(1-3):247–255, 1994.
- [6] O. Knill. The dimension and Euler characteristic of random graphs. <http://arxiv.org/abs/1112.5749>, 2011.
- [7] O. Knill. A graph theoretical Gauss-Bonnet-Chern theorem. <http://arxiv.org/abs/1111.5395>, 2011.
- [8] O. Knill. The McKean-Singer Formula in Graph Theory. <http://arxiv.org/abs/1301.1408>, 2012.
- [9] O. Knill. The Euler characteristic of an even-dimensional graph. <http://arxiv.org/abs/1307.3809>, 2013.
- [10] O. Knill. Characteristic length and clustering. <http://arxiv.org/abs/1410.3173>, 2014.
- [11] O. Knill. Universality for barycentric subdivision. <http://arxiv.org/abs/1509.06092>, 2015.
- [12] O. Knill. Gauss-Bonnet for multi-linear valuations. <http://arxiv.org/abs/1601.04533>, 2016.
- [13] O. Knill. On Fredholm determinants in topology. <https://arxiv.org/abs/1612.08229>, 2016.
- [14] O. Knill. On primes, graphs and cohomology. <https://arxiv.org/abs/1608.06877>, 2016.
- [15] O. Knill. On Helmholtz free energy for finite abstract simplicial complexes. <https://arxiv.org/abs/1703.06549>, 2017.
- [16] O. Knill. Sphere geometry and invariants. <https://arxiv.org/abs/1702.03606>, 2017.
- [17] R. Stanley. The number of faces of a simplicial convex polytope. *Advances in Mathematics*, 35:236–238, 1980.
- [18] R. Stanley. *Enumerative Combinatorics, Vol. I*. Wadworth and Brooks/Cole, 1986.
- [19] R. Stanley. *Combinatorics and Commutative Algebra*. Progress in Math. Birkhaeuser, second edition, 1996.

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