A SIMPLE SPHERE THEOREM FOR GRAPHS

OLIVER KNILL

Abstract. A finite simple graph G is declared to have positive curvature if every in G embedded wheel graph has five or six vertices. A d-graph is a finite simple graph G for which every unit sphere is a $(d-1)$-sphere. A d-sphere is a d-graph G for which there exists a vertex x such that $G-x$ is contractible. A graph G is contractible if there is a vertex x such that $S(x)$ and $G-x$ are contractible. The empty graph 0 is the (-1)-sphere. The 1-point graph 1 is contractible. The theorem is that for $d \geq 1$, every connected positive curvature d-graph is a d-sphere. A discrete Synge result follows: a positive curvature graph is simply connected and orientable. For every $d > 1$, there are only finitely many positive curvature graphs. There are six for $d = 2$ and all have diameter ≤ 3. To prove the theorem, we use a "geomag lemma" which shows that every geodesic in G can be extended to an immersed 2-graph S of positive curvature and must so be a 2-sphere with positive curvature. As none of these has diameter larger than 3, also G has a diameter 3 or less. This can be used to show that $G-x$ is contractible and so must be a sphere.

1. The result

1.1. A finite simple graph $G = (V, E)$ is called a ***d*-graph**, if for all $x \in V$, the unit sphere $S(x)$ (the graph generated by the vertices attached to x) is a $(d-1)$-sphere. A graph G is called a ***d*-sphere** if it is a d-graph and removing one vertex x renders $G-x$ (the graph G with x and all connections to x removed) is contractible. A graph G is called **contractible** if there exists a vertex x so that $S(x)$ and $G-x$ are both contractible. These inductive definitions define d-graphs, which are discrete manifolds or d-spheres which are discrete spheres. The assumptions are primed by the assumption that the empty graph 0 is a (-1)-sphere and that the 1-point graph 1 is contractible. A wheel graph S, the unit ball of a point in a 2-graph, is called **embedded** in

Date: October 6, 2019.

1991 *Mathematics Subject Classification.* 05Cxx, 57M15, 68R10 32Q10.

Key words and phrases. Positive curvature, graphs, sphere theorem, Mickey mouse theorem.

G if it is a sub-graph of G and if its vertex set W generates S. In other words, S is embedded if every simplex in G built from vertices in W is a simplex of S. A d-graph is declared to have **positive curvature**, if all embedded wheel graphs S have less than 6 boundary points. This means that the wheel graph S itself including the center has 5 or 6 vertices. These rather strong curvature assumption allow for a rather strong conclusion:

Theorem 1 (Simple sphere theorem). *For $d \geq 1$, every connected positive curvature d-graph G is a d-sphere.*

1.2. One could also call it the **Mickey Mouse sphere theorem**. According to [16], Raoul Bott once asked Richard Stanley why he wanted to work on a "Mickey Mouse subject". Bott obviously was teasing as he showed also great respect for Rota-style combinatorics (still according to [16]). A reference to Mickey Mouse appears also in [18] but in the context of hyperbolic surfaces, which can have the shape of a mouse. The name is fitting in the positive curvature case, as there are very few graphs which have positive curvature and they are all very small.

1.3. We need $d > 0$ as for $d = 0$, the connectedness condition gives the 1-point graph $G = 1$, which is not a 0-sphere. For $d = 1$ also, the curvature condition is mute, but all connected 1-graphs are 1-spheres. For $d = 2$, the curvature of a positive curvature graph is $K(x) = 1 - \deg(x)/6 \in \{1/3, 1/6\}$. The largest positive curvature graph, the icosahedron with constant curvature $K(x) = 1 - 5/6 = 1/6$, has diameter 3. The smallest positive curvature graph, the octahedron with constant curvature $K(x) = 1 - 4/6 = 1/3$, has diameter 2. As we will see, there are exactly six positive curvature graphs in dimension 2.

1.4. In higher dimensions, we argue with a **geomag lemma**: any geodesic arc C between two points A, B can be extended to an embedded 2-sphere S. This surface is by no means unique in general. It might surprise that no orientation assumption as in the continuum is needed. It turns out that the strong curvature condition does not allow for enough room to produce a discrete projective plane. As the completion of the arc C to a 2-dimensional discrete surface has diameter ≤ 3, also G has diameter ≤ 3. As 2-dimensional spheres are simply connected, there are no shortest geodesic curves which are not homotopic to a point. Actually, a half sphere containing the closed geodesic defines the homotopy deformation, as it collapses a closed loop on the equator to a point on the pole. This is a discrete Synge theorem. But the simple sphere theorem for graphs holds without orientability assumption.

1.5. To see that G must be a d-sphere, we first note that in a positive curvature graph, the union of all unit balls centered at the vertices of a unit ball is a ball or then is already a sphere. In other words, the set of vertices in distance ≤ 2 form a ball. (This statement can fail if the positive curvature assumption is dropped, as then $B_2(x)$ can become non-simply connected already). Now pick a point and take $B_2(x)$, the graph generated by all vertices in distance 2 or less from x. This is either a ball or the entire G. In the later case, just remove one vertex z with maximal distance to x so that B has a sphere boundary and B is contractible. In the former case, cover every vertex $y \in B_2(x)$ with a ball $B_x(y)$ in G but always avoid a fixed vertex z. We have now covered $G - x$ in a way to see that it is contractible. By definition, G is then a d-sphere.

1.6. Negative curvature graphs can be defined similarly. But the definition shows more limitations there: we have so far not seen any negative curvature graphs, if **negative curvature** means that all embedded wheel graphs in G have more than 6 boundary points. One can exclude them easily in dimensions larger than 2:

Remark. There are no negative curvature graphs for $d > 2$.

Proof. For $d > 2$, any collection of $(d-2)$ intersections of neighboring unit spheres produces a 2-sphere S. As S has Euler characteristic 2, Gauss-Bonnet leads to some positive curvature and so at least 6 and maximally 12 wheel graphs with less than 6 vertices. For a 3-graph for example, the unit spheres are 2-spheres which must contain at least 6 positive curvature wheel graphs. For a 4-graph, a discrete analogue of a 4-manifold, the unit spheres are 3-spheres which by assumption have to have positive curvature too. As we have established already, this 3-graph contains then 2-spheres and so some positive curvature. □

1.7. What remains to be analyzed is the case $d = 2$. There are a priori only finitely many connected negative curvature 2-graphs G with a given genus $g > 1$ because the genus defines the Euler characteristic $\chi(G) = 2 - 2g < 0$. As the curvature $K(x)$ of every vertex $x \in V$ of a negative curvature graph $G = (V, E)$ is $\leq -1/6$, we see from the Gauss-Bonnet formula $\chi(G) = \sum_{x \in V} K(x)$ that the number of vertices in V is bounded above by $|\chi(G)| * 6 = (2 - 2g) * 6 = (12 - 12g)$.

1.8. We currently believe it should be not too difficult to prove that there are no negative curvature graphs in the case $d = 2$ too. The first case is genus $g = 2$, in which case $\chi(G) = -2$. As the negative curvature closest to 0 is $-1/6$, the number of vertices must be 12 or less. So, the question is whether there is a 2-graph with 12 vertices and negative curvature. In that particular case $g = 2$, there is none. In general, it appears that there not enough vertices to generate the g "holes" needed. We have not a formal proof of this statement for larger g but believe it can go along similar lines as in the case $g = 2$: if we look at the wheel graph centered at some vertex, then we already use 8 vertices. Only 4 vertices are left to build a 2-graph. But they are already needed to satisfy the degree 7 requirement. Having no vertices (magnetic balls) any more to continue building, we have to identify boundary points. Any such identification produces loops of length 2 which is not compatible with having a 2-graph (every unit circle must be a circular graph with 4 or more vertices).

2. Two-dimensional graphs

2.1. For $d = 2$, we know that the curvatures are constant in the octahedron (curvature is constant 1/3) and icosahedron case (curvature is constant 1/6) and that:

Lemma 1 (Positive curvature 2-graphs)**.** *There are exactly six positive curvature 2-graphs which are connected.* [1]

Proof. Because the curvature is $\geq 1/6$ and the Euler characteristic can not be larger than 2 for a connected 2-graph (the Betti vector being $(1, b_1, 1)$ in the orientable case and $(1, b_1, 0)$ in the non-orientable case), the vertex cardinality v has a priori to be in the set $\{6, 7, 8, 9, 10, 11, 12\}$. Given such a vertex cardinality v, the edge and face cardinalities $e = (v-2)3$, $f = (v-2)2$ are determined by Gauss-Bonnet and the Dehn-Sommerville relation $2e = 3f$ holding for 2-graphs. The number of curvature-(1/6) vertices has to be in the set $\{0, 2, 4, 6, 8, 10, 12\}$ because the Euler handshake formula $2e = \sum_{x \in V} \deg(x)$ implies that the total vertex degree is even and curvature 1/6 vertices are the only odd degree vertices possible in a positive curvature 2-graph. Note that in all cases, except for the case $v = 12$, two unit discs centered around a curvature 1/6 vertex always intersect. (This holds simply by looking at the total cardinality as such a wheel graph has 6 vertices and two disjoint discs have 12 vertices. It is only in the icosahedron case that we have two disjoint disks with curvature 1/6.) That restricts the possibilities. For $v = 6$, and $v = 12$, we have Platonic solids. For $v = 7, v = 8$ and $v = 12, v = 11$, we must have two adjacent vertices with different

[1]"Arithmetic is being able to count up to twenty without taking off your shoes." – Mickey Mouse

curvature. In the case $v = 9$, the three degree 5 vertices have to be adjacent. That fixes the structure. The vertex cardinality 11 is missing. There is no positive curvature graph with 11 vertices because such a graph has exactly one degree- 4 vertex and otherwise only degree-5 vertices. Look at the point x with cardinality 4 and then look at the spheres $S_r(x)$ around it: the sphere $S_2(x)$ of radius 2 must have 4 entries and so define an other degree-4 vertex. □

2.2. When this list was was first compiled on July 2 of 2011, its derivation used the built-in polyhedral graph libraries of degree $6 - 12$ in Mathematica 8. For vertex size smaller or equal to 9, we then searched with brute force over 215 suitable adjacency matrices and then cross-referenced all two-dimensional ones for graph isomorphism.

2.3. There are always finitely many 2-graphs with fixed vertex cardinality and so finitely many also with non-negative curvature. How many are there? We believe there are none. The number can grow maximally polynomially in r as we must chose 12 vertices with curvature $1/6$ from n or 10 vertices of curvature $1/6$ and 1 curvature $1/3$ etc or then 6 curvature $1/6$ vertices. While there are infinitely many fullerene type graphs with non-negative curvature, we also have only finitely many graphs of non-negative curvature which have no flat disc of fixed radius r. Also here, we do not have even estimates about the number of such graphs depending on r. We only know that it must grow polynomially in r because $12\pi r^2 \geq v$ and the number of positive curvature graphs with vertex cardinality v is polynomial in v. We will in the last section comment on the case when the scalar curvature for 2-graphs is replaced by the Ricci curvature, which is a curvature on edges. Even so Ricci curvature does not satisfy a Gauss-Bonnet formula, there are only finitely many Ricci positive curvature 2-graphs but we do not know how many there are.

2.4. From the list of six positive curvature graphs, we see that all positive curvature graphs have diameter 2 or 3. Two of them have diameter 3, the icosahedron with 12 vertices, as well as the graph with 10 vertices.

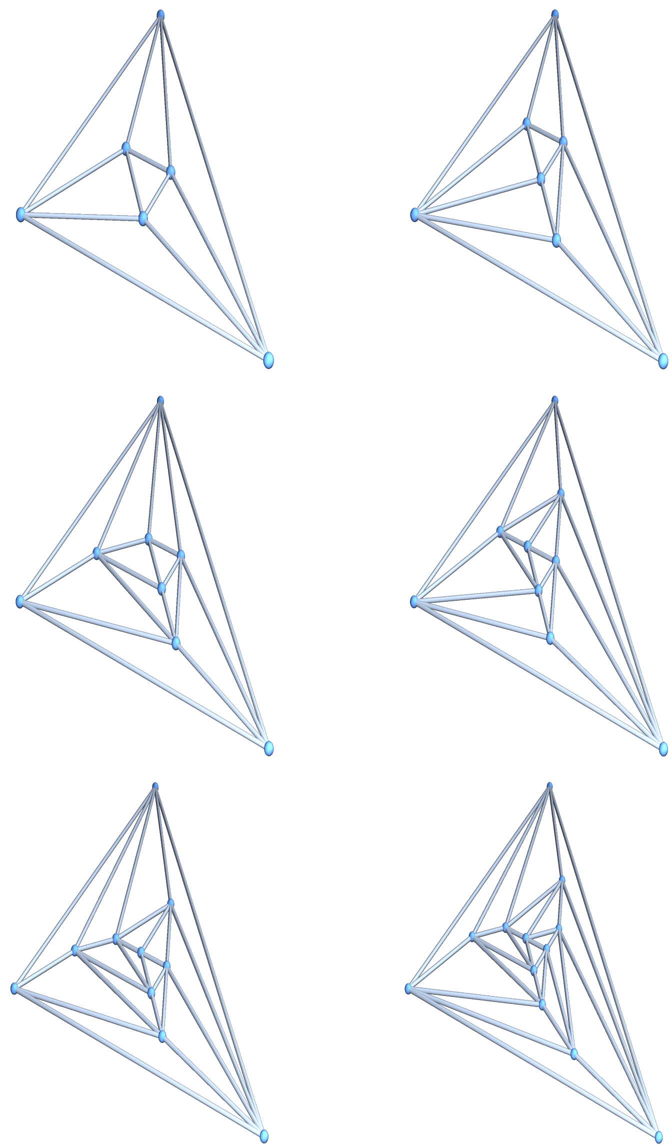

FIGURE 1. The 6 positive curvature 2-graphs of dimension 2. These are "six little mice".

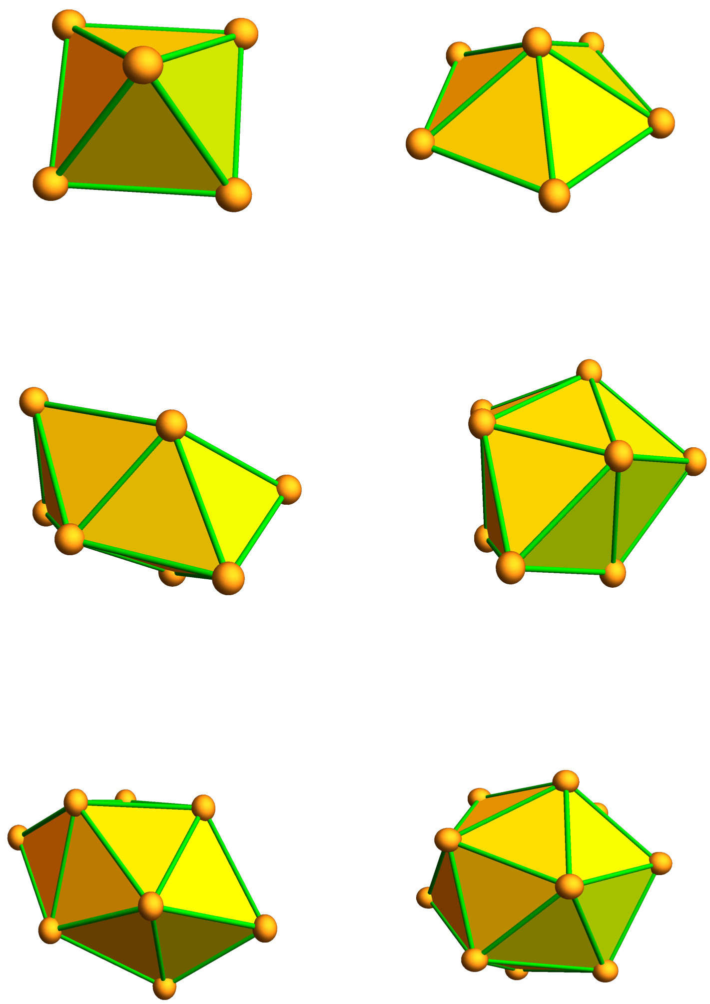

FIGURE 2. The same six graphs are displayed when embedded in space and realized as polyhedra. Note however that the simple sphere theorem is a combinatorial result which does not rely on any geometric realization. Unlike in discrete differential geometry frame works like Regge calculus, we do not care about angles, lengths or other Euclidean notions.

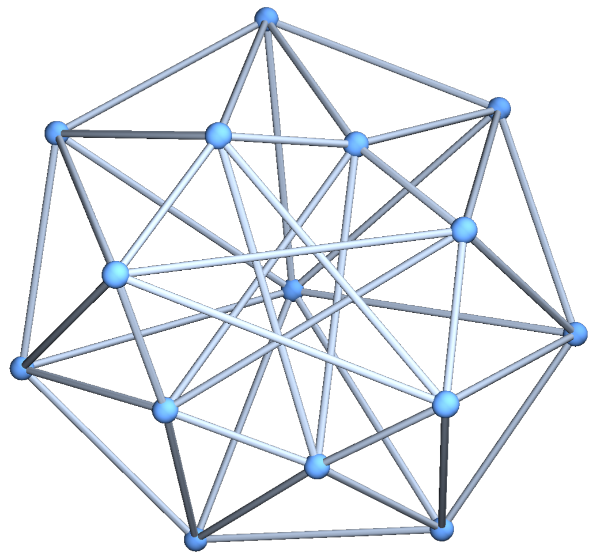

FIGURE 3. This figure shows a projective plane G with 15 vertices. I learned about this particular graph from Jenny Nitishinskaya, who constructed this graph as an example of a 2-graph with chromatic number 5. This chromatic number property is related to the fact that G can not be the boundary of a simply connected 3-graph [9]. The curvature of G is negative at one point. There is no room to implement a projective plane as a positive curvature graph: as the Euler characteristic of a projective plane is 1 and the curvature is at last 1/6 at every point, there can only be 6 vertices. This means that the graph is a wheel graph with 5 boundary points. The non-existence of positive curvature 2-graphs which implement projective planes makes the orientability condition unnecessary in the simple sphere theorem.

3. Geomag lemma

3.1. The proof of the simple sphere theorem is elementary and constructive in any dimension. Given a positive curvature graph G, we will show that why after removing a vertex x, we have a contractible graph $G - x$. The reason is that if we take away a unit ball, the remaining graph is small of radius 3 so that it can be covered by a ball of radius 2, for which every boundary point is covered with a ball of radius 1. To say it in other words, one can see that G is a union of two balls. This is known in the continuum: a n dimensional smooth complete manifold which is the union of two balls is homeomorphic to a sphere [13]. This characterization of spheres also holds in the discrete.

3.2. A graph is called a d-ball, if it is of the form $G - x$, where G is a d-sphere. By definition, the boundary of a d-ball (the set of vertices y where $S(y)$ is not a $(d-1)$-sphere), is a $(d-1)$-sphere because it agrees with $S(x)$ of the original d-sphere G. By induction in dimension the unit sphere $S(y)$ of a boundary point y is a $(d-1)$-ball because $S(y) + x$ is the unit sphere $S_G(y)$ in G.

Lemma 2. *A d-graph G which is the union of two d-balls is a d-sphere.*

Proof. Let $G = A \cup B$, where A, B are balls. Take a vertex x at the boundary of B. By definition, as B is contractible, we can remove vertices of B until none is available any more and we have only vertices of A left. Now we again by definition can remove vertices of A, until nothing is left. ☐

3.3. This is a Lusternik-Schnirelmann type result [7]. The **Lusternik-Schnirelmann category** or simply **category** $\text{cat}(G)$ of a graph G is the minimal number of contractible graphs needed to cover G. Define $\text{crit}(G)$ as the minimal number of critical points which an injective function f can have on G. The Lusternik-Schnirelmann theorem is $\text{cat}(G) \leq \text{crit}(G)$. Especially, d-spheres can be characterized as as the d-graphs with category 2. Related is the Reeb sphere theorem which tells that d-graphs which admit a function with two critical points is a sphere [10]. This implies the 2-ball theorem.

3.4. For a d-graph G, and two vertices x, y, a curve C connecting x with y is called a **geodesic arc** if there is no shorter curve in G connecting x and y. A **closed curve** C in G is called a **geodesic loop** if for any two vertices x, y in C, there is a geodesic arc from x, y which is contained in C.

3.5. The key is the following **geomag lemma**:

Lemma 3 (Geomag)**.** *a) Given any* 2*-dimensional surface* S *with boundary embedded in a* 2*-graph* G *and a point* x *on the boundary* S*, there exists a wheel graph centered at* x *which extends the surface. b) Given a geodesic arc* C *from* x *to* y*, there exists a two-dimensional embedded surface* S *(a 2-graph with boundary) which contains* C*.*

Proof. a) Let x be a boundary point of the surface S, (a boundary point is a point where $S(x) \cap S$ is an arc on the $(d-1)$-sphere $S(x)$ and not a circle). We can now build a geodesic on $S(x)$ connecting the arc $S(x) \cap S$ but which is disjoint from the arc AB (see the lemma below). This circle completion extends the surface S to a larger surface by including the "magnets" from the arc $S(x) \cap S$.
b) Start with the boundary point x of the geodesic arc C and build a wheel graph H centered at x. Now extend the surface as in part a) at $H \cap C$. Continue extending the surface until reaching y. Now, we have a two-dimensional surface S with boundary. □

3.6. Here is an other lemma which is inductively used to extend a surface.

Lemma 4. *Given two vertices* x, y *in a* d*-sphere* G *and a geodesic arc* xy*, then there this arc can be extended to a circle* C *in* G *(a circle in* G *is an embedded* 1*-graph in* G*).*

Proof. We remove a distance 1 neighborhood N of non-boundary points of xy in G. The graph N can be constructed as the union of all unit-balls centered at vertices of xy different from $\{x, y\}$. Now just take a geodesic in the graph $G - N$. It completes the graph and does not touch xy in any place different from $\{x, y\}$. □

3.7. The example of the octahedron graph G and two an arc connecting two antipodal vertices x, y in G is a situation, where the circle completion is unique.

3.8. A consequence is:

Corollary 1 (Loop extension)**.** *Any geodesic loop is part of a* 2*-graph* S *which is immersed in* G*.*

3.9. More importantly, we have Bonnet-Myers theorem type result which bounds the diameter of positive curvature d-graphs:

Corollary 2. *The diameter of any positive curvature graph is* ≤ 3*.*

3.10. There is no obvious analogue of the classical Cheng rigidity theorem characterizing positive curvature manifolds with maximal diameter as "round spheres". In the graph case, a maximal diameter 3 positive curvature graph can already in dimension 2 lead to different graphs. There are exactly two positive curvature 2-graphs with maximal diameter 3. And they are not-isometric 2-spheres.

3.11. A d-graph is **simply connected** if every closed path C in G can be deformed to a point. A **deformation step replaces** two edges in C contained in a triangle with the third edge in the triangle (2-simplex) or then does the reverse, replaces an edge with the complement of a triangle. This is equivalent to the continuum. We can define an addition of equivalence classes of closed curves and get $\pi_1(G)$, the fundamental group. It is the same graph as when looking at the classical fundamental group of a geometric realization but we do not need a geometric realization). A d-graph is simply connected, if the fundamental group is the trivial group.

Corollary 3 (Synge)**.** *Every d-graph of positive curvature is simply connected.*

Proof. A geodesic loop in G can be extended to a 2-sphere S. This 2-sphere S is simply connected. We can deform the loop to a point by making the deformation on $S \subset G$. □

4. Classical results

4.1. In this section, we mention some history. The combinatorial version can help to understand a major core point of differential geometry: "local conditions like positive curvature can have a global topological effect". A special question is to see how positive curvature relates to the diameter or injectivity radius. An other question is how it affects cohomology. The ultimate question is to relate it to a particular class of manifolds like spherical space forms (quotients of a sphere by a finite subgroup of the orthogonal group) or spheres and in which category the relation is done (like continuous or diffeomorphism classes). A panoramic view over major ideas of differential geometry is given in [3]. For history, see [2].

4.2. The topic of search for local conditions enforce global conditions is central in differential geometry. To cite [2]: *"since Heinz Hopf in the late 20's the topic of curvature and topology has been and still remains the strongest incentive for research in Riemannian geometry".* The theme is that positive curvature produces some sort of "sphere" and that negative curvature produces spaces which have universal covers which are Euclidean spaces. The former are sphere theorems pioneered by Hopf and Rauch, the later Hadamard-Cartan type results. In dimension 1, where curvature assumptions are mute, one has both, a compact and connected 1-manifold is always a circle, which is a sphere and the universal cover is the real line. In the continuum, we need an orientability assumption to get Synge or a pinching condition to get a sphere theorem. Local-global statements appear also in combinatorics: the 4-color theorem is a global statement about the maximal number of colors if the local injectivity condition is satisfied. The 4-color theorem is equivalent to the statement that 2-spheres have chromatic number 3 or 4.

4.3. Synge's theorem of 1936 is one of the oldest general results about positive curvature Riemannian manifolds [17, 6]. It already used a general bound on the diameter $L \leq \pi/\sqrt{K}$ which is called Myer's theorem in terms of the minimal value K of the curvature. Synge's theorem states that a compact orientable and connected positive curvature manifold is simply connected. It is not as deep as the sphere theorem of Rauch-Berger-Klingenberg [14], Berger and Klingenberg (see [3, 13]), which assures that a sufficiently pinched orientable positive curvature manifold is a sphere. More recent are differentiable sphere theorems, in particular the theorem of Brendle and Schoen [4] which assures that a complete, simply connected, quarter-pinched Riemannian manifold is diffeomorphic to the standard sphere. By Synge theorem, one can replace the simply connectedness assumption with orientability.

4.4. Synge's result [17] telling that a positive curvature manifold is simply connected is appealing as it mixes positive curvature, a local differential geometric notion with orientability and simply connectedness which are both of a global and topological nature. Synge already uses isometry argument. In [11], Synge's theorem is proven using a theorem of Weinstein which tells that for an even-dimensional positive curvature manifold, an orientation preserving isometry $f : M \to M$ has a fixed point. For an odd-dimensional positive curvature manifold, an orientation reversing orientation has a fixed point. Weinstein's theorem follows for spheres or projective spaces from the Lefschetz fixed point theorem but it is more general and is also remarkable as it only uses the positive curvature assumption. Synge's theorem uses calculus of variations: a minimal geodesic connecting p with $f(p)$ has as a second variation a linear operator which is positive definite and an isometry of a compact oriented even dimensional manifold has a fixed point.

4.5. As for the beginnings of the sphere theorem, ([4] and the introduction to [5, 15] give overviews), it was Heinz Hopf who, motivated largely by physics, conjectured, starting in 1932 that a sufficiently pinched positive curvature space must be a sphere. After Rauch visited ETH in 1948/1949, he proved the first theorem assuming a pinching condition of a bout 3/4. Rauch's theorem is remarkable as it is the first of this kind. It introduced a "purse string method" which can be seen as a continuous version of a geomag argument. The topological sphere theorem of 1960, proven by M. Berger and W. Klingenberg proves that under the optimal 1/4 pinching condition an orientable positive curvature manifold has to be a sphere. In 2007, R. Schoen and S. Brendle got then the differential case, using newly available Ricci flow deformation methods.

5. Questions

5.1. Can one use the geomag idea to design a proof of the classical Synge theorem by some sort of approximation? It would require some more technical things. Start with a geodesic two-dimensional surface patch and extend it to a two-dimensional surface along a geodesic. Let γ be a closed curve in M which can not be contracted and let p, q be two points in γ of maximal distance apart. Extend γ to a surface and smooth it out, still making sure the positive curvature surface S remains immersed in M. As it is a connected, compact two-dimensional Riemannian manifold, it must be the projective plane or the 2-sphere. In the case when M is orientable, S is orientable too and must be a sphere. One can now contract γ in the contractible 2-dimensional surface S.

5.2. In order to weaken positive curvature to get closer to the continuum sphere theorems, one could use the notion of **Forman-Ricci curvature** [1]. Lets just call it Ricci curvature. Ricci curvature of a 2-graph is a function attached to edges. Positive Forman curvature means that for every edge $e = (a, b)$, the Ricci curvature $K(e) = 1 - \deg(a, b)/6$ is positive, where $\deg(a, b) = (\deg(a) + \deg(b))/2$. The Ricci curvature does not satisfy any Gauss-Bonnet formula but it is likely to lead to a sphere theorem. We have not counted the number of 2-graphs with that positive Ricci curvature.

5.3. By averaging curvatures differently, one can get other notions of curvature. Still according to Forman, one can assign a curvature to triangles $f = (a, b, c)$ by $K(f) = 1 - \deg(a, b, c)/6$ where $\deg(a, b, c)$ the average of the vertex degrees of the vertices a, b, c. There is no doubt that there is a sequence of sphere theorems in a combinatorial setting which captures more aspects the continuum and still relies only on the geomag construction idea. This still needs to be done and there will be a threshold, where projective planes will be allowed, forcing to include the orientability condition, as in the continuum.

5.4. Various definitions of curvature have been proposed in the discrete. Some of them are quite involved. Looking at second order curvatures $S_2(x) - 2S_1(x)$ does not work well and are computationally complicated. Already in the planar case [8], this Puiseux type curvature already leads to sensitive issues when proving an Umlaufsatz. Ollivier type curvatures [12, 1] are even tougher to work with. We take the point of view that the definition of curvature should be simple and elementary.

5.5. An interesting open combinatorial problem is to enumerate all positive curvature graphs in d, when taking the curvature assumption of this paper. Similarly, we would like to get all 2-dimensional genus $g > 1$ graphs of negative curvature (we believe that there are none). All 2-dimensional positive curvature cases are obtained by doing edge refinements starting with the octahedron, We can ask whether every d-dimensional positive curvature graph can be obtained from edge refinements.

5.6. One can define notions of Ricci curvatures different than what Forman did. The following definition gives the scalar curvature in the case $d = 2$. Can we prove as in the continuum that positive Ricci curvature, a quantity assigned to edges given as the average of all curvatures of wheel graphs containing e, imply a bound on the diameter of G? This would be a more realistic Myers theorem.

5.7. What possible vertex cardinalities do occur of positive curvature graphs. In two dimensions, we see that for $d = 2$, the icosahedron, the graph with a maximal number of vertices is unique. Is this true in dimension d also? If not how many are there with maximal cardinality. We believe that the suspensions of icosahedron are the largest positive curvature graphs in any dimension. These are the cross polytopes of Schläfli. In the sense defined here, these cross polytopes are the only Platonic d-spheres in dimension $d \geq 5$ (the other two classically considered, the hyper-cubes or the hyper-simplices are not d-graphs.)

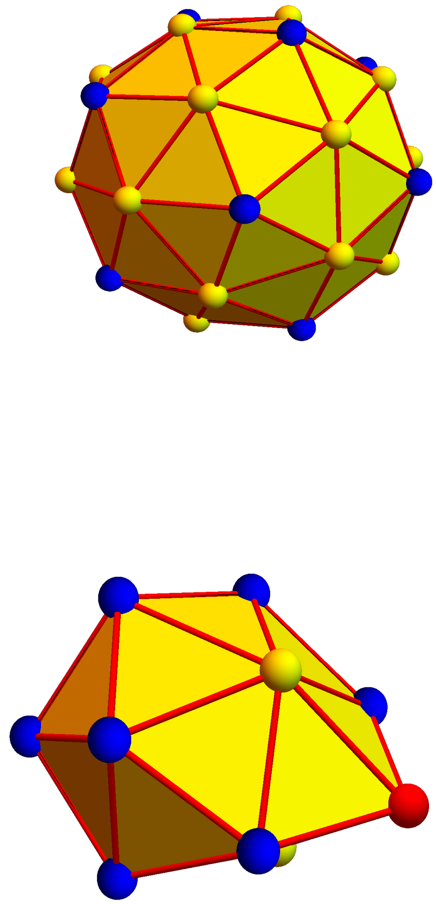

FIGURE 4. The Pentakis dodecahedron has vanishing Ricci curvature at some edges. A once edge refined icosahedron has positive Ricci curvature everywhere but two vertices with zero curvature. Unlike the curvature $1 - \deg(x)/6$ for 2-graphs, the Ricci curvature does not satisfy any Gauss-Bonnet identity.

5.8. In classical differential geometry dealing with d-manifolds M, there are a couple of notions of intrinsic curvature, curvature which does not depend on M being embedded in a higher dimensional space. The theorema egregia of Gauss allows to see sectional curvatures as independent of the embedding and use it to define the Riemann curvature tensor, averaging sectional curvature over 2-planes intersecting in a line gives Ricci curvature, averaging all Ricci curvatures gives scalar curvature which enters the Hilbert action. Then there is the Euler curvature, a Pfaffian of the Riemann curvature tensor appears in Gauss-Bonnet-Chern.

Afterword

5.9. The topic relating local properties like curvature with global topological features is an interesting theme also in physics. The reason is that basic fundamental laws in physics are local by nature if information needs time to propagate. Curvature in particular is a fundamental local quantity. Relativity relates it to mass and energy. Having a definite sign of curvature is desirable for various reasons. Positive curvature and orientability implies simply-connectedness. Negative curvature is often dubbed anti-de-Sitter and assures no conjugate points for the geodesic flow. This simplifies physics.

5.10. The original investigations by Heinz Hopf have been motivated by physics. As cited in [15], Hopf wrote in 1932: "The problem of determining the global structure of a space form from its local metric properties and the connected one of metrizing - in the sense of differential geometry - a given topological space, may be worthy of interest for physical reasons". At that time, the geometrization of gravity due to Einstein had been a major drive to investigate more of differential geometry.

5.11. Cosmological questions related to curvature about space-time been investigated early, in particular by Willem de Sitter. A de Sitter space is a positive curvature analogue of the Minkowski space. The Synge result that it is simply connected is important as a non-simply connected manifold produces twin paradox problems, where traveling along a geodesic coming back to the same point in an equivalent reference frame produces serious causality issues.

Figure 5. The 6 positive curvature 2-graphs physically built with the magnetic building tool "geomag".

References

[1] F. Bauer, B. Hua, J. Jost, S. Liu, and G. Wang. The geometric meaning of curvature: Local and nonlocal aspects of ricci curvature. In L. Najman and P. Romon, editors, *Modern Approaches to Discrete Curvature*, volume 2184 of *Lecture Notes in Mathematics*. Springer, 2017.

[2] M. Berger. *Riemannian Geometry During the Second Half of the Twentieth Century*. AMS, 2002.

[3] M. Berger. *A Panoramic View of Riemannian Geometry*. Springer, 2003.

[4] S. Brendle and R. Schoen. Sphere theorems in geometry. In *Handbook of geometric analysis, 3*, volume 14 of *Adv. Lect. Math*, pages 41–75. Int. Press, 2010.

[5] Simon Brendle. *Ricci Flow and the Sphere Theorem*. Graduate Studies in Mathematics. American Mathematical Society, 2010.

[6] M.P. do Carmo. *Differential forms and applications*. Universitext. Springer-Verlag, Berlin, 1994. Translated from the 1971 Portuguese original.

[7] F. Josellis and O. Knill. A Lusternik-Schnirelmann theorem for graphs. http://arxiv.org/abs/1211.0750, 2012.

[8] O. Knill. A discrete Gauss-Bonnet type theorem. *Elemente der Mathematik*, 67:1–17, 2012.

[9] O. Knill. Coloring graphs using topology. http://arxiv.org/abs/1410.3173, 2014.

[10] O. Knill. A Reeb sphere theorem in graph theory. https://arxiv.org/abs/1903.10105, 2019.

[11] A. Mathew. Synge-Weinstein theorems in Riemannian geometry. http://math.uchicago.edu/ amathew/syngew.pdf, 2011.

[12] Y. Ollivier. Ricci curvature of Markov chains on metric spaces. *J. Funct. Anal.*, 256:810–864, 2009.

[13] P. Petersen. *Riemannian Geometry*. Springer Verlag, second edition, 2006.

[14] H. E. Rauch. A contribution to differential geometry in the large. *Ann. of Math. (2)*, 54:38–55, 1951.

[15] R. Schoen. How curvature shapes space. Talk at ETH Zuerich, 2017.

[16] R. Stanley. How the upper bound conjecture was proved. *Ann. Comb*, 18:533–539, 2014.

[17] J.L. Synge. On the connectivity of spaces of positive curvature. *Quarterly Journal of Mathematics*, 7:316–320, 1936.

[18] W. Thurston. The geometry and topology of three-manifolds. http://www.msri.org/publications/books/gt3m, 2002.

Department of Mathematics, Harvard University, Cambridge, MA, 02138