

THE MCKEAN-SINGER FORMULA IN GRAPH THEORY

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ABSTRACT. For any finite simple graph $G = (V, E)$, the discrete Dirac operator $D = d + d^*$ and the Laplace-Beltrami operator $L = dd^* + d^*d = (d + d^*)^2 = D^2$ on the exterior algebra bundle $\Omega = \oplus \Omega_k$ are finite $v \times v$ matrices, where $\dim(\Omega) = v = \sum_k v_k$, with $v_k = \dim(\Omega_k)$ denoting the cardinality of the set \mathcal{G}_k of complete subgraphs K_k of G . We prove the McKean-Singer formula $\chi(G) = \text{str}(e^{-tL})$ which holds for any complex time t , where $\chi(G) = \text{str}(1) = \sum_k (-1)^k v_k$ is the Euler characteristic of G . The super trace of the heat kernel interpolates so the Euler-Poincaré formula for $t = 0$ with the Hodge theorem in the real limit $t \rightarrow \infty$. More generally, for any continuous complex valued function f satisfying $f(0) = 0$, one has the formula $\chi(G) = \text{str}(e^{f(D)})$. This includes for example the Schrödinger evolutions $\chi(G) = \text{str}(\cos(tD))$ on the graph. After stating some immediate general facts about the spectrum which includes a perturbation result estimating the Dirac spectral difference of graphs, we mention as a combinatorial consequence that the spectrum encodes the number of closed paths in the simplex space of a graph. We give a couple of worked out examples and see that McKean-Singer allows to find explicit pairs of nonisometric graphs which have isospectral Dirac operators.

1. INTRODUCTION

Some classical results in differential topology or Riemannian geometry have analogue statements for finite simple graphs. Examples are Gauss-Bonnet [17], Poincaré-Hopf [19], Riemann-Roch [2], Brouwer-Lefschetz [18] or Lusternik-Schnirelman [13]. While the main ideas of the discrete results are the same as in the continuum, there is less complexity in the discrete.

We demonstrate here the process of discretizing manifolds using graphs for the **McKean-Singer formula** [24]

$$\chi(G) = \text{str}(e^{-tL}),$$

where L is the Laplacian on the differential forms and $\chi(G)$ is the Euler characteristic of the graph G . The formula becomes in graph theory an elementary result about eigenvalues of concrete finite matrices. The content of this article is therefore teachable in a linear algebra course. Chunks of functional analysis like elliptic regularity which are needed in the continuum to define the eigenvalues are absent, the existence of solutions to discrete partial differential equations is trivial and no worries about smoothness or convergence are necessary. All we do here is to look at eigenvalues of a matrix D defined by a finite graph G . The discrete

Date: January 6, 2013.

1991 Mathematics Subject Classification. Primary: 05C50, 81Q10 .

Key words and phrases. Graph theory, Heat kernel, Dirac operator, Supersymmetry.

approach not only recovers results about the heat flow, we get results about unitary evolutions which for manifolds would require analytic continuation: also the super trace of $U(t) = e^{itL}$ where L is the Laplace-Beltrami operator is the Euler characteristic. While $U(t)f$ solves $(id/dt - L)f = 0$, the Dirac wave equation $(d^2/dt^2 + L) = (d/dt - iD)(d/dt + iD)f = 0$ is solved by

$$(1) \quad f(t) = U(t)f(0) = \cos(Dt)f(0) + \sin(Dt)D^{-1}f'(0),$$

where $f'(0)$ is on the orthocomplement of the zero eigenspace. Writing $\psi = f - iD^{-1}f'$ we have $\psi(t) = e^{iDt}\psi(0)$. The initial position and velocity of the real flow are naturally encoded in the complex wave function. That works also for a discrete time Schrödinger evolution like $T(f(t), f(t-1)) = (Df(t) - f(t-1), f(t))$ [15] if D is scaled.

Despite the fact that the Dirac operator D for a general simple graph $G = (V, E)$ is a natural object which encodes much of the geometry of the graph, it seems not have been studied in such an elementary setup. Operators in [5, 14, 25, 4] have no relation to the Dirac matrices D studied here. The operator D^2 on has appeared in a more general setting: [23] builds on work of Dodziuk and Patodi and studies the combinatorial Laplacian $(d + d^*)^2$ acting on Čech cochains. This is a bit more general: if the cover of the graph consists of unit balls, then the Čech cohomology by definition agrees with the graph cohomology and the Čech Dirac operator agrees with D . Notice also that for the Laplacian L_0 on scalar functions, the factorization $L_0 = d_0^*d_0$ is very well known since d_0 is the incidence matrix of the graph. The matrices d_k have been used by Poincaré already in 1900.

One impediment to carry over geometric results from the continuum to the discrete appears the lack of a Hodge dual for a general simple graph and the absence of symmetry like Poincaré duality. Do some geometric conditions have to be imposed on the graph like that the unit spheres are sphere-like graphs to bring over results from compact Riemannian manifolds to the discrete? The answer is no. The McKean-Singer supersymmetry for eigenvalues holds in general for any finite simple graph. While it is straightforward to implement the Dirac operator D of a graph in a computer, setting up D can be tedious when done by hand because for a complete graph of n nodes, D is already a $(2^n - 1) \times (2^n - 1)$ matrix. For small networks with a dozen of nodes already, the Dirac operator acts on a vector space on vector spaces having dimensions going in to thousands. Therefore, even before we had computer routines in place which produce the Dirac operator from a general graph, computer algebra software was necessary to experiment with the relatively large matrices first cobbled together by hand. Having routines which encode the Dirac operator for a network is useful in many ways, like for computing the cohomology groups. Such code helped us also to write [13]. Homotopy deformations of graphs or nerve graphs defined by a Čech cover allow to reduce the complexity of the computation of the cohomology groups.

Some elementary ideas of noncommutative geometry can be explained well in this framework. Let $H = L^2(\mathcal{G})$ denote the Hilbert space defined by the simplex set \mathcal{G} of the graph and let $B(H)$ denote the Banach algebra of operators on H . It is a Hilbert space itself with the inner product $(A, B) = \text{tr}(A^*B)$. The operator $D \in B(H)$ defines together with a subalgebra A of $B(H)$, a **Connes distance** on

\mathcal{G} . Classical geometry is when $A = C(\mathcal{G})$ is the algebra of diagonal matrices. In this case, the Connes metric $\delta(x, y) = \sup_{l \in A, |[D, l]| \leq 1} (l, e_x - e_y)$ between two vertices is the geodesic shortest distance but δ extends to a metric on all simplices \mathcal{G} . The advantage of the set-up is that other algebras A define other metrics on \mathcal{G} and more generally on **states**, elements in the unit sphere X of A^* . In the classical case, when $A = H = C(\mathcal{G}) = R^v$, all states are **pure states**, unit vectors v in H and correspond to states $l \rightarrow \text{tr}(l \cdot e_v) = (lv, v)$ in A^* . For a noncommutative algebra A , elements in X are in general **mixed states**, to express it in quantum mechanics jargon. We do not make use of this here but note it as additional motivation to study the operator D in graph theory.

The spectral theory of Jacobi matrices and Schrödinger operators show that the discrete and continuous theories are often very close but that it is not always straightforward to translate from the continuum to the discrete. The key to get the same results as in the continuum is to work with the right definitions. For translating between compact Riemannian manifolds and finite simple graphs, the Dirac operator can serve as a link, as we see below. While the result in this paper is purely mathematical and just restates a not a priori obvious symmetry known already in the continuum, the story can be interesting for teaching linear algebra or illustrating some ideas in physics. On the didactic side, the topic allows to underline one of the many ideas in [24], using linear algebra tools only. On the physics side, it illustrates classical and relativistic quantum dynamics in a simple setup. Only a couple of lines in a modern computer algebra system were needed to realize the Dirac operator of a general graph as a concrete finite matrix and to find the quantum evolution in equation (1) as well as concrete examples of arbitrary members of cohomology classes by Hodge theory. The later is also here just a remark in linear algebra as indicated in an appendix.

2. DIRAC OPERATOR

For a finite simple graph $G = (V, E)$, the exterior bundle $\Omega = \bigoplus_k \Omega_k$ is a finite dimensional vector space of dimension $n = \sum_{k=0} v_k$, where $v_k = \dim(\Omega_k)$ is the cardinality of \mathcal{G}_k the set of K_{k+1} simplices contained in G . As in the continuum, we have a super commutative multiplication \wedge on Ω but this algebra structure is not used here. The vector space Ω_k consists of all functions on \mathcal{G}_k which are antisymmetric in all $k + 1$ arguments. The bundle splits into an even $\Omega_b = \bigoplus_k \Omega_{2k}$ and an odd part $\Omega_f = \bigoplus_k \Omega_{2k+1}$ which are traditionally called the bosonic and fermionic subspaces of Ω . The exterior derivative $d : \Omega \rightarrow \Omega, df(x_0, \dots, x_n) = \sum_{k=0}^{n-1} (-1)^k f(x_0, \dots, \hat{x}_k, \dots, x_{n-1})$ satisfies $d^2 = 0$. While a concrete implementation of d requires to give an orientation of each element in \mathcal{G} , all quantities we are interested in do not depend on this orientation however. It is the usual choice of a basis ambiguity as it is custom in linear algebra.

Definition 1. Given a graph $G = (V, E)$, define the **Dirac operator** $D = d + d^*$ and **Laplace-Beltrami** operator $L = D^2 = dd^* + d^*d$. The operators $L_p = d_p^*d_p + d_{p-1}d_{p-1}^*$ leaving Ω_p invariant are called the Laplace-Beltrami operators L_p on p -forms.

Examples. We write $\lambda^{(m)}$ to indicate multiplicity m .

1) For the complete graph K_n the spectrum of D is $\{-\sqrt{n}^{(2^{n-1}-1)}, 0, \sqrt{n}^{(2^{n-1}-1)}\}$.

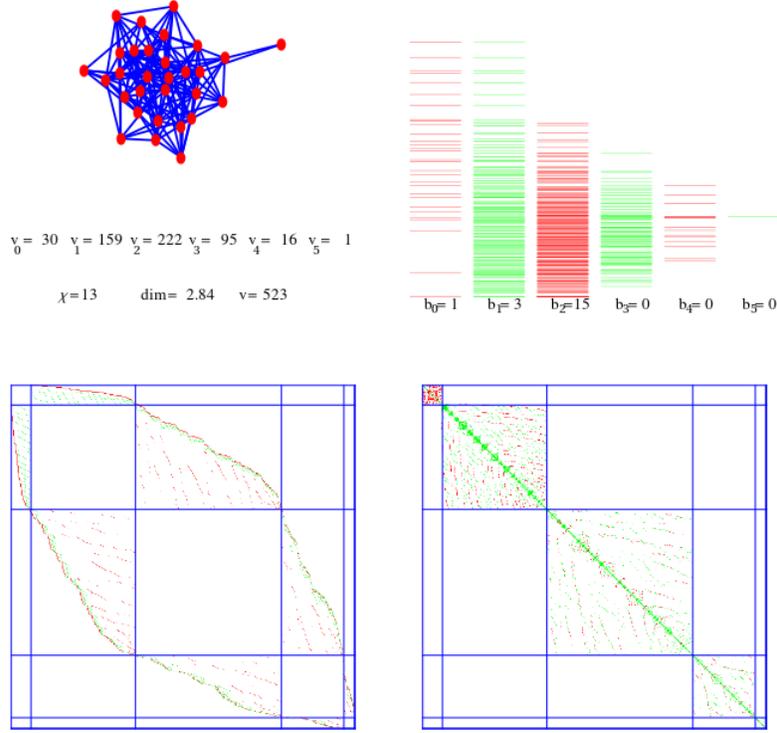


FIGURE 1. The lower part shows the Dirac operator $D = d + d^*$ and the Hodge Laplace-Beltrami operator $L = dd^* + d^*d$ of a random graph with 30 vertices and 159 edges. The Dirac operator D of this graph is a 523×523 matrix. The Laplacian decomposes into blocks L_p , which are the Laplacians on p forms. The eigenvalues of L_k shown in the upper right part of the figure are grouped for each p . The multiplicity of the eigenvalue 0 is by the Hodge theorem the p 'th Betti number b_p . In this case $b_0 = 1, b_1 = 3, b_2 = 15$. Nonzero eigenvalues come in pairs: each fermionic eigenvalue matches a bosonic eigenvalue. This super symmetry discovered by McKean and Singer shows that $\text{str}(e^{itL}) = \sum_{\lambda} (-1)^{\deg(\lambda)} e^{it\lambda} = \chi(G) = \sum_p (-1)^p b_p = \sum_p (-1)^p v_p$. While all this is here just linear algebra, the McKean formula mixes spectral theory, cohomology and combinatorics and it applies for any finite simple graph.

- 2) For the circle graph C_n the spectrum is $\{\pm\sqrt{2 - 2\cos(2\pi k/n)}\}, k = 0, \dots, n-1$, where the notation understands that 0 has multiplicity 2. The product of the nonzero eigenvalues, a measure for the complexity of the graph is n^2 .
- 3) For the star graph S_n the eigenvalues of D are $\{-\sqrt{n}, -1^{(n-1)}, 0, 1^{(n-1)}, \sqrt{n}\}$. The product of the nonzero eigenvalues is n .
- 4) For the linear graph L_n with n vertices and $n-1$ edges, the eigenvalues of D are the union of $\{0\}$, and $\pm\sigma K$, where K is the $(n-1) \times (n-1)$ matrix which has 2 in the diagonal and 1 in the side diagonal. The product of the nonzero eigenvalues

is $-n$.

Remarks.

1) The scalar part L_0 is one of the two most commonly used Laplacians on graphs [6]. The operator L generalizes it to all p -forms, as in the continuum.

2) When d_k is implemented as a matrix, it is called a signed **incidence matrix** of the graph. Since it depends on the choice of orientation for each simplex, also the Dirac operator D depends on this choice of basis. Both L and the eigenvalues of D do not depend on it.

3) The Laplacian for graphs has been introduced by Kirkhoff in 1847 while proving the matrix-tree theorem. The computation with incidence matrices is as old as algebraic topology. Poincaré used the incidence matrix d already in 1900 to compute Betti numbers [9, 28].

4) While D exchanges bosonic and fermionic spaces $D : \Omega_b \rightarrow \Omega_f, \Omega_f \rightarrow \Omega_b$, the Laplacian L is the direct sum $L_k : \Omega_k \rightarrow \Omega_k$ of Laplacians on k forms.

5) The kernel of L_k is called the vector space of **harmonic k forms**. Such harmonic forms represent cohomology classes. By Hodge theory (see appendix) the dimension of the kernel is equal to the k 'th Betti number, the dimension of the k 'th cohomology group $H^k(G) = \ker(d_k)/\text{im}(d_{k-1})$.

6) Especially, L_0 is the graph Laplacian $B - A$, where B is the diagonal matrix containing the vertex degrees and A is the adjacency matrix of the graph. The matrix L_0 is one of the natural Laplacians on graphs [6].

7) While $d_{p-1}d_{p-1}^*$ always has $p+1$ in the diagonal, the diagonal entries $d_p^*d_p(x, x)$ counts the number of K_{p+1} simplices attached to the p simplex x and $d_p d_p^*(x, x)$ is always equal to $p+1$. The diagonal entries $L_k(x, x)$ therefore determine the number of $k+1$ dimensional simplices attached to a k dimensional simplex x . We will see that the Dirac and Laplacian matrices are useful for combinatorics when counting closed curves in \mathcal{G} in the same way as adjacency matrices are useful in counting paths in G .

8) The operator D generalizes as in the continuum to bundles. The simplest example is to take a 1-form $A \in \Omega^1$ and to define $d_A f = df + A \wedge f$. This is possible since Ω is an algebra. We have then the generalized Dirac operator $D_A = d_A + d_A^*$ and $L_A = D_A^2$. The **curvature operator** $Fg = d_A \circ d_A g$ is no more zero in general. McKean-Singer could generalize to this more general setup as D_A provides the super-symmetry.

9) A 1-form A defines a field $F = dA$ which satisfies the **Maxwell equations** $dF = 0, d^*F = j$. Physics asks to find the electromagnetic field F , given j . While this corresponds to the 4-current in classical electro magnetism which includes charge and electric currents, it should be noted that no geometric structure is assumed for the graph. The Maxwell equations hold for any finite simple graph. It defines the evolution of light on the graph. The equation $d^*dA = j$ is in a Coulomb gauge $d^*A = 0$ equivalent to $L_1 A = j$. where L_1 is the Laplacian on 1 forms. Given a current j , we can get A and so F by solving a system of linear equations. This is possible if j is perpendicular to the kernel of L_1 which by Hodge theory works if G is simply connected. Linear algebra determines then the field F , a function on triangles of a graph. As on a simply connected compact manifold, a simply connected graph does not feature light in vacuum since $LA = 0$ has only the solution $A = c$ for some constant 1-form.

10) Dirac introduce the Dirac operator D in the continuum to make quantum mechanics Lorentz invariant. The quantum evolution becomes wave mechanics. Already in the continuum, there is mathematically almost no difference between the wave evolution with given initial position and velocity of the wave the Schrödinger evolution with a complex wave because both real and imaginary parts of $e^{iDt}(u - iD^{-1}v) = \cos(Dt)u + \sin(Dt)D^{-1}v$ solve the wave equation. The only difference between Schrödinger and wave evolution is that the initial velocity v of the wave must be in the orthocomplement of the zero eigenvalue. (This has to be the case also in the continuum, if we hit a string, the initial velocity has to have average zero momentum in order not to displace the center of mass of the string.) A similar equivalence between Schrödinger and wave equation holds for a time-discretized evolution $(u, v) \rightarrow (v - Du, u)$.

11) An other way to implement a bundle formalism is to take a unitary group $U(N)$ and replace entries 1 in d with unitary elements U and -1 with $-U$. Now $D = d + d^*$ and $L = D^2$ are selfadjoint operators on $L^2(\mathcal{G}, C^N)$ for some N and can be implemented as concrete $v \cdot N$ matrices. The gauge field U defines now curvatures, which is a $U(N)$ -valued function on all triangles of \mathcal{G} . Again, the more general operator D is symmetric and supersymmetry holds for the eigenvalues. Actually, as $\text{tr}(D^n)$ can be expressed as a sum over closed paths, the eigenvalues of D_A are just N copies of the eigenvalues of D if the multiplicative curvatures are 1 (zero curvature) and the graph is contractible. Since $\text{tr}(f(D)) = 0$ for odd functions f and $\text{tr}(D^2)$ is independent of U , the simplest functional on U which involves the curvature is the Wilson action $\text{tr}(D^4)$ which is zero if the curvature is zero. It is natural to ask to minimize this over the compact space of all fields A . More general functionals are interesting like $\det^*(L)$, the product of the nonzero eigenvalues of L ; this is an interesting integral quantity in the flat $U = 1$ case already: it is a measure for complexity of the graph and combinatorially interesting since $\det^*(L_0)$ is equal to the number of spanning trees in the graph G .

3. MCKEAN-SINGER

Let $G = (V, E)$ be a finite simple graph. If K_n denotes the complete graph with n vertices, we write \mathcal{G}_n for the set of K_{n+1} subgraphs of G . The set of all simplices $\mathcal{G} = \bigcup_n \mathcal{G}_n$ is the super graph of G on which the Dirac operator lives. The graph G defines \mathcal{G} without addition of more structure but the additional structure is useful, similarly as tangent bundles are useful for manifolds. It is useful to think of elements in \mathcal{G} as elementary units. If v_n is the cardinality of \mathcal{G}_n then the finite sum

$$\chi(G) = \sum_{n=0}^{\infty} (-1)^n v_n$$

is called the Euler characteristic of G .

The set Ω_n of antisymmetric functions f on \mathcal{G}_n is a linear space of k -forms. The exterior derivative $d : \Omega_n \rightarrow \Omega_{n+1}$ defined by

$$df(x_0, x_1, \dots, x_{n+1}) = \sum_k (-1)^k f(x_0, \dots, \hat{x}_k, \dots, x_{n+1})$$

is a linear map which satisfies $d \circ d = 0$. It defines the cohomology groups $H^n(G) = \ker(d_n)/\text{im}(d_{n-1})$ of dimension $b_n(G)$ for which the Euler-Poincaré formula $\chi(G) = \sum_n (-1)^n b_n(G)$ holds. The direct sum $\Omega = \oplus_n \Omega_n$ is the discrete analogue of the exterior bundle.

McKean and Singer have noticed in the Riemannian setup that the heat flow e^{-tL} solving $(d/dt + L)f = 0$ has constant super trace $\text{str}(L(t))$. The reason is that the nonzero bosonic eigenvalues can be paired bijectively with nonzero fermionic eigenvalues. It can be rephrased that D is an isomorphism from $\Omega_b^+ \rightarrow \Omega_f^+$, where Ω_b^+ is the orthogonal complement of the zero eigenspace in Ω_b and Ω_f^+ the orthogonal complement of the zero eigenspace in Ω_f . In the limit $t \rightarrow \infty$ we get the dimensions of the harmonic forms; in the limit $t \rightarrow 0$ we obtain the simplicial Euler characteristic.

Theorem 1 (McKean-Singer). *For any complex valued continuous function f on the real line satisfying $f(0) = 0$ we have*

$$\chi(G) = \text{str}(\exp(f(D))) .$$

Especially, $\text{str}(\exp(-tL)) = \chi(G)$ for any complex t .

Proof. Since $D = d + d^*$ is a symmetric $v \times v$ matrix, the kernel of D and the kernel of $L = D^2$ are the same. The functional calculus defines $f(D)$ for any complex valued continuous function. Since D is normal, $DD^* = D^*D = -L$, it can be diagonalized $D = U^*ED$ with a diagonal matrix E and defines $f(D) = U^*f(E)U$ for any continuous function. Because f can by Weierstrass be approximated by polynomials and the diagonal entries of D^{2k+1} are empty, we can assume that $f(D) = g(D^2) = g(L)$ for an even function g . Let Ω^+ be the subspace of Ω spanned by nonzero eigenvalues of L . Let Ω_p^+ be the subspace generated by eigenvectors to nonzero eigenvalues of L on Ω_p and Ω_f^+ be the subspace generated by eigenvectors to nonzero eigenvalues of L on Ω_f . Since D commutes with L , each eigenvector f of $g(L)$ on Ω_f has an eigenvector Df of $g(L)$ on Ω_p . Since $D : \Omega_p^+ \rightarrow \Omega_m^+$ is invertible, there is a bijection between fermionic and bosonic eigenvalues. Each nonzero eigenvalue appears the same number of times on the fermionic and bosonic part. \square

Remarks.

1 By definition, $\text{str}(1) = \sum_{k=0} (-1)^k v_k$ agrees with the Euler characteristic. It can be seen as an **analytic index** $\text{ind}(D)$ of the restricted Dirac operator $\Omega_b \rightarrow \Omega_f$ because $\ker(D)$ is the space of harmonic states in Ω_b and $\text{coker}(D) = \ker(D^*)$ is the dimension of the fermionic harmonic space.

2) The original McKean-Singer proof works in the graph theoretical setup too as shown in the Appendix.

3) In [15] we have for numerical purposes discretized the Schroedinger flow. This shows that we can replace the flow e^{itD} by a map $T(f, g) = (g - Df, f)$ with a suitably rescaled D which is dynamically similar to the unitary evolution and has the property that the system has **finite propagation speed**. The operator $T^2(f, g) = (f - D(g - Df), g - Df) = (f + Lf, g) - (Dg, Df)$ has the super trace $\text{str}(T^2) = \text{str}((1, 1)) = 2\chi(G)$ so that also this discrete time evolution satisfies the McKean-Singer formula.

4) The heat flow proof interpolating between the identity and the projection onto

harmonic forms makes the connection between Euler-Poincaré and Hodge more natural. While for $t = 0$ and $t = \infty$ we have a Lefschetz fixed point theorem, for $0 < t < \infty$ it can be seen as an application of the Atiyah-Bott generalization of that fixed point theorem. In the discrete, Atiyah-Bott is very similar to Brouwer-Lefschetz [18]. For $t = 0$ the Lefschetz fixed point theorem expresses the Lefschetz number of the identity as the sum of the fixed points by Poincaré-Hopf. In the limit $t \rightarrow \infty$, the Lefschetz fixed point theorem sees the Lefschetz number as the signed sum over all fixed points which are harmonic forms. For positive finite t we can rephrase McKean-Singer's result that the Lefschetz number of the Dirac bundle automorphism e^{-tL} is time independent.

4. THE SPECTRUM OF D

We start with a few elementary facts about the operator D :

Proposition 2. *Let $\vec{\lambda}$ denote the eigenvalues of D and let deg denote the maximal degree of G . If λ is an eigenvalue, then $-\lambda$ is an eigenvalue too so that $E[\lambda] = \sum_i \lambda_i = 0$.*

Proof. If we split the Hilbert space as $\Omega = \Omega_p \oplus \Omega_f$ then $D = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$, where $A = d + d^*$ is the annihilation operator and A^* which maps bosonic to fermionic states and A is the creation operator. Define $P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $L = D^2$, $P^2 = 1$, $DP + PD = 0$ is called supersymmetry in 0 space dimensions (see [29, 7]). If $Df = \lambda f$, then $PDPf = -Df = -\lambda f$. Apply P again to this identity to get $D(Pf) = -\lambda(Pf)$. This shows that Pf is an other eigenvector. \square

Proposition 3. *Every eigenvalue λ is contained in $[-\sqrt{2\text{deg}}, \sqrt{2\text{deg}}]$ so that every eigenvalue of $L = D^2$ is contained in the interval $[0, 2\text{deg}]$ and $\text{Var}[\lambda] = \sum_{i=1}^v \lambda_i^2/v \leq 2\text{deg}$.*

Proof. Because all matrix entries of D are either -1 and 1 and because there are deg such entries in each row or column, each row or column has maximal length $\sqrt{\text{deg}}$. That the standard deviation is $\leq \sqrt{2\text{deg}}$ follows since the "random variables" λ_j take values in $[-\sqrt{2\text{deg}}, \sqrt{2\text{deg}}]$. Also a theorem of Schur [26] confirms that $\sum_i \lambda_i^2 \leq \sum_{i,j} |D_{ij}|^2 \leq n \cdot 2\text{deg}$. \square

Remarks.

1) It follows that the spectrum of D is determined by the spectrum of L . If $\mu_j \in [0, \text{deg}]$ are the eigenvalues of L , then $\pm\sqrt{\mu_j}$ are the eigenvalues of D . It follows already from the reflection symmetry of the eigenvalues of D that the positive eigenvalues of L appear **in pairs**. The McKean-Singer statement is stronger than that. It tells that if one member of the pair is in the bosonic sector, the other is in the fermionic one.

2) The Schur argument given in the proof is not quite irrelevant when looking at spectral statistics. Since the average degree in the Erdős-Rényi probability space $E(n, 1/2)$ is of the order $\log(n)$ the standard deviation of the eigenvalues is of the order $2\log(n)$ even so we can have eigenvalues arbitrarily close to $\sqrt{2(n-1)}$.

3) Numerical computations of the spectrum of D for large random graphs is computationally expensive because the Dirac matrices are much larger than the adjacency matrices of the graph. It would be interesting however to know more about the

distribution of the eigenvalues of D for large matrices.

4) Sometimes, an other Laplacian K_0 defined as follows for graphs. Let $V_0(x)$ denote the degree of a vertex x . Define operator $K_{xx} = 1$ if $V_0(x) > 0$ and $K_{xy} = -(V_0(x)V_0(y))^{-1/2}$ if (x, y) is an edge. This operator satisfies $K = CL_0C$, where C is the diagonal operator for which the only nonzero entries are $V_0(x)^{-1/2}$ if $V_0(x) > 0$. While the spectrum of L_0 is in $[0, 2\text{deg}]$ the spectrum of K is in $[0, 2]$. Since L_0 has integer entries, it is better suited for combinatorics. In the following examples of spectra we use the notation $\lambda^{(n)}$ indicating that the eigenvalue λ appears with multiplicity n .

For the complete graph K_{n+1} the spectrum of K_0 is $\{0, (n+1)/n\}^{(n)}$ while the spectrum of L_0 is $\{0, n\}^{(n)}$. For a cycle graph C_n the spectrum of K_0 is $\{1 - \cos(2\pi k/n)\}$ while the spectrum of L_0 is $2 - 2\cos(2\pi k/n)$. This square graph C_4 shows that the estimate $\sigma(L) \subset [0, 2\text{deg}]$ is optimal. For a star graph S_n which is an example of a not a vertex regular graph, the eigenvalues of K_0 are $\{0, 1^{(n-2)}, 2\}$ while the spectrum of L_0 is $\{0, 1^{(n-2)}, n\}$.

Proposition 4. *The number of zero eigenvalues of D is equal to the sum of all the Betti numbers $\sum_k b_k$.*

Proof. This follows from the Hodge theorem (see Appendix): the dimension of the kernel of L_k is equal to b_k . \square

Definition 2. The **Dirac complexity** of a finite graph is defined as the product of the nonzero eigenvalues of its Dirac operator D .

The Euler-Poincaré identity assures that $\sum_{i=0}^{\infty} (-1)^i (v_i - b_i) = 0$. If we ignore the factor $(-1)^i$, we get a number of interest:

Lemma 5. *The number of nonzero eigenvalue pairs in D is the **sign-less Euler-Poincaré number** $\sum_{i=0}^{\infty} (v_i - b_i)/2$.*

Proof. The sum $\sum_i v_i = v$ is the total number of eigenvalues and by the previous proposition, the sum $\sum_i b_i$ is the total number of zero eigenvalues. Since the eigenvalues of D come in pairs $\pm\lambda$, the number of pairs is the sign-less Euler-Poincaré number. \square

Corollary 6. *The sign-less Euler-Poincaré number is even if and only if the Dirac complexity is positive.*

Proof. Arrange the product of nonzero eigenvalues of D as a product of pairs $-\lambda_j^2$. \square

Corollary 7. *If G is a triangularization of a sphere and which has an even number of edges, then the Dirac complexity is positive.*

Proof. We have $\chi(G) = v_0 - v_1 + v_2 = 2$ and since $b_0 = b_2 = 1$ and $b_1 = 0$ we have $\sum_i b_i = 2$. We can express now the sign-less Euler-Poincaré number in terms of the number of edges:

$$[(v_0 + v_1 + v_2) - (b_0 + b_1 + b_2)]/2 = [(2 + 2v_1) - 2]/2 = 2v_1/2 = v_1.$$

\square

The following explains why the dodecahedron or cube have negative complexity:

Corollary 8. *If G is a connected graph without triangles which has an even number of vertices, then the Dirac complexity is negative.*

Proof. $\chi(G) = v_0 - v_1 = b_0 - b_1 = 1 - b_1$. The sign-less Euler-Poincaré number is $[(v_0 + v_1) - (b_0 + b_1)]/2 = [1 - b_1 + 2v_1 - (1 + b_1)]/2 = v_1 - b_1 = v_0 - b_0 = v_0 - 1$. \square

For cyclic graphs C_n the complexity is n^2 if n is odd and $-n^2$ if n is even. For star graphs S_n , the complexity is n if n is odd and $-n$ if n is even.

Corollary 9. *For a tree, the Dirac complexity is positive if and only if the number of edges is even.*

Proof. The Dirac complexity is still $v_1 - b_1$ as in the previous proof but now $b_1 = 0$ so that it is v_1 . \square

We have computed the complexity for all Platonic, Archimedean and Catalan solids in the example section. All these 31 graphs have an even number v_1 of edges and an even number v_0 of vertices.

5. PERTURBATIONS OF GRAPHS

Next we estimate the distance between the spectra of different graphs: We have the following variant of Lidskii's theorem which I learned from [22]:

Lemma 10 (Lidskii). *For any two selfadjoint complex $n \times n$ matrices A and B with eigenvalues $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \dots \leq \beta_n$, one has*

$$\sum_{j=1}^n |\alpha_j - \beta_j| \leq \sum_{i,j=1}^n |A - B|_{ij}.$$

Proof. Denote with $\gamma_i \in \mathbb{R}$ the eigenvalues of the selfadjoint matrix $C := A - B$ and let U be the unitary matrix diagonalizing C so that $\text{Diag}(\gamma_1, \dots, \gamma_n) = UCU^*$. We calculate

$$\begin{aligned} \sum_i |\gamma_i| &= \sum_i (-1)^{m_i} \gamma_i = \sum_{i,k,l} (-1)^{m_i} U_{ik} C_{kl} U_{il} \\ &\leq \sum_{k,l} |C_{kl}| \cdot \left| \sum_i (-1)^{m_i} U_{ik} U_{il} \right| \leq \sum_{k,l} |C_{kl}|. \end{aligned}$$

The claim follows now from Lidskii's inequality $\sum_j |\alpha_j - \beta_j| \leq \sum_j |\gamma_j|$ (see [26]) \square

This allows to compare the spectra of Laplacians L_0 of graphs which are close. Lets define the following metric on the Erdos-Rényi space $G(n)$ of graphs of order n on the same vertex set. Denote by $d_0(G, H)$ the number of edges at which G and H differ divided by n . Define also a metric between their adjacency spectra $\vec{\lambda}, \vec{\mu}$ as

$$d_0(\vec{\lambda}, \vec{\mu}) = \frac{1}{n} \sum_{j=1}^n |\lambda_j - \mu_j|.$$

Corollary 11. *If the maximal degree in either G, H is deg , then the adjacency spectra distance can be estimated by*

$$d_0(\vec{\lambda}, \vec{\mu}) = 2\text{deg} \cdot d_0(\sigma_0(G), \sigma_0(H)).$$

Proof. We only need to verify the result for $d_0(G, H) = 1$, since the left hand side is the l^1 distance between the vectors λ, μ and the triangle inequality gives then the result for all $d_0(G, H)$. For $d_0(G, H) = 1$, the matrix $C = B - A$ differs by 1 only in 2deg entries. The Lidskii lemma implies the result. \square

Remarks.

1) We take a normalized distance between graphs because this is better suited for taking graph limits.

2) As far as we know, Lidskii's theorem has not been used yet in the spectral theory of graphs. We feel that it could have more potential, especially when looking at random graph settings [16].

3) For the Laplace operator L_0 this estimate would be more complicated since the diagonal entries can differ by more than 1. It is more natural therefore to look at the Dirac matrices for which the entries are only 0, 1 or -1 .

To carry this to Dirac matrices, there are two things to consider. First, the matrices depend on a choice of orientation of the simplices which requires to chose the same orientation if both graphs contain the same simplex. Second, the Dirac matrices have different size because D is a $v \times v$ matrix if v is the total number of simplices in G . Also this is no impediment: take the union of all simplices which occur for G and H and let v denote its cardinality. We can now write down possibly **augmented** $v \times v$ matrices $D(G), D(H)$ which have the same nonzero eigenvalues than the original Dirac operator. Indeed, the later matrices are obtained from the augmented matrices by deleting the zero rows and columns corresponding to simplices which are not present.

Definition 3. Define the **spectral distance** between G and H as

$$\frac{1}{v} \sum_{j=1}^v |\lambda_j - \mu_j|,$$

where λ_j, μ_j are the eigenvalues of the augmented Dirac matrices G, H , which are now both $v \times v$ matrices.

Definition 4. Define the **simplex distance** $d(G, H)$ of two graphs G, H with vertex set V as $(1/v)$ times the number of simplices of G, H which are different in the complete graph on V . Here v is the total number of simplices in the union when both graphs are considered subgraphs of the complete graph on V .

Definition 5. The **maximal simplex degree** of a simplex x of dimension k in a graph G is the sum of the number of simplices of dimension $k + 1$ which contain x and the sum of the number of simplices of dimension $k - 1$ which are included in x .

In other words, the maximal simplex degree is the number of nonzero entries $d_{x,y}$ in a column x of the incidence matrix d . We get the following perturbation result:

Corollary 12. *The Dirac spectra of two graphs G, H with vertex set V satisfies*

$$d(\sigma(G), \sigma(H)) \leq 2\text{deg} \cdot d(G, H),$$

if deg is the maximal simplex degree.

Proof. Since we have chosen the same orientation for simplices which are present in both graphs, this assures that the matrix $D(G) - D(H)$ has entries of absolute

value ≤ 1 and are nonzero only at $D_{x,y}$, where an incidence happens exactly at one of the graphs G, H . The value $2\deg \cdot d(G, H)$ is an upper bound for $\frac{1}{v} \sum_{k,l} |D(G)_{kl} - D(H)_{kl}|$, which by Lidskii is an upper bound for the spectral distance $\frac{1}{v} \sum_{j=1}^v |\lambda_j - \mu_j|$. \square

Remarks.

1) While the estimate is rough in general, it can become useful when looking at graph limits or estimating spectral distances between Laplace eigenvalues of different regions. If we look at graphs of fixed dimensions like classes of triangularizations of a manifold M , then \deg is related to the dimension of M only and both sides of the inequality have a chance to behave nicely in the continuum limit.

2) For large graphs which agree in many places, most eigenvalues of the Laplacian must be close. The result is convenient to estimate the spectral distance between the complete graph K_n and G . In that case, $2 \cdot \deg = 2(2^{n-1} - 1) \leq 2^n$ and we have

$$d(\sigma(G), \sigma(K_n)) \leq d(G, K_n).$$

Since the Dirac spectrum of K_n is contained in the set $\{-\sqrt{n}, 0, \sqrt{n}\}$. It follows that Erdos-Renyi graphs in $G(n, p)$ for probabilities p close to 1 have a Dirac spectrum concentrated near $\pm\sqrt{n}$. If G has m simplices less than K_n , then $d(G, K_n) \leq m/2^n$.

Examples.

1) Let G be the triangle and H be the line graph with three vertices. Then $v = 7$. We have $\deg = 3$. Since the graphs differ on the triangle and one edge only, we have $d(G, H) = 2/7$ and $2\deg \cdot d(G, H) = 12/7 = 1.71\dots$. The augmented Dirac matrices are

$$D_G = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix}, D_H = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The spectrum of the Dirac matrix of G is $\lambda = \{-\sqrt{3}, -\sqrt{3}, -\sqrt{3}, 0, \sqrt{3}, \sqrt{3}, \sqrt{3}\}$. The spectrum of the augmented Dirac matrix of H is $\mu = \{-\sqrt{3}, -1, 0, 0, 0, 1, \sqrt{3}\}$. The spectral distance is $d(\sigma(G), \sigma(H)) = (2 + 2\sqrt{3})/7 = 0.781\dots$

2) If G is K_n and H is K_{n-1} then $v = 2^n - 1$. We have $d(G, H) = [(2^n - 1) - (2^{n-1} - 1)]/(2^n - 1) = 2^{n-1}/(2^n - 1) \sim 1/2$ and $2\deg = (2(2^{n-1} - 1)) \sim 2^n$ and so $2\deg d(G, H) \sim 2^{n-1}$. The Dirac eigenvalues differ by $|\sqrt{n} - \sqrt{n-1}|$ at $(2^{n-1} - 2)$ places and by \sqrt{n} at $(2^n - 1) - (2^{n-1} - 1) = 2^{n-1}$ places. The spectral distance is $|\sqrt{n} - \sqrt{n-1}| \frac{(2^{n-1}-2)}{2^n-1} + \sqrt{n} \frac{2^{n-1}}{2^n-1}$ which is about $\sqrt{n}/2$. Taking away one vertex together with all the edges is quite a drastic perturbation step. It gets rid of a lot of simplices and changes the dimension of the graph.

3) Let G be the wheel graph W_4 with 4 spikes. It is the simplest model for a planar region with boundary. Let H be the graph where we make a pyramid extension over one of the boundary edges. This is a homotopy deformation. The graph G has 17 simplices and the graph H has $v = 21$ simplices. The graph H has one vertex, two edges and one triangle more than G so that $d(G, H) = 4/21 = 0.190\dots$. The maximal degree is $\deg = 4$ so that the right hand side of the estimate is $32/21 = 1.52\dots$

The eigenvalues of the Dirac operator $D(H) =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a & a & a & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & a & a & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & a & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & a & 0 & 0 \\ a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & a & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & a & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 1 \\ 0 & 0 & 0 & a & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 1 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(where, we wrote $a = -1$ for typographic reasons) of H are $\{-2.370, -2.302, -2.266, -2.191, -1.913, -1.839, -1.732, -1.5, -1.302, -0.9296, 0, 0.9296, 1.302, 1.529, 1.732, 1.839, 1.913, 2, 2.266, 2.302, 2.370\}$. The eigenvalues of the augmented Dirac operator of G are $\{-2.236, -2.236, -2.236, -1.732, -1.732, -1.732, -1.732, -1, 0, 0, 0, 0, 0, 1, 1.732, 1.732, 1.732, 1.732, 2.236, 2.236, 2.236\}$. The spectral differences is $0.337998\dots$. We were generous with the degree estimate. The newly added point adds simplices with maximal degree 3 so that the proof of the perturbation result could be improved to get an upper bound $8/7 = 1.142\dots$. It is still by a factor 3 larger than the spectral difference, but we get the idea: if H is a triangularization of a large domain and we just move the boundary a bit by adding a triangle then the spectrum almost does not budge. This will allow us to study spectral differences $\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{j=1}^n |\lambda_j - \mu_j|$ of planar regions in terms of the area of the symmetric difference.

6. COMBINATORICS

Adjacency matrices have always served as an algebraic bridge to study graphs. The entry A_{ij}^n has the interpretation as the number of paths in G starting at a vertex i and ending at a vertex j . From the adjacency matrix A , the Laplacian $L_0 = B - A$ is defined. Similarly as for L_0 , we can read off geometric quantities also from the diagonal entries of the full Laplacians $L = D^2$ on p forms.

Definition 6. The number of $(p + 1)$ -dimensional simplices which contain the p -dimensional simplex x is called the **p-degree** of x . It is denoted $\text{deg}_p(x)$.

Examples.

- 1) For $p = 0$, the degree $\text{deg}_0(x)$ is the usual degree vertex degree $\text{deg}(x)$ of the vertex x .
- 2) For $p = 1$ the degree $\text{deg}_1(x)$ is the number of triangles which are attached to an edge x .

Proposition 13 (Degree formulas). For $p > 0$ we have $\text{deg}_p(x) = L_p(x, x) - (p+1)$. For $p = 0$ we have $\text{deg}_0(x) = L_0(x, x)$.

Proof. Because d_p has $p + 1$ nonzero entries 1 or -1 in each row x , we have $d_p^* d_p(x, x) = p + 1$. The reason for the special situation $p = 0$ is that L_0 is the only Laplacian which does not contain a second part $d_0 d_0^*$ because d_0^* is zero on scalars:

$$L_0 = d_0^* d_0, L_1 = d_1^* d_1 + d_0 d_0^*, L_2 = d_2^* d_2 + d_2 d_2^* \text{ etc. .}$$

□

Remarks.

1) The case $p = 0$ is special because L_0 only consists of $d_0^*d_0$ while L_p with $p > 0$ has two parts $d_{p-1}d_{p-1}^* + d_p^*d_p$. The degree formulas actually count closed paths of length 2 starting at x . For $p > 0$, paths can also lower the dimension. For example, for $p = 1$, where we start with an edge, then there are two paths which start at the edge, chose a vertex and then get back to the edge. This explains the correction $p + 1$.

2) For $p = 1$, we can count the number of triangles attached to an edge x with $\deg_1(x) = L_1(x, x) + 2$.

3) A closed path interpretation of the diagonal entries will generalize the statement. Paths of length 2 count adjacent simplices in \mathcal{G} .

We can read off the total number v_p of p -simplices in G from the trace of L_p . This generalizes the Euler handshaking result that the sum of the degrees of a finite simple graph is twice the number of edges:

Corollary 14 (Handshaking). $\text{tr}(L_p) = (p + 2)v_{p+1}$.

Proof. Summing up $\deg_p(x) = L_p(x, x) - (p + 1)$ gives $v_{p+1} = \text{tr}(L_p) - (p + 1)v_{p+1}$. □

We write just 1 for the identity matrix. The formula

$$\text{str}(L + 1) = \chi(G)$$

follows from McKean-Singer because $\text{str}(1) = \chi(G)$, $\text{str}(L) = 0$. Combinatorically it is equivalent to a statement which we can verify directly and manifests in cancellations of traces:

$$\begin{aligned} \text{tr}(L_0 + 1) &= 2v_1 + v_0 \\ \text{tr}(L_1 + 1) &= 3v_2 + 3v_1 \\ \text{tr}(L_2 + 1) &= 4v_3 + 4v_2 \\ &\dots \end{aligned}$$

Adding this up gives $\text{str}(L+1) = \text{tr}(L_0+1) - \text{tr}(L_1+1) + \dots = v_0 - v_1 + v_2 - v_3 + \dots = \chi(G)$.

An other consequence is

$$\begin{aligned} \text{tr}(L_0)/2 &= v_1 \\ \text{tr}(L_1 - 2)/3 &= v_2 \\ \text{tr}(L_2 - 3)/4 &= v_3 \\ &\dots \end{aligned}$$

Remarks.

1) Each identity $\text{str}(L^k + 1) = \chi(G)$ produces some "curvatures" on the super graph \mathcal{G} satisfying

$$\chi(G) = \sum_{x \in \mathcal{G}} \kappa(x)$$

the case $k = 0$ being trivial giving $\kappa(x) = (-1)^{\dim(x)}$ for which Gauss-Bonnet is the definition of the Euler characteristic and where $k = 1$ is the case just discussed.

While the adjacency matrix A of a graph has $\text{tr}(A^k)$ as the number of closed paths in G of length k , the interpretation of $\text{tr}(L_0^k) = \text{tr}((B - A)^k)$ becomes only obvious when looking at it in more generally when looking the full Laplacian L as we do here. Instead of finding an interpretation where we add loops to the vertices, it is more natural to look at paths in \mathcal{G} :

Definition 7. A **path** in $\mathcal{G} = \bigcup \mathcal{G}_k$ is a sequence of simplices $x_k \in \mathcal{G}$ such that either x_k is either strictly contained in x_{k+1} or that x_{k+1} is strictly contained in x_k and a path starting in \mathcal{G}_k can additionally to \mathcal{G}_k only visit one of the neighboring spaces \mathcal{G}_{k+1} or \mathcal{G}_{k-1} along the entire trajectory.

The fact that this random walk on \mathcal{G} can not visit three different dimension-sectors

$$\mathcal{G}_{k-1}, \mathcal{G}_k, \mathcal{G}_{k+1}$$

is a consequence of the identities $d^2 = (d^*)^2 = 0$; a path visiting three different sectors would have cases, where d or d^* appears in a pair in the expansion of $(d + d^*)^{2k}$. Algebraically it manifests itself in the fact that L^n is always is a block matrix for which each block L_k^n leaves the subspace Ω_k of k -forms invariant.

Examples.

- 1) For a graph without triangles, a path starting at a vertex v_0 is a sequence $v_0, e_1, v_1, e_2, v_2, \dots$. Every path in \mathcal{G} of length $2n$ corresponds to a path of length n in G .
- 2) For a triangular graph G , there are three closed paths of length 2 starting an edge $e = (v_1, v_2)$. The first path is e, v_1, e , the second e, v_2, e and the third is e, t, e where t is the triangle.
- 3) Again for the triangle, there are 6 closed paths of length 4 starting at a vertex: four paths crossing two edges v, e_i, v, e_j, v and two paths crossing the same edge twice v, e_i, v_1, e_i, v . There are 9 paths of length 4 starting at an edge. There are four paths of the form e, v_j, e, v_k, e and four paths of the form e, v_i, e_j, v_i, e and one path e, t, e, t, e .

Proposition 15 (Random walk in \mathcal{G}). *The integer D_{xy}^k is the number of paths of length k in \mathcal{G} starting at a simplex x and ending at a simplex y . The trace $\text{tr}(D^{2k})$ is the total number of closed paths in \mathcal{G} which have length $2k$.*

Proof. We expand $(d + d^*)^k$. For odd k we have expressions of the form $dd^*d \cdots dd^*$ or $d^*d \cdots d^*d$. For even k , we have expressions of the form $dd^* \cdots d^*$ or $d^*d \cdots d$. The second statement follows from summing over x . \square

As a consequence of the McKean-Singer theorem we have the following corollary which is a priori not so obvious because we do not assume any symmetry for the graph.

Corollary 16. *Let G be an arbitrary finite simple graph. The number of closed paths in \mathcal{G} of length $2k$ starting at even dimensional simplices is equal to the number of closed paths of length $2k$ starting at odd dimensional simplices.*

Proof. $\text{str}(D^{2k}) = \text{tr}(L^k|\Omega_b) - \text{tr}(L^k|\Omega_f) = 0$. \square

For example, on a triangle, there are 6 closed paths of length 4 starting at a vertex and 9 closed paths of length 4 starting at an edge and 9 closed paths of length 4 starting at a triangle. There are $3 \cdot 6 + 9 = 27$ paths starting at an even dimensional

simplex (vertex or triangle) and $27 = 3 \cdot 9$ paths starting at an odd dimensional simplex (edge).

7. CURVATURE

Finally, we want to write the curvature $K(x)$ of a graph using the operator D . For a vertex $x \in V$, denote by $V_k(x)$ the number of K_{k+1} graphs in the unit sphere $S(x)$. Define also $V_{-1}(x) = 1$. The curvature at a vertex x is defined as

$$K(x) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{V_{k-1}(x)}{k+1} = \sum_{k=0}^{\infty} (-1)^k \frac{V_{k-1}(x)}{k+1} .$$

It satisfies the **Gauss-Bonnet theorem** [17]

$$\sum_{x \in V} K(x) = \chi(G) ,$$

an identity which holds for any finite simple graph. The result becomes more interesting and deeper, if more geometric structure on the graph is assumed. For example, for geometric graphs, where the unit spheres are discrete spheres of fixed dimension with Euler characteristic like in the continuum, then $K(x) = 0$ for odd dimensional graphs. This result [20] relies on discrete integral geometric methods and in particular on [21] which assures that curvature is the expectation of the index of functions.

Definition 8. Denote by $L_p(x)$ the operator L_p restricted to the unit sphere $S(x)$. It can be thought of as an analogue of a **signature** for differential operators. Define the linear operator $A_p \rightarrow A'_p(x) = A_{p-1}(x)/(p+1)$ so that $A''_p(x) = A_{p-2}(x)/(p(p+1))$.

Corollary 17. *The curvature $K(x)$ at a vertex x of a graph satisfies*

$$K(x) = \text{str}(L(x)'') .$$

Proof. From $\text{tr}(L_p(x)) = (p+2)V_{p+1}$ we get $V_{p-1}(x) = \text{tr}(L_{p-2}(x))/p$ and so

$$K(x) = \sum_{p=0}^{\infty} \frac{V_{p-1}(x)}{p+1} = \sum \frac{\text{tr}(L_{p-2}(x))}{p(p+1)} = \text{str}(L''(x)) .$$

□

We have now also in the discrete an operator theoretical interpretation of Gauss-Bonnet-Chern theorem in the same way as in the continuum, where the **Atiyah-Singer index theorem** provides this interpretation.

If we restrict D to the even part $D : \Omega_b \rightarrow \Omega_f$, the Euler characteristic is the **analytic index** $\ker(D) - \ker(D^*)$. The **topological index** of D is $\sum_{x \in X} \text{str}(D^2(x)'')$. As in the continuum, the discrete Gauss-Bonnet theorem is an example of an index theorem.

8. DIRAC ISOSPECTRAL GRAPHS

In this section we describe a general way to get Dirac isospectral graphs and give example of an isospectral pairs. In the next section, among the examples, an other example is given. There is an analogue quest in the continuum for isospectral nonisometric metrics for all differential forms. As mentioned in [3], there are various examples known. Milnor's examples with flat tori lifts to isospectrality on forms for example.

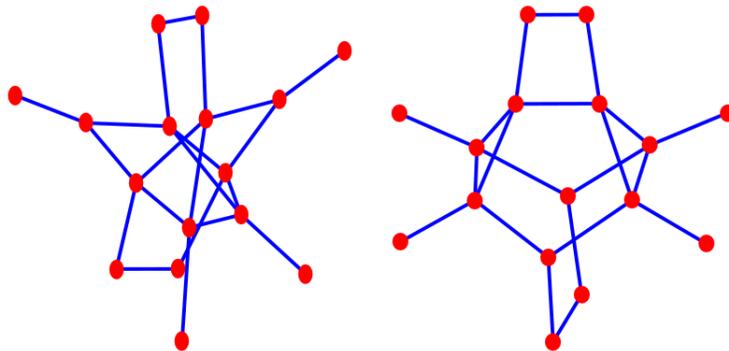


FIGURE 2. The Halbeisen-Hungerbühler pair is an example of a pair of isospectral graphs for the Dirac operator D . Both graphs have $v_0 = 16$ vertices, $v_1 = 21$ edges and $v_2 = 2$ triangles so that $\chi(G) = v_0 - v_1 + v_2 = -3$. The Betti numbers are $b_0 = 1, b_1 = 4, b_2 = 0$. Since the spectrum of L_2 is $\{3, 3\}$ for both graphs and Sunada methods in [11] have shown that the Laplacians $L_0(G_i)$ on functions are isospectral, the McKean-Singer theorem in the proof of Proposition 18 implies that also L_1 are isospectral so that the two graphs are isospectral for D .

Proposition 18. *Two connected finite simple graphs G_1, G_2 which have the same number of vertices and edges and which are triangle free and isospectral with respect to L_0 are isospectral with respect to D .*

Proof. Since $\chi(G_i) = v_0 - v_1 = b_0 - b_1 = 1 - b_1$ are the same, Hodge theory shows that L_1 has b_1 zero eigenvalues in both cases. Because the graphs are connected, L_0 have $b_0 = 1$ zero eigenvalues. The McKean-Singer theorem implies that the nonzero eigenvalues of L_0 match the nonzero eigenvalues of L_1 . \square

This can be applied also in situations, where triangles are present and where the spectrum on L_2 is trivially equivalent. An example has been found in [11]. That paper uses Sunada's techniques to construct isospectral simple graphs. We show that such examples can also be isospectral with respect to the Dirac operator.

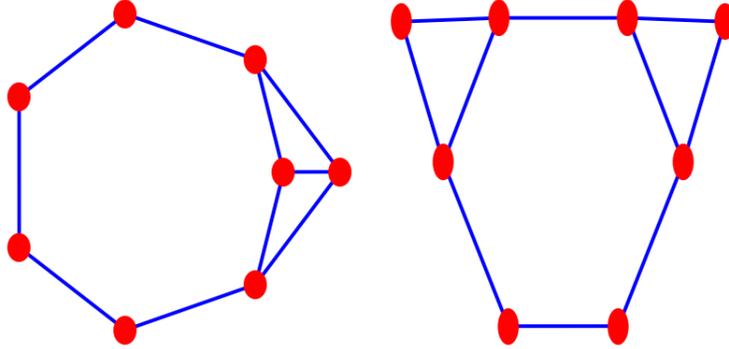


FIGURE 3. An example of a cospectral pair for the Laplace operator L_0 given by Haemers-Spence [10]. Both graphs have $v_0 = 8$ vertices, $v_1 = 10$ edges and $v_2 = 2$ triangles so that $\chi(G) = v_0 - v_1 + v_2 = 0$. The Betti numbers are $b_0 = 1, b_1 = 1, b_2 = 0$. While the spectra of L_0 agree, the spectra of L_2 are not the same. For the left graph, the eigenvalues are 4, 2, for the right graph, the eigenvalues are 3 and 3. We would not even have to compute the eigenvalues because $\text{tr}(L_2^2)$, which has a combinatorial interpretation as paths of length 4 in \mathcal{G} does not agree. We conclude from McKean-Singer (without having to compute the eigenvalues) that also the eigenvalues of L_1 do not agree.

These examples are analogues to examples of manifolds given by Gornet [3] where isospectrality for functions and differential forms can be different.

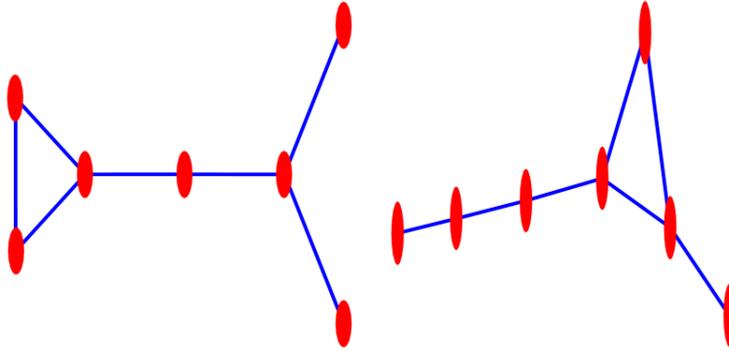


FIGURE 4. Two isospectral graphs for L_0 found in [27]. Both are contractible and have Euler characteristic 1. Since they have the same operator L_2 , by McKean Singer also the spectra of L_1 are the same. The two graphs are isospectral with respect to D .

The rest of the article consists of examples.

9. EXAMPLES

9.1. **Cyclic graph.** Let G be the cyclic graph with 4 elements.

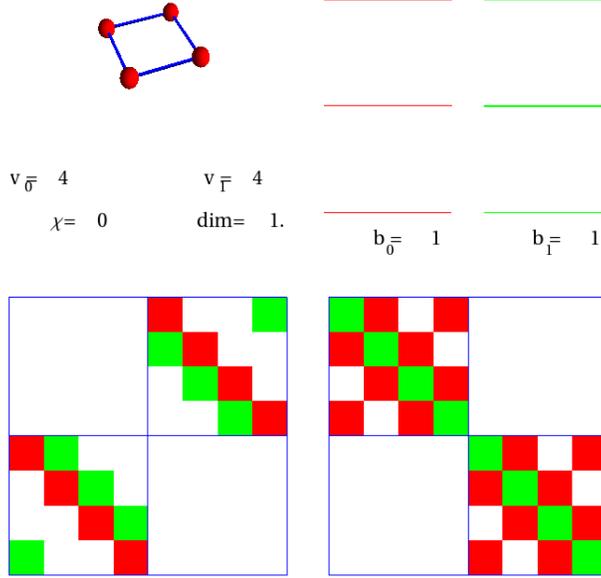


FIGURE 5.

The exterior derivative d is a map from $\Omega_0 = R^4 \rightarrow \Omega^1 = R^4$. It defines $L_0 = d^*d$. The operator $L_1 : \Omega_1 \rightarrow \Omega_1 = dd^*$ is the same than L_0 and $\text{tr}(\exp(-tL_0) - \text{tr}(\exp(-tL_1))) = 0$ for all t . We have

$$L_0 = d_0^*d_0 = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, L_1 = d_0d_0^* = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

and so

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}, L = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}.$$

The bosonic eigenvalues are $\{4, 2, 2, 0\}$, the fermionic eigenvalues are $\{4, 2, 2, 0\}$. The example generalizes to any cyclic graph C_n . The operators $L_0 = L_1$ are Jacobi matrices $2 - \tau - \tau^*$ which Fourier diagonalizes to the diagonal matrix with entries $\lambda_k = 2 - 2 \cos(2\pi k/n), k, 1, \dots, n$.

9.2. **Triangle.** Let G be the triangle.

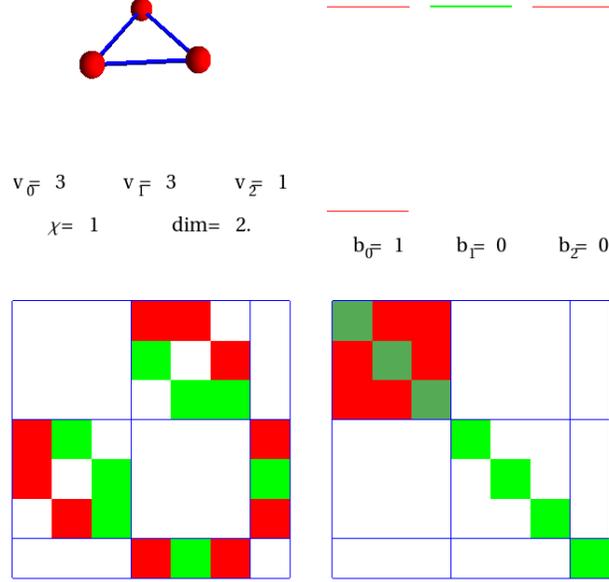


FIGURE 6.

The exterior derivative $d : \Omega_0 \rightarrow \Omega_1$ is

$$d_0 = \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ -1 & 1 & 0 & b \\ 1 & 0 & -1 & c \\ 0 & 1 & 1 & \end{array} \right], d_1 = [-1 \quad 1 \quad -1]$$

and

$$L_0 = d_0^* d_0 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, L_1 = d_1 d_1^* + d_0^* d_0 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, L_2 = d_1 d_1^* = [3] .$$

The Dirac and Laplace operator is

$$D = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 \end{bmatrix}, L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} .$$

The later has the bosonic eigenvalues $\{3, 3, 0, 3\}$ and the fermionic eigenvalues $\{3, 3, 3\}$. We have

$$\text{tr}(e^{-tL_0}) = e^{-3t} (e^{3t} + 2), \text{tr}(e^{-tL_1}) = 3e^{-3t}, \text{tr}(e^{-tL_2}) = e^{-3t};$$

and so $\text{str}(e^{tL}) = 1$.

9.3. **Tetrahedron.** For the tetrahedron K_4 , the exterior derivative d_0 is a map from $\Omega_0 = R^4 \rightarrow \Omega^1 = R^6$.

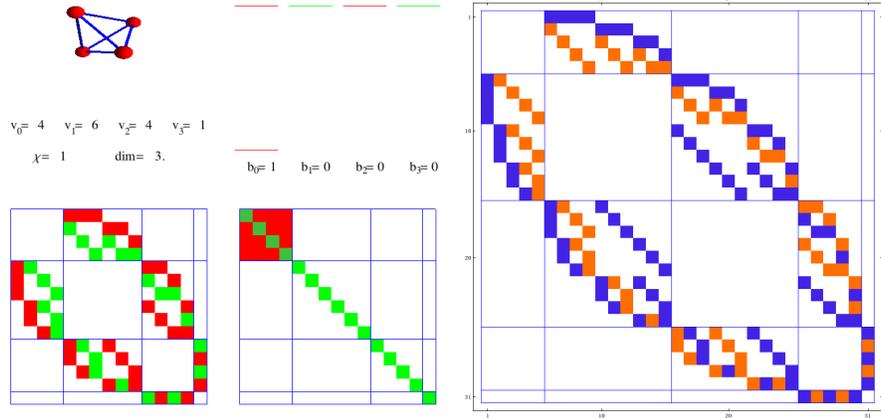


FIGURE 7. The Dirac operator of the tetrahedron K_4 is shown to the left. To the right, we see the Dirac matrix of the hyper tetrahedron K_5 .

$$d_0 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \text{ and } L_0 = d_0^* d_0 = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}. \quad d_1 =$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix} \text{ and } d_1^* d_1 = \begin{bmatrix} 2 & -1 & -1 & 1 & 1 & 0 \\ -1 & 2 & -1 & -1 & 0 & 1 \\ -1 & -1 & 2 & 0 & -1 & -1 \\ 1 & -1 & 0 & 2 & -1 & 1 \\ 1 & 0 & -1 & -1 & 2 & -1 \\ 0 & 1 & -1 & 1 & -1 & 2 \end{bmatrix} \text{ and}$$

$$d_0 d_0^* = \begin{bmatrix} 2 & 1 & 1 & -1 & -1 & 0 \\ 1 & 2 & 1 & 1 & 0 & -1 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ -1 & 1 & 0 & 2 & 1 & -1 \\ -1 & 0 & 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 & 1 & 2 \end{bmatrix} \text{ which sums up to } L_1 = d_1^* d_1 + d_0 d_0^* = 4Id.$$

$$\text{Finally, } d_2 = [1, 1, 1, 1] \text{ and } L_2 = d_2^* d_2 + d_1 d_1^* = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}. \text{ For a complete}$$

graph K_n , the eigenvalues of L_0 are $\{0, n, \dots, n\}$. The other operators L_k with $k \geq 1$ are diagonal with entries n . The Dirac operator D restricted to p -forms is a $B(n+1, p+1) \times B(n+1, p)$ matrix, where $B(n, k) = n!/(k!(n-k)!)$ is the binomial coefficient.

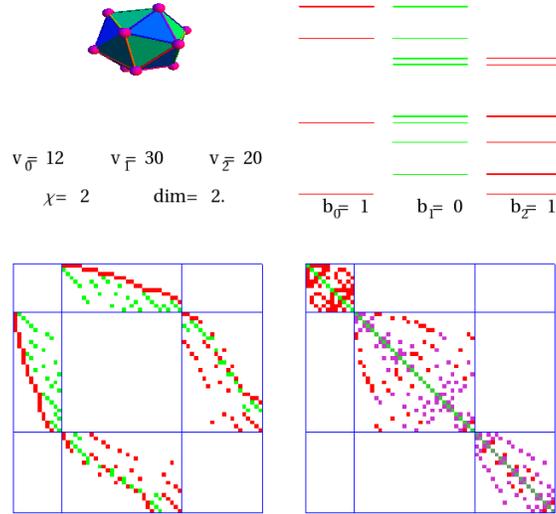


FIGURE 9.

9.5. **Icosahedron.** We write λ^k if the eigenvalue λ appears with multiplicity k . The bosonic eigenvalues of $L = D^2$ are

$$0^2, 2^5, 3^4, 5^4, 6^5, (3 - \sqrt{5})^3, (5 - \sqrt{5})^3, (3 + \sqrt{5})^3, (5 + \sqrt{5})^3.$$

The fermionic eigenvalues of L are

$$2^5, 3^4, 5^4, 6^5, (3 - \sqrt{5})^3, (5 - \sqrt{5})^3, (3 + \sqrt{5})^3, (5 + \sqrt{5})^3.$$

The eigenvalues 0 appear only on bosonic parts matching that the Betti vector $\beta = (1, 0, 1)$ has its support on the bosonic part. The picture colors the vertices, edges and triangles according to the ground states, the eigenvectors to the lowest eigenvalues of L_0, L_1 and L_2 .

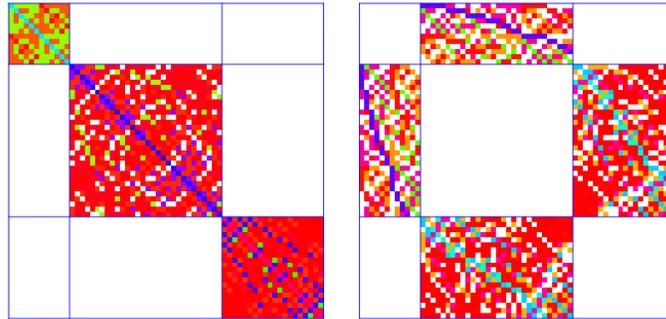


FIGURE 10. The matrices $\text{Re}(\exp(iD)) = \cos(D)$ and $\text{Im}(\exp(iD)) = \sin(D)$ which appear in the solution of the wave equation on the graph. Having the matrix D in the computer makes it easy to watch the wave evolution on a graph.

9.6. **Fullerene.** The figure shows a fullerene type graph based on an icosahedron. It has $v_0 = 42$ vertices, $v_1 = 120$ edges and $v_2 = 80$ triangles. The Dirac operator D is a 242×242 matrix

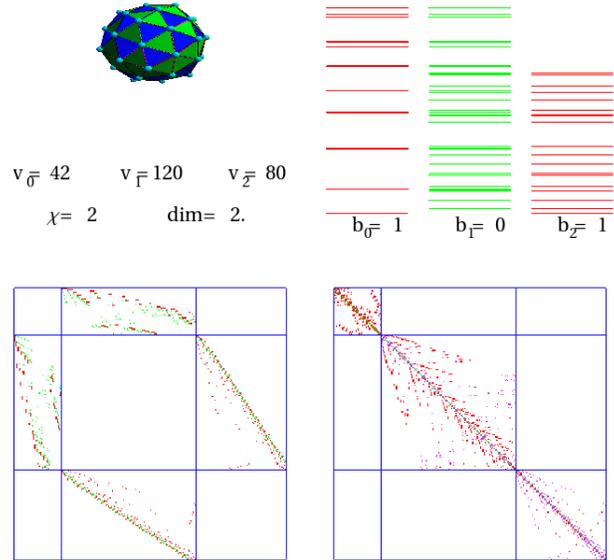


FIGURE 11.

In the next picture we see the three sorted spectra of L_0, L_1 and L_2 .

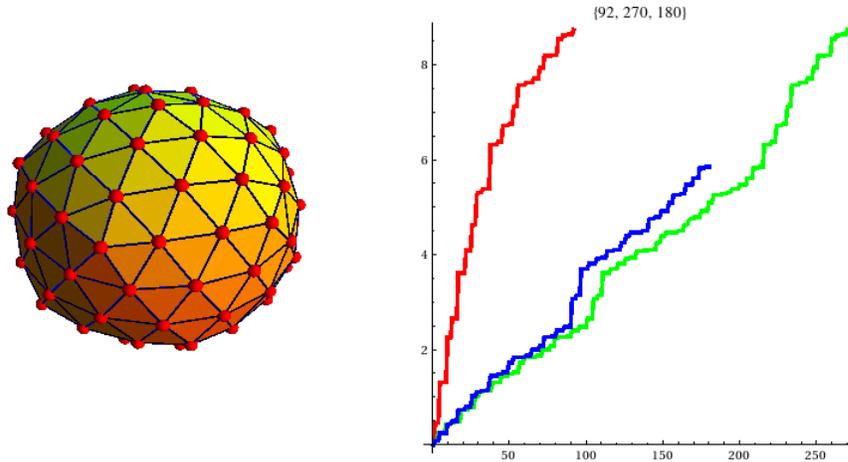


FIGURE 12.

9.7. **Torus.** Here is the spectrum of a two dimensional discrete torus

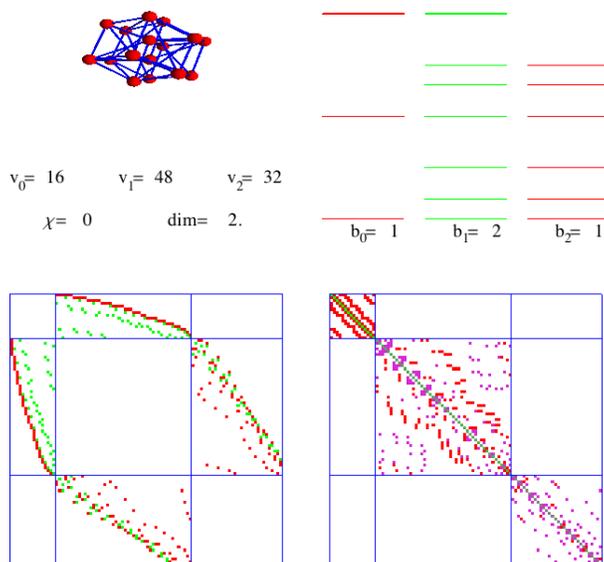


FIGURE 13.

and a discrete Klein bottle:

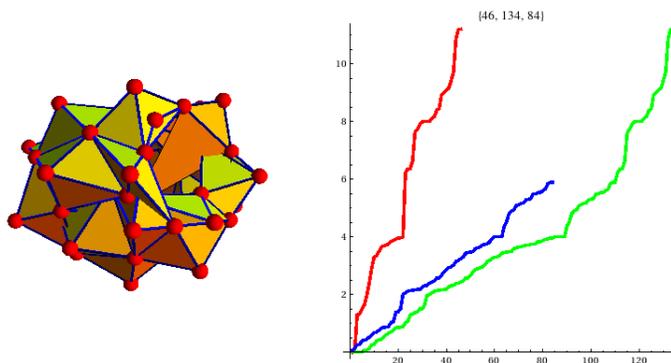


FIGURE 14.

9.8. **Dunce hat and Petersen.** The dunce hat is an example important in homotopy theory. It is a non-contractible graph which is homotopic to a one point graph.

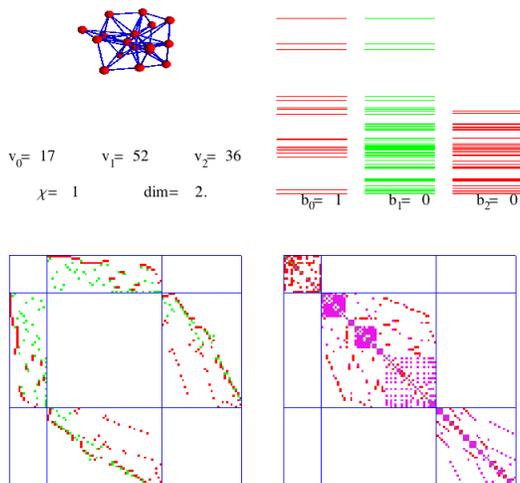


FIGURE 15.

Petersen graph

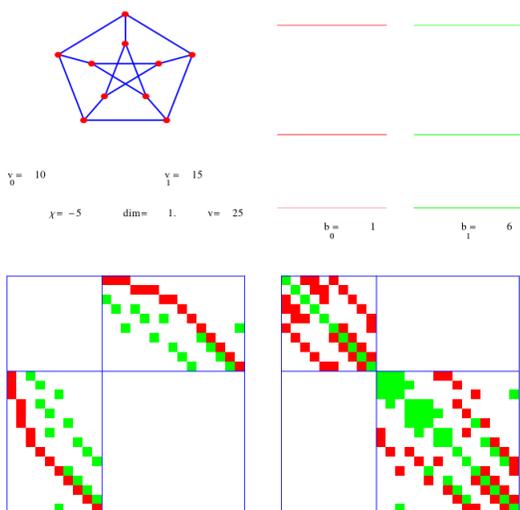


FIGURE 16.

9.9. Isospectral graphs. As mentioned in the previous section, some known isospectral graphs for the graph Laplacian are also isospectral for the Dirac operator.

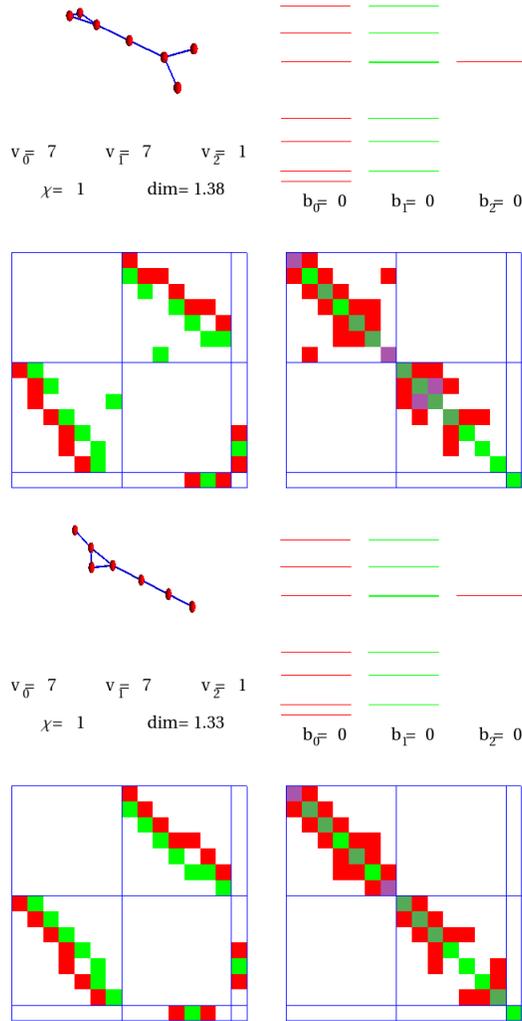


FIGURE 17.

In [27], all isospectral connected graphs up to order $v_0 = 7$ have been computed. For order $v_0 = 7$, there are already examples which are also isospectral for all p -forms and for the Dirac operator. Some of them lead to Dirac isospectral graphs: $\sigma(D) = \{-\sqrt{3 + \sqrt{2}}, \sqrt{3 + \sqrt{2}}, -\sqrt{2 + \sqrt{3}}, \sqrt{2 + \sqrt{3}}, -\sqrt{3}, -\sqrt{3}, \sqrt{3}, \sqrt{3}, -\sqrt{3 - \sqrt{2}}, \sqrt{3 - \sqrt{2}}, -1, 1, -\sqrt{2 - \sqrt{3}}, \sqrt{2 - \sqrt{3}}, 0\}$. The Laplacian eigenvalues are $\sigma(L_0) = \{3 + \sqrt{2}, 2 + \sqrt{3}, 3, 3 - \sqrt{2}, 1, 2 - \sqrt{3}, 0\}$ and $\sigma(L_1) = \{3 + \sqrt{2}, 2 + \sqrt{3}, 3, 3, 3 - \sqrt{2}, 1, 2 - \sqrt{3}\}$ as well as $\sigma(L_2) = \{3\}$.

9.10. **Planar regions.** One of the questions which fueled research on spectral theory was whether one “can hear the shape of a drum”. The discrete analogue question is whether the Dirac operator can hear the shape of a discrete hexagonal region.

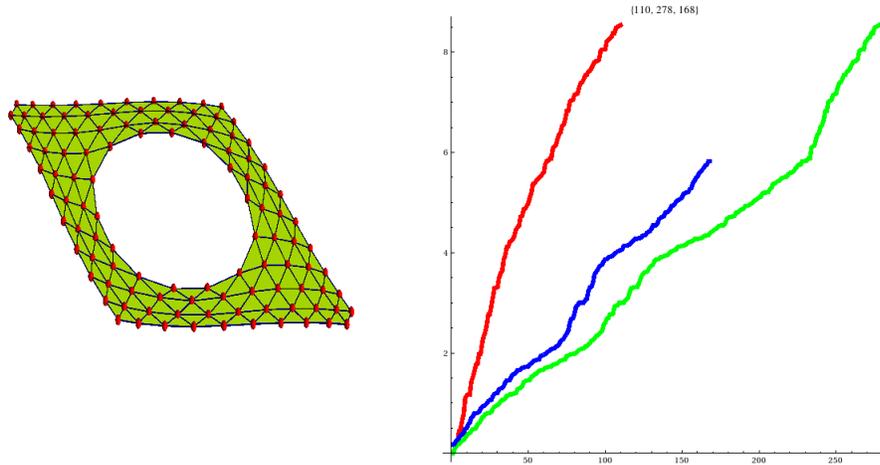


FIGURE 18.
A hexagonal region and the spectra of the Laplacians L_0, L_1, L_2 .

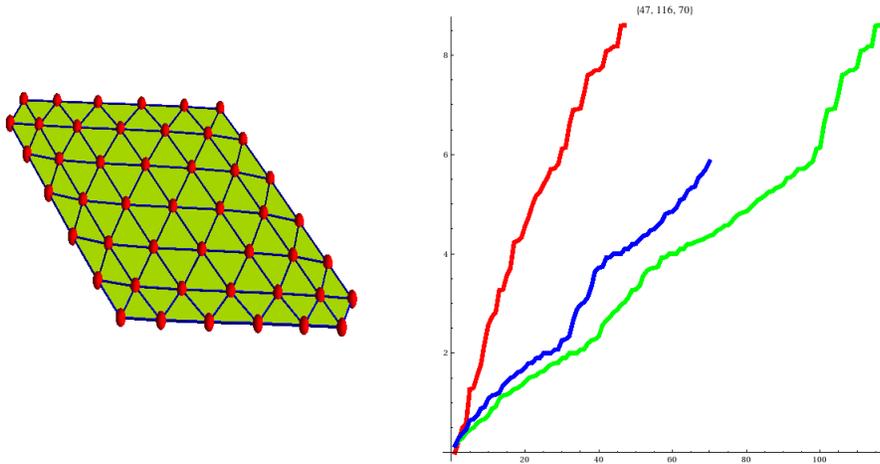


FIGURE 19.
A convex hexagonal region and the spectra of L_0, L_1, L_2 .

As far as we know, no isospectral completely two dimensional convex graphs are known.

9.11. **Benzenoid.** Isospectral benzenoid graphs were constructed in [1]. They are isospectral with respect to the adjacency matrix A , but not with respect to the Laplacian $L = B - A$. After triangulating the hexagons, the first pair of graphs have Euler characteristic 0, the second Euler characteristic 1. But after triangularization, the graphs are not isospectral with respect to A , nor with respect to L .

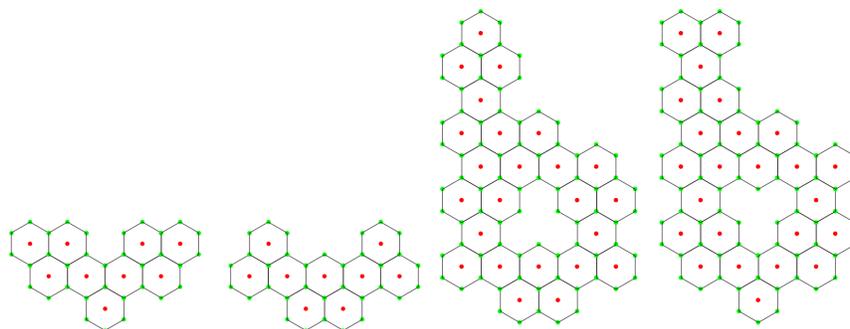


FIGURE 20. Two examples of isospectral Benzenoid pairs which are isospectral for the adjacency matrix A . These graphs are one dimensional since they contain no triangles. McKean-Singer shows that the nonzero spectrum of L_0 is the same than the nonzero spectrum of L_1 .

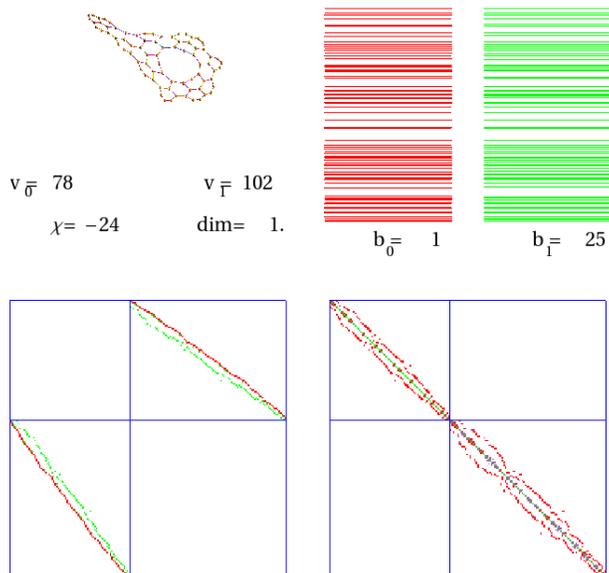


FIGURE 21. The Dirac spectrum of the first Benzenoid.

9.12. **Gordon-Webb-Wolpert.** Isospectral domains in the plane were constructed with Sundada methods by Gordon, Webb and Wolpert [8]. A simplified version consists of two squares and three triangles.

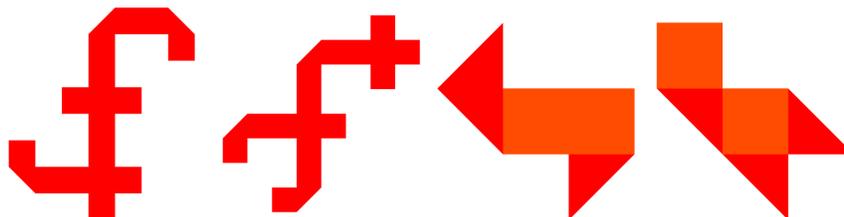


FIGURE 22.

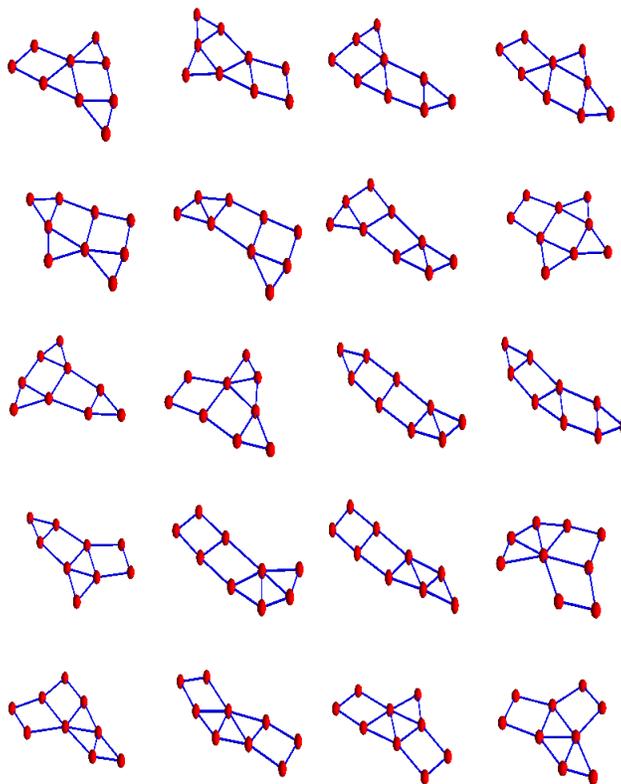


FIGURE 23.

To try out graph versions of this, we computed the spectra of a couple of graphs of this type. They are all homotopic to the figure 8 graph but are all not isospectral. This shows that discretizing Sunada can not be done naively. The next section mentions examples given in [8] which were obtained by adapting Sunada type methods to graphs.

9.13. **Halbeisen-Hungerbühler.** Here are the picture of the Dirac matrices and their spectrum of the isospectral graphs constructed in [8].

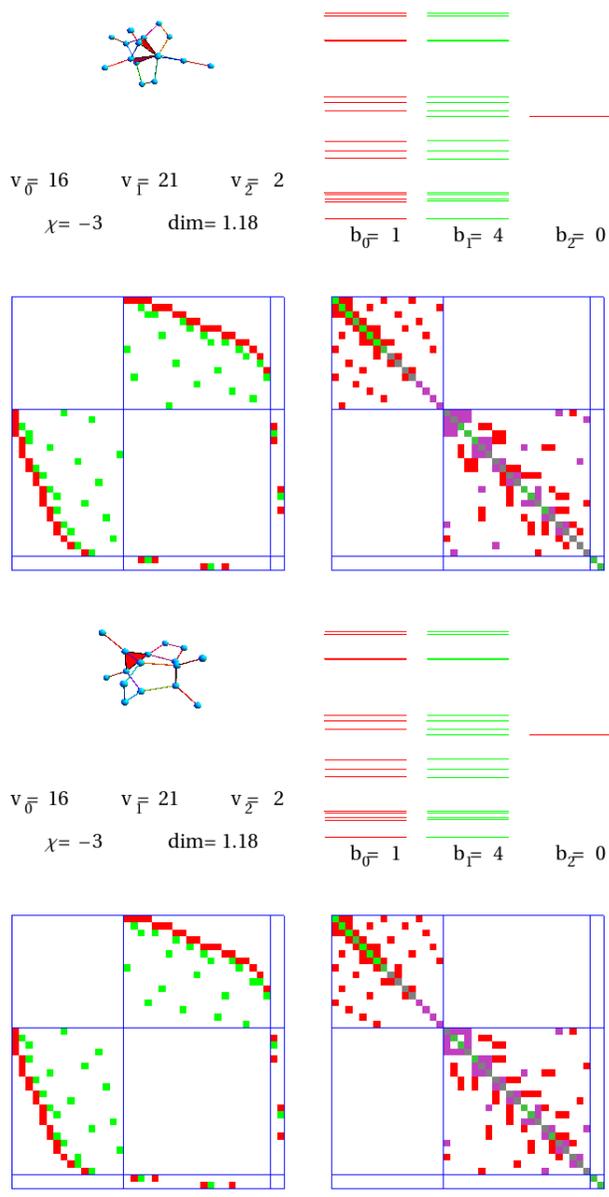


FIGURE 24.

9.14. **Polyhedra.** For the 5 platonic solids, we have the following Dirac spectra and complexities $\prod_{\lambda \neq 0} \lambda$, where we write λ^n for λ with multiplicity n .

Solid	Non-negative Dirac Spectrum	Complexity
Tetrahedron	$\{2^7, 0^1\}$	-2^{14}
Octahedron	$\{\sqrt{6}^3, \sqrt{2}^3, 2^6, 0^2\}$	$2^{18} \cdot 3^3$
Cube	$\{\sqrt{6}, 2^3, \sqrt{2}^3, 0^6\}$	$-2^{10} \cdot 3$
Dodecahedron	$\{\sqrt{2}^5, \sqrt{3}^4, \sqrt{5}^4, \sqrt{3 - \sqrt{5}}^3, 0^{12}\}$	$-2^{11} \cdot 3^4 \cdot 5^4$
Icosahedron	$\{\sqrt{2}^5, \sqrt{3}^4, \sqrt{5}^4, \sqrt{6}^5, \sqrt{5 \pm \sqrt{5}}^3, \sqrt{3 \pm \sqrt{5}}^3, 0^2\}$	$2^{22} \cdot 3^9 \cdot 5^7$

The following table confirms that for two-dimensional geometric graph for which the unit sphere is a circular graph, the Dirac complexity is positive. We actually came up with the statement after watching the following tables:

$A =$	<table border="1" style="width: 100%; text-align: center;"> <thead> <tr> <th>dim(G)</th> <th>complexity</th> <th>$\chi(G)$</th> </tr> </thead> <tbody> <tr><td>2</td><td>-2.6121×10^{10}</td><td>-4</td></tr> <tr><td>1</td><td>-2.6147×10^{42}</td><td>-60</td></tr> <tr><td>1</td><td>-5.9608×10^{17}</td><td>-24</td></tr> <tr><td>2</td><td>-2.1859×10^{25}</td><td>-10</td></tr> <tr><td>3/2</td><td>-4.2166×10^{40}</td><td>-40</td></tr> <tr><td>3/2</td><td>-4.7406×10^{16}</td><td>-16</td></tr> <tr><td>2</td><td>-2.1456×10^{28}</td><td>-4</td></tr> <tr><td>2</td><td>-8.2030×10^{69}</td><td>-10</td></tr> <tr><td>5/3</td><td>-5.1018×10^{12}</td><td>-4</td></tr> <tr><td>5/3</td><td>-1.0423×10^{30}</td><td>-10</td></tr> <tr><td>1</td><td>-2.2518×10^{22}</td><td>-30</td></tr> <tr><td>1</td><td>-2.4277×10^9</td><td>-12</td></tr> <tr><td>5/3</td><td>-5.8320×10^6</td><td>-2</td></tr> </tbody> </table>	dim(G)	complexity	$\chi(G)$	2	-2.6121×10^{10}	-4	1	-2.6147×10^{42}	-60	1	-5.9608×10^{17}	-24	2	-2.1859×10^{25}	-10	3/2	-4.2166×10^{40}	-40	3/2	-4.7406×10^{16}	-16	2	-2.1456×10^{28}	-4	2	-8.2030×10^{69}	-10	5/3	-5.1018×10^{12}	-4	5/3	-1.0423×10^{30}	-10	1	-2.2518×10^{22}	-30	1	-2.4277×10^9	-12	5/3	-5.8320×10^6	-2
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$, C =$	<table border="1" style="width: 100%; text-align: center;"> <thead> <tr> <th>dim(G)</th> <th>complexity</th> <th>$\chi(G)$</th> </tr> </thead> <tbody> <tr><td>1</td><td>-4.6449×10^6</td><td>-10</td></tr> <tr><td>2</td><td>3.5323×10^{84}</td><td>2</td></tr> <tr><td>2</td><td>1.9246×10^{35}</td><td>2</td></tr> <tr><td>1</td><td>-6.6871×10^{15}</td><td>-28</td></tr> <tr><td>1</td><td>-1.2496×10^{31}</td><td>-58</td></tr> <tr><td>1</td><td>-7.8275×10^{12}</td><td>-22</td></tr> <tr><td>1</td><td>-3.4164×10^{15}</td><td>-22</td></tr> <tr><td>1</td><td>-4.0315×10^{37}</td><td>-58</td></tr> <tr><td>3</td><td>3.7873×10^{26}</td><td>2</td></tr> <tr><td>3</td><td>1.4404×10^{64}</td><td>2</td></tr> <tr><td>2</td><td>2.7042×10^{44}</td><td>2</td></tr> <tr><td>2</td><td>3.4381×10^{18}</td><td>2</td></tr> <tr><td>3</td><td>-1.6987×10^{14}</td><td>1</td></tr> </tbody> </table>	dim(G)	complexity	$\chi(G)$	1	-4.6449×10^6	-10	2	3.5323×10^{84}	2	2	1.9246×10^{35}	2	1	-6.6871×10^{15}	-28	1	-1.2496×10^{31}	-58	1	-7.8275×10^{12}	-22	1	-3.4164×10^{15}	-22	1	-4.0315×10^{37}	-58	3	3.7873×10^{26}	2	3	1.4404×10^{64}	2	2	2.7042×10^{44}	2	2	3.4381×10^{18}	2	3	-1.6987×10^{14}	1
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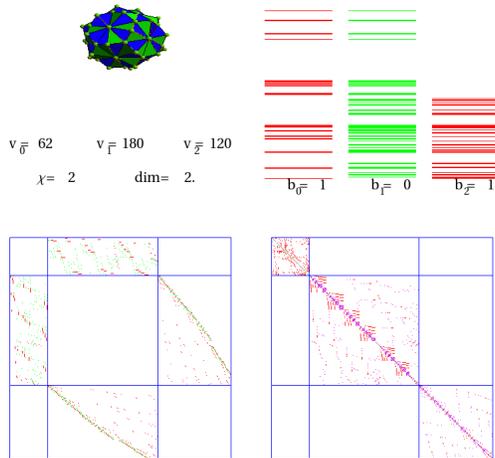


FIGURE 25. The DisdyakisTriacontahedron is the Catalan solid with maximal complexity.

APPENDIX: THE ORIGINAL PROOF

While the super symmetry argument is elegant, the original proof is illustrative and shows better the link to Hodge theory. We follow [24] but change notation and expand some of the steps slightly. Denote by E_λ^p the eigenspace of the eigenvalue λ of L on $\Omega^p(G)$.

Proposition 19 (McKean-Singer). $\sum_p (-1)^p \dim(E_\lambda^p) = 0$ for all $p \geq 0$.

Proof. If λ is a positive eigenvalue, we just write E^p instead of E_λ^p .

(i) $E^p = dE^{p-1} \oplus d^*E^{p+1}$

Proof. That the right hand side is included in the left hand side follows from $[d, L] = [d^*, L] = 0$. For example: if $f = dg$ with $Lg = \lambda g$, then $Lf = Ldg = dLg = d\lambda g = \lambda dg = \lambda f$ and $f \in E^p$.

Given $f \in dE^{p-1}, g \in d^*E^{p+1}$, then $\langle f, g \rangle = \langle dh, d^*k \rangle = \langle d^2h, k \rangle = 0$ so that the two linear spaces on the right hand side are perpendicular. Assume we are given $f \in E^p$ which is perpendicular to both subspaces. Then $0 = \langle f, dE^{p-1} \rangle = \langle d^*f, E^{p-1} \rangle$ and $0 = \langle f, d^*E^{p+1} \rangle = \langle df, E^{p+1} \rangle$ so that $d^*f = df = 0$ and $Lf = 0$ which together with $f \in E^p$ implies $f = 0$.

(ii) Summing the identity $\dim(E^p) = \dim(dE^{p-1}) + \dim(d^*E^{p+1})$ obtained in (i) gives $\sum_p (-1)^p \dim(E^p) =$

$$\begin{aligned} &= \sum_{p \text{ even}} \dim(E^p) - \sum_{p \text{ odd}} (\dim(dE^{p-1}) + \dim(d^*E^{p+1})) \\ &= \sum_{p \text{ even}} \dim(E^p) - \dim(dE^p) - \dim(d^*E^p) \\ &= \sum_{p \text{ even}} \dim(dE^{p-1}) + \dim(d^*E^{p+1}) - \dim(dE^p) - \dim(d^*E^p) \\ &= \sum_{p \text{ even}} \dim(dE^{p-1}) + \dim(d^*E^{p+1}) - \dim(dd^*E^{p+1}) - \dim(d^*dE^{p-1}) \\ &= \sum_{p \text{ even}} [\dim(dE^{p-1}) - \dim(dd^*E^{p+1})] + [\dim(d^*E^{p+1}) - \dim(d^*dE^{p-1})] \geq 0. \end{aligned}$$

(iii) $\sum_p (-1)^p \dim(E^p) \leq 0$

From (ii), we recycle $\sum_p (-1)^p \dim(E^p) = \sum_{p \text{ even}} \dim(E^p) - \dim(dE^p) - \dim(d^*E^p)$. The claim follows now because each term is ≤ 0 :

$$\begin{aligned} &\dim(E^p) - \dim(dE^p) - \dim(d^*E^p) \\ &= \dim(LE^p) - \dim(dE^p) - \dim(d^*E^p) \\ &\leq \dim(dd^*E^p) + \dim(d^*dE^p) - \dim(dE^p) - \dim(d^*E^p) \\ &= [\dim(dd^*E^p) - \dim(dE^p)] + [\dim(d^*dE^p) - \dim(d^*E^p)] \leq 0. \end{aligned}$$

The two parts (ii) and (iii) imply the proposition. \square

Corollary 20. $\text{str}(e^{-tL}) = \chi(G)$.

Proof. Proposition (19) implies $\text{str}(L^k) = 0$ for $k > 0$. A Taylor expansion of the super trace of the heat kernel using the proposition gives

$$\text{str}(e^{-tL}) = \text{str}\left(1 - t\frac{L}{1!} + t^2\frac{L^2}{2!} - \dots\right) = \text{str}(1) = \chi(G)$$

\square

APPENDIX: HODGE THEORY FOR GRAPHS

For $t = 0$ we have by definition the Euler characteristic and for $t \rightarrow \infty$, the supertrace of the identity on harmonic forms. The proof of the Hodge theorem stating that $H^p(G)$ is isomorphic to the space $E_0^p(G)$ of harmonic p -forms is clearly visible in McKean-Singers proof:

- Lemma 21.** a) $d^*L = Ld^* = d^*dd^*$ and $dL = Ld = dd^*d$.
 b) $Lf = 0$ is equivalent to $df = 0$ and $d^*f = 0$.
 c) $f = dg$ and $h = d^*k$ are orthogonal. $\text{im}(d) \cup \text{im}(d^*)$ span $\text{im}(L)$.

Proof. a) is clear from $d^2 = (d^*)^2 = 0$.

b) if $Lf = 0$ then $0 = \langle f, Lf \rangle = \langle d^*f, d^f \rangle + \langle df, df \rangle$ shows that both $d^*f = 0$ and $df = 0$. The other direction is clear.

c) $\langle f, h \rangle = \langle dg, d^*k \rangle = \langle ddg, k \rangle = \langle g, d^*d^*k \rangle = 0$. □

Any p -form can be written as a sum of an exact, a coexact and a harmonic p -form.

Lemma 22 (Hodge decomposition). *There is an orthogonal decomposition*

$$\Omega = \text{im}(L) + \ker(L) = \text{im}(d) + \text{im}(d^*) + \ker(L) .$$

Any g can be written as $g = df + d^*h + k$ where k is harmonic.

Proof. The operator $L : \Omega \rightarrow \lambda$ is symmetric so that image and kernel are perpendicular. We have seen before that the image of L splits into two orthogonal components $\text{im}(d)$ and $\text{im}(d^*)$. □

Theorem 23 (Hodge-Weyl). *The dimension of the vector space of harmonic k -forms on a simple graph is b_k . Every cohomology class has a unique harmonic representative.*

Proof. If $Lf = 0$, then $df = 0$ and so $Lf = d^*df = 0$. Given a closed n -form g then $dg = 0$ and the Hodge decomposition shows $g = df + k$ so that g differs from a harmonic form only by df and so that this harmonic form is in the same cohomology class than f . We can assign so to a cohomology class a harmonic form and this map is an isomorphism. □

APPENDIX: ČECH COHOMOLOGY FOR GRAPHS

We have mentioned that the Dirac operator has appeared when computing Čech cohomology of a manifold and investigating the relation between the spectrum of the graph and the spectrum of the manifold [23]. We want to show here that Čech cohomology is equivalent to graph cohomology. This is practical: while graph cohomology is easy to implement in a computer, it can be tedious to compute the operator D and so cohomology groups. If the zero eigenvalues and so the cohomology is the only interest in D , then one can proceed in two different but related ways: One possibilities to reduce the complexity of computing the kernel of D is to deform the graph using homotopy steps to get a simpler graph. Another possibility is to look at the Dirac operator of the nerve. The following definition is equivalent to a definition of Ivashchenko:

Definition 9. A graph is called **contractible in itself**, if there is a proper subgraph H which is contractible and which is the unit sphere of a vertex. The graph G without the vertex x and all connections to x is called a contraction of G . For a

contractible graph, there is a sequence of such contraction steps which reduces G to a one point graph.

This allows now to consider Čech cohomology for graphs:

Definition 10. A Čech cover of G is a finite set of subgraphs U_j which are contractible and for which any finite intersection of such subgraphs is either empty or contractible. A Čech cover defines a new graph N called the **nerve** of the cover. The vertices of N are the elements of the cover and there is an edge (a, b) if both the subgraphs a, b have a nonempty intersection. **Čech cohomology** is then defined as the graph cohomology of the nerve of G .

Remark. The number of vertices of the nerve of G is an upper bound for the geometric category $\text{gcat}(G)$ of G and so to strong category $\text{Cat}(G)$ [13].

Proposition 24. *The nerve of a Čech cover of G is homotopic to G .*

Proof. In a first step we expand each of the U_j so that in each U_j , the intersections $U_k \cap U_j$ are in a neighborhood $B(x_j)$ of a point $x_j \in U_j$. Now homotopically shrink each U_j to $B(x_j)$. \square

Corollary 25. *Čech cohomology on a finite simple graph is equivalent to graph cohomology on that graph.*

Proof. Graph cohomology obviously is identical to Čech cohomology if we chose the cover $U_j = B(x_j)$ for the vertices x_j which has the property that the nerve of G is equal to G . Because the nerve N of G is homotopic to G and graph cohomology of homotopic graphs is the same (as already proven by Ivashchenko [12]), we are done. \square

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