ON GRAPHS, GROUPS AND GEOMETRY

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ABSTRACT. A metric space (X, d) is declared to be natural if (X, d) determines an up to isomorphism unique group structure $(X, +)$ on the set X such that all the group translations and group inversion are isometries. A group is called natural if it emerges like this from a natural metric. A simple graph X is declared to be natural if (X, d) with geodesic metric d is natural. We look here at some examples and some general statements like that the graphical regular representations of a finite group is always a natural graphs or that the direct product on groups or the Shannon product of finite graphs preserves the property of being natural. The semi-direct product of finite natural groups is natural too as they are represented by Zig-Zag products of suitable Cayley graphs. It follows that wreath products preserve natural groups. The Rubik cube for example is natural. Also free products of finitely generated natural groups are natural. A major theme is that non-natural groups often can be upgraded to become natural by extending them to become Coxeter groups. Examples of non-natural groups are cyclic groups whose order is divisible by 4, the quaternion group, the integers, the lamplighter group, the free groups or the group of p-adic integers. The prototype and starting point to all this is to extend the integers to the infinite dihedral group, replacing the single generator by two free reflections. We conclude with a short discussion of the hypothesis of using the dihedral group as a physical time in dynamical system theory.

1. NATURAL METRIC SPACES, GROUPS AND GRAPHS

1.1. Define a **metric space** (G, d) to be **natural** if it carries up to isomorphisms a unique group structure $(G, +, 0)$ such that for every g in G, the **group translations** $x \to r_g(x) = x + g$ and $x \to l_g(x) = g + x$ as well as the **inversion** $I : x \to i(x) = -x = x^{-1}$ are all **isometries** of the metric space (G, d).

1.2. A **group** written additively as $(G, +, 0)$ or multiplicatively as $(G, *, 1)$ is called **natural** if it emerges as the group structure coming from a natural metric space (G, d). For a natural group, we only need to give the metric space to get a multiplication table which is unique up to isomorphism. The algebra is forced by the geometry, space determines time.

1.3. A **simple graph** $X = (V, E)$ is declared to be **natural** if it is a natural metric space (X, d) with the geodesic metric d. We will also look at infinite graphs or **weighted graphs** in which the edges E are assigned a length such that the geodesic distance produces a metric space (V, d). Any finite metric space can be seen as a weighted simple complete graph (V, E, d).

1.4. A natural metric space naturally upgrades to a **metric group** $(G, d, +, 0)$. The algebraic structure can be recovered after applying the **forgetful functor** $(G, d, +, 0) \to (G, d)$ from **metric groups** to **metric spaces**. Metric groups are special **topological groups**. The trivial topology with discrete metric $d(x, y) = 1$ if $x \neq y$ on a set X is never natural if X is infinite, because every group structure on X would be compatible and because there are many non-isomorphic groups of any non-finite cardinality.

1.5. In order to see that a metric space (X, d) is natural, one has to be able to construct the group structure from it. In order to see that a group is natural, one has to find the corresponding natural metric space. To show that a metric space (X, d) is not natural, one has either to establish that no metric-compatible group structure is possible on X or then that there are at least two non-isomorphic group structures compatible with the metric. In order to see that a group is not natural, one has either to see that there is no compatible metric which generates it or to verify that for any compatible metric, there are at least two different compatible group structures possible.

1.6. Here is the general result for finite groups. If $S = S^{-1}$ is a generator set of a group G, denote by $\Gamma(G, S)$ the corresponding **Cayley graph**. One can either stick with the geodesic metric or change to a more general metric space (G, d). By definition, any natural finite group $(G, +)$ comes from a natural metric space (G, d) and so from a weighed Cayley graph for G. If $(H, S), (K, T)$ are finitely presented groups, then the **semi-direct product** $H \rtimes K$ with multiplication $(k_1, h_1) * (k_2, h_2) = (k_1 k_2, h_1 h_2^{k_1})$ [1] has as Cayley graph the Zig-Zag product $\Gamma(H, S) \rtimes \Gamma(K, T)$ which is the Cayley graph of $H \rtimes K$ with generators $\{t_1 s t_2, t_1 \in T, s \in S \cup \{1\}, t_2 \in \{T \cup 1\}\}$. [2] One of the observations we will point out here is that the semi-direct product of two natural finite groups is natural. The metric space emerges from a weighted zig-zag product of the corresponding weighted Cayley graphs. In particular, the direct product of natural groups is natural, coming from a weighted Shannon product of weighted Cayley graphs.

2. First examples

2.1. An example of a natural graph is the **complete graph** K_p for which the number p of vertices is prime. The corresponding metric generates the **discrete topology** on a finite set with p elements. There is only one group structure because there is up to isomorphism only one finite group of prime order p. It is the **cyclic group** $Z_p = \mathbb{Z}/(p\mathbb{Z})$ with p elements. [3] An unweighted complete graph with a countable set of vertices would already be non-natural: it features both Abelian as well as non-Abelian group structures.

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[1] As K is the base of a "fiber bundle" and the normal group N is serves as "normal fibers", we write the base first, as custom in fiber bundles

[2] As the zig-zag product for Cayley graphs relates to the semi-direct product for the groups [53], we reuse the notation for graphs.

[3] The notation Z_p is common. There will be no danger to confuse it with the p-adic group of integers $\mathbb{Z}_p$ which will appear later.

2.2. The smallest example of a non-natural group is the cyclic group C_4. Any metric space with four points which admits this group as a symmetry group also allows to host the **Klein 4-group** $D_2 = Z_2 \times Z_2$. The group D_2 is the smallest **dihedral group** and itself is natural. In order to see that, we need a metric that breaks the symmetry. We replace therefore the geodesic metric on the cyclic graph C_4 so that only $Z_2 \times Z_2$ remains as a possible group structure. More generally, none of the groups Z_{2^k} is natural for $k \geq 2$. We will rephrase this that all Abelian finite groups not containing Z_4 as a subgroup are natural. The case $p = 2$ is very special because all groups Z_{p^k} for odd primes p turn out to be natural. The "make it Coxeter!" construction like $C_4 = \langle a, a^4 = 1 \rangle \to D_4 = \langle a, b, a^2 = b^2 = (ab)^2 = 1 \rangle$ is not available as a tool to build an other non-equivalent but compatible group.

2.3. Up to isomorphism, there are two groups of order 6. Both are natural. Let us look first at the **symmetric group** S_3 of all permutations on a 3-point set. This non-Abelian group is also the smallest non-Abelian dihedral group and a semi-direct product $D_3 = Z_3 \rtimes Z_2$. In the case D_3, the **zig-zag product** of C_3 and Z_2 with a suitable metric defines the natural metric space. The zig-zag product of C_3 and Z_2 is isomorphic to the **utility graph** $K_{3,3}$. While the utility graph both admits D_3 and $Z_2 \times Z_3$ group structures and therefore is not natural, it carries a modified metric that breaks the symmetry. Excluding Z_6 as such, it forces the group structure D_3. There are simpler metric spaces which work in this case: the **connection graph** C_3' of C_3 already is a natural graph. Without having to place special weights, it only admits the dihedral group D_3 as a group structure.
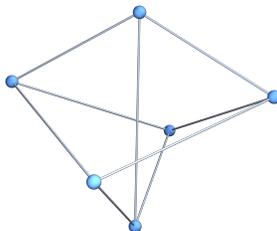

FIGURE 1. The zig-zag product of the Cayley graphs of Z_3 and Z_2 belonging to the semi-direct product $D_3 = Z_3 \rtimes Z_2$ is the bipartite graph $K_{3,3}$ which is also known as the **utility graph**. The graph itself is not natural as it features both the direct product $Z_2 \times Z_3$ as well as the semi-direct product D_3. However, it can be weighted to admit only the D_3 group structure.

2.4. The second group of order 6 is the cyclic group Z_6 which is the **direct product** of Z_2 with Z_3. That this decomposition is possible is part of the **fundamental theorem of Abelian groups**. In the direct product case $Z_3 \times Z_2$, the **Shannon product** of the Cayley graphs C_3 and C_2 of Z_3 and Z_2 with a weighted metric can serve as the natural metric space. [4] Together with a group completed disjoint addition, it produces the **Shannon ring**, a commutative ring structure on **signed graphs** (see [47] for more on this). Note that the Shannon product of C_3 (which

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[4]The Shannon product is also called **strong product**, but Shannon [65] was the first who defined the product in 1956.

NATURAL SPACES

is also the complete graph K_3 as a graph) with K_2 is K_6, a graph which is not yet
natural; it admits both the Abelian Z_6 as well as the non-Abelian $Z_3 \rtimes Z_2$ group
structures. [5] Note that Z_6 is natural too as a product of two natural groups Z_2 and
Z_3. We need to adapt the metric to force the Z_6 group structure: one can write
the grid graph $C_2 \times C_3$, take different distances in the two directions then take the
Euclidean distance invariant under translations.

2.5. The **connection graph** G' of a graph G is defined to be the graph in which
the complete sub-graphs of G are the vertices and where two points are connected if
they intersect. It has as a sub-graph the Barycentric refinement (which has the same
vertex set but less edges). In the Barycentric refinement, two pints are connected if
one is contained in the other. In general, the connection graph C'_n of a cyclic graph
C_n or the connection graph $\mathbb{Z}'$ of the integer graph $\mathbb{Z}$ are natural. The natural
group defined by C'_n is the dihedral group D_n because the automorphism group
of C'_n is D_n. Indeed, the graph C'_n is a **graphical regular representation** of
D_n. Completely analogue, the group defined by $\mathbb{Z}'$ is the **infinite dihedral group**
D_∞ which is natural. Connection graphs are not only natural in the new sense
considered here. They are also natural in that they are associated to Laplacians
which have a spectral gap which does not close in the infinite volume limit. See
[43].

2.6. A finite simple graph (V, E) for which $|V| = |\mathrm{Aut}(V, E)|$ is natural is called
the **graphical regular representation** of its automorphism group. A character-
ization of Cayley graphs which are graphical regular representations is not known.
Because a potential group structure on the vertices is part of the automorphism
group of the graph, graphs with $|\mathbf{Aut}(V, E)| < |V|$ never are natural. The **com-
plete graph** $K_p = (V, E)$ for an odd prime p is an example of a natural graph
with $p = |V| < |\mathrm{Aut}(V, E)| = p!$. It is a natural graph because p being prime forces
a cyclic group structure. The graph K_p is not a graphical regular representation
because the automorphism group $\mathrm{Aut}(K_p) = S_p$ is the symmetric group S_p which
has larger order $p!$ in comparison to p for $p > 2$.

2.7. The **integers** $\mathbb{Z}$ are not natural. Given any translation invariant metric $(\mathbb{Z}, d)$
(there are many like for example the metric induced when placing the integers on
a **helix** $n \to [\cos(an), \sin(an), bn] \in \mathbb{R}^3$ and taking the induced Euclidean metric
from $\mathbb{R}^3$), there is a second, non-Abelian group structure on this metric space. It
is the **infinite dihedral group** $D_\infty = \langle a, b, a^2 = b^2 = 0 \rangle$. The later preserves
the metric space but it is non-isomorphic to $\mathbb{Z}$ because it is non-Abelian. The zero
element is the **empty word**, the addition is done **concatenating** words, but there
are two branches. But we can place a metric on the set $\mathbb{Z}$ in which $d(0, a) \neq d(0, b)$.
Now, one has only one group structure which is compatible. Indeed, such a group
has to be a subgroup of $D_\infty = \mathrm{Aut}(\mathbb{Z}, d)$ and $\mathbb{Z}$ does not qualify as the metric has
broken the translation symmetry on a half integer level. We have just proven that
the infinite dihedral group is a natural group. The transition of going from $\mathbb{Z}$ to
D_∞ is an example of a Higman-Neumann-Neumann (HNN) extension. Unlike for
finite groups, for infinite groups this is possible without changing the cardinality;
a pop culture picture for the cardinality point is the **Hilbert hotel metaphor**.

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[5]We write C_3 for the graph K_3 in which we take the 1-dimensional skeleton complex rather
than the Whitney complex of K_3 (a two-dimensional complex). C_3 equipped with the **1-skeleton
complex** is with notation distinguished from K_3 even so as graphs $C_3 = K_3$.

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This example was the starting point for this research. It was fascinating to see that we can have an infinite metric space that determines a unique group structure.

2.8. Fundamental groups of surfaces can be both natural or non-natural. The **projective plane** $\mathbb{P}^2$ has the natural fundamental group $Z_2 = \mathbb{Z}/(2\mathbb{Z})$. The d-torus has the fundamental group $\mathbb{Z}^d$, which is non-natural. The fundamental group G of the **Klein bottle** is $G = \langle a, b | abab^{-1} \rangle$ which is the semi-direct product $H \rtimes K = \mathbb{Z} \rtimes \mathbb{Z}$, where H is the normal subgroup generated by b and K is the subgroup generated by a. While every element in G can be written as $b^n a^m$, the multiplication is not the direct product. The later would lead to the fundamental group $\mathbb{Z}^2$ and not $\mathbb{Z} \rtimes \mathbb{Z}$. Indeed, the Klein bottle is a **non-trivial fiber bundle** of the circle, while the 2-torus is a **trivial fiber bundle**. The group G is not natural because any metric which forces the group is translation invariant in a. We can however **dihedralise** it and get a natural group. [6] The dihedralisation is a special **HNN extension**. Write $a = ts, t^2 = s^2 = 1$ to et a group $G' = \langle t, s, b | tsbtsb^{-1}, t^2, s^2 \rangle$ which now emerges from the metric.

2.9. The product of a **compact connected Lie group** G with some Euclidean space $\mathbb{R}^n$ is natural. The reason is that these are known precisely to be the **Lie groups** on which there exists a **bi-invariant metric**. The group G then has a **bi-invariant Riemannian metric** d and is therefore a topological (even metric) group in which the metric and the arithmetic are compatible. By Gleason and Montgomery-Zippin, a topological group $(X, d, +)$ for which (X, d) is a topological manifold, has a unique Lie group structure, because any other compatible group operation again will be a Lie group so that there is an isomorphism. (Gleason-Montomery-Zippin more generally establish that if a locally compact topological group G is a projective limit of Lie groups and G has no small subgroups then G is a Lie group).

2.10. The fact that a **compact connected Lie group is natural** is also constructive. Given such a metric space, the group structure can actually be constructed: while there can be different metrics, if G is not simple, any choice of a **Riemannian manifold** (G, d) for which d is bi-invariant fixes the group structure: pick a point 0 and call it "zero". Now we use the metric (using the **fundamental theorem of Riemannian geometry**) to construct a unique connection ∇ and so get a **Lie algebra structure** $[X, Y] = \nabla_X Y - \nabla_Y X$. The Lie algebra determines the group operation on a small ball $B_r(0)$ (with r smaller than the radius of injectivity in the Riemannian manifold) by the **Baker-Campbell-Hausdoff formula**. Knowing how to add group elements in a small ball $B_r(0)$ defines then the group structure globally, because the set of group elements in $B_r(0)$ generate G. We have not found an example yet of a locally compact (positive dimensional) Lie group for which we know that it is not natural. Good examples to look for are the **Heisenberg groups** H_n or Euclidean symmetry groups $\mathbb{R}^n \rtimes O(n)$. (Note that we only have established so far that the semi-direct product preserves natural groups in the finite case. We actually believe that it could be true in general.)

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[6]This is a non-standard notation. "Coxeterize" would work too as the result is a Coxeter group.

2.11. Among the **Euclidean spheres** in $\mathbb{R}^n$, the 0-sphere, 1-sphere and 3-sphere are natural. For the 0-sphere Z_2, things are settled because this is a finite group of prime order forcing the unique group Z_2 of order 2. The group structure on positive-dimensional cases $S^1 = U(1), S^3 = SU(2)$ comes from the multiplicative group structure given by the list of **real associative normed division algebras** $\mathbb{R}, \mathbb{C}, \mathbb{H}$ classified by the **Frobenius theorem** [23]. Since they are compact Lie groups, the multiplication on them is unique. All other Euclidean spheres are not natural metric spaces because they do not admit a topological group structure at all [64]. The fact that the circle $\mathbb{T}^1$ is natural but the integers $\mathbb{Z}$ is not natural shows that the **Pontryagin duality operation** $G \to \overline{G}$ does not preserve the class of natural groups.

2.12. We will look at some finite groups and see that the **alternating groups** A_n and **symmetric groups** S_n are all natural. One can get the metric by realizing the metric space as a weighted Cayley graphs using two generators in the group. The small A_n and S_n ones are especially interesting. It appears that non-natural non-Abelian finite groups are harder to find. But there are some: the **quaternion group** Q_8 is not natural. We discuss this below and see that any metric an 8-point set which is compatible with Q_8 also will allow for the group $Z_2^3 = Z_2 \times Z_2 \times Z_2$.

2.13. We will see that the direct product of natural groups is natural and that Z_2 and Z_{p^n} are natural for odd primes p and $n \geq 1$. From the **fundamental theorem of Abelian groups** follows then that any finite Abelian group, whose order n is not divisible by 4, must be natural. In other words, any finite Abelian group containing Z_4 as a subgroup is non-natural. For example, Z_{24} is not natural as it factors into $G = Z_8 \times Z_3$ and Z_8 is not natural. The fact that a natural $G \times H$ implies both factors G, H to be natural is easier to see because the quotient metric space provides invariant metrics on G and H. These metrics must be natural as otherwise there would be two non-equivalent group structures say on G, producing two non-equivalent group structures on $G \times H$.
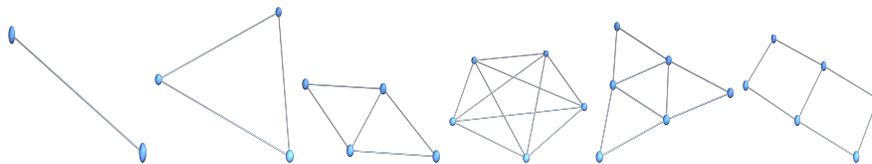

FIGURE 2. Some natural metric spaces defining small groups $Z_2, Z_3, Z_2^2, Z_5, Z_2 \times Z_3, S_3$. Up to order 6, only the finite group $Z_4 = \mathbb{Z}/(4\mathbb{Z})$ is non-natural group. All other groups of order 6 are natural.

2.14. Any dense subgroup of a natural group is natural and a completion of a natural group is natural. We will discuss this in a separate section. For example, the rational numbers $\mathbb{Q}$ or the **dyadic rational numbers** (a set which is in bijection with $\mathbb{Q}$ via the **Minkowski question mark function**) or the **rational circle** (the rational numbers on the circle $\mathbb{Q}/\mathbb{Z}$) or the **Prüfer group** $\{e^{2\pi ik/2^n}, k, n \in \mathbb{Z}\}$ which are the **dyadic rational numbers** modulo 1, are all natural. If the Prüfer group was not natural, then then also $\mathbb{T}$ would be not natural as the two different

group structures would by continuity produce two group structures on the closure $\mathbb{T}$ with respect to the Euclidean distance. On the other hand, we have seen that if the completion of G is not natural, then G is not natural. As the integers are not natural, this implies that the dyadic integers are not natural. Indeed, the dyadic integers $\mathbb{Z}_2$ and more generally the p-adic integers $\mathbb{Z}_p$ contain the rational integers $\mathbb{Z}$ as a dense set. The group addition $x \to x+1$ is the **adding machine**, a uniquely ergodic system on $\mathbb{Z}_p$. The subgroup $\mathbb{Z}$ is just the orbit of the element 0. Translation is uniquely ergodic because there is only group invariant measure on the compact topological group $\mathbb{Z}_p$, the **Haar measure**.

2.15. None of the **free groups** F_n are natural. In each of the cases $F_n, n \geq 1$, one can define an alternative **Coxeter group structure** on F_n. The reason, why F_n is not natural is indeed very similar to the reason why $F_1 = \mathbb{Z}$ is not natural: we can just replace any single generator g in the group F_n with two reflections a, b satisfying $g = ab, a^2 = b^2 = 1$. If one makes this **dihedral change** with all the generators, one obtains the free product FD_n of **infinite dihedral groups**. This **dihedration** is now a Coxeter group and natural. One can change the metric on the Cayley graph so that the weighted Cayley graph as a metric space forces the group structure.

2.16. In general, the **free product** $G * H$ of two finitely generated natural groups is natural. If both are not the trivial group, then the free product is an infinite group acting on a tree. This is part of **Bass-Serre theory** which is relevant for fundamental groups of surfaces, where more generally the free product with amalgamation appears. In particular, the free product of natural finite groups is natural and the graph is the free product of the Cayley graphs.

2.17. The simplest example of a free product of finite groups is the **infinite dihedral group** which is the free product $Z_2 * Z_2$. Also the **modular group** $PSL(2, \mathbb{Z})$ is natural, as it is the free product $Z_2 * Z_3$ [2]. The proof is obtained by noticing that $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ generate $PSL(2, \mathbb{Z})$ so do $x = BA$ and $y = B$ but $x^3 = 1$ in $PSL(2, \mathbb{Z})$ and $y^2 = 1$ in $PSL(2, \mathbb{Z})$ and no words in x, y produces the identity in $PSL(2, \mathbb{Z})$ so that no further relations exist and $PSL(2, \mathbb{Z}) = \langle x, y | x^2 = y^3 = 1 \rangle = Z_2 * Z_3$.
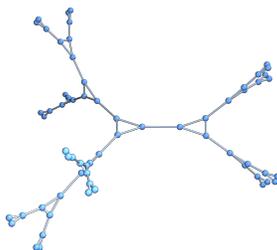

FIGURE 3. The figure shows part of the Cayley graph of $PSL(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$.

2.18. None of the p-adic group $G = \mathbb{Z}_p$ with p-adic metric are natural metric spaces. For any metric which is invariant under these Abelian groups G, there are different non-Abelian group structures which still preserve the metric. In the case $p = 2$, the construction is similar to the $\mathbb{Z} \to D_\infty$ transition: one can see $\mathbb{Z}_2$ as the boundary of an infinite rooted Bethe lattice tree. [7] If the generator a flips the two main sub-trees and the generator b flips the sub-trees and additional induces the group addition by 1 on the right tree and group addition by -1 on the left tree, then $a^2 = b^2 = 1$ and $ab = T$ is the group addition by 1 on half of the space and subtraction by 1 on the other half. The construction for $\mathbb{Z}_p$ is similar. There is no other metric on $\mathbb{Z}_p$ which forces the group structure, because any such G-invariant metric is also invariant under a non-Abelian modification.

3. Dynamical aspects

3.1. Metric groups G are examples of **topological groups** [51, 55, 32], objects located in the heart of harmonic analysis, physics or operator algebras. Examples are **compact Lie groups** G which are special as they feature a bi-invariant Riemannian metric d. A finite group G with a selected set of generators $S = \{g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}\}$ satisfying $S^{-1} = S$ define an **undirected Cayley graph** with **geodesic distance** as metric, where the distance between two group elements $d(g, h)$ is the minimal **word length** of $g^{-1}h$.

3.2. A metric group defines a **topological dynamical system** if the group $(G, +)$ is thought of as **"time"** acting on the **metric space** (G, d) which is considered **"space"**. In this context time is associated with the algebraic structure and space with the topological structure. By definition, any metric group G acts on itself in an isometric way. Both the left translation $L_g(x) = gx$ as well as right translations $R_g(x) = xg$ are **transitive** by definition. For a natural metric, the metric alone produces a **natural group of symmetries on its own metric space** without any need to define the group operation. The **group multiplication table** is determined up to isomorphisms from the topology. One can now define notions like the **periodic part of the group** and the complement, the **aperiodic part of** G: the periodic part of G is the set of $x \in G$ for which there exists an $n > 0$ such that $x^n = 1$. The aperiodic part is the complement in G.

3.3. In the case of a **compact natural metric space** the topological dynamical system can be upgraded to a canonical **measure theoretical dynamical system**. Compact topological groups are **uniquely ergodic** if seen as topological dynamical systems: there is a unique G-invariant probability measure m on the **Borel σ-algebra** of G. It is the normalized **Haar measure** and invariant under automorphisms of the probability space given by the multiplication with an element in G. Asking the group left and right translations and inversions to be isometries identifies natural compact metric groups as **uniquely ergodic transformation groups** for which the metric alone determines the arithmetic as well as all the expectations of random variables.

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[7] This picture is often invoked as a more intuitive picture of the abstract **projective limit** construction. A non-standard analysis trained person would take n non-standard large and deal the sphere of distance n to the origin of the root. It is a finite group but every element in the Cantor set is infinitesimally close to an element in that finite group.

3.4. For a natural group G, the **group of isometries** $\text{Aut}(G, d)$ of the metric space (G, d) defining $(G, +)$ must contain the group G. But it can be larger: in the case of the complete graph (K_p, d) for example with the discrete metric $d(x, y) = 1$ for all $x \neq y$, the isometry group is the **symmetric group** S_p while for the cyclic graph (C_p, d) the isometry group is the **dihedral group** D_p. So, K_7 is an example of a natural graph, even so it is not a graphical regular representation of a group. We don't know yet of a general way to decide naturalness. [8]

3.5. Since almost all finite groups are **class I**, groups admitting a graphical regular representation, we expect also, (that is for not necessarily finite groups), that a general group is natural. Among non-natural cases are popular groups like the **integers** $\mathbb{Z}$, the group of **dyadic integers** $\mathbb{Z}_2$ or the free groups F_n. We will see that for many non-natural groups, one can define on their metric spaces additional non-Abelian group structures which as upgrades are expected to be natural. Interestingly a **Coxeter group upgrade** often works.

3.6. We have just seen that a natural compact metric space (X, d) defines a natural dynamical system (G, X, d, m), where $G = X$ acts on X as isometries, preserving the unique **Haar probability measure** m. Because the action is **transitive**, it is **ergodic** (every G invariant Borel set has measure 0 or 1). It is even uniquely ergodic (there is only one invariant measure). Given a specific point x in the metric space (G, d), we have left and right transformations $L_x, R_x : X \to X$ which are measure-preserving.

3.7. Let us call a point $x \in X$ **ergodic**, if L_x is ergodic with respect to the invariant measure m. L_x is ergodic if and only if R_x is ergodic if and only if the subgroup generated by x is dense in the entire group. If the orbit of L_x is dense, then by the **Krylov-Bogolyubov theorem**, the Birkhoff functionals $f \to \limsup_n n^{-1} \sum_{k=1}^{n} f(T_x^n y)$ on the Banach space $C(G)$ of continuous functions on G define an invariant measure, the Haar measure on G. The metric space X obviously splits into an ergodic and a non-ergodic part. For $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ for example, all rational x are non-ergodic, while the irrational x are ergodic.

3.8. As an other example is obtained by looking at the k**-dimensional torus** $X = \mathbb{T}^k$. Almost all points $x \in X$ are ergodic. A point $x = (x_1, x_2, \dots x_k) \in X$ is called ergodic if the coordinates are **rationally independent** meaning that there is no non-zero integer vector $n = (n_1, n_2, \dots, n_k)$ such that $n \cdot x = n_1 x_1 + n_2 x_2 + \cdots + n_k x_k = m$ is an integer. This assures transitivity and so ergodicity. Lebesgue almost all points x are ergodic because the set of non-ergodic x is a countable union of sub-mananifolds in $\mathbb{T}^k$ which all have measure zero so that also the union has zero measure. In the case $\mathbb{T}^2$ for example, the set of non-ergodic points is a union of projections from $\mathbb{R}^2 \to \mathbb{T}^2$ of countably many lines $ax + by = c$, where a, b, c are integers and $(x, y) \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

3.9. If (V, E) is a natural finite simple graph with vertex set V, then the Haar probability measure is the **uniform measure** which assigns to every point the weight $1/|V|$. Once we have fixed a specific implementation of the by assumption unique group structure, every vertex $v \in V$ has an associated automorphism $T_x = L_x$, the left translation. The **Lefschetz number** $\chi(G, T_x)$ of such a transformation

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[8]It would be interesting to know the complexity of deciding whether a finite group of order n is natural or not.

is defined as the **super trace** of T_x on the cohomology groups $H^k(G) = \ker(L_k)$, where L_k are the blocks in the **Hodge Laplacian** $L = D^2 = (d+d^*)^2 = dd^* + d^*d$. We write this as

$$\chi(G, T_x) = \sum_{k=0}^{d} (-1)^k \operatorname{tr}(T_x|H^k(G)) .$$

Note that by Hodge theory, the vector spaces $H^k(G)$ are just the kernels of concrete matrics L_k acting on k-forms (functions on all k-dimensional simplices of (V,E). The **Lefschetz fixed point theorem** for graphs [36] [9] assures then that $\chi(G, T_x) = \sum_{y \in Fix(T_x)} i_T(y)$, where $\operatorname{Fix}(T_x)$ is the **fixed point set** consisting of complete subgraphs of G which are fixed by T_x.

3.10. We also know that the average Lefschetz number over all automorphisms of a graph is 1:

$$\frac{1}{|V|} \sum_{x \in V} \chi(G, T_x) = \chi(G/G) = 1 .$$

This is essentially a Riemann-Hurwitz formula (see [36]). We have now a necessary algebraic condition which allows to probe whether a graph is natural:

Proposition 1. *Let (V, E) be a natural graph. The sum over all Lefschetz numbers over all transformations T_x is equal to $|V|$.*

3.11. The Lefschetz fixed point theorem for graphs also allows to get a grip on counting simplices which are fixed by the automorphisms. These fixed simplices become points in the Barycentric refinement and correspond to fixed points of homeomorphisms in continuum frame works. For a complete subgraph y that is fixed by T_x and a vertex x, let $i_x(y)$ be the **index** of T_x, defined as

$$i_x(y) = (-1)^{\dim(x)} \operatorname{sign}(T_x|y) .$$

Corollary 1. *For a natural graph, $\sum_{x \in G} \sum_{y \in \operatorname{Fix}(T_x)} i_x(y) = |G|$.*

3.12. For the cyclic graph C_n for example, where the isometry group is is the natural group $G = D_n$, every non-zero group element x defines a translation which has zero index and the identity has total index 0, the Euler characteristic of the graph. There are always m reflections which have index 2 and m reflections which have index 0. We see that the total index $\sum_x \sum_{y \in \operatorname{Fix}(T_x)} i_x(y) = 2n = |D_n|$.

4. GRAPHICAL REGULAR REPRESENTATIONS

4.1. A simple criterion for a graph $G = (V, E)$ to be natural involves the **automorphism group** $\operatorname{Aut}(G)$ of G. It is the group of all **graph automorphisms** of G. An automorphism of a graph could also be defined as a bijections on the vertex set V which preserves the set E of edges and the set E^c of non-edges.

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[9]It holds of course for any finite abstract simplicial complex. The Whitney complex of a graph is just one example and the most intuitive one.

4.2. A simple graph (V, E) is called a **graphical regular representation** of a group G if it is a **Cayley graph** of **generators** $S = S^{-1} \subset G$ of G with the property that the automorphism group of (V, E) is G. A group G is said to have a **graphical regular representation**, if it is the automorphism group of a graph with a graphical regular representation.

Proposition 2. *If (V, E) is a finite simple graph for which the automorphism group $G = \mathrm{Aut}(V, E)$ has order $|V|$, then the graph (V, E), the group $(G, +)$ as well as the metric space (V, d) are all natural.*

Proof. Because left translations in a natural group are graph isomorphisms of the Cayley graphs, a natural group G must be a subgroup of the finite group $\mathrm{Aut}(V, E)$. Of course, there is only one group with $|V|$ elements which is a subgroup $\mathrm{Aut}(V, E)$ and which has $|Aut(V, E)| = |V|$ elements: it is $G = \mathrm{Aut}(V, E)$. In order to attach group elements to vertices of the graph, choose one of the vertices and call it 0. Now for any automorphism $T \in Aut(V, E)$, attach it to the vertex $x = T(0)$. This pairs up $G = \mathrm{Aut}(V, E)$ with the vertex set V. $\square$

4.3. For example, for the connection graph C_n' of C_n which has $|V| = 2n$ elements, the automorphism group is D_n, a group with $2n$ elements. Therefore C_n' and D_n and (V, d) are natural.

4.4. Almost equivalent is the case of finite metric spaces. The automorphism group now becomes **the group of isometries**:

Proposition 3. *Assume (G, d) is a finite metric space for which the group of isometries has order $|G|$, then (G, d) is natural.*

Proof. Any group compatible with the metric must be a subgroup of the group of isometries and if it is the group of symmetries, it is uniquely determined. We can build up the group structure by labeling any point as 0, then take an isometry T in the isometry group and assign the isometry group element T to the point $T(0)$ in the metric space. $\square$

4.5. One can also reverse this. We see that most graphs are non-natural because most graphs lack enough symmetry. Especially, every graph with trivial automorphism group (meaning there is no symmetry except the identity) is non-natural if it has more than two vertices. For example, the linear graph L_2 of diameter 2 with 3 vertices can not be natural because its automorphism group is $Z_2 = \mathbb{Z}/(2\mathbb{Z})$. This is not large enough to host a group structure with 3 group elements.

Proposition 4. *If $|V|$ does not divide the order of the automorphism group of a finite simple graph (V, E) then (V, E) is not natural.*

Proof. By the **Lagrange theorem** in group theory, the order of a subgroup of $\mathrm{Aut}(V, E)$ must divide the order $|\mathrm{Aut}(V, E)|$ of $\mathrm{Aut}(V, E)$. For a natural graph (V, E), the order of the induced group is $|V|$. $\square$

4.6. While the topic of natural groups has relations with graphical regular representations of a group, it is different. Graphical regular representations of a finite group render the group natural but there are natural graphs which do not have a graphical regular representation and there are natural groups which do not come from graphs alone but need weighted graphs. The group $Z_2 \times Z_3$ for example needs a finite metric space which is weighted Cayley graph. A general, finite metric space

can always be described by a weighted Cayley graph as we are allowed to take $S = G$ as the generating set.

Proposition 5. *If a finite Cayley graph is a graphical regular representation, then it is natural. If a finite group has a graphical regular representation then it is natural.*

Proof. This follows from the definitions as the Cayley graph defines then a metric space (V, d) on which the group G has the same order $|V|$ which is a subgroup of the automorphism group. $\square$

4.7. Many graphs are natural even so they have no graphical regular representation. For natural graphs, the automorphism group can be much larger than the number of vertices of the graph. For example, every complete graph K_p with prime p is natural despite that it is far from being a graphical regular representation. In the case of $(V, E) = K_p$, we have $|Aut(K_p)| = p!$ which is for $p > 2$ always larger than $|V|$. Still, there is a unique group which can be planted on K_p because there is only one group of prime order p.

4.8. If G has a graphical regular representation, then only the identity element fixes the set S of generators. But even if G induces non-trivial symmetries on S, it is still possible that the graph is natural. We just put different distances on the different generator edges which breaks the symmetries and so that G induces no symmetry any more on S. Still, we can use results from the theory of graphical regular representation, [49] for example establishes a result implying that all **metacyclic** p**-groups** are natural.

4.9. As the above proposition shows, the concept of graphical regular representation comes close to the notion of "naturalness". There is a difference because a group can be natural without coming from a bare graph. It can be natural with respect to a metric space like for example a weighted Cayley graph.

4.10. The concept of graphical regular representations started with Sabidussi [63], who calls a graph **vertex transitive** if the automorphism group is transitive. At that time, Cayley graphs were still called **Cayley color-groups** as one can imagine the different generators assigned different colors. Examples of follow up work is Nowitz in 1968 and Imrich, Watkins, Bannai [5] in the 1970ies [70]. Modern sources are [31, 49].

5. Graph operations

5.1. The **graph complement** of (V, E) is the graph (V, E^c), where E^c is the complement of the set of all possible edges on the complete graph $K_{|V|}$ with $|V|$ elements. It is here convenient to extend graph an isometry also to the **disconnected case**, where $d(x, y) = \infty$ rewrites that x, y are in different connectivity components. With this assumption, every permutation of the vertex set V is a graph isometry of the complete graph K_n as well as of the graph complement $P_n = K_n^c$, the **point graph** with n vertices and no edges.

5.2. The graph complement of a Cayley graph $\Gamma(G, S)$ is sometimes a Cayley graph again with $\Gamma(G, S^c)$ like if $S^c = G - S - \{0\}$ generates the same graph. The group $G = Z_7$ can be generated by $a(x) = x + 1, a^{-1} = x - 1$ leading to the Cayley graph C_7, the cyclic graph with 7 elements. Its graph complement C_y^c which is a discrete Moebius strip generated by $a(x) = x + 2, a^{-1}(x) = x - 2$ and $b(x) = x + 3, b^{-1}(x) = x - 3$ on $\mathbb{Z}/(7\mathbb{Z})$. Graph complements of cyclic graphs have a rich structure [46]. Let us not use the extended definition for metric spaces allowing $d(x, y) = \infty$ and restrict to graphs for which both graph and complement are connected.

Proposition 6. *If (V, E) is a connected natural finite simple graph such that its graph complement is connected, then then his graph complement (V, E^c) is natural too.*

Proof. Every isometry of a finite simple graph equipped with the geodesic metric necessarily maps edges into edges and non-edges to non-edges. Graph automorphism and isometries are the same. Now we can use the fact that the group of graph automorphisms of a graph G and the group of graph automorphisms of the graph complement G^c is the same. If G is natural, there is an up to isomorphism unique group structure on G so that all group operations are isometries. This group structure is also possible on the graph complement. Any different, non-isomorphic group structure on the complement would produce on on (V, E). $\square$

5.3. We know that a complete graph K_p is natural if and only if $p = 1$ or p is prime. This implies for example that for any prime p, the p-point graph without edges is natural. There is only one group structure on a set with p elements and we can also take the trivial metric on the graph complement of K_p. It is just not any geodesic metric.

5.4. Let us dwell a bit more on the disjoint union of two metric spaces which is some sort of problematic notion as it requires to define the distance between two separate points to be infinite. In the case when both metric spaces have finite diameter, we can postulate a finite distance between disconnected parts. The **disjoint union** of two compact metric spaces is a metric space again, if we allow the distance function to be extended. But it is important to note that the disjoint union of two natural metric spaces $(X_1, d_1) \cup (X_2, d_2)$ is not necessarily natural in the strict sense simply because we would need to extend the distance function to take infinite values. The disjoint union of two copies of $\mathbb{R}$ for example is not natural. Place 0 on one branch, then look at the point a of minimal distance to 0 on the second branch. Now because of the isometry assumption $a = -a$ and every point x on one branch has a mirror point $x + a$. This allows for the group structure $\mathbb{R} \times Z_2$ as well as a semi-direct non-abelian group structure $\mathbb{R} \rtimes \mathbb{Z}_2$ where every element on the mirror branch is a reflection.

5.5. Also the **join** of two natural graphs is in general not natural. While we could put a weighted distance on the cyclic graph C_4 such that it becomes natural, just produce a rectangular shape. The group then is the **Klein four group** $Z_2 \times Z_2 = D_2$ which is natural. The graph $K_2 \oplus K_2 = K_4$ is not natural (as a graph) while K_2 is natural and $K_2 \oplus K_2$ carries a weighted metric that is natural.

5.6. Is the **discrete 3-sphere graph** $C_4 \oplus C_4$ natural? It is the smallest discrete 3-sphere and has 8 vertices. It is **not natural** as it admits both the quaternion group as well as the elementary Abelian group Z_2^3. Also the 2-sphere, the **octahedron graph** $C_4 \oplus C_2$, which is a 2-sphere with 6 vertices, is not natural. There are two groups of order 6. The cyclic group $Z_6 = \mathbb{Z}/(6\mathbb{Z})$ and the symmetric group $S_3 = D_3$. The automorphism group of the octahedron is $S_4 \times Z_2$ which has 48 elements and this admits both $C_6 \sim C_3 \times C_2$ as well as S_3. If we look at the graph as a metric space and break the symmetry of the metric by changing the length of a single triangle, then we have only the S_3 symmetry left so that S_3 is natural.
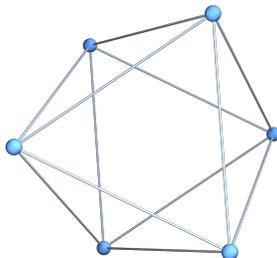

FIGURE 4. The octahedron graph is not natural as it admits two different group structures. One can change the metric however to produce a natural metric space: shrink the lengths of a single triangle without changing any other lengths. This produces a metric space which is natural and induces the group S_3. We knew already that $S_3 = D_3$ is natural because it is a dihedral group which is natural.

5.7. There are various graph multiplications. For all of them, the vertex set of the product is the **Cartesian product** $V(G) \times V(H)$ of the vertex sets of G and H. Among Abelian multiplications are the **weak product**, the **strong product** (=Shannon product [65]), the **large product** (the dual of the strong product [62]) and the **tensor product** in which $E(G) \times E(H)$ is the edge set. A non-abelian product is the **lexicographic product**. A general question is to determine which graph products preserve natural graphs. An other question is which graph products allow for a new metric on the product so that it becomes a natural metric space.

5.8. Each graph multiplication can be extended to weighted graphs and produce again a weighted graph. An edge $((a, b), (c, d))$ in the product has then the weight $d(a, c) + d(b, d)$. Any finite natural group defines weighted natural graphs and by adapting the weights one can assure that the **direct product** of two groups is natural. We will come to that.

5.9. For the subclass of Cayley graphs (in particular for graphs which must be vertex transitive), there is an other product, the **Zig-Zag product**. It is associated with the **semi-direct product** of the corresponding groups and when weighted can be used to see that the semi-direct product of two finite natural groups is natural.

5.10. There is also a **non-Abelian** multiplication, the **lexicographic product of graphs** $G \cdot H$ (introduced by Hausdorff in 1914 and further studied by Harary and Sabidussi). The edges in the later are $((a,b),(c,d))$ if $(a,c) \in E(G)$ or $a = c$ and $(b,d) \in E(H)$. The non-Abelian lexicographic product is interesting because it is **self dual** in that the product of the graph complements is the complement of the product and that the **factorization problem** for graphs is complexity-wise equivalent to the graph isomorphism problem. This means that testing whether a graph of size n is a lexicographic product of two smaller graphs is polynomially equivalent to decide whether two graphs of size n are isomorphic. It has been asked in [71], whether prime factors of the strong product can be computed in polynomial time.

5.11. The lexicographic product of P_n with a graph G is the n fold copy of G. However, the lexicographic product of G with P_n is connected. The graphs $K_n \cdot P_2$ are $(n-1)$-spheres by induction because $K_n \cdot P_2$ is the suspension of $K_{n-1} \cdot P_2$. The Betti vectors are $b = (1,0,0,\ldots,1)$ leading to Euler characteristic 0 or 2 depending on parity. Interesting is the case $K_n \cdot P_m$ which leads to n-dimensional graphs with Betti vectors $(1,0,0,\ldots,(m-1)^n)$. This produces explicit examples of graphs, where the Euler characteristic grows exponentially with the dimension and the vertex size. The f-vector $f = (f_0, f_1, f_2, \ldots, f_d)$ of $K_n \cdot P_m$ is $f_k = B(n, k+1)m^k$ and the f-function $f(t) = 1 + f_0 t + f_1 t^2 + \ldots f_d t^{d+1}$ is $(1 + mx)^n$.

Proposition 7. *The Lexicographic products $A \cdot B$ and $B \cdot A$ of two finite simple connected natural graphs can be weighted to become a natural metric space.*

Proof. Again just take the product metric where the distances in the first coordinate and the second coordinate are different. We need connected in order to have a natural metric space in the product. We can not conclude that spheres $K_n \cdot P_2$ are natural for example. □
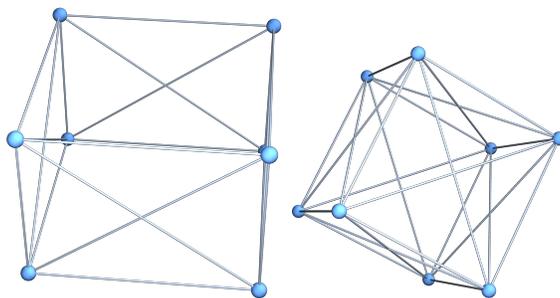

FIGURE 5. The lexicographic product $C_4 \cdot K_2$ of C_4 with K_2 and the lexicographic product $K_2 \cdot C_4$ of K_2 with C_4

6. COMPLETIONS

6.1. We note here that a metric space is natural if and only if its completion is natural. This will imply that $\mathbb{Q}$ is natural because $\mathbb{R}$ is natural. It also implies that the p-adic group of integers Z_p is not natural, because the non-natural $\mathbb{Z}$ is dense in Z_p. But like the real numbers $\mathbb{R}$, also each of the additive groups in the p-**adic numbers** $\mathbb{Q}_p$ are natural.

6.2. By Ostrowski's theorem, the only topological completion of $\mathbb{Q}$ are $\mathbb{Q}_2$ or $\mathbb{Q}_p$ with odd p or $\mathbb{R} = \mathbb{Q}_0$. It turns out that all these groups are natural. The argument is to show first that a dense set of a natural group is natural, then see from $\mathbb{R}$ being natural that $\mathbb{Q}$ is natural. The group operation is forced by any metric. It is also forced by the p-adic distance metric and so assures that the p-adic numbers $\mathbb{Q}_p$ are natural. Similarly as the rational integers are not natural in $\mathbb{Q}$, also the p-adic integers are not natural. Again, one can use the same completion argument. The p-adic integers Z_p are the completion of Z with respect to the p-adic norm.

Proposition 8. *A topological group G is natural if and only if its topological completion $\mathbb{G}$ is natural.*

Proof. (i) First assume that G is a natural topological group. As $\overline{G}$ admits a group operation by taking limits, we only have to show that it does not admit two non-isomorphic group structures. If it would, restricting the metric of the completion $\overline{G}$ to the group G produces a natural metric space because two different group structures could be extended to two different group structures on the completion rendering the completion non-natural.
(ii) Now assume that $\overline{G}$ is natural. Its restriction to G is a group structure on G and also defines a metric. This metric space (G, d) is natural. If it were not, there would be two compatible group structures on (G, d). Both of them would extend to $\overline{G}$ and produce two group structures on $\overline{G}$. $\qquad\square$

6.3. A consequence is that the rational numbers $\mathbb{Q} \subset \mathbb{R}$ or finite field extensions like the Gaussian rationals $\mathbb{Q}(i) \subset \mathbb{C}$ are natural. Here are examples:

Corollary 2. *The additive group of rational numbers $\mathbb{Q}$ is natural. The p-adic integers $\mathbb{Z}_p$ are not natural. The additive group of any algebraic number field is natural. The additive group of the algebraic numbers is natural. The vector field $\mathbb{Q}^n$ is natural.*

Proof. This all follows from the proposition. For example, for $G = \mathbb{R}$, as the metric $d(x, y) = |x - y|$ allows for only one group structure, this also holds for the dense set $\mathbb{Q}$. $\qquad\square$

7. Direct products

7.1. The **Shannon product** [10] of two arbitrary graphs G, H is the graph $G * H$ in which the Cartesian product is the vertex set and where two pairs $(a, b), (c, d)$ are connected if the projection on both factors is either an edge or vertex. There are three possibilities therefore, connections of the form $((a, b), (a, c))$ with (b, c) being an edge in H. Or then, a connection is of the form $((a, c), (b, c))$ where (a, b) is an edge in G. Or then a connection is of the form $((a, b), (c, d))$ where (a, c) is an edge in G and (b, d) is an edge in H.

7.2. In the special case of a product of two Cayley graphs, we get again a Cayley graph. The Shannon product of the Cayley graphs $\Gamma(G, S_G), \Gamma(H, S_H)$ is the Cayley graph $\Gamma(G \times H, S_G S_H \cup S_G \cup S_H)$. If the generator set is **enhanced** by adding 1 $\overline{S}_G = S_G \cup \{1\}$ then $\overline{S}_G \overline{S}_H$ is an enhanced generating set for the Cayley graph. The enhanced generator set is only used for the multiplication. It is not that the Cayley

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[10]Also called strong product

graphs get equipped with a self-loop so that Cayley graphs remain undirected simple graphs.

7.3. The next proposition gives an other reason why the notion of Shannon product of graphs is important. It led to the notion of Shannon capacity [65] (see [42] for a more expository account). It produces a natural arithmetic [40, 47]. It is compatible with curvature [45] and cohomology, also when looking at Lefschetz numbers.

Proposition 9. *The Cayley graph $\Gamma(G \times H, \overline{S}_G \overline{S}_H)$ is the Shannon product of the Cayley graphs $\Gamma(G, \overline{S}_G), \Gamma(H, \overline{S}_H)$.*

Proof. Given two points $(a, b), (c, d) = (xa, yb)$ where $x \in \overline{S}_G$ and $y \in \overline{S}_H$. They are connected if either (a, c) is connected in G and (b, d) is connected in H or then $a = c$ and (b, d) is connected in H or $b = d$ and (a, c) are connected in G. This is exactly what the Shannon product does. $\square$

7.4. The **direct product** $G \times H$ of two groups G, H is a special case of the **semi-direct product** $G \rtimes H$. In the direct product case, the action of the base on the fibers is trivial.

Lemma 1. *The direct product $G_1 \times G_2$ of two natural finite groups G_1, G_2 is natural.*

Proof. Let (G_i, d_i) be the metric spaces inducing G_i for $i = 1, 2$. The group of isometries of G_i contains G_i as a subgroup. Take the product metric $G_1 \times G_2, d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + kd_2(x_2, y_2)$ with k larger than any minimal non-zero distance in G_1. An isometry T of $G_1 \times G_2$ must then be a direct product of isometries T_i of G_i. As the number of elements in the metric space $(G_1 \times G_2, g)$ matches the order of the isometry group of the product, it is natural. $\square$

7.5. This generalizes to non-necessarily finite groups and to **semi-direct products** $G = H \rtimes K$ which is a group in which H is a normal subgroup. Think of K as the base and the group H as the fiber. The base elements $k \in K$ now introduce group automorphisms $\phi_k : H \to H$ on the fibers and $(k_1, h_1) * (k_2, h_2) = (k_1 k_2, h_1 \phi_{k_1} k_2)$ is the group operation on the semi-direct product.

7.6. The prototype of a semi-direct product is the **Euclidean group** $G = \mathbb{R}^n \rtimes O(n)$ generated by orthogonal transformations (rotations, reflections) and translations. This is a vector bundle with base manifold $O(n)$ where at every point the fiber is a vector space $\mathbb{R}^n$. The operation is best seen when writing a group element as $a : x \to Ax + b$ with $A \in O(n)$ and $b \in \mathbb{R}^n$. If $b : x \to Bx + c$ is an other group, then $ab(x) = ABx + Ac + b$, showing how the base rotation group has acted on the fibers. Now, this Euclidean group is already interesting as it is a non-Abelian Lie group which does not admit a bi-invariant metric. Is it natural? We do not think so because any metric d on G renders it a locally compact topological group and G would admit a bi-invariant metric. We know however that direct products of compact Lie groups with $\mathbb{R}^n$ admit a bi-invariant metric. More generally for any normed vector space V the isometry group is $V \rtimes O(V)$. (i.e. [28]).

7.7. For finite groups, the Cayley graphs of the semi-direct product is the **Zig-Zag product** $\Gamma(H, S) \rtimes \Gamma(K, T)$ for the corresponding Cayley graphs $\Gamma(H, S), \Gamma(K, T)$. This is defined as the Cayley graph of $H \rtimes K$ generated by $\{t_1 s t_2, t_1, t_2 \in T, S \in S\}$. The name is justified in that the three group operations t_1, s, t_2 generate a zig-zag curve.

7.8. There is a condition which is satisfied if the direct product (X, d) of two metric spaces $(X_1, d_1), (X_2, d_2)$ is natural [28]: an isometry T of (X, d) decomposes as a product of isometries if for every $x_1 \in X_1$, there is a point $T_1(x_1)$ such that $T(x \times X_2) = (T_1(x) \times X_2)$.

7.9. Not only the direct product preserves natural groups. Also the semi-direct product does:

Proposition 10. *The semi-direct product of arbitrary natural finite groups is natural.*

Proof. Just extend the Zig-Zag product to a weighted Cayley graph metric. In general, we need for a more rich metric structure, as we need for example to distinguish a graph like the **Frobenius group** $Z_7 \rtimes Z_3$ and the direct product group $Z_{21} = Z_7 \times Z_3$ which are the two only groups of order 21. $\qquad \square$

7.10. The **fundamental theorem for finite Abelian groups** states that every finite Abelian group is a direct product of cyclic groups. If G is the product of cyclic prime groups or prime power groups of odd primes, the product is natural: for any sequence of primes p_j for example, the Abelian group $Z_{p_1} \cdots Z_{p_n}$ is natural. Also the group Z_{p^k} is natural if p is an odd prime. Proof: a metric hosting it must have a cyclic symmetry. A dihedral symmetry group can not be planted on a metric space with and odd number of vertices. So, we have:

Proposition 11. *Any finite Abelian group which does not contain $\mathbb{Z}_4$ as a subgroup is natural.*

Proof. By the fundamental theorem of Abelian groups, a finite Abelian group is of the form $Z_{p_1^{n_1}} \times \cdots Z_{p_m^{n_m}}$, where the primes p_j are not necessarily disjoint. For odd primes p, the group Z_{p^n} is natural because a metric space defining the group must have a cyclic symmetry and so have a cyclic or dihedral symmetry. In the odd case however, we can not implement a dihedral group structure on the vertices as this would require the order of the group to be even. Taking direct products with Z_2 is possible as the Klein four group $Z_2 \times Z_2$ is a dihedral group and so natural. Now use induction with respect to the number of factors and that the product of a natural group with a cyclic group Z_{p^n} of prime order $p \neq 2$ is natural for odd primes p. $\qquad \square$

8. More examples

8.1. A **Bethe lattice** can be seen as a Cayley graph. Lets look at the Bethe lattice of degree 4 which is the **Cayley graph** of the **free group** $F_2 = \{a, b|\}$ with 2 generators. It allows for an additional group structure $DF_2 = G = \{a, b, c, d | a^2 = b^2 = c^2 = d^2 = 1\}$ so that F_2 is not natural. But the non-Abelian upgrade group DF_2 is natural. We can define a metric d on the Bethe lattice (G, d) by assigning different lengths to the edges belonging to the four generators a, b, c, d, breaking so the symmetry and preventing the F_2 group structure. This is exactly how the infinite dihedral group $D_\infty = DF_1$ has emerged as a natural extension of the integers, although it is non-Abelian. Now, we can fix a point in the broken Bethe lattice (seen as a metric space without group structure) and call it $0 = \{\}$. Attach different lengths to the four edges leaving 0. The geometry forces now that the four points a, b, c, d in the unit sphere of 0 are generators of the group. Because for any

18

invariant group, the inverses $a^{-1}, b^{-1}, c^{-1}, d^{-1}$ also need to have the same length, we must have $a = a^{-1}, b = b^{-1}, c = c^{-1}, d = d^{-1}$. We are therefore forced to have the non-Abelian dihedral group structure DF_2. An example of a group element in DF_2 is **the word** $x = ababcdcdad$. Its inverse is $-x = dadcdcbaba$. Elements satisfying $x + x = 0$ are **palindromes**. Similarly as in the infinite dihedral group case, elements of DF_2 are words with **non-repeated letters**. We like to see this as a sort of **Pauli principle**.

8.2. In the case of finite groups, the question whether G is natural or not can be decided in finite time. The reason is that we have only finitely many metric structure types which are possible. We can speculate that the computational effort to decide is exponential in the group order. For small orders, we can do it case by case. For example, from the 5 groups of order 8, the groups D_4, $Z_2 \times D_2, Z_2^3$ are natural (using already that products of natural groups are natural) while the cyclic group $Z_8 = \mathbb{Z}/(8\mathbb{Z})$ and the **quaternion group** Q_8 are both not natural. If the cyclic group C_8 would come from a metric (X, d), it would not only host the Abelian group structure but also the dihedral group structure D_4. Like in the case $\mathbb{Z}$, the groups Z_{2n} for $n \geq 2$ happen also to be examples of **non-simple** groups which **do not split**.

8.3. The groups Z_{p^n} with **odd prime** p are non-simple, non-split groups. They are natural: for a finite group with p^2 elements, there are only two group structures, the cyclic group and the group $Z_p \times Z_p$. There is a natural metric: take a product metric with different lengths in order to eliminate the cyclic group. Let us look at the prime $p = 3$ and the cyclic group Z_9 for example. We claim that the cyclic graph C_9 with geodesic metric does not admit a $Z_3 \times C_3$ symmetry. The group $\mathbf{Aut}(C_9, d)$ of isometries of (C_9, d) is D_9, a group with 18 elements. But the group $Z_3 \times Z_3$ is not a subgroup of this dihedral group because a subgroup of index 2 would be a normal subgroup and every subgroup of a dihedral group is either cyclic or dihedral (see [16] Theorem 3.1). This argument goes over to any group Z_{p^n} for odd primes p. But this argument obviously fails for Z_{2^n} with $n \geq 2$.

8.4. Among all the groups of order 27, we know that the groups $Z_{27}, Z_3 \times Z_9, Z_0 \rtimes Z_3, Z_3^3$ are natural. There is also the **Burnside group** $B(2, 3)$, the quotient of the free group F_2 by the subgroup generated by all cubes of that group. We know that $B(2, 3)$ is natural. The reason is that $B(2, 3)$ is also known as the **Heisenberg group modulo 3** and so the semi-direct product of the **translation group** (in the two-dimensional plane defined by the finite group Z_3) with the shear group (which is isomorphic to Z_3. In total, $B(2, 3) = (Z_3 \times Z_3) \rtimes Z_3$. Burnside already knew in 1902 that all groups $B(d, 3)$ are finite. We do not know yet which $B(d, 3)$ are natural.

8.5. As for finite Abelian groups, the structure is clear from the fundamental theorem of Abelian groups. We have seen already that every finite group G of prime order p is natural: it comes from the discrete metric, which is the geodesic metric on the complete graph K_p with p vertices. Because all groups of prime order p must be cyclic Z_p, there are more possibilities for a metric to generate the group, One could take the geodesic metric of the cyclic graph C_p or its graph complement $(C_p)^c$ for example. While the groups Z_{2^n} are not natural for $n \geq 1$, the groups Z_{p^n} are natural.

19

8.6. The **quaternion group** Q_8 has as automorphism group the symmetric group S_6, a group with 6! elements. This large symmetry suggests that G can host itself and become natural. However, the metric of the Cayley graph of Q_8 generated by $S = \{i, j, k, -i, -j, -k\}$ (we add the inverses S^{-1} to get an undirected graph) allows only to choose 4 lengths as parameters. And these are the multiplications by $i, j, k, -1$. Any such a metric also admits the group structure $Z_2^3 = Z_2 \times D_2$, the product of the 2-point group Z_2 and the **Klein four group** $D_2 = Z_2 \times Z_2$. The quaternion group Q_8 is not natural. Additionally, we know it is an example of a non-simple group which does not split.

8.7. The property of being a non-simple but non-split group is shared also by higher **dicylic groups** Dic_n and we have $Q_8 = Dic_2$. Even-so $1 \to Z_{2n} \to Dic_n \to Z_2 \to 1$ is a short exact sequence, we do not have $Dic_n = Z_{2n} \rtimes Z_2$. The reason is that the possible choices of semi-direct products are either the dihedral group D_{2n} or then the direct product $Z_{2n} \times Z_2$. Dic_n has order $4n$ and can be presented as $\langle a, b | a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle$. In the quaternion case $n = 2$ one has the familiar generators $a = i, b = j, ab = k$. We believe that all Dic_n are non-natural but have not yet shown this.
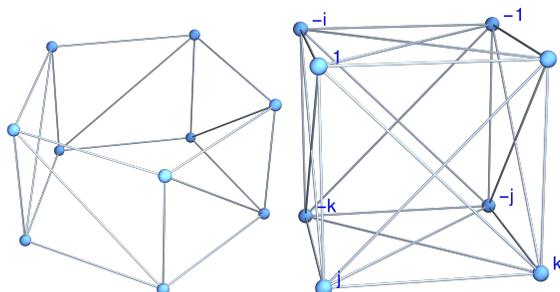

FIGURE 6. The graph to the left generates the natural group D_5. The quaternion group $Q_8 = \langle i, j, k | i^2 = j^2 = k^2 = ijk = -1 \rangle$ is not natural. We see to the right the Cayley graph generated by $i, j, k, -1$. For a metric space generating this group, we can chose the base length $d(0, i)$, the height $d(0, j)$, the side diagonal lengths $d(0, k)$ and the top bottom diagonal length $d(0, -1)$.

8.8. The **Burnside groups** $B(d, n) = F_d / \{x^n, x \in F_d\}$ generalize the cyclic groups $B(1, n) = Z_n = F_1 / \{x^n, x \in F_1 = \mathbb{Z}\}$ but they can be infinite as Golod and Shafarevich established in 1964. We do not know yet which Burnside groups are natural. The **general Burnside problem** asking to identify the Burnside groups $B(d, n)$ which are finite, is still open and appears to be a tough problem in combinatorial group theory. One already does not know whether $B(2, 5)$ is finite.

8.9. Related to the **Burnside problem** is the recursively defined **Grigorchuk group** G, a subgroup of the automorphism group of the rooted Bethe lattice T, an infinite tree with a degree 2 root and all other vertex degrees being 3. It was historically the first example of a finitely generated group with **intermediate growth rate** $|B_n(x)|$ of the size of balls of radius n in the **Cayley graph** of the group. It is between **polynomial growth**, happening for example for the free Abelian

OLIVER KNILL

group $\mathbb{Z}^n$ and **exponential growth**, happening for example for the free group $\mathbb{F}_n$. The tree naturally defines a hierarchy of trees $T = T_1 \cup T_2, T_1 = T_{11} \cup T_{12}$ etc. If $T = T_1 \cup T_2$ splits T at the root into two trees, then a flips $T_1 \leftrightarrow T_2$, b flips $T_{11} \leftrightarrow T_{12}$ flips $T_{211} \leftrightarrow T_{212}$ and acts in the same way on T_{222}; the generator c flips $T_{11} \leftrightarrow T_{12}$, leaves T_{21} and lets b act on T_{22}. The fourth generator d leaves T_1, flips T_{211} with T_{212} and lets b act on T_{222}. The four generators all are involutions satisfying $a^2 = b^2 = c^2 = d^2 = bcd = 1$. The Cayley graph for G equipped with different distances to the generators a, b, c, d then produces a natural group. [21] produced a continuum of such examples with a continuum of non-equivalent growth functions and all featuring isospectral Laplacians. The group G was originally defined as an interval exchange transformation on $[0, 1]$ which has the property that the action of the stabilizer of a dyadic subinterval $[k-1, k]/2^n$ is after a natural identification with $[0, 1]$ the same group action (see [8]). It is an example of a finite automatic group and a source for fractals. The pro-finite completion of G has fractal Hausdorff dimension $5/8$. Grigorchuk's original definition on $\{0, 1\}^{\mathbb{N}}$ was $(0x)^a = 1x, (1x)^a = 0x, (0x)^b = 0x^a, (1x)^b = 1x^c, (1x)^c = 0x^a, (1x)^c = 1x^d$ and $(1x)^d = 0x, (1x)^d = 1x^b$ is an interval exchange transformation exchanging dyadic intervals similar as the **von Neumann-Kakuktani system** (=adding machine) generated by the single transformation $(0x)^a = 1x, (1x)^a = 0x^a$.
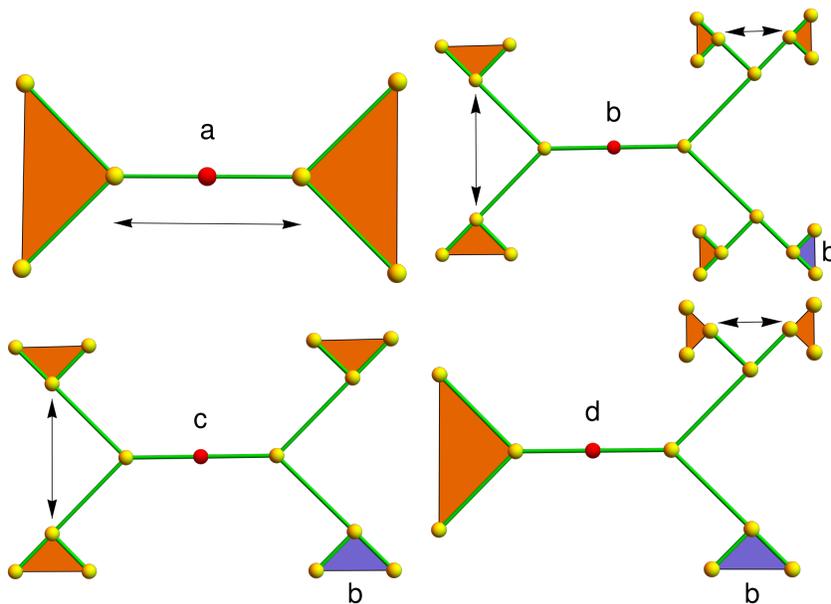

FIGURE 7. The four generators of the Grigorchuk group G acting on a rooted Bethe lattice.

8.10. Motivated by Grigorchuk is the **Gupta-Sidki group**. It is an infinite 3-group which produced a simple example for the **weak Burnside problem** and the first known 2-generator infinite 3-group. The construction obviously generalizes from $p = 3$ to any odd prime p. The case $p = 3$ is featured prominently in [9], where Baumslag mentions that he was told that the group is not finitely presentable but

that he would not know of a proof. [11] The Gupta-Sidki group acts on a rooted tree of base vertex degree 3 and otherwise having constant degree 4. Pick an arbitrary root. The first generator a rotates the three main branches T_1, T_2, T_3 of the tree. The second generator b rotates the first branch T_1 in one direction, the second branch T_2 in the other direction and plants its action recursively on the third branch T_3. One has $a^3 = b^3 = 1$ and all elements of G have order which is a power of 3. To see that the group is natural, pick G itself as the set of generators. Pick the from the $\{a, b, a^{-1}, b^{-1}\}$ generated geodesic metric $d(x,y) = d(0,s)$ where s is written as the shortest word in the letters a, b, a^{-1}, b^{-1}. Now perturb the metric so that all generators have rationally independent lengths. The geodesic distance in this weighted Cayley graph X produces now a metric space (X, d) which only features the group G. The reason is because for odd p, the groups Z_p^n are all natural.
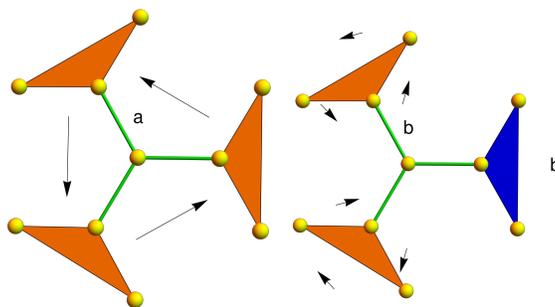

FIGURE 8. The two generators of the Gupta-Sidki group acting on the rooted tree of degree 4 and base degree 3.

8.11. Unlike the group $\mathbb{Z}_2$ of **dyadic integers**, the dihedral non-abelian upgrade version is natural. One can generate it as a subgroup of the automorphism group of the tree X as before. Define $U(x,y) = (y, U(x)), V(x,y) = (V(x), b)$, then define $A(x,y) = (y,x)$ and $B(x,y) = (V(y), U(x))$. Then $A^2 = 1$ and because $VU = UV = Id$ also $B^2 = 1$. (Just check $B^2(x,y) = B(V(y), U(x)) = (VU(x), UV(y)) = (x,y)$) We also have $AB(x,y) = (U(x), U(y))$ and $BA(x,y) = (V(x), V(y))$. So, $AB = T$ adds 1 on both branches while BA subtracts 1 on both branches. We have written the von Neumann Kakutani system as a product of two involutions.

```
(* nonabelian group on dihedral group of integers      *)
U[x_]        :=x;         V[x_] :=x;                      n=8;
U[{X_, Y_}]:={Y ,U[X]}; V[{X_, Y_}]:={V[Y] ,    X};
A[{X_, Y_}]:={Y , X};    B[{X_, Y_}]:={V[Y],U[X]};
P[x_]:=Partition[x, 2]; F[x_]:=ListPlot[Flatten[x]];
u=Last[NestList[P,Range[2^(n+1)],n]];
GraphicsGrid[{{F[A[u]],F[B[u]]},{F[A[B[u]]],F[B[A[u]]]}}]
```

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[11]Baumslag mentioned this as an open problem in his course from 1987 and as a student of that course I have at that time made attempts to prove this. Sidki [66] proves in 1984 that Grigorchuk gave in 1980 already an argument. [9] was published in the year 1993, where the presentation problem is still stated an open problem but this must have been the state in 1987. It appears as if Sidki's proof needed some time to be accepted.
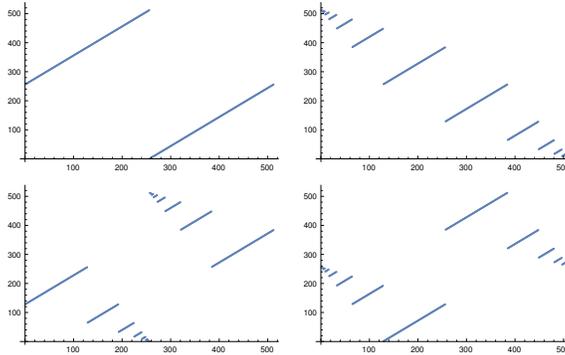
FIGURE 9. The permutations A, B and $T = AB, T^{-1} = BA$ leave the spheres $S_k(0)$ in the tree X in variant. The transformations A, B induce on each sphere $S_k(0)$ a dihedral group action. The pro-finite limit of dihedral groups D_{2^n} produces the dihedral dyadic group.

8.12. When trying to find a group which corresponds to Barycentric refinement in two dimensions, one can look at the following recursively defined group on a rooted tree X with constant degree 4 except at the origin where the degree is 3. We have $T(a, b, c) = (T(b), c, a), S(a, b, c) = (b, S(c), a)$ and $U(a, b, c) = (b, c, U(c))$. We can check that $T(S(U))) = Id$. But the group is not commutative, $ST \neq TS$. With $J(a, b, c) = (J(b), J(c), J(a))$ and its inverse $J^{-1} = K(a, b, c) = (K(c), K(a), K(b))$, we have $J^{-1}SJ = U$ and $J^{-1}TJ = S$. The group is generated by any two members of $\{T, S, J\}$ like $\langle S, J \rangle$. If we take the Cayley graph with these generators and take different distances $d(0, S(0))$ and $d(0, J(0))$, we have a metric space which only admits this group action.
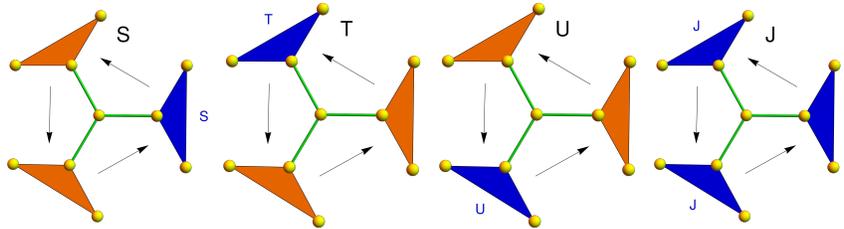

FIGURE 10. S, T, U generate the dihedral 3-adic integer group. Each rotates the three main branches and induces itself on one of the branches. G is also generated by S, J as $J, J^2 = J^{-1}$ conjugate the three generators. From $STU = Id$ one can see that any two of the four actions generate the full group.

8.13. Here is the Mathematica code which produces the graphs of T, S, U as well as of J, J^2, J^3.

```
n = 5; T[x_] := x; S[x_] := x; U[x_] := x; J[x_]:=x; K[x_]:=x;
T[{a_, b_, c_}] := {T[b], c, a};
```
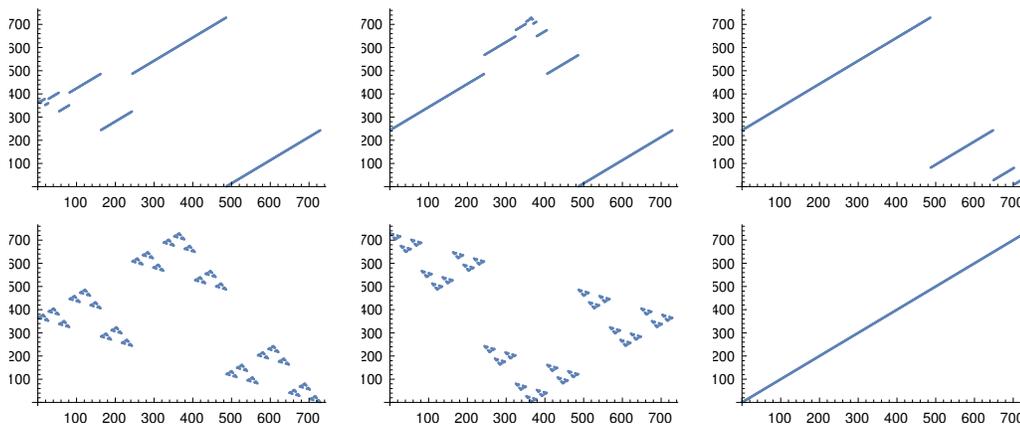
FIGURE 11. The graphs of S,T,U as well as J, J^2 and $S^3 = 1$. We have $\langle S^3 \rangle \sim \langle T^3 \rangle \sim \langle U^3 \rangle = \mathbb{Z}_3$ so that $\langle S, T, U \rangle$ is the non-abelian $\mathbb{Z}_3 \rtimes Z_3$. It shows that $(\mathbb{Z}_3, d)$ with dihedral metric is not natural.

```
S[{ a_ , b_ , c_ }]  :=  {b , S[c] , a};
U[{ a_ , b_ , c_ }]  :=  {b , c , U[a]};
J[{ a_ , b_ , c_ }]  :=  {J[b] ,J[c] ,J[a]};
K[{ a_ , b_ , c_ }]  :=  {K[c] ,K[a] ,K[b]};
P[ x_ ]  :=  Partition[x , 3];   F[ x_]:= ListPlot[Flatten[x]];
R[X_]  :=  Last[NestList[P , X , n]];   u = R[Range[3^(n + 1)]];
GraphicsGrid[{{F[T[u]] ,F[S[u]] ,F[U[u]]} ,
             {F[J[u]] ,  F[J[J[u]]] ,  F[J[J[J[u]]]]}}];
```

8.14. The non-Abelian **dihedral group** $D_n = D/(n\mathbb{Z})$ is natural. In order to force the group structure, we need a metric space with $(2n)$ elements which has a n-fold symmetry and a 2-fold symmetry but we have to avoid a $2n$-symmetry. In the later case, with a $2n$-fold symmetry, the product of the groups $\mathbb{Z}/(2n\mathbb{Z})$ and $D_n = D/(n\mathbb{Z})$ can be admitted as a group structure. A concrete metric space (X, d) which forces the dihedral groups is a prism graph. With the metric coming from the embedding of the graph, the symmetry $Z_n \times Z_2 = \mathbb{Z}/(n\mathbb{Z}) \times \mathbb{Z}/(2\mathbb{Z})$ is not allowed as a group structure. The group inversion is an isometry but the two neighbors a, b of 0 are not inverses of each other. There is a "twist" or asymmetry in the graph.

8.15. We can also take the cyclic graph $X = C_{2n}$, where is the distance d between the vertices $2k$ and $2k+1$ is 1 and the distance between $2k+1$ and $2k+2$ is 2. We can also take the **connection graph** of C_n. It is natural and admits only the D_n group structure. The group operations on the edges are reflections at the edges, the group operations on the vertices are translations.

8.16. Let K_p be the complete graph. We show that the connection graph K'_p is never natural. Its connection graph K'_p still has the symmetric group S_p as automorphism group. The connection graph K'_p has the same automorphism group but has $|V'| = 2^p - 1$ vertices. But there is no subgroup of this order of the
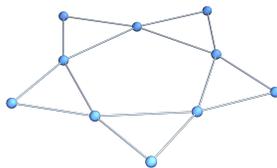
24

FIGURE 12. The connection graph of a cyclic graph, (here C_5') is always natural. There is only one group structure possible, the dihedral group.

automorphism group because $2^p - 1$ never factors $p!$. Most connection graphs are not natural: if we take a graph G with a prime p number of nodes which has a trivial automorphism group, then the connection graph G' has a trivial automorphism group too and G' is not natural. elements.

8.17. Unlike $(\mathbb{R}, |x - y|)$, the metric space $(\mathbb{Z}, |x - y|)$ is not natural. It admits both $(\mathbb{Z}, +)$ as well as the infinite dihedral group $(D, +)$ structure and these two groups are not isomorphic. The group $(D, +)$ is natural because it comes from a metric on the half integers which admits only one group structure. Again, as in the finite cyclic case, a concrete metric space (X, d) which forces the group structure D are the half integers $X = \mathbb{Z} + \mathbb{Z}/2$ with the geodesic distance generated by an asymmetric distance a between 0 and $1/2$ and b between 0 and $-1/2$.

8.18. A **product** $(X_1 \times X_2, d_1 \times d_2) = (X_1, d_1) \times (X_2, d_2)$ of two metric spaces can be defined by taking an l^∞ norm $d_r(x, y) = \max(d_1(x_1, y_1), rd_2(x_2, y_2))$ on the Cartesian product. In the case of graphs this product produces a product of (distance weighted) graphs.

8.19. The following result is the metric space analog to product result for finite natural groups

Proposition 12. *For $r \neq 1$, the direct product of two natural compact metric spaces (X_i, d_i) is a natural metric space.*

Proof. Look at the group of isometries on the product. It contains the G_i, the unique groups which are induced on X_i and so the product $G_i \times G_j$. If T is an isometry of the product space $X = X_1 \times X_2$, then T has to split as $T(x, y) = (T_1(x), T_2(y))$. To see this apply transformations in G_i such that $T(0, 0) = (0, 0)$. Now T preserves every sphere $d(0, x) = const$. This forces a group operation associated to T_1 to be a translation on each slice $G_1 \times \{y\}$ and a group operation associated to T_2 to be a translation in $\{x\} \times G_2$. This forces then $T(x, y) = (T_1(x), T_2(y)) = (x + g, y + h)$ with group elements $g \in G_1, h \in G_2$. $\square$

8.20. As a consequence, the tori $\mathbb{T}^n$ with the product metric are natural. We do not think that the compactness really matters in general.

8.21. The **infinite dihedral group** D_∞ is the finitely presented group $\langle a, b | a^2 = b^2 = 0 \rangle$. Addition in the group is given by concatenating words. The neutral element 0 is the empty word. We can write $1 = a, 2 = ab, 3 = aba, 4 = abab, ...$ and $-1 = b, -2 = ba, -3 = bab, -4 = baba$ showing that no artificial negative elements have to be introduced here. One could label elements also as translation

and reflection symmetries R_n, S_n with $n \in \mathbb{Z}$. The translations are one one branch, the reflections on the other.

8.22. In the context of calculus we like to think of R_n as the integers and of S_n as the half integers. The R_n are translations and the S_n are reflections at the edges of $\mathbb{Z}$. If D_∞ is seen as acting on itself as a set represented by the half integers $\mathbb{Z}/2$, then we can take the reflections $a(x) = -x + 1/2$ and $b(x) = -x - 1/2$ as generators. We have $ab(x) = a(-x - 1/2) = x + 1/2 + 1/2 = x + 1$ and $ba(x) = b(-x + 1/2) = x - 1/2 - 1/2 = x - 1$. Both a and b switching the pair of integer lines but ab translates us forward. Now, if we take a metric on the Cayley graph of $D_\infty = F_2/\langle a^2, b^2 \rangle = \langle a, b | a^2 = b^2 = \rangle$ such that $d(0, 1/2) \neq d(-1/2, 0)$ then this is a natural metric space.

8.23. Similarly as the **finite dihedral groups** $D_n = Z_n \rtimes Z_2$ are semi-direct products of cyclic groups Z_n with Z_2, the infinite dihedral group is the semi-direct product $\mathbb{Z} \rtimes Z_2$ of the integers with the 2-point group Z_2. The integers $\mathbb{Z}$ are then a normal subgroup and Z_2 is $D_\infty/\mathbb{Z}$. We like to think about a semi-direct product as a fiber bundle over the base group Z_2. The normal fibers are the integers $\mathbb{Z}$. The group operation is $(s, n) + (t, m) = (s + t, n + (-1)^s m)$ which means that on the first branch (the integers), we have the usual addition, while on the second branch (the half integers), we have a reverse group operation.

Proposition 13. *The infinite dihedral group is natural.*

Proof. Take the weighted Cayley graph (X, d) of the finitely presented group $\langle a, b | a^2 = b^2 = 1 \rangle$ such that the edges corresponding to a have different lengths than the edges corresponding to b. Declare $d(0, a) = \alpha > 0$ and $d(0, b) = \beta > 0$ with $\alpha \neq \beta$. There are no other neighbors. This defines a metric space which as a set is just the integers $\mathbb{Z}$ (or half-integers $\mathbb{Z}/2$ after a Hilbert hotel identification). We claim that this metric space admits only one group structure such that all translations and inversions are isometries. Pick an element in the metric space and call it 0. There is exactly one point in distance α. Call it a. Because also $d(0, a^{-1}) = \alpha$ we have $a^{-1} = a$ or $a^2 = 1$. There is also exactly one point in distance β. Call it b. The same argument shows $b^2 = 1$. Now, assume an other relation holds, then some shortest word $ababab \cdots a$ is equal to 1. Then this means that the metric space (X, d) has a closed path which is not contractible. But (X, d) is simply connected. So, the group $\langle a, b | a^2 = b^2 = \rangle$ □

8.24. Also the discrete complex plane $D_\infty \times D_\infty$ is natural if we take the product metric. The metric space comes from the Shannon product of weighted graphs. If we take a different scale in the two components, we have no additional symmetry. The symmetry group is then $D \times D$ and every strict subgroup is not transitive. The **discrete complex plane** $D_\infty \times D_\infty$ is natural if we take the product metric. The metric space comes from the Shannon product of weighted graphs. If we take a different scale in the two components, we have no additional symmetry. The symmetry group is then $D \times D$ and every strict subgroup is not transitive.

8.25. The **lamplighter group** is the finitely generated group $G = \langle l, r | l^2 = 1, lr^n l r^{-n} l r^n l r^{-n} = 1 \rangle$, where l stands for "light change" and r stands for "right translation". We will see below that it is not natural. A group element can be represented as a pair $(\omega, n$, where ω is a configuration in $\oplus_{k \in \mathbb{Z}} \{0, 1\}$ of configurations

with finitely many non-zero elements (lights are on) as well as a position position $n \in \mathbb{Z}$ (the lamplighter). The addition is $(n, \omega) + (m, \eta) = (n + m, \omega + \eta + e_n)$ where e_n is the basis vector with support at n. The set of finite configurations can also be written as the countable group $\Omega = \oplus_{k \in \mathbb{Z}} Z_2$ or the group of polynomials $\Omega = Z_2[t, t^{-1}]$. The generator l lights the lamp at 0. The generator r moves the lamplighter one step to the right. The move $r^n l r^{-n}$ lights or closes the light at position $n \in \mathbb{Z}$. The move $l r^n l r^{-n} l r^n l r^{-n}$ changes a light at 0, then at n, then at 0 again and then at n again. After such a double round tour, the lamplighter is at the same spot and all the lights in the same state than before the tour.

8.26. The lamplighter group can also be considered in finite set-ups, where $\mathbb{Z}$ is replaced by the cyclic group Z_n. The world on which the lamplighter lives is now cyclic. Now, for all even n different from 2, the group is natural. There is a HNN extensions of the Lamplighter group which appears as the fundamental group of a 7-manifold and which shows that bounded torsion is a necessary condition for the strong Atiyah conjecture on L^2 cohomology of Riemannian manifolds. The conjecture of Atiyah that closed manifolds of bounded torsion have rational L^2 Betti numbers is still open.
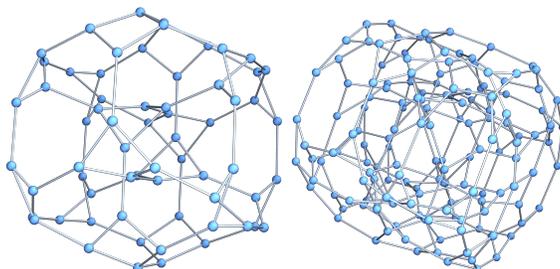

FIGURE 13. The lamplighter group over Z_4 is not natural. The lamplighter group over Z_5 is natural.

8.27. The lamplighter group is a non-Abelian group with countably many elements. How large is the ball $B_k(0)$ of radius k? After k steps, only k lamps can have been lighted and the lighter can only be in distance k away. This means that there are maximally k^2 group elements in $B_k(0)$. The group G has **polynomial growth**. The group G is finitely generated but not finitely presentable.

8.28. The lamplighter group can be written as the semi-direct product $\oplus_k Z_2 \rtimes \mathbb{Z}$ or the wreath product $Z_2 \wr_{\mathbb{Z}} \mathbb{Z}$. While the group $\oplus_k Z_2$ is natural, the group $\mathbb{Z}$ is not. Define the **dihedral lamplighter group** as $Z_2 \wr_{\mathbb{Z}} D_\infty$. It is obtained by replacing the walk $\mathbb{Z}$ of the lamplighter by the non-Abelian group $D_\infty = \langle a, b | a^2 = b^2 = 1 \rangle$. Figuratively speaking, the lamplighter's left and right step can now have different length. We could call it the **limping lamplighter group**. It is presented as

$$\langle l, a, b | l^2 = a^2 = b^= 1, lwlw'lwlw' = 1 \rangle \,,$$

where w runs over all words in a, b where two a or b's are not neighbors and where w' is the word read backwards.
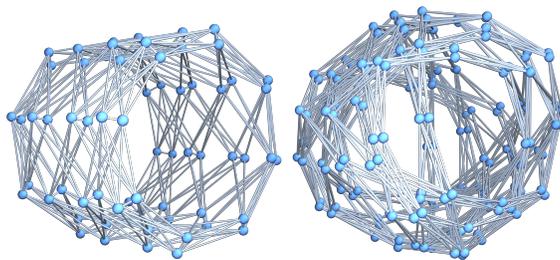
FIGURE 14. The Zig-Zag product of the Cayley graphs in the case of the lamplighter group over Z_4 and Z_5.

8.29. Similarly as we had to go from the Abelian group $\mathbb{Z}$ to the non-abelian group $\mathbb{D}$ to get a natural group, we have to go from the lamplighter group to the limping lamplighter group.

Proposition 14. *The lamplighter group is not natural. On the other hand, the limping lamplighter group is natural.*

8.30. More general lamplighter groups have been considered [3]. For any commutative ring R, one can look at $R[t, t^{-1}] \rtimes \mathbb{Z}$ for example. The action of the base $\mathbb{Z}$ onto the fibers has to be specified. One can also make the base higher dimensional, like the Baumslag-Remeslennikov example $\mathbb{Z}[x, x^{-1}(1+x)^{-1}] \rtimes \mathbb{Z}^2$, where $(t, s) \in \mathbb{Z}$ acts on the fibers by multiplying the Laurent polynomial with x or $(1+x)$, the first example of a finitely presented group with an abelian normal subgroup of infinite rank.

8.31. A **metabelian group** which by definition is a group for which there is an Abelian normal subgroup H such that G/H is Abelian. The lamplighter group, all dihedral groups, the infinite dihedral group, all finite groups of order smaller than 24 are metabelian. Also the limping lamplighter group is metabelian. The symmetric group S_4 of order 24 is the smallest non-metabelian finite group. A metabelian group G has $[G, G]$ as an Abelian normal subgroup and since Abelianization $G/[G, G]$ is naturally Abelian, the metabelian property is equivalent to having the commutator subgroup $[G, G]$ as a normal subgroup.

8.32. The **dyadic group of integers** $\mathbb{Z}_2$ is the dual group of the **Pruefer group** which is the group of **rational dyadic numbers** $e^{2\pi i k/2^n}$. The group $\mathbb{Z}_2$ is the unit ball in the field $\mathbb{Q}_2$ of 2-adic numbers with 2-adic norm. A dyadic integer is given by a sequence $x = a_1 a_2 a_3 \dots$ with $a_k \in \{0, 1\}$. The addition is point-wise with carry-over. If only finitely many a_k are non-zero, we can associate with x an integer $\sum_k a_k 2^k$. The group $\mathbb{Z}_2$ is not natural as it is the **pro-finite limit** of non-natural groups Z_{2^k}. It can be extended to a **dihedral dyadic group** which is the pro-finite limit of dyadic groups D_{2^n} and which contains the infinite dihedral group D_∞ as a dense subgroup.

8.33. In ergodic theory, one can represent a dyadic integer also as a real number $x = \sum_k a_k 2^{-k} \in [0, 1]$ and the presentation is unique for all but a countable set. The addition $x \to x + 1$ is the **von Neumann-Kakutani system**, a measure preserving transformation which has the Pruefer group as the spectrum. We will see that the dyadic group of integers is not natural. While the p-**adic group**

of numbers is natural for odd primes p, this does not apply for the subgroup of integers. This mirrors that $\mathbb{Z}$ is not natural, while $\mathbb{R}$ is natural. We have seen this already from the fact that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$ and as $\mathbb{Z}$ is not natural, also $\mathbb{Z}_p$ is not natural. Let us see this an other way

Proposition 15. *For any prime p, the group of p-adic integers $\mathbb{Z}_p$ is not natural.*

Proof. As a set, $X = \mathbb{Z}_p$ is a Cantor set, the boundary of a rooted Bethe lattice with one central vertex degree p and where all other vertex degrees are $p+1$. This tree is self-similar as every of its p branches is again the same tree. The boundary of this tree is the set X. Every element at the boundary can be encoded by a unique path $a_1 a_2 \cdots, a_k, \cdots$ with $a_j \in Z_p$. There is a smallest group translation in $\mathbb{Z}_p$. The group structure on the dense subset $\mathbb{Z} \subset \mathbb{Z}_p$ defines the group structure on $\mathbb{Z}_p$. We only have to say what the addition $x \to T(x) = x + 1$ looks like. As an action T on the tree it rotates the p base branches by 1 and additionally induces the same action T on the isomorphic sub-tree given by the first branch.
Assume (X, d) is a metric space on which the group $\mathbb{Z}_p$ acts by isometries as translation, we will build an other group structure which is non-Abelian and so not isomorphic but still preserves the metric space. In the case $p = 2$, let A be the transformation which flips the two main branches. Let B be the transformation which induces T on the first main branch and T^{-1} on the second main branch. This non-Abelian group preserves the original metric space and each group element can be matched thanks to the self-similarity with an element in X. It is a non-Abelian group operation on X for which all additions as well as inversions are isometries. The group $\mathbb{Z}_2$ is not natural. For $p > 2$, let A be the transformation which rotates the main branches and which induces the same transformation A on the first sub-branch. Let B be the transformation which rotations the main branches and induces B on each sub-branch. Now $B^p = 1$ and A^p induces addition by A on the first subgraph. The group generated by A, B is an alternative group structure on $\mathbb{Z}_p$. $\qquad\square$

8.34. Every Euclidean space $(G, d) = (\mathbb{R}^n, |x - y|)$ is natural. Also here, we know the isometry group as it is the **Euclidean group** $\mathbb{R}^n \rtimes O(n)$ but the later group can not be planted on $\mathbb{R}^n$ itself as the later has only one translation invariant group structure. Assume (G, d) admits an other group structure. That means every x defines both left and right translations T_x which preserve the distance. Distance preserving transformations on $\mathbb{R}^n$ are the Euclidean group $\mathbb{R}^n \rtimes O(n)$. So, G must be a normal subgroup of this Euclidean group and since $O(n)$ has no normal subgroups beside its center $\{-1, 1\}$ we must have $G = \mathbb{R}^n$.

8.35. Let us look at the compact **Lie group** G, where we have a left and right invariant metric g coming from the Killing form. The identity component of the group of isometries of G is $G \times G/Z(G)$, where $Z(G)$ is the **center** of G. The reason is that G acts by left and right multiplication and that the center of G is the set of actions where left and right multiplication is the same. Now, if G is Abelian, then $Z(G) = G$ and the component of the identity of the isometry group is equal to G. This is the reason why the tori $\mathbb{T}^n$ are natural.

Proposition 16. *Any Lie group which is the product of a compact Lie group with $\mathbb{R}^n$ is natural.*

Proof. The main point is that such a group G admits a left and right invariant (and so bi-invariant) Riemannian metric and also must be a manifold so that (G, d) must be a Riemannian manifold. There is a unique torsion-free connection on it which allows to define a Lie algebra structure. This in turn defines, via the exponential map the group structure in some neighborhood B of a point 0 declared to be the zero element. As any group element is a product of finitely many elements in B the group structure is determined everywhere. □

8.36. Let us look at the example of the compact Lie group $G = SU(2)$. It is natural coming from the natural metric space given by the unit sphere $S^3 = \{x \in \mathbb{R}^4, |x| = 1\}$ with the induced rotational invariant metric. The group of isometries is the Lie group $H = SU(2) \times SU(2)/Z_2$ as the center of the group is $Z_2 = \{1, -1\}$. As the only connected subgroup of H of dimension 3 is $SU(2)$, we see that $SU(2)$ is natural.

Proposition 17. *Among the spheres in Euclidean spaces, the 0-sphere Z_2, the 1-sphere T^1 and the 3-sphere $SU(2)$ are the only natural metric spaces.*

Proof. This is related to the classification of list of normed real division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ classified by Frobenius. As they are compact Lie groups the multiplication is unique. Any other Euclidean sphere does not admit a topological group structure [64]. □

8.37. One can ask which compact Riemannian manifolds are natural. We have seen that the only spheres which are natural are the zero, one and three dimensional spheres $Z_2, T^1, SU(2)$. It follows for example that the 2-sphere S^2 is not natural. A natural Riemannian manifold needs a large group of isometries. The group of isometries of an even-dimensional Riemannian manifold M of **positive curvature** is a finite dimensional Lie group and the fixed point set N is again a positive curvature manifold with each component having even codimension [14, 48]. Interesting is already the case when the group of isometries contains a circle. This restricts the structure of even dimensional positive curvature manifolds considerably. For example, if the fixed point set has co-dimension 2, then a theorem of Grove and Searle applies [27, 44].

8.38. The **symmetric groups** S_n are natural for $n > 3$. The group S_4 has 24 elements. We could start with the octahedron graph which has an automorphism group of 48 and somehow extend it to a metric space with 24 elements still keeping an automorphism group with 24 elements. We were not yet able to get a metric space which generates S_4. As S_1, S_2, S_3 are all natural, it is natural to suspect that also S_n is natural for $n > 3$.

8.39. An other class of graphs which look promising to decide are **dicylic groups** Dic_n which for $n = 2$ is the quaternion group $Dic_2 = Q_8$ and for $n = 1$ is the cyclic group $Dic_1 = C_4$ (which usually is not considered dicylic as it is already cyclic). As both Dic_1 and Dic_2 are non-natural, it is natural to suspect that Dic_n are non-natural also for $n > 2$. More generally, one can look at **generalized dicyclic groups** $Dic(A, y)$ which is defined by an abelian group A and an element $y \in A$. Now create an additional "imaginary" element x satisfying $x^2 = y$ and look at the group generated by A and x. The complex numbers are an example of a generalized dicyclic group because $Dic(\mathbb{R}, -1) = \mathbb{C}$.

30

8.40. The alternating group A_4 is the only non-simple alternating group. It is also known as the **tetrahedral group**. It is a semi-direct product of the **Klein group** $Z_2 \times Z_2$ and the cyclic group Z_3. A natural metric space can be given by the Shannon product of Cayley graphs. One can also see directly that it is the only group structure which can be planted on the truncated tetrahedron. Since the truncated tetrahedron graph $G = (V, E)$ has $|V| = 12$ vertices and an automorphism group $Aut(V, E)$ of order 12, the natural nature is established. It follwos that the graph complement G^c of the truncated tetrahedron is natural. (By the way, G has Euler characteristic -2 and constant curvature $-1/6$ everywhere, G^c has Euler characteristic 6 and constant curvature $1/2$ everywhere.)
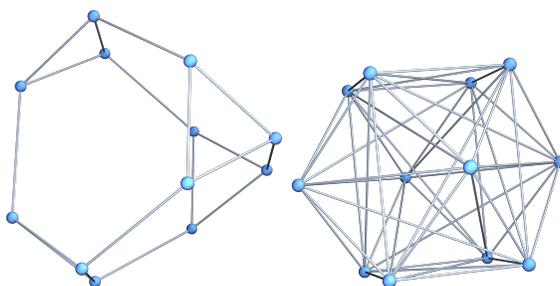

FIGURE 15. The truncated tetrahedron graph is a natural graph. It admits only one group structure: the alternating group A_4. It therefore is natural. To the right we see the graph complement.

8.41. The **alternating group** A_5 is the smallest non-solvable group and the smallest simple non-Abelian simple group. As one could suspect that we need a nontrivial normal subgroup in order that the group is natural, it is an interesting example. Still, the group is natural. It comes from the truncated icosahedron graph, a graph with 60 vertices which is an Archimedean solid.
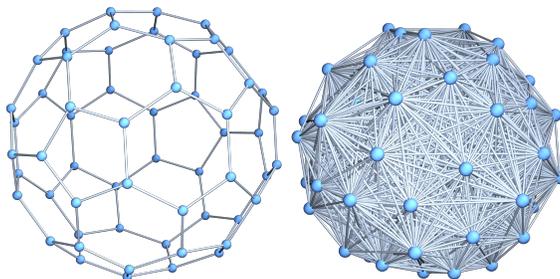

FIGURE 16. The truncated icosahedron graph is natural. It produces the natural group A_5, the smallest non-abelian simple group. To the right, we see the graph complement of the truncated icosahedron.

9. EXTENSIONS

9.1. Given a group G and a normal subgroup N, one of the questions in the **Jordan-Hölder program** aiming to understand the structure of groups, is to determine whether G **splits**, that is whether one can write G as a semi-direct product $G = N \rtimes H$, where $H = G/N$ is the factor group. The examples $Z_4, \mathbb{Z}, Q_8$ show that this is not possible in general, even so there are normal subgroups in each case. Remarkably, these groups are also non-natural.

9.2. Given a group $(G, +, 0)$ on can look at the new set $G \times Z_2$ on which there is a **dihedral doubling**: let $a : x \to x'$ denote the identification from one branch to the other satisfying $a^2 = 1$. Also assume $aga = g^{-1}$. For every group element g define the new group element $b_g = ag$. It is an involution. The new dihedral upgrade $\phi(G, d, +, 0)$ of the group is the group generated by these involutions. It is an example of a **semi-direct product**. Now, if $d(0, a) \neq d(0, b_x)$ for all x, then the metric alone determines the action of a. For now, we need the technical assumption that there is a distance in G not assumed: $\bigcup_{x \in G} d(0, x) \neq [0, \infty)$. This is the case if G is discrete or compact.

Proposition 18. *If $(G, d, +, 0)$ is a natural discrete or compact group, then $\phi(G, d, +, 0)$ is a natural group.*

Proof. We have to show that with a suitable metric on (G, d), there is no other group structure possible on $\phi(G, d)$ than the dihedral doubling construction. By assumption we can find a distance k which does not appear in G. Define $d(0, 0') = k$. By group invariance, we have $d(x, x') = k$. Given an element $x \in G$ we can find its dihedral double x' because the sphere of radius k consists of only one element. For the subgroup (G, d), we have no choice. On the conjugate part (G', d'), we have an isomorphic structure G'. There is an image $0'$ of 0 and the only possible identifications are $x \to x'$ or $x \to -x'$. With the former identification, there are two group structures which work, with the later, we have to define a metric on the double cover which selects the twisted one. If we chose a metric satisfying $d(0, ag) \neq d(0, ga)$, this excludes the product group structure $G \times Z_2$. $\square$

9.3. This is related to the **generalized dicylic group construction**. If y is an element of order 2 in an Abelian group, define an other element with $x^2 = y$ and postulate $x^{-1}ax = a^{-1}$.

9.4. There are many questions: what are the fixed points of the doubling operation ϕ on the category of locally compact metric groups? In the compact case, we also have a renormalization of dynamical systems: given a natural metric space (X, d), there is a unique probability measure m, the Haar measure on the compact topological group $(X, d, +, 0)$. The corresponding σ-algebra $\mathcal{A}$ produces now a probability space $(X, \mathcal{A}, m)$ which is a Lebesgue space. We can therefore arrange that $X = [0, 1]$ and that G is a group of measure preserving transformations on X. Under the doubling map, the operations are rewritten on $X_1 = [0, 1/2], X_2 = [1/2, 1]$ and a symmetry $A : X_1 \to X_2, A : X_2 \to X_1$ added. As the set of measure preserving transformations of $[0, 1]$ has a complete metric $d(T, S) = m(\{x \in [0, 1], T(x) \neq S(x)\}$ we can look at limit points or fixed points. In particular, we can find, whether any fixed point of ϕ is natural.

9.5. As a comparison to the dicyclic extension, the 2:1 integral extension for measure theoretical dynamical system has a unique fixed point, the **von Neumann-Kakutani system** which is conjugated to a group translation on the group $\mathbb{Z}_2$ of dyadic integers. This operation also works on the class of compact metric spaces: define the distance between two points x, y' as $1 + d(x', y')/2$ and normalize so that the maximal distance is 1.

9.6. Let us write $Z_n = \mathbb{Z}/(n\mathbb{Z})$ for the **cyclic group** with n elements and $\mathbb{Z}_p$ for the **group of p-adic integers**. The p-adic integers are a **profinite group** obtained as an **inverse limit** $\varprojlim Z_{p^n}$ of finite cyclic groups. In general, we have

Proposition 19. *If G is the profinite group obtained as an inverse limit of non-natural Abelian groups G_n where each group admits also a non-abelian group structure, then the also the profinite limit G is non-natural.*

Proof. Each of the groups G_n comes from a metric space (G_n, d_n) which admits both an abelian group G_n as well as a non-ablian group H_n. Now the profinite limits produce two group structures on the limit, one Abelian and one non-abelian. $\square$

9.7. As an application, one can see that the 2-adic group of integers are non-natural because the groups Z_{2^n} are not natural for $n > 1$: a metric space producing the group must have cyclic symmetry and so also have dihedral symmetry D_n. Let us write D_2 for the projective limit of the dihedral groups D_n. It is the dihedral extension of the dyadic group $\mathbb{Z}_2$. More generally, we can look at the **dihedral p-adic group** of integers

$$D_p = \varprojlim D_{p^n} \ .$$

One has more however. The self-similar nature does not allow natural groups to survive the pro-finite limit. We have seen earlier that the p-adic group of integers $\mathbb{Z}_p$ are all non-natural.

9.8. Because for odd primes p, the cylic groups Z_{p^n} are natural for $n \geq 1$ unlike Z_{2^n} which is not natural for $n > 1$. As the group of p-adic integers $\mathbb{Z}_p$ is not-natural we see that the profinite limit of finite natural groups is not necessarily natural.

Proposition 20. *The profinite limit of natural groups is not necessarily natural.*

9.9. The group $\mathbb{Z}$ is the **free group** with one generator F_1. We can also see it generated by $\{a, a^{-1}\}$. With both a and its inverse as generators, the Cayley graph s the undirected graph $(V = \mathbb{Z}, E = \{(n, n+1), n \in \mathbb{Z}\})$. We can also see it as the **Bethe lattice** $B_2 = \text{Bethe}_2$ for which the vertex degree is constant 2. What about non-abelian free groups or higher vertex degree Bethe lattices. They turn out to be non-natural. But we can modify the metric on a Bethe lattice so that it becomes a natural metric space hosting a natural group, the **free dihedral group** FD_n generalizing infinite dihedral group $FD_2 = D_\infty$. They are also known as the simplest **Coxeter groups**, groups which are generated by reflections and relations given by pairs of generators. The groups FD_n are Coxeter groups where the Coxeter matrix is the identity matrix. Important examples of Coxeter groups are Weyl groups in the theory of Lie algebras.

9.10. Lets look at a few small n. The Bethe lattice B_2 hosts $\mathbb{Z}$ and $FD_2 = \mathbb{D}_\infty$. The Bethe lattice B_3 hosting the two groups $FD_3 = \langle a, b, c | a^2 = b^2 = c^2 = 1 \rangle$ and $\langle a, a^{-1}, b | b^2 = 1 \rangle$ and therefore is not natural. However, with a metric in which we have three different lengths for each of the generators, the group FD_3 becomes natural. The Bethe lattice B_4 hosts $FD_4 = \langle a, b, c, d | a^2 = b^2 = c^2 = d^2 = 1 \rangle$, the **mixed free dihedral group** $\langle a, a^{-1}, b, c | b^2 = c^2 = 1 \rangle$ as well as the **dihedral free group** $F_2 = \langle a, a^{-1}, b, b^{-1} | \rangle$. The later is natural.

9.11. Define the **free dihedral group** FD_n as

$$F_n = \langle a_1, a_2, \cdots, a_n | a_1^2 = a_2^2 = ... a_n^2 = 1 \rangle .$$

It generalizes $\mathbb{Z}_2 = FD_1$ and the dihedral group FD_2.

Proposition 21. *For any degree $n \geq 2$ Bethe lattice B_n with geodesic distance is not natural. We can change metric however so that it becomes a natural metric space. The emerging group is the free dihedral group FD_n which is natural.*

10. Barycentric renormalization

10.1. Given a finite graph G_0 we can look at the sequence of **Barycentric refinements** G_n. The Graph G_n is obtained from G_{n-1} by taking the complete subgraphs as vertices and connecting two if one is contained in the other. The eigenvalues $L(G_n)$ of the Kirchhoff Laplacian scaled to the interval $[0, 1]$ converge to a universal limit function F which only depends on the maximal dimension d of G_0 [38, 39]. In the case $d = 1$, where $F(x) = 4 \sin(\pi x/2)^2$ satisfies $T(F(x)) = F(2x)$ with the quadratic map $T(z) = z(4 - z)$, the **integrated density of states** $F^{-1}(x) = (2/\pi) \arcsin(\sqrt{x}/2)$ and density of states $f(x) = \pi^{-1}/\sqrt{x(4 - x)}$ on $[0, 4]$ which is the **arcsin-distribution**. In the case $d = 2$, the function $F(x)$ appears to be non-smooth, even have a fractal nature. We expect that there is an almost periodic operator on a compact topological group which produces this spectrum. We still have not yet identified this group.

10.2. Let us revisit the Barycentric renormalization story in the one-dimensional case but focus on its relation to **group theory**. Let X be the **rooted Bethe lattice**. It is an **infinite tree** with constant vertex degree 3 except at the root $0 \in V(X)$, where the vertex degree is 2. Let T be the recursively defined **graph automorphism** of the tree X which flips the two main branches and on one of the two branches induces the same transformation T. The transformation preserves the **spheres** $S_k(0)$ of radius k centered at the origin 0 of X. It induces there a cyclic permutation of the 2^k vertices in $S_k(0)$. The boundary of the tree X is the **dyadic group of integers** and T extends to this **compactification** where it induces the translation $x \to x + 1$, the **adding machine**. This dynamical system is measure theoretically conjugated to an interval map with a countable set of intervals of $[0, 1]$ which is also called the **von Neumann-Kakutani system**. One can abstractly get this from the ergodic system by computing the group of eigenvalues $\{e^{2\pi k/2^n}\}$ which is called the **Prüfer group** $\hat{\mathbb{Z}}_2$, which is the group of dyadic rationals modulo 1 and the **dual group** of the compact topological group $\mathbb{Z}_2$ of dyadic integers. In general, any ergodic automorphism of a probability space is conjugated to a group translation on a compact topological group (e.g. [17]).

10.3. From a group theoretical point of view, the permutation induced on the spheres $S_k(0)$ are cyclic permutations. The corresponding **Cayley graphs** are also known as **Schreier coset graphs** (Nebengruppenbilder) of the stabilizer groups G_k which fix the trees on level k. They are cyclic graphs of length 2^k. One can visualize the permutation T_k on these 2^k points by plotting the graph of p mapping $S_k(0) = \{1, 2, 3, \cdots, 2^k\}$ to $T_k(p) \in S_k(0)$. For every k, we have a discretization of the **interval map** of von-Neumann-Kakutani. In the limit we get the interval map on $[0, 1]$. The group generated by T restricted to the boundary $\mathbb{Z}_2$ of X is a **pro-finite limit** of finite cyclic groups. In the case $d = 2$, we expect a similar thing to happen. The problem is however that the graphs belonging to Barycentric refined graphs are in dimension larger than 1 never Cayley graphs, nor Schreier graphs because the both Cayley and Schreier graphs are vertex transitive and so have constant vertex degree.

```
(* The adding machine on the dihedral group of integers  *)
k=8; T[x_]:=x;T[{X_,Y_}]:={Y, T[X]};  P[x_]:=  Partition[x,2];
p=Last[NestList[P,Range[2^(k+1)],k]]; ListPlot[Flatten[T[p]]]
```
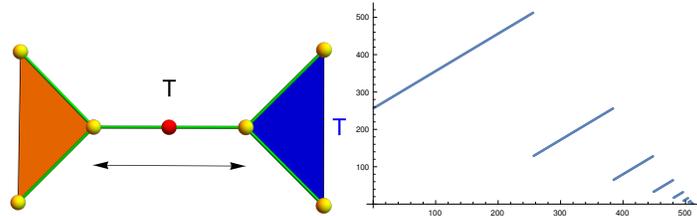

FIGURE 17. The inductively defined transformation T on the rooted binary tree X induces cyclic permutations T_k on the spheres $S_k(x)$. In the limit $k \to \infty$ it produces the von Neumann-Kakutani system T which is conjugated to a group translation on the dyadic group of integers $\mathbb{Z}_2$.
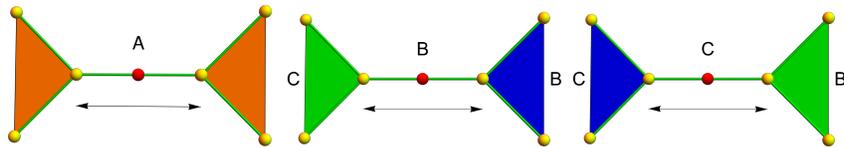
FIGURE 18. The transformations a, b, c which generate a non-abelian group structure on the metric space $\mathbb{Z}_2$. We have $a^2 = 1$ and $c = b^{-1}$. Now, b^2 generates the von-Neumann Kakutani adding machine $x \to x + 1$ on the right branch and the inverse on the left branch.

11. REMARKS

11.1. The first investigation on natural spaces was triggered by an observation we made in 2008 that the group structure on the integers $(\mathbb{Z}, +)$ is not determined

by the metric. This clashed with the case $(\mathbb{R}, +)$, where the metric determines the group. A second group structure on the integers was then seen as the dihedral infinite group $(\mathbb{D}, +)$. This group then turns out to be natural and fixed by a metric space. The question of naturalness appeared to have affinities with the question which non-simple groups do not split. The topic was for us also an opportunity to learn a bit more about groups and visit old friends like the Rubik or Gupta-Sidki groups. It appears that the group theory of finitely generated but not necessarily finitely presented groups could also help us in the future to answer a question about the Barycentric limit of simplicial complexes [38, 38] which in the one-dimensional case is an **Abelian story**, with an underlying group given by the dyadic group of integers. In the general case, the limiting dynamical system is almost certainly a compact non-commutative self-similar group.

11.2. Even earlier, we got interested in the subject of group actions in the context of **dynamical systems theory**, the area of mathematics we have worked initially. Dynamical systems theory is naturally a **theory of time** as a group G acting on a space X defines a **time evolution** on X. In the Abelian case $G = \mathbb{R}$, one deals with evolution equations like ordinary or partial differential equations (ODE's or PDE's). In the case $\mathbb{Z}$ one deals with invertible discrete dynamical systems. One can now ask what happens if G acts on itself as automorphism. Are there more cases where only one group structure appears? It came to a surprise to us that the integers $\mathbb{Z}$ are not natural, as any metric admitting one group structure must admit two non-isomorphic group structures. But then we realized that the **dihedral integers** $D = \langle a, b, a^2 = b^2 = 1 \rangle$ are natural. This group is very close to the integers and one can see it as a pair of two integer lines or **half integers**. The group D_{infty} has a translation subgroup generated by ab, a product of two non-commuting involutions. As an index 2 subgroup, $\mathbb{Z}$ is automatically normal in D_∞. We can see D as a vector bundle over $\mathbb{Z}_2$. This generalizes to higher dimensions like $DF_2 = \langle a, b, c, d | a^2 = b^2 = c^2 = d^2 = 1 \rangle$ which is a dihedral version of the free group with two generators. The Grigorchuk group is a 2-group which is a factor of this group DF_2.

11.3. Changing from $\mathbb{Z}$ to D_∞ originates from reflection geometry. One can see **reflections** as **generalized points** similarly as **ideals** are **generalized numbers**. This also works in Euclidean spaces. Rather than talking about points x in $\mathbb{R}^n$, we can talk about the reflections a_x at these points. This defines a non-Abelian group satisfying $a_x a_y = -a_y a_x$. Similarly as D is a **double cover** of $\mathbb{Z}$, the **Euclidean reflection group** covers the translation group. Also similarly as the **Hilbert hotel map** defines a bijective correspondence between $\mathbb{Z}$ and D, the reflections a_x correspond in a 2-1 way to points x but the group structure is different. The subgroup generated by all pairs $a_x a_y$ is the usual translation group, an Abelian group. Remarkably, the reflection group is a **non-Abelian additive group** structure. Reflection groups are important in mathematics in general. Coxeter groups and in particular Weyl groups are generated by reflections. The theory of **Coxeter groups** started with Coxeter in 1934. See [26].

11.4. Let (V, E) be a finite simple graph. Take two involutions $a, b : V \to E \to V$ with the property that $a(x) \in x$ or $x \in a(x)$. As in [41], the map $F = ab : V \to V$ defines now a discrete **vector field** as it maps a vertex x to a neighboring vertex $F(x) = ab(x)$ passing through the edge $b(x)$ connecting them. This is an example on

OLIVER KNILL

how we have replaced the $\mathbb{Z}$ action $x \to F(x)$ with an action of D_∞ as time. The set-up can be generalized with an arbitrary set of reflections $a_i : V \to E \to V$. A word in this finite alphabet produces a path in the graph. Replacing the commutative time $\mathbb{Z}$ to the non-commutative time D_∞ was no problem.

11.5. One can see the space D_∞ also as the set of **half-integers** $-1 = ba, -1/2 = b, 0, 1/2 = a, 1 = ab, 3/2 = aba...$ a group where **word concatenation** is the addition. This non-Abelian group is more natural written as a multiplication group. If $d(0,a) \neq d(0,b)$ the weighted graph is natural allowing the dihedral structure. While $x \to x^{-1}$ is an isometry in the group, there is an **arrow of time**, $x \to -x$ as seen on the number line when identifying edge points as half integers, is no more an isometry.

11.6. The **arrow of time** is a one-dimensional feature. It disappears in higher dimensions as we can now have different reflections a_i. A combination of three reflections produces in a flat space the zero translation. Doing this is now possible without violating a Pauli principle. This generalizes on the dihedral free group DF_n where we can not return to the origin. On the factor group $DZ^n = DF_n/\{a_i a_j = a_j a_i\}$ however for $n > 1$ we can turn around without violating that Pauli principle: $abababcbababac$ is the 0 element.

11.7. Writing motion as a product of reflections can generalized to Riemannian manifold with non-positive curvature. As then, the reflection T at a point a is well defined thanks to the non-existence of caustics (Hadamard). To define $T(x)$ build the geodesic to a then continue that geodesic through a until the same distance is done again. The combination of two such reflections is now a translation on the manifold. If the curvature is non-flat the successive reflections $T(a_1)T(a_2)T(a_3)T(a_4)$ is a translation which is non-zero in general. In the flat case, the translation is zero. If all possible reflections $a_j : V \to E$, $x \in a_j(x)$ appear, and the a_j are chosen randomly still satisfying the Pauli exclusion preventing $a_j^2(x) = x$, we have a path $a_1 a_2 a_3...$ on the graph without backtracking.

11.8. In the compact case, where the **Haar measure** is a unique group bi-invariant measure a natural metric space (X,d) alone defines a unique **measure theoretical dynamical system** $(X, \mathcal{A}, G, m)$ in which the group G emerges as the "time evolution" on itself. It so provides a natural **probability space**. The metric space especially defines a unique representation of the unique G in the unitary group of $L^2(X, \mathcal{A}, m)$. The interplay between topological dynamics and measure theoretical dynamics is rich [20]. It is especially interesting if a topological dynamical system has a unique invariant measure, is uniquely ergodic. One can then use both the measure theoretical world as well as the topological world. In the first case one can be interested in things which are true almost everywhere (like convergence of Birkhoff sums) or then in things which are true Baire generically. Also recurrence questions can be asked both in the topological as well as in the ergodic setting [24].

11.9. One can now attach properties to points of the metric space. For example, define the **ergodic set** in a natural compact metric space (X,d) as the set of points x for which the corresponding transformation $T_x(y) = y + x$ defines an ergodic dynamical system $(X, \mathcal{A}, T_x, m)$. For the circle $X = \mathbb{T}_1$ for example with that natural metric, the ergodic set is set of points x for which $d(x,0)/diam(X)$ is

37

irrational. We can define spectral types to every point x depending on the spectral type of the unitary $f \to f(T_x)$ on $L^2(X, \mathcal{A}, m)$.

11.10. The question of whether a given structure determines an other categorical structure can be asked for any pair of categories. The set-up can not be reversed in our case: we can not switch topology and algebra and ask whether the group determines a metric. The reason is that there are in general many metrics on a group which render all the group operations are isometries; the **trivial metric** $d(x, y) = \delta_{x-y}$ is always an example. It is always a metric that is invariant on a given group.

11.11. Finally, we should mention that the non-Abelian dihedral structure of the integers is already present in the very first mathematical artifacts we know. The marks of a **tally stick** can be identified with a. To make the marks visible, they have to be separated. The gaps can be identified with b. Our 10 thousand year old ancestors have not seen the need for negative numbers, but a gap is needed if we want to determine the sign. A number like 1 given by ab gives the direction. The symbol $baba$ with the gap to the left would mean that the mark is red backwards and means -2. Reading a number backwards gives negative numbers $bababa$ for example is -3. In the case of the tally sticks, the gaps are of course only implicit, but in some sense, the quartz stone planted at the end tells in which direction to read the bone.

11.12. The **dihedral number line** D_∞ is in in many ways much more natural than the integers $\mathbb{Z}$. We have seen a mathematical explanation by identifying the former as a natural group and the later as a non-natural group. But there are other ways why the structure is natural: we do not have rather awkwardly introduce a group completion $\mathbb{N} \to \mathbb{Z}$ to get the negative numbers, we have them already given as words, without having to introduce a negative sign. We have $ab = 1$ and $ba = -1$ for example. We could replace the letters a, b with other symbols. In computer science, it is the binary notation with 0 and 1 which is the basic way to write but there, no Pauli principle disallowing aa or bb is in place so that much more information can be filled. But such a system would have been impractical for the cave mathematicians writing on bones as it is would be hard to count the number of gaps. Our binary encoding is a place value system which historically was first introduced by Sumerians who used a hexagesimal system when writing Clay tablets. The dihedral number line given as alternating sequences of two symbols faded and only reappeared when geometers started to look at symmetry groups.

11.13. It was Felix Klein who shifted the interest from the geometric metric spaces to its group of symmetries. Euclidean symmetries are generated by reflections leading to "reflection geometry" [4]. One can identify for example the set of translations in Euclidean space with the space itself. This group of translations has a cover in the form of the group of all point reflections. It is a non-Abelian group which should be thought of as a dihedral double cover of Euclidean space.

11.14. We can look at a dihedral versions of the Cayley graph of $\mathbb{Z}^n$ for example, with generating set $S = S^{-1} = \{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\}$ with a generating set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ and adding the relations $a_i^2 = b_i^2 = 1, a_i a_j = a_j a_i, a_i b_j = b_j a_i$ for all $0 \le i \ne j \le n$. If we postulate that an eligible path in this lattice is one where no Pauli principle violation occurs (no unnecessary $a_i a_i$ terms for example),

OLIVER KNILL

FIGURE 19. The earliest known mathematical documents are tally
sticks like the Ishango bone. If we think of a scratch as an a
and as a gap as a symbol b, then we have a dihedral description.
Reading things backwards gives negative numbers $ababab = 3$ and
$bababa = -3$. Of course the dihedral non-abelian group structure
was not on the mind of the earliest mathematicians but marks on
bones are a way of counting which in some sense already involves
two type of symbols, the mark and the gap between two marks.

then in one dimensions we are forced to move in one direction, while in for $n \geq 2$
we can pretty much move around freely in the space.

11.15. Our first experimentations in this area relating metric spaces and groups
origins from 2008. It had at that time partly been motivated by **inquiry based
learning pedagogy**. Already years before 2008, after seeing experts teaching with
such methods and attending conferences on the subject of IBL, I felt at that time
that inquiry based learning can only work well if **new and unstudied topics** are
covered. In a well connected world with search engines and social network discussion
groups like stack exchange, maintaining an inquiry based climate has become very
difficult if not impossible The pioneers of inquiry based learning worked initially
with new set theoretical topology, which at that time was not saturated and also
had a gentle entry point. Today, for many classical topics, one can find answers to
known questions by just asking the question verbatim on the internet. There are
data bases of graphs, groups, topological spaces, books on counter examples etc etc.
14 years have passed since 2008 and the power of search has increased enormously
since. We are at a point, where a standard math question can be entered as such
into the search box and with a high probability one gets to a reasonable answer.
Rather than thinking about a proof for hours, days or weeks, a proof emerges in a
fraction of a second. So, we need a topic which is completely unexplored and the
theme studied here would have been a good example. Of course, once this file is
on ArXiv, a search for a question might bring it up. We would need an other new
topic.

11.16. Arithmetic usually starts with the **integers**, a commutative additive group.
From it, commutative or non-commutative ring or more sophisticated structures
are built. Could there be something more fundamental than the integers? We
argue here that the answer could be yes. In order to do so, we have to specify
what "fundamental" is. We propose to look at arithmetic structures which emerge
uniquely from a metric space, a set equipped with a distance function. It turns
out that the non-commutative dihedral group of integers is more fundamental than
the integers. Unlike the integers, the group structure of the dihedral numbers is
determined from a metric space. The same also holds for the finite dihedral groups.

While the cyclic groups C_{2n} are not natural in the above sense, the dihedral groups are.

11.17. Leopold Kronecker was probably the first **ultra-finitist**. His famous statement that the *"dear lord has created the integers while every thing else is made by humans"*, indicates this. Challenging this statement, one can ask now whether there is a more fundamental additive arithmetic than the usual "cave" or "pebble" arithmetic $(\mathbb{Z}, +)$ we know from elementary school. Of course, this question depends on what one considers "natural" or "fundamental". But the dihedral structure (as we tried to explain above) is actually very natural. It introduces counting in a minimalistic way without number system, without place-value just using two symbols and defining $1 = ab, 2 = abab, 3 = ababab, \ldots$ having negative numbers without group completion built in too as $-1 = ba, -2 = baba, -3 = bababa, \ldots$. The numbers are doubled with half integers $1/2 = a, 3/2 = aba, etc$. One of the surprises was that the group structure is forced from the metric, if one takes a metric space in the form of a weighted graph such that $d(0, a) \neq d(0, b)$.

11.18. An other motivation comes if one looks at the overall structure of mathematics. One can muse about **which categories are the champions in a major field**. If one looks at algebra, the concept of a **"group"** appears to be the most intriguing. In a topological setting, the concept of **"metric space"** is most fundamental. When looking at order structures the **"posets"** are among the top. Partially ordered sets immediately define "graphs", where the nodes are the sets and where two are connected if one is contained in the other. Alternatively, one can look at the graph in which the connections are taken if two sets intersect. This is interesting from the spectral point of view for example [43]. What happens if one asks whether one champion defines an other? For Lie groups for example, one can look at the problem of finding a metric structure in the form of a bi-invariant metric.

11.19. Double cover structures are often observed in mathematics. An example is given by **projective spaces** which are double covered by **spheres**, **spin groups** are double covers of **special orthogonal groups**. **Pin groups** are double covers of **orthogonal groups**. It can make sense therefore to generalize "points" like the integers $\ldots, -1, -1, 0, 1, 2, \ldots$ to objects generated by "involutions" a, b leading to $\ldots baba, ba, 0, ab, abab \ldots$. There is an analogy in number theory, where one transitioned from "points" to "prime ideals". Involutions are very natural objects. When going from $\mathbb{Z}$ to $\mathbb{D}$ one moves from points (numbers) to elements of the symmetry group (reflections). Natural means that the elements of the symmetry group of a space can be identified with the space itself.

11.20. An early draft of this document from 2008 gained my attention more recently when looking at complex analysis in a **discrete setting**. Natural arithmetic comes up when looking at **ultra-finite single variable calculus models** in a situation, where **time** is one-dimensional. For ultra-finite calculus on $\mathbb{Z}$ with derivative $df(x)/dx = f(x + 1) - f(x)$ and polynomials $[x]^n = x(x - 1) \cdots (x - n + 1)$, one has $d[x]^n/dx = n[x]^{(n-1)}$. If we want to make this algebra work well also on the negative axes, we need $df(x)/dx = f(x + 1/2) - f(x - 1/2)$ and use $[x]^n = (x - n/2)(x - n/2 + 1) \ldots (x + n/2 - 1)(x + n/2)$. Again, one has $d[x]^n/dx = n[x]^{(n-1)}$. The use of half integers reminds of dihedral structures. A doubled lattice appears

naturally when constructing the dihedral integers D. It appears also as a Barycentric refinement.

11.21. There is an other link. In geometry, other dihedral type structure appear: if we have a geometry and an exterior derivative $a = d^*$ for example we can also look at the dual $b = d$, then $ab = d^*d$ and $ba = d^*d$. While in geometry these are Laplacians, in the case of dihedral numbers, both are considered a translation. In the algebra of **creation and annihilation operators** in quantum field theories we have $aa^* = -a^*a$ and $aa^* - a^*a = 2$. The effect of the combination aa^* is to increase the number of particles by 1 (mathematically the eigenvalue increases). This is exactly what the word ab does in the dihedral group. $x \to x + ab$ means adding 1. The words of even length form a **normal subgroup** of the dihedral group of integers isomorphic to $\mathbb{Z}$. Indeed, $\mathbb{D} = \mathbb{Z} \rangle \mathbb{Z}_2$ is a bundle over the two point space, where each fiber is the integer line.

12. Dihedral time

12.1. This section is a bit different. It touches on a topic which is very popular in the physics literature because it has also philosophical connections. It is the question **"What is time?"**. For a mathematician, the answer is simple: time is a group or semi-group acting on a space. The puzzling riddle is when we look at the role of time how it is observed in nature: while we can move freely in space, we can not move freely in time. There is something fundamentally different about time which geometric notions of **space-time manifolds** do not address. Postulating a **semi-group** like $\mathbb{N}$ rather than $\mathbb{Z}$ does not solve this problem as it just puts the non-invertibility as an axiom. This is Prigogines approach but it feels like a cheap stunt similarly as "deus ex machina", "antropic principle" or "multiverse landscape" ideas [57].

12.2. A major enigma in physics is the **arrow of time**. We observe the inability to influence the development of time, despite the fact that all fundamental physical laws are invariant under group inversion possibly with modification like in the CPT symmetry case where a time change comes with charge and parity change. The phenomenon is often formulated within a thermodynamic or cosmological frame works. But it is something, we experience in small laboratories and in daily life. Already on a small scale experiments, we have no "force field" which allows us to change the direction in time. There are external force fields which allow us to change the direction of a moving particle, we can use physical obstacles to prevent going an object to a specific place but we can not build obstacles which slow down, stop or reverse time. We can slow down time by moving around or to placing the experiment in a gravitational field but that is just a consequence of relativity.

12.3. Explanations which have been proposed are entropy [22] or postulating irreversibility [56]. Both are problematic if we accept that fundamental interactions can be reversed. There is no evidence that any of the known fundamental forces lead to dynamics that is impossible to reverse. There is a CPT symmetry observed for example. **Time machines** are possible in principle in the frame work of general relativity [69] if one postulates the existence of exotic matter producing a mass tensor for which the metric of the Einstein equations allows travel backwards in time. Similarly, exotic small matter (like in an ant-men (Marvel comics)

universe) would allow to build a **Maxwell demon** violating the second law of thermodynamics. A more practical implementation of the Maxwell Demon idea is the Feynman-Smuluchowski ratchet first proposed by Marian Smoluchowski. As Feynman illustrated in his lectures, the idea does not work because the ratchet itself is part of the system. As a mathematician however, we can think about a model in which shrinking of objects like in the Marvel universe is possible. While ludicrous from a physics standpoint, it is completely consistent with most of the physics we know. An other argument against entropy is the Poincaré recurrence. We just have to wait long enough to have a system come back close to the situation it has started with. Already in 1896, Ernst Zermelo argued against such a mechanical system but the argument is just that the time span is unreasonably long. This is not a mathematical argument, it is an engineering argument. If we have to wait trillions of year until something comes back, it is an argument which fails for practical reasons and not fundamentally. Non-recurrence can occur in infinite dimensional Hamiltonian systems where no invariant probability measure might exist. An example is Vlasov dynamics as formulated in a Hamiltonian frame work as a dynamical system on diffeomorphisms [35].

12.4. While postulates like the irreversibility postulate of Prigogine are bold, they resemble the Greek atomic hypothesis. Also there, the suggestion had been based on experiments and experience (we can not crush matter arbitrarily small) with limited effort) [12] but these limitations were removed more and more until reaching the length scale of the constituents of the nucleus. We have perfect reversibility for the fundamental equations which describe the motion of particles but have to deal with friction phenomena breaking this. Democritus was not able to crush very small quartz sand particles and postulated that there are smallest units. While we can not read the mind of Democritus any more, it is likely that his postulate was motivated by the empirical fact like that sufficiently refined powder became impossible to be refined any further. There was no evidence at the time of Democritus which supported an atomic hypothesis. Experimental evidence came only much, later with experiments like Millikan's establishing that some particles like the electron have quantized fundamental charges.

12.5. Could a dihedral time-line explain the special nature of time: once we have a direction of time and the time evolution is obtained by stacking reflective operations on top of each other, then we can only move in the direction in which we are already going. In some sense, the sequence of letters a, b is like the evolution of a wave, the a playing the role of the position and b of momentum. In Hamiltonian systems one has both position and momentum. A wave does not change direction because it has also momentum. If we accept that fundamentally (similarly as the Pauli principle), that no pairs aa or bb are allowed to occur, then we can only move in one direction. The finitely presented Coxeter group $\mathbb{D} = \langle a, b | a^2 = b^2 = 1 \rangle$ is in some sense more fundamental in comparison to the free group $\mathbb{Z} = \langle a | \rangle$. Maybe this is too philosophical but in some sense the group $\mathbb{D}$ is more natural than $\mathbb{Z}$ as the generator structure is simpler, consisting of involutions.

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[12]A gypsum particle has the size of maybe 25 -35 micro meters which is 0.025 mm. This is far from the size of an atom which is of average size 0.1 nano meters which is about 10^{-7} mm.

12.6. Much as been written about the nature of space of time. Already when looking at the book literature we see an obsession with the topic: [22, 30, 59, 50, 7, 52, 56, 54, 58, 1, 13, 12, 11, 19, 67, 68, 25, 18, 60, 6]. Much of these writings are also philosophical, cosmological or interpretative. But they show what a vast pool of ideas have been proposed. Some examples:

Zeno	Motion needs to be explained properly
Laplace	The future is determined from the present
Newton	Time is an absolute and given by a real time axes
Eddington	Entropy drives time, makes processes irreversible
Einstein	Time is part of a space-time geometry
Prigogine	Determinism is flawed on a fundamental level
Mueller	Expansion of the universe creates more time
Smolin	Time is an illusion. Physics evolves with time
Rovelli	Time emerges in a statistical context
Barbour	Time evolves towards rich complexity

12.7. As pointed out in [6], the problem why we can not reverse time is not compatible with the fundamental laws of physics for the simple reason that the later laws are all reversible. Laws in statistical mechanics are not really fundamental as they emerge as an idealized limit of a **microcanonical description** that is a reversible n-body problem. All fundamental n-body problems are deterministic. Indeterministic phenomena like collisions in the Newtonian n-body problem or simultaneous multi-particle collision singularities for billiards are just limitations of the model. Gas particles are not billiard models and celestial bodies have a radius. As for the evolution of space-time singularities we have even no model which tells for example if a black hole evaporates to the point that it loses the property of being a singularity. This is a question of t'Hooft. It illustrates very well, what difficulties we have when combining the quantum and the gravitational world.

12.8. The problem of the arrow of time is neither a cosmological nor a statistical mechanics problem because we can look at the problem also in a very small laboratory and study it involving only very few particles. There is no fundamental force which allows us to change the direction of time. While we can slow down time by placing an apparatus into a gravitational field or observe the experiment in motion, we can not reverse it without some exotic matter. Not that imagination lacks: there are many stories and movies dealing with time machines and time travel.

12.9. The puzzle is why it we have no ability to speed up or slow down time willfully for a part of physical space, while keeping time the same in the rest. This is especially enigmatic if one looks at mathematical models of space-time which treat space and time on the same footing as geometry. We can for example use electric or magnetic field to reverse the velocity of a moving particle but we can not use force fields to reverse the velocity of the time evolution for the particle. The inability to reverse time is something we can experience in small scale laboratories already. The loss of information due to motion happens in any direction, whether we move forward or backwards.

12.10. Eddington explained the arrow of time with "entropy". One must note however that a mathematical definition of entropy of the "universe" is beyond mathematical rigor. In a cosmological setting it is questionable as the universe

is not a system in equilibrium. Different parts of space which have no causal connection. Mathematically, entropy has been defined by Bolzmann or Shannon for finite probability spaces like finite σ-algebras of a probability space. The notion can also make sense for smooth enough probability distributions as **differential entropy** $-\int f(x) \log(f(x))\ dx$ in probability theory, but entropy is already not defined if f is not smooth enough. Entropy can become negative for continuous distributions. Most notions of entropy assume the entropy of some sort of finite σ-algebra $\mathcal{A} = \{A_1, \ldots, A_n\}$ with $P[A_i] = p_i$ or quantum mechanically as von Neumann entropy $-\operatorname{tr}(\rho \log(\rho))$, where ρ is a finite dimensional **density matrix**. The mathematics has been pushed to dynamical systems [61] using dynamically defined entropy in an ergodic setting (metric entropy of Sinai) or topological setting (topological entropy).

12.11. It is an empirical fact that any fundamental dynamical system we know is reversible. Any dynamical system defined by fundamental process therefore **preserves entropy**. All fundamental particle or wave motions are **Hamiltonian systems** and so reversible, whether they are classical, relativistic or quantum. If one has an invariant probability measure, one has Poincaré-recurrence. A good model problem is the **Vlasov gas** in a finite container. This is an infinite dimensional Hamiltonian system of the form $f''(x) = \int \nabla V(f(x) - f(y))\ dy$. As a Hamiltonian system, it is obviously a reversible system and generalizes the n-**body problem** with potential V. The n-body problem is the case when the measure $\mu = dy$ is replaced with a measure supported by finitely many points. What happens is that if we put such a gas into a container like a box, then initially this macroscopic system will move (the gas shakes the container leading to a non-trivial coupled system of a infinite and finite dimensional Hamiltonian system). The system however will settle down asymptotically (at least this is what we expect. No mathematical proof has been done even so there is no doubt about this, especially if the billiard dynamics in the container is ergodic). The reason is that time evolution $f(t)$ in a space of smooth functions moves towards parts of space where f becomes more and more complicated. This happens even in integrable situations like a piston in a cylindrical container on which we have a Vlasov gas on both sides. We observe an arrow of time, even-so the system is Hamiltonian. For more details see [34] or the Chapter on Vlasov in [35]. For a Vlasov gas, we can look at the particle density which is $\rho(x) = \int_Y f(x, y)\ dy$. This is a conditional expectation and of course, the entropy of $\rho_t(x)$ in time will increase to the maximum where the density is constant. The reason for the increase of entropy is that we take a conditional expectation. The particle **position density** $\rho(x)$ contains much less information than the particle **phase space density**. Looking at ρ is looking at the infamous shadows of Plato's cave but the allegory is different as we **know** what the phase space system is. It is perfectly nice infinite dimensional Hamiltonian system.

12.12. Despite all the statistical mechanical or cosmological diversions, the conundrum remains why we we can freely move around in space, but not in time. The statistical mechanical point of view is obviously based on the fact that smooth quantities evolving under simple evolutions become more and more complicated. Mathematically this can be expressed that their derivatives grow over time. This leads to **loss of information** as we have to go to finer and finer σ-algebras to keep the initially known information. This mechanism is often dubbed "entropy

increases". But fundamentally, it is just our inability to keep track. This losing track happens very fast if the system shows **sensitive dependence on initial conditions**. Iterating a map $T(x, y) = (2x - y + 2\sin(x), x)$ mod 1 a 100 times forward and then 100 times backwards does not bring us to the same point if the calculation is done in real arithmetic. Time is obviously built in a completely different way than space even so general relativity treats space time as a **space-time manifold**. Let us mention, besides **Vlasov dynamics**, an other Hamiltonian system which features some sort of arrow in time. It deals with a Hamiltonian system which features an expansion of space.

12.13. A spontaneous expansion of a geometry like a Riemannian manifold or a graph can be explained by moving in the symmetry group of the geometry: if we allow the Dirac operator $D = d + d^*$ of a Riemannian manifold or simplicial complex deform isospectrally, this leads to an expansion if one postulates distances is coming from the Dirac operator $d + d^*$. The deformed operator $D(t) = d + d^* + b$ will develop a diagonal part but still feature exterior derivatives $d(t)$ which keeps the same cohomology. As d defines distance, the deformation affects distance. The origin of the inflation (the Janus point in the terminology of Barbour) is derived mathematically as the time in which the Dirac operator D has no diagonal part. It does not matter in what direction the system moves. It always expands when moving away from the Janus point. If we pick the time 0 when the Dirac operator has no diagonal parts, we can evolve in any direction and experience expansions in any direction [37]. Why does the Dirac operator move at all? Because not moving in the isospectral set has probability zero (similarly that there is zero probability that a stone moving in space does not rotate or that any system with continuum symmetry is in a certain particular state. Still, also this Dirac expansion model postulates that time is the real axes. Time could be multi-dimensional like in a Kaluza-Klein situations. Also, having time as a discrete non-commutative structure similar as in **Connes-Lott model** of the standard model is not excluded. Going to a dihedral time is a non-commutative notion of time which actually is very close to a discrete time $\mathbb{Z}$.

12.14. In physics, one has seen again and again that the role of **symmetries** is extremely important. By **Noether's theorem**, symmetries are related to **invariants**. Translation symmetry is related to momentum conservation, time symmetry related to energy conservation, rotational symmetry to angular momentum conservation etc. Also, symmetries seem to dictate what processes are possible and what processes are not. We also have to accept the empirical fact that all **fundamental processes** are reversible. Postulates like Prigogine's that fundamentally things are not reversible is not supported yet by any fundamental model: as pointed out by Barbour for example, all fundamental models in classical or relativistic mechanics, quantum mechanics, general relativity, electrodynamics, quantum field theories and especially the standard model allow for a time reversal. In fundamental interaction, one knows also the **TCP symmetry** but no fundamental process is known which does not work backwards in time (possibly changing charge and parity). What happens with the **dihedral time hypothesis** is that we have a time structure which is postulated to be the infinite dihedral group. The justification so far to look at this postulate is purely mathematical and not based on physics. The dihedral group is a natural metric space in which group translations and reflections are isometries

45

but where time reversal is not an isometry if the distance to the origin to a and the distance of the origin to b is not the same.

12.15. The **dihedral time hypothesis** is the suggestion that time reversal is not possible, because doing so would **break of symmetry**. The use of a **non-Abelian time in the discrete**, changes from an unnatural space $\mathbb{Z}$ to a natural space D_∞. There are two steps which are related: The postulate of a **quantized time** replacing the Lie group $\mathbb{R}$ with the discrete group $\mathbb{Z}$. But this loses naturalness. In order for it to become natural, it requires to use $\mathbb{D}$ and having **time to become non-commutative**. This breaks the time reversal symmetry leading to an **arrow of time**. In the group $\mathbb{D} = \{a^2 = b^2 = 1\}$, going from x to x^{-1} does not change anything on a microscopic level as $a = a^{-1}$ and $b = b^{-1}$. But if the group comes from a natural metric (forcing the algebraic structure uniquely from the metric) then the time reversal symmetry $x \to -x$ is an isometry when restricting to strings of even length (the **integers**) but it is not an isometry when restricting to strings of odd length (the **half integers**): the distances $d(0, 1/2) = d(0, a) \neq d(0, b) = d(0, -1/2)$ have different length. Similarly, $d(0, 3/2) = d(0, aba) \neq d(0, bab) = d(0, -3/2)$.

12.16. Questions.

12.17. Having seen examples like C_4, $\mathbb{Z}$ or Q_8 both as non-natural groups and also see these groups appear as examples of **non-simple, non-split groups** begs for the question whether there are more relations. The class of non-simple, non-split groups and the class of non-natural groups are not the same. There are non-split, non-simple groups like Z_9 which are natural. There are also products like $Z_4 \times Z_4$ which are non-natural but which split by definition. Still, there seems to be an unusual amount of overlap.

Question A: Do all non-natural groups H have a normal subgroup N such that either N or H/N are non-natural?

12.18. We have seen that Lie groups with a bi-invariant metric are natural. One can ask which Lie groups are natural, which finitely presented groups are natural, which Burnside groups are natural, whether natural groups can be characterized by other means whether the group problem is solvable for natural groups, or how the fraction of natural finite groups of order $\leq n$ compares with the number of all groups of order $\leq n$. While we know for compact Lie groups or $\mathbb{R}^n$ that we have natural groups, there are still many non-compact groups for which things are undecided.

Question B: Is there a non-natural Lie group?

12.19. A Cayley graph is undirected if one takes for every generator also its inverse. Which **symmetric Cayley graphs** are natural? We can say that if a finite subgroup of a permutation group is generated by cyclic groups of prime order, then it is natural. The Cayley graph of a group admits then a simply transitive action of G by graph automorphisms and as it can not have any other symmetry, it must be natural.

Question C: Can one characterize finite non-natural Cayley graphs? How can one characterize non-natural finite groups.

12.20. Having seen that all symmetric groups S_n or alternating groups are natural and that the semi-direct group $N \rtimes H$ of two natural groups N, K natural establishes that the **Rubik cubes** is natural. The most standard $3 \times 3 \times 3$ Rubik cube G is

$$G = N \rtimes H$$

, where $N = Z_3^7 \times Z_2^{11}$ is the normal subgroup (the fiber) and where $H = S_8$ is the base. The $2 \times 2 \times 2$ Rubik cube is $N \rtimes H$, where $N = \mathbb{Z}_3^7 \times \mathbb{Z}_2^{11}$ and $H = (A_8 \times A_{12}) \rtimes \mathbb{Z}_2$.
[13]

12.21. The following question deals with general groups, not necessarily with finite groups.

> **Question D:** Is it true in general that the semi-direct product of two general natural groups A, B is natural? (Not only the finite group case?)

12.22. In the finite case, where A, B are both finite and where A comes from the weighted Cayley graph $C(A, \alpha)$ and B comes from the weighted Cayley graph $C(B, \beta)$ then $A \rtimes B$ comes from weighted **zig-zag product** $C(A \rtimes B, \gamma) = C(A, \alpha) \circ C(B, \beta)$. (see Theorem 2.3 in [53]). For Lie groups already, we have the Euclidean group $\mathbb{R}^n \rtimes O(n)$ which does not admit a bi-invariant Riemannian metric. Still, it could in principle exist an other metric which is not a Riemannian metric that is invariant. One would have to establish that every left and right invariant metric comes from a Riemannian metric. This is not clear. There could be Finsler metrics for example.

12.23. If the answer to question D is affirmation, then the **Euclidean group** $\mathbb{R}^n \rtimes O(n)$ would be natural. Similarly, if the **generalized Lorentz groups** $O(1, n)$ would be natural. If the semi-direct product always preserves natural groups, then then the **generalized Poincaré group** would be natural. A special case of question D is the question whether a semi-direct product of two natural Lie groups A, B is natural.

13. About this document

13.1. In June of 2008 we wondered for which topological groups, the topology determines the arithmetic. It was a bit of a surprise to see hat the group $\mathbb{Z}$ of rational integers does not qualify in this respect as it also carries also a non-abelian $\mathbb{D}$ group structure. However, the non-commutative dihedral group $\mathbb{D}$ is natural in this respect. There is a metric on $\mathbb{D}$ as a set such that the group structure $\mathbb{D}$ is determined. The general question then appeared which groups are natural in this respect. Finite groups of prime order are natural simply because there is only one group structure. Already the cyclic groups $\mathbb{Z}_n$ are not natural in general for $n \geq 4$ while all non-commutative dihedral groups $\mathbb{D}_n$ are natural. The real line $\mathbb{R}$ is natural. There is only one group operation on the usual Euclidean metric $d(x, y) = |x - y|$ such that all group translations and taking inverses are isometries.

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[13]A $2 \times 2 \times 2$ version of the Pocket cube was suggested by Larry Nichols in 1970 while he was a graduate student at Harvard. It was based on magnets and did not feature colors. Nichols then founded the Moleculon Research Corporation. He lives since 1959 in Arlington, MA.

13.2. The additive group $(\mathbb{Z}, +)$ is the foundation for virtually all the arithmetic we know. Since the beginning of mathematics in the Paleolithic era, illustrated by artefacts like the Ishango bone found near the Semliki River, the addition of numbers and especially the commutativity of the basic additive arithmetic has been developed all over the world. The pair of arithmetic operations, addition and multiplication, defines a ring, bound together by distributivity. The additive operation on integers as the most fundamental operation has never been questioned and Kronecker was the first who formulated ultra finite ideas. The quotation of Kronecker has been popularized in [10] and the book title of the collection [29].

13.3. Could it be that some non-commutative additive structure is more fundamental than the known additive structure on integers? I could argue here that the answer is "yes", despite the fact that historically, serious steps into non-commutative mathematics [15] have emerged much, much later than the counting on integers. Non-commutative multiplicative structures most likely first appeared as **symmetry groups of geometries** and especially in permutation groups. The symmetry group of a polygon is already non-commutative.

13.4. But much earlier, already Euclid must have known (at least intuitively as Euclid had no group theory at hand) that the affine group of the plane which contains rotations and translations, is not commutative. It is generated by all reflections on lines as the composition of two reflections at parallel lines produce a translation perpendicular to the lines and the composition of two reflections at intersecting lines produces a rotation around the intersection point. In both cases, the order in that Coxeter group picture matters. There have been attempts to use reflection geometry early in geometry classes [4, 33] [14]

13.5. When **matrix algebra** and later also **representation theory** emerged, one has linked groups with operator algebras. Non-commutativity in the multiplication turned out to be fundamental. In the very small, the multiplication of observables in quantum theories is no more commutative. This is illustrated by the commutation relations in quantum mechanics which is now part of pop culture and motivated fundamental developments like non-commutative differential geometry [15] by pushing ideas from non-commutative measure theory and commutative topology to differential geometry. Non-commutative measure theory is an extended theory of von-Neumann algebras and non-commutative topology is an extended theory of C^* algebras. Non-commutative differential geometry uses **spectral triples** (D, A, H), in the form of a Hilbert space H, an algebra of operators and a selfadjoint operator D satisfying for which $[a, D]$ has finite norm for all $a \in A$. The operator D allows a polar decomposition $D = |D|U$ which then defines a metric space (X, d) where X is the set of pure states on the norm closure of A.

13.6. The question of a more fundamental arithmetic than the "cave" arithmetic $(\mathbb{Z}, +)$ of course depends on what is considered "natural" or "fundamental". We do not mean to weaken or change the axioms but insist on keeping groups and metric spaces. Using only a **monoid structures** for example and to look for the natural numbers $(\mathbb{N}, +)$, would not be an option for us. We want to have a criterion which makes a **group** natural and try to achieve this by linking the group structure

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[14]Max Jeger was my first geometry teacher in college who would heavily use "Spiegelungs geometrie".

with a **different category of objects** outside of arithmetic. This can be an **order structure**, a **topological structure**, a **measure theoretical structure**, a **simplicial complex structure**, a **differential structure** of a **symplectic structure**. In **Lie group theory** for example, one asks the group structure to be compatible with a given **differential structure** on the manifold: the addition and inversions need to be a smooth operation.

13.7. What we start to explore here could be done between any two algebraic and topological categories. It turns out however that things often do not lock together nicely. Given a too large algebraic structure like the category of **groupoids**, it does not force a link in the sense that the algebraic structure determine the topological structure or that the topological structure decides the algebraic structure. Also taking something more general than a metric space like **topological spaces** is harder: the reason is that the **symmetry group of a topological space**, the **group of homeomorphisms** of a topological space is in general much larger than the set itself. The group of homeomorphisms on a circle $\mathbb{T}^1$ for example is infinite-dimensional, while the circle itself is finite-dimensional. But the group of transformations which are compatible with the metric, the **isometries** of the metric space have a chance to be a finite-dimensional space. The group of isometries on any **compact Riemannian manifolds** for example is known to be a Lie group (possibly discrete or even trivial) and so finite-dimensional.

13.8. This file, like many other projects has not seen the light for more than a decade. In the summer of 2021, after a minor health scare, I became aware that I have reached an age, where for pure statistical reasons the chances of termination increase and I started to work already then on this file after realizing the odds. [15] It would be a nightmare to observe from a possible after-life (probably working as a junior data research assistant to the Almighty who tries desperately to decrease the odds that the "humans on earth experiment" would end up in failure), that rather interesting files like this one would have been overwritten by a teenager playing GTA 5 on a recycled SSD drive of my workstation, that has ended up in a garage sale. I post this on the ArXiv. There are no plans to send this document for publication. It definitely would need a complete rewrite, starting entirely from scratch. Any attempts in the last months to shorten the paper made things only longer. Any comments or feedback are welcome, especially since I have not seen the theme appear anywhere else. It would be especially good to hear about literature I might have missed.

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[15]In the US, the death rate of men per year in my age group is about $p = 1/100$ which is 10 times higher than for a 20 year old. (https://www.ssa.gov)

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